HOMOLOGICAL CODIMENSION OF MODULAR RINGS OF INVARIANTS AND THE KOSZUL COMPLEX

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SUMMARY: Let ρ : G ⊆→ GL(n, IF) be a representation of a finite group over the field IF of characteristic p, and h₁,..., hₘ ∈ IF[V]ᵢvariant polynomials that form a regular sequence in IF[V]. In this note we introduce a tool to study the problem of whether they form a regular sequence in IF[V]ᵢ. Examples show they need not. We define the cohomology of G with coefficients in the Koszul complex

(κ, ∂) = (IF[V] ⊗ E(s⁻¹h₁,..., s⁻¹hₙ), ∂(s⁻¹hᵢ) = hᵢ : i = 1,..., n),

which we denote by H*(G; (κ, ∂)), and use it to study the homological codimension of rings of invariants of permutation representations of the cyclic group of order p, for p ≠ 0, and to answer the above question in this case.

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Let $G$ be a finite group and $\rho : G \to \text{GL}(n, \mathbb{F})$ a representation of $G$ over the field $\mathbb{F}$. Suppose that $h_1, \ldots, h_k \in \mathbb{F}[V]^G$ are invariant polynomials (we assume familiarity with the basic ideas, definitions, and notations of invariant theory of finite groups as found for example in [22]) that form a regular sequence in $\mathbb{F}[V]$. We pose the question: do they form a regular sequence in $\mathbb{F}[V]^G$? In general the answer will be no. For example if $\mathbb{F}$ is a Galois field of characteristic $p$ and $d_{n,0}, \ldots, d_{n,n-1} \in \mathbb{F}[V]^G$ the Dickson polynomials (see [22] chapter 8), then $d_{n,0}, \ldots, d_{n,n-1}$ are certainly a regular sequence in $\mathbb{F}[V]$, but, are a regular sequence in $\mathbb{F}[V]^G$ if and only if $\mathbb{F}[V]^G$ is Cohen–Macaulay, and this certainly need not be the case (see e.g. [22] chapter 6 and [23] §4).

This study began in an attempt to verify the depth conjecture of Landweber and Stong in some concrete examples by using the methods (not the results) of [8], even though the depth conjecture has been proved by other methods by Borguiba and Zarati [3] (see also [23] §6). These computations appear in §3 and §4.

In contrast to [8], where the ground field is algebraically closed, we take advantage of the fact that, over a finite field $\mathbb{F}$ there is a universal ring of invariants for representations of degree $n$, namely the Dickson algebra $D^*(n)$. Since $D^*(n) \subseteq \mathbb{F}[V]^G$ is always a finite extension, the homological codimension of $\mathbb{F}[V]^G$ as a ring is the same as the homological codimension of $\mathbb{F}[V]^G$ as a $D^*(n)$-module. The Dickson algebra $D^*(n)$ is the polynomial algebra $\mathbb{F}[d_{n,n-1}, \ldots, d_{n,0}]$ and hence has finite global dimension, so a famous equality of Auslander and Buchsbaum [1] allows us to convert the computation of the homological codimension of $\mathbb{F}[V]^G$ over $D^*(n)$ into an equivalent computation of the homological (i.e. projective) dimension of $\mathbb{F}[V]^G$ over $D^*(n)$. For this we introduce a spectral sequence, which, loosely speaking, is a Koszul–Serre dual to the one used by Ellingsrud and Skjelbred in [8]. We hope in this way to make this circle of ideas available to a larger audience then seems to have been attracted by [8] alone.

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§1. A Motivational Example

Let $\rho : G \to \text{GL}(n, \mathbb{F})$ be a representation of a finite group $G$ over the field $\mathbb{F}$. If $|G| \in \mathbb{F}^\times$ then the Reynolds operator

$$\pi^G = \frac{1}{|G|} \text{Tr}^G : \mathbb{F}[V] \to \mathbb{F}[V]^G$$

defines a splitting of the inclusion $\mathbb{F}[V]^G \subseteq \mathbb{F}[V]$. If $h_1, \ldots, h_k \in \mathbb{F}[V]^G$ form a regular sequence in $\mathbb{F}[V]$, then they are algebraically independent, and $\mathbb{F}[V]$ is a free $\mathbb{F}[h_1, \ldots, h_k]$-module. Since $\mathbb{F}[V]^G \subseteq \mathbb{F}[V]$ is an $\mathbb{F}[V]^G$-direct summand ($\pi^G$ is an $\mathbb{F}[V]^G$-linear map (see [22] §2.4)), it is also an $\mathbb{F}[h_1, \ldots, h_k]$-direct summand, and hence projective as an $\mathbb{F}[h_1, \ldots, h_k]$-module. In this graded connected context, projective and free are the same thing, so $\mathbb{F}[V]^G$ is a free $\mathbb{F}[h_1, \ldots, h_k]$-module, and therefore (see e.g. [22] 6.2) $h_1, \ldots, h_k$ is
SC and the last conclusion

