Brown representability and flat covers  

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Abstract. We exhibit a surprising connection between the following two concepts: Brown representability which arises in stable homotopy theory, and flat covers which arise in module theory. It is shown that Brown representability holds for a compactly generated triangulated category if and only if for every additive functor from the category of compact objects into the category of abelian groups a flat cover can be constructed in a canonical way. The proof also shows that Brown representability for objects and morphisms is a consequence of Brown representability for objects and isomorphisms.

In this note we exhibit a surprising connection between the following two concepts:

- Brown representability which arises in stable homotopy theory, and
- flat covers which arise in module theory.

Let $\mathcal{T}$ be a compactly generated triangulated category, for example the stable homotopy category of CW-spectra, and denote by $\mathcal{F}$ the full subcategory of compact objects (an object $X$ in $\mathcal{T}$ is compact if the representable functor $\text{Hom}(X, -)$ preserves arbitrary coproducts).

We prove that Brown representability holds for $\mathcal{T}$ if and only if for every additive functor $\mathcal{F}^{\text{op}} \to \text{Ab}$ a flat cover can be constructed in a canonical way. The proof also shows that Brown representability for objects and morphisms is a consequence of Brown representability for objects and isomorphisms.

Let us recall the relevant definitions. For every object $X$ in $\mathcal{T}$ consider the functor

$$H_X = \text{Hom}(-, X) \mid_{\mathcal{F}} : \mathcal{F}^{\text{op}} \to \text{Ab}.$$ 

This gives rise to a functor $\mathcal{T} \to (\mathcal{F}^{\text{op}}, \text{Ab})$ into the category of additive functors $\mathcal{F}^{\text{op}} \to \text{Ab}$. Every functor of the form $H_X$ is exact, that is, it sends triangles in $\mathcal{F}$ to exact sequences in $\text{Ab}$. In some cases also the converse is true. More precisely, one says that Brown representability holds for $\mathcal{T}$, if

- every exact functor $\mathcal{F}^{\text{op}} \to \text{Ab}$ is isomorphic to $H_X$ for some object $X$ in $\mathcal{T}$, and
- every natural transformation $H_X \to H_Y$ is of the form $H_f$ for some map $f : X \to Y$.

A classical theorem due to Brown and Adams states that Brown representability holds for the stable homotopy category [1]. This has been generalized in recent work by Neeman, which also contains an example of a compactly generated triangulated category where Brown representability fails [8]. For the relevance of Brown representability in the representation theory of finite groups we refer to recent work of Benson and Gnacadja [3].

Now consider the functor category $(\mathcal{F}^{\text{op}}, \text{Ab})$. Recall that there exists a tensor product

$$(\mathcal{F}^{\text{op}}, \text{Ab}) \times (\mathcal{F}, \text{Ab}) \to \text{Ab}, \quad (F, G) \to F \otimes_\mathcal{F} G$$

where for any functor $F : \mathcal{F}^{\text{op}} \to \text{Ab}$, the tensor functor $F \otimes_\mathcal{F} -$ is determined by the fact that it preserves colimits and $F \otimes_\mathcal{F} \text{Hom}(X, -) \cong F(X)$ for all $X$ in $\mathcal{F}$. A functor $F : \mathcal{F}^{\text{op}} \to \text{Ab}$ is flat if the tensor functor $F \otimes_\mathcal{F} -$ is exact. Our interest in flat functors is motivated by the elementary fact that a functor is flat if and only if it is exact [7]. A map $p : E \to F$ between functors is called a flat cover of $F$, if

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$E$ is flat and every map $E' \to F$ such that $E'$ is flat factors through $p$;

- every endomorphism $e: E \to E$ satisfying $pe = p$ is an isomorphism.

This is a useful concept because flat covers usually exist whereas projective covers rarely exist. In fact, Enochs conjectured in [5] that every module over an associative ring has a flat cover. Note that a module category is a special case of a functor category. In any case, Enochs’ conjecture seems to be hard; it is still open [9]. Let us construct for every functor $F: \mathcal{F}^{\text{op}} \to \text{Ab}$ an object $X \in \mathcal{T}$ and a map $p_F: H_X \to F$ which is a good candidate for being a flat cover of $F$. Choose a minimal injective copresentation of the form

$$0 \to F \to H_I \xrightarrow{Hf} H_J$$

in $(\mathcal{F}^{\text{op}}, \text{Ab})$. This is possible by Lemma 1 below. The map $f$ can be completed to a triangle

$$X \xrightarrow{e} I \xrightarrow{f} J \xrightarrow{g} \Sigma X,$$

and the map $H_e: H_X \to H_I$ induces a map $p_F: H_X \to F$. Note that $p_F$ is uniquely determined by $F$, up to isomorphism. We are now in a position to state the main result of this note.