By contrast the five Dickson polynomials of highest degrees

\[ d \]

a regular sequence in \[ \mathbb{F}_\ell \]

sections.

Therefore the space of invariant linear forms has dimension 3, the space of invariant quadratic forms dimension 12, the space of invariant cubic forms dimension 28, etc. In degrees 1 and 2 it is relatively easy to find bases for the invariant forms:

\[ Z^1, Z^2, Z^3, x_1^1, x_2^1, x_3^1, y_1^1, y_2^1, y_3^1, z_1^1, z_2^1, z_3^1 \]

is a basis for the invariant linear forms, and the 6 products

\[ x_1 x_2, x_1 x_3, x_2 x_3, y_1 y_2, y_1 y_3, y_2 y_3 \]

of characteristic 2. The Poincaré series of \[ \mathbb{F}_\ell \]

using either Molien’s theorem to compute the Poincaré series over

\[ \mathbb{F}_\ell \]

by letting

\[ Z \]

\[ x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \]

is acyclic. For the sake of clarity here is the definition of the Koszul complex and notation that we are using (see \[ \text{[22]} \] §6.2, \[ \text{[21]} \] part II §1).
DEFINITION: Let \( A \) be a graded connected commutative algebra over a field \( \mathbb{F} \) and \( a_1, \ldots, a_n \in A \). The Koszul complex of \( A \) with respect to \( a_1, \ldots, a_n \) is the differential graded commutative algebra

\[
\mathcal{K} = \mathcal{K}(a_1, \ldots, a_n) = A \otimes E(s^{-1}a_1, \ldots, s^{-1}a_n)
\]

where \( E(s^{-1}a_1, \ldots, s^{-1}a_n) \) denotes a graded 1 - exterior algebra with \( \deg(s^{-1}a_i) = -1 + \deg(a_i) \), and the differential \( \partial \) is defined by requiring

\[
\begin{align*}
\partial |_A &= 0 \\
\partial(s^{-1}a_i) &= a_i \quad \text{for} \ i = 1, \ldots, n \\
\partial(x \cdot y) &= \partial(x)y + (-1)^{\deg(x)}x\partial(y) \quad \forall x, y \in \mathcal{K}.
\end{align*}
\]

Introduce the Koszul complex

\[
\mathcal{K} = \mathbb{F}[V] \otimes E(s^{-1}h_1, \ldots, s^{-1}h_k)
\]

\[
\partial(s^{-1}h_i) = h_i \text{ for } i = 1, \ldots, k.
\]

The group \( G \) acts on \( \mathcal{K} \) via the representation \( \rho \) on \( \mathbb{F}[V] \) and trivially on \( E(s^{-1}h_1, \ldots, s^{-1}h_k) \). Moreover \( \mathcal{K} = \mathcal{K}^G \). By hypothesis \( h_1, \ldots, h_k \in \mathbb{F}[V] \) is a regular sequence, so \((\mathcal{K}, \partial)\) is acyclic. Hence the question at hand becomes: is \((\mathcal{K}^G, \partial)\) also acyclic?

More generally we would want to relate \( H^\ast(\mathcal{K}, \partial)^G \) and \( H^\ast(\mathcal{K}^G, \partial) \). One way to do so is to note that \( M^G = H^0(G; M) \) for any \( G \)-module \( M \). Since the functor \( H^0(G; -) \) is not exact in general it is not surprising that the higher derived functors (see e.g. [10]) of \( H^0(G; -) \) enter into the discussion.

§2. Koszul Cohomology of a Group

As a tool to deal with the problems encountered in the introduction and §1 we set up a spectral sequence\(^2\) relating the cohomology of a \( G \)-cocomplex to the cohomology of the fixed cocomplex. In order to keep the discussion as simple as possible\(^3\) we suppose that \( \rho : G \hookrightarrow \text{GL}(n, \mathbb{F}) \) is a representation of a finite group \( G \) over the field \( \mathbb{F} \) and that \( h_1, \ldots, h_k \in \mathbb{F}[V]^G \) are invariant polynomials which form a regular sequence in \( \mathbb{F}[V] \). Let \((\mathcal{K}, \partial)\) denote the Koszul complex

\[
\mathcal{K} = \mathbb{F}[V] \otimes E(s^{-1}h_1, \ldots, s^{-1}h_k)
\]

\[
\partial(s^{-1}h_i) = h_i \text{ for } i = 1, \ldots, k
\]

with the extended action of \( G \) as in the previous section. We denote by \( I\!F(G) \) the group algebra of \( G \) over \( I\!F \) and let \( \mathcal{P}(G) \twoheadrightarrow I\!F(G) \) denote a projective resolution of \( I\!F(G) \) regarded as an \( I\!F(G) \)-module via the augmentation homomorphism \( \varepsilon : I\!F(G) \twoheadrightarrow I\!F \), such as for example the bar construction \( \mathcal{B}(G) \) of \( G \) over \( I\!F \) (see e.g. [4] or [6]). Introduce the double complex\(^4\)

\[
\mathcal{C} = \text{Hom}_{I\!F(G)}(\mathcal{P}(G), \mathcal{K}).
\]

A bit of care is needed with the gradings to turn this into an acceptable double complex, i.e., one satisfying the standard grading conventions (see e.g. [15]). The differential \( d^\ast \) coming

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1. This is the totalization of the Koszul complex bigraded as in [21].
2. In fact this spectral sequence is a form of Koszul-Serre dual, or local cohomology dual, to the spectral sequence employed by Ellingsrud and Skjelbred in [8].
3. It will be immediately clear to the experts that a much broader discussion is possible, however, my aim here is not generality, but to provide a tool to make concrete computations in invariant theory.
4. See also [4] chapter VII section 5 for a similar construction in connection with the cohomology of semidirect products.
construction differential) and then the Koszul complex differential will become our main technical tool. If we denote this spectral sequence by \( \{ E_r, d_r \} \) then as a consequence of 2.1 (\( \Rightarrow \) denotes converges to)

\[
E_r \Rightarrow H^* \left( G; \frac{IF[V]}{(h_1, \ldots, h_k)} \right).
\]

Since \( G \) acts trivially on \( E(s^{-1}h_1, \ldots, s^{-1}h_k) \) the term \( E_1 \) of the spectral sequence takes the form

\[
E_1 = H^*(G; IF[V]) \otimes E(s^{-1}h_1, \ldots, s^{-1}h_k).
\]

Therefore the term \( E_2 \) may be identified with the cohomology of yet another Koszul complex, namely for the elements

\[
h_1, \ldots, h_k \in H^0(G; IF[V]) = IF[V]^G \subseteq H^*(G; IF[V]).
\]
Therefore, recalling that \( h_1, \ldots, h_k \in \text{IF}[V]^G \) are algebraically independent, we see
\[
E_2^{s,t} = \text{Tor}^{s}_{\text{IF}[h_1, \ldots, h_k]}(H^t(G; \text{IF}[V]), \text{IF})
\]
where \( H^*(G; \text{IF}[V]) \) is regarded as an \( \text{IF}[h_1, \ldots, h_k] \)-module via the inclusion
\[
\text{IF}[h_1, \ldots, h_k] \subseteq \text{IF}[V]^G = H^0(G; \text{IF}[V]) \subseteq H^*(G; \text{IF}[V])
\]
and the product in group cohomology. To summarize, we have shown:

**Proposition 2.2:** With the preceding hypotheses and notations there is a convergent second quadrant spectral sequence \( \{ E_r, d_r \} \) with
\[
E_r \Rightarrow H^*(G; \text{IF}[h_1, \ldots, h_k])
\]
\[
E_2^{s,t} = \text{Tor}^{s}_{\text{IF}[h_1, \ldots, h_k]}(H^t(G; \text{IF}[V]), \text{IF}).
\]

**Proof:** We need only remark that convergence is a consequence of the fact that \( E_2^{s,t} = 0 \) if \( s < -k \). □

This spectral sequence is a precursor of Grothendieck’s local cohomology spectral sequence [11].

As a simple application of this spectral sequence we reprove a result of Landweber and Stong [14], that serves as a model for further applications.