**Theorem.** Let $\mathcal{T}$ be a compactly generated triangulated category and denote by $\mathcal{F}$ the full subcategory of compact objects in $\mathcal{T}$. Then the following conditions are equivalent:

1. Brown representability holds for $\mathcal{T}$.
2. Every exact functor $\mathcal{F}^{\text{op}} \to \text{Ab}$ is isomorphic to $H_X$ for some object $X$ in $\mathcal{T}$, and every isomorphism $H_X \to H_Y$ is of the form $H_f$ for some map $f: X \to Y$ in $\mathcal{T}$.
3. A functor $F: \mathcal{F}^{\text{op}} \to \text{Ab}$ is flat if and only if $\text{Ext}^2(-, F) = 0$.
4. For every functor $F: \mathcal{F}^{\text{op}} \to \text{Ab}$ the map $p_F: H_X \to F$ is a flat cover of $F$.

For a local version of this result we refer to the proposition at the end of this note. The theorem is based on the interplay between the triangulated structure of $\mathcal{T}$ and the abelian structure of the functor category $(\mathcal{F}^{\text{op}}, \text{Ab})$. A similar approach can be found in recent work of Christensen, Strickland, and Neeman [4, 8]. In fact, the connection between Brown representability and the vanishing of $\text{Ext}^2(-, F)$ for exact functors $F$ has been noticed by these authors. The new ingredient which leads to the connection with flat covers, is the existence of injective envelopes in $(\mathcal{F}^{\text{op}}, \text{Ab})$, see [6]. Recall that a monomorphism $i: F \to Q$ in any abelian category is an injective envelope of $F$, if $Q$ is injective and every endomorphism $e: Q \to Q$ satisfying $ei = i$ is an isomorphism.

The rest of this paper is devoted to proving the main result. To this end we need to recall our assumptions on the triangulated category $\mathcal{T}$:

- $\mathcal{T}$ has arbitrary coproducts;
- the isomorphism classes of compact objects in $\mathcal{T}$ form a set;
- $H_X = 0$ implies $X = 0$ for every object $X$ in $\mathcal{T}$.

The proof of the theorem is divided into several lemmas. The first one is taken from [7]. We include the short proof for the convenience of the reader.

**Lemma 1.** Let $Q$ be an injective object in $(\mathcal{F}^{\text{op}}, \text{Ab})$. Then there exists, up to isomorphism, a unique object $I$ in $\mathcal{T}$ such that $Q \cong H_I$. Moreover, the map

$$\text{Hom}(X, I) \to \text{Hom}(H_X, H_I), \quad f \mapsto Hf,$$

is an isomorphism for all $X$ in $\mathcal{T}$.

**Proof.** Consider the exact functor $F: \mathcal{T}^{\text{op}} \to \text{Ab}$, $X \mapsto \text{Hom}(H_X, Q)$. It sends coproducts in $\mathcal{T}$ to products in $\text{Ab}$ and is therefore representable by Brown’s classical theorem [2]. Thus
$F \cong \text{Hom}(-, I)$ for some object $I$ in $\mathcal{T}$. The induced map $\text{Hom}(I, I) \cong F(I) = \text{Hom}(H_I, Q)$ sends $\text{id}_I$ to a map $p: H_I \to Q$ which is an isomorphism since

$$H_I(C) = \text{Hom}(C, I) \cong \text{Hom}(H_C, Q) \cong Q(C)$$

for every compact object $C$ by Yoneda’s lemma. The inverse $p^{-1}: Q \to H_I$ induces an isomorphism

$$\text{Hom}(X, I) \cong \text{Hom}(H_X, Q) \cong \text{Hom}(H_X, H_I)$$

which is precisely the map $f \mapsto H_f$. This finishes the proof.

**Lemma 2.** Let $F \in (\mathcal{F}^{\text{op}}, \text{Ab})$. Then $\text{Ext}^2(-, F) = 0$ if and only if the canonical map $p_F: H_X \to F$ is an isomorphism.