**Proposition 2.3:** Suppose that \( \rho : G \hookrightarrow \text{GL}(n, \text{IF}), n \geq 2, \) is a representation of a finite group \( G \) over the field \( \text{IF} \). If \( h_1, h_2 \in \text{IF}[V]^G \) are a regular sequence in \( \text{IF}[V] \), then they are a regular sequence in \( \text{IF}[V]^G \) also.

**Proof:** The polynomial algebra \( \text{IF}[h_1, h_2] \) has global dimension 2, so the functors \( \text{Tor}_{\text{IF}[h_1, h_2]}^{s}(-, -) \) are identically zero for \( s < -2 \). The following diagram representing \( E_2 \)

```
\[
\begin{array}{ccc}
0 & & 0 \\
\vdots & & 0 \\
\vdots & & 0 \\
\vdots & & 0 \\
\vdots & & 0 \\
\end{array}
\]
```

\[
\begin{array}{cccc}
0 & & 0 & 3 \\
\vdots & & 0 & 2 \\
\vdots & & 0 & 1 \\
\vdots & & 0 & E_{2}^{-1,0} \\
\vdots & & 0 & 0 \\
\end{array}
\]

\*

**Figure 2.1:** \( E_2 \) in the dimension 2 case

shows that there is no way that a nonzero differential can either arrive at or leave from \( E_{2}^{-1,0} \).

The elements of \( E_{2}^{-1,0} \) have negative total degree. \( H^*(G; \text{IF}[V]_{(h_1, h_2)}) \) is zero in negative degrees, and hence \( E_{\infty}^{s,t} \) is also zero in negative degrees. Therefore
\[
0 = E_{2}^{-1,0} = \text{Tor}_{\text{IF}[h_1, h_2]}^{-1}(H^0(G; \text{IF}[V]), \text{IF}) = \text{Tor}_{\text{IF}[h_1, h_2]}^{-1}(\text{IF}[V]^G, \text{IF})
\]
and the result follows. □
In this section we will apply the spectral sequence of 2.2 to rings of invariants in characteristic 2. To see why this case is particularly amenable we begin with a number of general remarks.

If $\rho: G \hookrightarrow \text{GL}(n, \mathbb{F})$ is a representation of a finite group $G$ over the field $\mathbb{F}$ of characteristic $p$ and $P = \text{Syl}_p(G) \leq G$ is a $p$-Sylow subgroup of $G$ then one has by [23] 4.6:

$$\text{hom} - \text{codim}(\mathbb{F}[V_G]) \geq \text{hom} - \text{codim}(\mathbb{F}[V_P]).$$

So to establish lower bounds for the homological codimension we may restrict ourselves to the case of $p$-groups, and henceforth we assume $G = P$ to be a finite $p$-group. We also assume that $\mathbb{F}$ is a Galois field with $q = p^m$ elements. The Dickson algebra (see e.g. [22] §8.1) $D^* (n)$ is a subalgebra of $\mathbb{F}[V_P]$ and the extensions $D^* (n) \subseteq \mathbb{F}[V_P] \subseteq \mathbb{F}[V_P]$ are finite. Therefore (see the do it yourself kit [5] exercise 1.2.26)

$$\text{hom} - \text{codim} D^* (n)(\mathbb{F}[V_P]) = \text{hom} - \text{codim}(\mathbb{F}[V_P]).$$

The Auslander±Buchsbaum equality [5] 1.3.3

$$n = \text{hom} - \text{codim} D^* (n)(\mathbb{F}[V_P]) + \text{hom} - \text{dim} D^* (n)(\mathbb{F}[V_P])$$

in turn allows us to convert the original homological codimension computation to one of the homological dimension of $\mathbb{F}[V_P]$ as $D^* (n)$-module. It is here that the spectral sequence $E_r \Rightarrow H^* / \text{char} \ P; \mathbb{F}[V_P] \text{GL}(n, \mathbb{F}) / \text{char} 1$ can be of use. (We have written $\mathbb{F}[V_P] \text{GL}(n, \mathbb{F})$ for $\mathbb{F}[V_P] \otimes D^* (n) \mathbb{F}$ for the ring of coinvariants of the group $\text{GL}(n, \mathbb{F})$ which coincides with standard notation.)