**Proof.** We fix the minimal injective copresentation

$$0 \to F \to H_I \xrightarrow{H_f} H_J$$

and denote by $e: X \to I$ the fiber of $f: I \to J$. We have $\text{Ext}^2(-, F) = 0$ if and only if $H_f$ is an epimorphism, and this is equivalent to $H_e$ being a monomorphism since the fiber $e$ gives rise to an exact sequence

$$H_{\Sigma^{-1}J} \xrightarrow{H_{\Sigma^{-1}g}} H_X \xrightarrow{H_f} H_I \xrightarrow{H_J} H_J \xrightarrow{H_g} H_{\Sigma X}.$$ 

Furthermore, $H_e$ is a monomorphism if and only if $H_X$ is a kernel of $H_f$. We conclude from the construction of the map $p_F$ that $\text{Ext}^2(-, F) = 0$ if and only if $p_F$ is an isomorphism.

**Lemma 3.** Let $X \xleftarrow{u} I \xrightarrow{f} J \xrightarrow{g} \Sigma X$ be a triangle in $\mathcal{T}$ and suppose that the induced map $\text{Im} H_f \to H_J$ is an injective envelope. Then every endomorphism $u: X \to X$ satisfying $ue = e$ is an isomorphism.

**Proof.** Choose a map $v: J \to J$ which completes the following commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{e} & I \\
\downarrow u & & \| \\
X & \xrightarrow{e} & I \\
\end{array} \quad \begin{array}{ccc}
I & \xrightarrow{f} & J \\
\| & & \| \\
I & \xrightarrow{f} & J \\
\downarrow u & & \downarrow u \\
\Sigma X & \xleftarrow{\Sigma u} & \Sigma X \\
\end{array}$$

The assumption on $u$ and $f$ imply that $H_e$ is an isomorphism. Thus $v$ is an isomorphism, and we conclude that $u$ is an isomorphism.

**Lemma 4.** Let $Y \in \mathcal{T}$ and $p = p_F: H_X \to H_Y$ be the canonical map for $F = H_Y$. Then the following are equivalent:

1. $\text{Ext}^2(-, H_Y) = 0$.
2. Every isomorphism $H_X \to H_Y \amalg Z$ which corestricts to $p$ is of the form $H_u$ for some map $u: X \to Y \amalg Z$.
3. There exists a map $v: X \to Y$ such that $p = H_v$.

**Proof.** We denote by $c: Y \to I$ the map which induces the first map of the minimal injective copresentation

$$0 \to H_Y \to H_I \xrightarrow{H_f} H_J;$$

it exists by Lemma 1. The map $c$ factors through the fiber $e: X \to I$ of $f$ via some map $w: Y \to X$ since $fc = 0$. Note that $pH_w = \text{id}_{H_Y}$.

(1) $\Rightarrow$ (2) If $\text{Ext}^2(-, H_Y) = 0$, then $p$ is an isomorphism by Lemma 2. Moreover, $H_w$ is an inverse for $p$, and therefore $w$ is an isomorphism. The inverse $w^{-1}: X \to Y$ gives a map such
that $H_{w^{-1}} = p$. This proves (2) since any isomorphism $H_X \to H_Y \| Z$ which corestricts to $p$ has $Z = 0$.

(2) $\implies$ (3) Complete the map $w: Y \to X$ to a triangle $Y \to X \to Z \to \Sigma Y$. This gives a split short exact sequence

$$0 \to H_Y \to H_X \to H_Z \to 0$$

since $pH_w = \text{id}_{H_Y}$. We obtain an isomorphism $H_X \to H_Y \| Z$ which corestricts to $p$. Condition (3) is now an immediate consequence of (2).

(3) $\implies$ (1) Let $p = H_v$. The composition $wv$ is an endomorphism of $X$ satisfying $ewv = e$. Therefore $wv$ is an isomorphism by Lemma 3. In particular $p = H_v$ is a monomorphism. On the other hand, $p$ is an epimorphism since $pH_w = \text{id}_{H_Y}$. It follows from Lemma 2 that $\text{Ext}^2(-, H_Y) = 0$. \hfill \square

**Proposition.** The following conditions are equivalent for an object $X$ in $\mathcal{T}$:

1. $\text{Ext}^2(-, H_X) = 0$.
2. Every map $H_{X'} \to H_X$ is of the form $H_u$ for some map $u: X' \to X$.
3. Every split epimorphism $H_{X'} \to H_X$ is of the form $H_u$ for some map $u: X' \to X$.

**Proof.** (1) $\implies$ (2) Fix a map $H_{X'} \to H_X$. Using Lemma 1, we choose injective envelopes $H_e: H_X \to H_I$ and $H_{e'}: H_{X'} \to H_{I'}$. The maps $e$ and $e'$ can be completed to triangles

$$X \xrightarrow{e} I \xrightarrow{f} J \xrightarrow{g} \Sigma X \quad \text{and} \quad X' \xrightarrow{e'} I' \xrightarrow{f'} J' \xrightarrow{g'} \Sigma X'$$

where $H_J$ is injective by our assumption on $X$. Using the injectivity of $H_I$, the map $h: H_{X'} \to H_X$ induces the following commutative diagram with exact rows

$$\begin{array}{c}
0 & \to & H_{X'} & \xrightarrow{H_{e'}} & H_{I'} & \xrightarrow{H_{f'}} & H_{I'} & \to & 0 \\
& & \downarrow{h} & & \downarrow & & \downarrow & & \\
0 & \to & H_X & \xrightarrow{H_e} & H_I & \xrightarrow{H_f} & H_I & \to & 0
\end{array}$$

An application of Lemma 1 gives the following commutative diagram

$$\begin{array}{ccc}
\Sigma^{-1}J' & \to & X' & \xrightarrow{e'} & I' & \xrightarrow{f'} & J' \\
\downarrow{\Sigma^{-1}w} & & \downarrow{v} & & \downarrow{w} & & \\
\Sigma^{-1}J & \to & X & \xrightarrow{e} & I & \xrightarrow{f} & J
\end{array}$$

since $H_I$ and $H_J$ are injective, and the fill-in map $u: X' \to X$ induces the map $h: H_{X'} \to H_X$ by construction.

(2) $\implies$ (3) Clear.

(3) $\implies$ (1) Use Lemma 4. \hfill \square

**Proof of the Theorem.** Throughout the proof we shall use freely the fact that flat and exact functors in $(\mathcal{F}^{\text{op}}, \text{Ab})$ coincide (see Lemma 2.7 in [7]).

(1) $\implies$ (2) Clear.

(2) $\implies$ (3) If $\text{Ext}^2(-, F) = 0$, then $F$ is exact by Lemma 2 and therefore flat. Conversely, suppose that $F$ is flat and therefore exact. Then $F \cong H_Y$ for some $Y \in \mathcal{T}$ by (2), and the second condition in (2) implies $\text{Ext}^2(-, F) = 0$ by Lemma 4.

(3) $\implies$ (4) Let $F' \in (\mathcal{F}^{\text{op}}, \text{Ab})$, and suppose that $E \to F$ is any map such that $E$ is flat. We need to show that $E \to F'$ factors through $pE$. We take the composition

$$H_Y \xrightarrow{pE} E \to F \to H_I$$
which is of the form $H_c$ for some map $c: Y \to I$ by Lemma 1. Clearly, $H_f H_c = H_{fc} = 0$, and therefore $fc = 0$, again by Lemma 1. Therefore $c$ factors through $e: X \to I$ via some map $u: Y \to X$. Condition (3) and Lemma 2 imply that $pE$ is an isomorphism, and the composition $H_u pE^{-1}: E \to HX$ gives the desired map.

Now let $v: H_X \to H_X$ be any endomorphism satisfying $pFv = pF$. We need to show that $v$ is an isomorphism. To this end observe that $v = H_u$ for some $u: X \to X$ by the Proposition. Moreover, $pFv = pF$ implies $Hu = H_c$, and therefore $eu = e$ by Lemma 1. An application of Lemma 3 shows that $v$ is an isomorphism, and this completes the proof of the fact that $pF$ is a flat cover of $F$.

(4) $\Rightarrow$ (1) Clearly, a flat cover $E \to F$ is an isomorphism if $F$ is flat. Therefore (4) implies that $P_F: H_X \to F$ is an isomorphism for every exact functor $F$. A combination of Lemma 2 and the Proposition shows that every map $H_X \to HY$ is induced by a map $X \to Y$ in $\mathcal{T}$. \qed

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REFERENCES


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