Notice that $H^* / \text{char} \ P; \mathbb{F}[V_P] \text{GL}(n, \mathbb{F}) / \text{char} 1$ is zero for $* < 0$ and therefore there are no elements of negative total degree in $E_{\infty}$. In other words, as the following diagram shows:

\[\begin{array}{c}
\vdots \\
E_{-2} \\
E_{-1} \\
E_0 \\
\vdots
\end{array}\]

\[\text{FIGURE 3.1: The vanishing area}

The terms on the border of the vanishing area, i.e., where $s + t = 0$ (referred to as the vanishing line) are connected with the stable invariants introduced in [12] and studied in [16].

To work with this spectral sequence we need to obtain information about $H^* (P; \mathbb{F}[V_P])$ and how the Dickson algebra acts on it. It would seem that since we do not know $\mathbb{F}[V_P] = H^0 (P; \mathbb{F}[V_P])$ that this would be a hopeless undertaking. However, in any case, it is natural to start with the smallest example $P = \mathbb{Z}/p\mathbb{Z}$, the cyclic group of order $p$. If $c \in \mathbb{Z}/p\mathbb{Z}$ is a generator then the periodic complex [6]
Then the representation of $Z$ spanned by the binomials $s_t$ is a free resolution of $I_F$ as $I_F$ is a free $F$-module. Therefore applying the functor $\text{Hom}_{IF}(\_\_ , \_\_)$ we see that $\text{Tor}_{i}^{IF}(Z,\_\_)$ for sums of the regular representation: the trivial part will cause no problems and from [24], as a consequence of the remarkable lemma 3.1 of [17], we have $\text{Tor}_{i}^{IF}(Z,\_\_)$ for $i$ even.

By using the Jordan form (or otherwise) one can see that a representation of $Z$ of characteristic 2 is always a permutation representation. A permutation representation of $Z$ is the group algebra of $Z$, and hence $\text{Hom}_{IF}(\_\_ , \_\_)$ for $\_\_ = (0, \_\_ )$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_)$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the cohomology of $\_\_ = (0, \_\_ )$.

For an index sequence $A$ where $A_{p}$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the cohomology of $\_\_ = (0, \_\_ )$.

With the aid of this result we can compute $\text{Tor}_{i}^{IF}(Z,\_\_)$ for $i$ even. (i) $\text{Im}(\text{Tr}_{IF}(\_\_ ) )$ for $\_\_ = (0, \_\_ )$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the cohomology of $\_\_ = (0, \_\_ )$. (ii) $\text{IF} [1, \ldots , a M ]$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the augmentation homomorphism, $\text{Tr}_{IF}(\_\_ )$ is the cohomology of $\_\_ = (0, \_\_ )$.

For integers $\_\_ = (0, \_\_ )$, $\text{Tor}_{i}^{IF}(Z,\_\_)$ for $i$ even, and need only determine the quotient of $\text{IF} [1, \ldots , a M ]$ for $\_\_ = (0, \_\_ )$.
min is a monomorphism. We may identify \( I F[\ldots] \) lies over \( \text{Im}(\text{Tr}) \) maps onto the subalgebra \( I F[\ldots] \) which is in turn a subalgebra of \( I F[\ldots] \).

Elements \( 1 \otimes \ldots \) of the cyclic group of \( \text{char} = 2 \) integers, and therefore \( I F[\ldots] \) is an ideal generated by a regular sequence.

**Theorem 3.4** (Ellingsrud and Skjelbred [8]): Let \( \rho : GL(n, \mathbb{Z}) : \mathbb{F} \to \mathbb{F} \) be a representation, and \( \rho_v \) for \( \rho \) be a representation. Then \( \rho_v \) occurs \( \text{dim}(\mathbb{F}) = n \) times in \( \rho \).

**Proposition 3.2**

**Corollary 3.3**

With the preceding hypotheses and notations we have for all \( \pi \):

\[
\text{Tor}^A_*(\pi, \pi) = \bigoplus_{i=1}^m \bigoplus_{s=1}^d \mathbb{F}[x_1, \ldots, x_m, y_1, \ldots, y_m]/(\pi)
\]

where \( \pi \) is a monomial of \( \text{deg} = -1 \) (and internal degree 3) for \( \mathbb{F}[x_1, \ldots, x_m, y_1, \ldots, y_m] \).
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PROOF: Let $V = I F^n$. Using the Jordan form (or otherwise) one sees that it is possible to choose a basis $x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_k$ for $V^*$ such that $\mathbb{Z}/2$ acts on $V^*$ by interchanging the vector variable $(x_1, \ldots, x_m)$ with the vector variable $(y_1, \ldots, y_m)$ and fixing $z_1, \ldots, z_k$. Therefore $IF[V]_{\mathbb{Z}/2} = IF[x_1, \ldots, x_m, y_1, \ldots, y_m]_{\mathbb{Z}/2} \otimes IF[z_1, \ldots, z_k]$ and $\text{hom±codim}(IF[V]_{\mathbb{Z}/2}) = \text{hom±codim}(IF[x_1, \ldots, x_m, y_1, \ldots, y_m]_{\mathbb{Z}/2}) + k$ and we may suppose $k = 0$ and $n = 2m$. The case $m = 1$ being elementary, we also may suppose $m > 1$.

Consider the spectral sequence of 2.2 in this case, for which we have $E_r^{s,t} \Rightarrow H^*(\mathbb{Z}/2; IF[V]_{\text{GL}(2m, IF)})$. $E_s^{s,t}_2 = \text{T}o \text{r}D^*(2m)(H^t(\mathbb{Z}/2; IF[V]), IF)$. From 3.3 it follows that $E_s^{s,t}_2 = 0$ for $s < -m$ and $t > 0$. This leads to the following picture for $E_2$:

```
                   0
                 s = -m + 2
               s = -m + 1
             s = -m
       E_{-m,1}^2       E_{-m,0}^2
          ------------------
              0
```

It follows that none of the terms $E_{-m,0}^2, \ldots, E_{-m+1,0}^2$ can be the target of a nonzero differential. Since all differentials starting on the $s$-axis are zero, if any of these terms were nonzero, it would represent nonzero elements of negative degree in $H^*(\mathbb{Z}/2; IF[V]_{\text{GL}(2m, IF)})$ which cannot exist. Hence we conclude $E_{-m,0}^2 = \cdots = E_{-m+1,0}^2 = 0$. The term $E_{-m,1}^2$ is nonzero and the indicated differential is the only possible nonzero differential either originating, or terminating, at $E_{-m,1}^2$. Since $E_{-m,1}^\infty = 0$ it follows that $d_2 : E_{-m,1}^2 \rightarrow E_{-m+2,0}^2$ must be an isomorphism. In particular $0 \neq E_{-m+2,0}^2 = \text{T}o \text{r}(-m+2)D^*(2m)(IF[V]_{\mathbb{Z}/2}, IF)$ so $\text{hom±dim}D^*(2m)(IF[V]_{\mathbb{Z}/2}) = m - 2$ from which it follows that $\text{hom±codim}(IF[V]_{\mathbb{Z}/2}) = m + 2$ as required.

§4. Permutation Representations of $\mathbb{Z}/p$ in Characteristic $p$

Let $p$ be an odd prime, $\mathbb{Z}/p$ the cyclic group of order $p$ and $X$ a finite $\mathbb{Z}/p$-set, i.e., a permutation representation of $\mathbb{Z}/p$. Since $p \in \text{IN}$ is a prime the only possible orbits of $\mathbb{Z}/p$ on $X$ are fixed points and free orbits: the action is semifree. Let $B$ denote the $\mathbb{Z}/p$-set with a free orbit is an orbit with $p$ elements cyclically permuted by $\mathbb{Z}/p$. 

\[9\]
To make use of 2.2 we need to compute \( \text{dim}_{\text{I}_F} \text{Hom}(\text{I}_F, \text{I}_F) \) notice that \( \text{I}_F \) is a field of characteristic \( p \). We are going to study the homological codimension of the ring of invariants \( \text{I}_F \). Buchsbaum equality converts a homological codimension computation into the computation confirming a formula of \( [8] \). Our strategy is the same as in §3: we use the Auslander±Buchsbaum equality to convert a homological codimension computation into the computation with the aid of the spectral sequence 2.2. We do this with the help of the standard resolution of Cartan±Eilenberg as described in §3. Recall (formula \( (\ast) \)) from §3.

The variables \( X_1, \ldots, X_m \) denote the underlying set the elements of \( \text{I}_F \), which is the corresponding \( \text{I}_F \)-linear representation. In this way the standard resolution of Cartan±Eilenberg as described in §3.

If \( \text{I}_F \) is a field we denote by \( \text{V}_{\text{I}_F} \) the free \( \text{I}_F \)-set with a basis for \( \text{I}_F \), and show that the representation \( \text{V}_{\text{I}_F} \) is defined over the prime subfield of \( \text{I}_F \) and \( \text{V}_{\text{I}_F} \) is identified as \( \text{V}_{\text{I}_F} \) as a module over the Dickson algebra \( \text{D} \). This we do with the help of the spectral sequence 2.2)

\[
\text{dim}_{\text{I}_F} \text{Hom}(\text{I}_F, \text{I}_F) = 2 + \text{dim}_{\text{I}_F} \text{Hom}(\text{I}_F, \text{I}_F) \]

another way to say this is that \( \text{I}_F \) is a field of characteristic \( p \).
is semifree. Let

\[ u \in U \]

for \( \alpha \in \hom_{\mathbb{Z}}(X) \), where

\[ \alpha \in U(\mathbb{Z}) \]

by cyclic permutation of \( \alpha \). Our next task is to describe the cohomology modules \( H_i \) of \( \mathbb{Z} \). With the preceding hypotheses and notations the maps

\[ d : \Rightarrow \]

de note the \( \alpha \)-module, so we conclude:

\[ \alpha \in \mathbb{Z} \]

and \( \alpha \in U \).
so we obtain

From these equations it is an easy matter to deduce the following criteria:

(iii)

PROPOSITION 4.4

PROPOSITION 4.3

(iv)

PROPOSITION 4.3

PROPOSITION 4.3

PROPOSITION 4.3

PROPOSITION 4.3
INVARIANT THEORY AND THE KOSZUL COMPLEX

THEOREM 4.5 (Ellingsrud and Skjelbred [8]): Let \( p \) be an odd prime, \( \mathbb{Z}/p \) the cyclic group of order \( p \) and \( X \) a finite \( \mathbb{Z}/p \)-set. If \( IF \) is a field of characteristic \( p \), then

\[
\text{hom} \pm \text{codim}(IF \cdot X, \mathbb{Z}/p) = 2 + \text{dim}IF(V_{\mathbb{Z}/p}X) = 2 + m + \left| \cdot \right|
\]

where \( m = \left| \cdot \right| \) is the number of free orbits of \( \mathbb{Z}/p \) on \( X \).

PROOF: Without loss of generality we may suppose that \( X_{\mathbb{Z}/p} = \emptyset \), \( m > 0 \) and \( IF \) is finite.

We consider the spectral sequence 2.2

\[
E_{r,s}^2 \Rightarrow H^s(\mathbb{Z}/p; IF \cdot X, IF)\text{GL}(mp, IF)
\]

\[
E_{s,t}^2 = \text{Tor}^s(D_\ast(mp)(IF \cdot X, IF), IF).
\]

By propositions 4.1 and 4.4 we have the precise vanishing line for \( E_{s,t}^2 \), \( t > 0 \), is

\[
s = -m(p-1) + 1.
\]

This leads to the following diagram for \( E_2 \):

\[
\begin{array}{cccccc}
0 & & & & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
E_{-m(p-1)+2} & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{array}
\]

which together with the fact that \( E_{s,t}^\infty = 0 \) for \( s + t < 0 \) implies that

\[
0 = E_{s,0}^2 = \text{Tor}^s(D_\ast(mp)(IF \cdot X, IF), IF) \text{GL}(mp, IF)
\]

for \( s < -m(p-1) + 1 \) and that the indicated differential is an isomorphism. Since

\[
E_{-m(p-1)+2}^2 \neq 0
\]

we conclude that

\[
\text{Tor}^s(D_\ast(mp)(IF \cdot X, IF), IF) \neq 0
\]

and hence \( \text{hom} \pm \text{dim}D_\ast(mp)(IF \cdot X, \mathbb{Z}/p) = m(p-1) - 2 \) and the result follows from the Auslander-Buchsbaum equality.

COROLLARY 4.6 (Fossum and Griffiths [9]): Let \( p \) be a prime, \( \mathbb{Z}/p \hookrightarrow \text{GL}(p, IF) \) the regular representation of the cyclic group \( \mathbb{Z}/p \) of order \( p \) over a field \( IF \) of characteristic \( p \).

Then \( IF \cdot X_{\mathbb{Z}/p} \) is Cohen-Macaulay if and only if \( p = 2 \) or \( 3 \).

PROOF: For \( p = 2 \) the result is clear. For odd \( p \), if the ring of invariants is Cohen-Macualy, substitute in the formula of 4.5 to obtain

\[
p = \text{hom} \pm \text{codim}(IF \cdot X, \mathbb{Z}/p) = 2 + \text{dim}IF(V_{\mathbb{Z}/p}X) = 2 + 1 = 3.
\]
By theorem 1, if \( \alpha \in I_{F[\bar{\ell}]} \) is a regular sequence in \( I_{F[\bar{\ell}]} \), then the dual vector space \( V^* \) is influenced by the cohomological computations of §3 and proposition 2.2. Choose any two quadratic invariants instead of the linear ones as in example 1 of §1. This is of course possible if we examine the motivational problem posed in §1 to good advantage in the case of permutation groups.

From the discussion of \( H^i \) \( \otimes \) \( \bar{\ell} \), we use the spectral sequence of proposition 2.2 in this case and find that the sequence of \( \text{Tor} \) of proposition 2.3 (refer to figure 2.1).

On the other hand, suppose we choose three polynomials \( f, h, \) and \( q_1, \ldots, q_m, f \), \( q_1, \ldots, q_m, f \), \( q_1, \ldots, q_m, f \) is a regular sequence in \( I_{F[\bar{\ell}]} \). Without loss of generality, we may suppose that the representation permutes a basis \( x_1, \ldots, x_m, y_1, \ldots, y_m \).

Hence we conclude that \( \xi \in E \) and to simplify the notations we furthermore suppose that \( x_i, y_j \), \( x_i, y_j \), \( x_i, y_j \) are the standard basis for the dual vector space \( V^* \) of \( V \). Note carefully, that we have chosen to work with the obvious representation \( \rho \). These polynomials are invariant and \( E = I_{F[\bar{\ell}]}(I_{F[\bar{\ell}]}) \) in §3.

We consider the spectral sequence of proposition 2.2 for the elements \( q_1, \ldots, f, h, q_1, \ldots, f, h, q_1, \ldots, f, h, \) but there are lots of other possible choices for \( Z \) such that \( Z = \text{Tor}_{s,t} I_{F[\bar{\ell}]} \) and \( \xi = 1 \), \( \xi = 1 \), \( \xi = 1 \), but because of §1 we know this choice could not lead to a regular sequence in \( I_{F[\bar{\ell}]} \).
result for regular sequences of maximal length.

the argument of M.D. Neusel used in example 1 of §1 we arrive at the following definitive sequence and the same argument would apply. Putting all these facts together, and adding $q$

not a regular sequence.

the field $\mathbb{F}$ of characteristic 2. Choose a basis for $\mathbb{F}$

contains. Corollary 5.2 implies:

$\text{Corollary 5.2:}$

$\text{Proposition 5.4:}$
in $\mathbb{IF}[X]$ of length $m + k + 2$ by choosing two appropriate invariant forms $f, h \in \mathbb{IF}[X]/\mathbb{Z}/p$; for example two linear forms in $\text{Span}_{\mathbb{IF}}(X_1 / \sqcup \ldots / \sqcup X_m)$. Then $N_1, \ldots, N_m, u_1, \ldots, u_k, f, h$ is a regular sequence in $\mathbb{IF}[X]/\mathbb{Z}/p$. It is not possible to extend $N_1, \ldots, N_m, u_1, \ldots, u_k$ to a regular sequence in $\mathbb{IF}[X]/\mathbb{Z}/p$ of length $m + k + 3$.

**Corollary 5.5**

With the notations of 5.4 we have that a sequence $h_1, \ldots, h_{m + k + 2} \in \mathbb{IF}[X]/\mathbb{Z}/p$ that is a regular sequence in $\mathbb{IF}[X]$ is also a regular sequence in $\mathbb{IF}[X]/\mathbb{Z}/p$ if and only if some $m + k$ of them form a regular sequence in $\mathbb{IF}[N_1, \ldots, N_m, u_1, \ldots, u_k]/\text{Im}(\text{Tr}_{\mathbb{Z}/p}) = \mathbb{IF}[N_1, \ldots, N_m, u_1, \ldots, u_k]$.

**Corollary 5.6**

Let $X$ be a finite $\mathbb{Z}/p$ set and $\rho: \mathbb{Z}/p \rightarrow \text{GL}(n, \mathbb{IF})$ the corresponding permutation representation over the field $\mathbb{IF}$ of characteristic $p$. Write $n = pm + k$ where $k = \mid X / \mathbb{Z}/p \mid$ and $m$ is the number of free orbits of $\mathbb{Z}/p$ on $X$. Assume $\rho$ is faithful so $m > 0$.

Then the ideal $\text{Im}(\text{Tr}_{\mathbb{Z}/p})$ has grade $\min(2, m)$. 

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References


