The product formula in unitary deformation K-theory

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Abstract. For finitely generated groups $\mathcal{G}$ and $\mathcal{H}$, we prove that there is a weak equivalence $\mathcal{KG} \wedge_{ku} \mathcal{KH} \simeq \mathcal{K}(\mathcal{G} \times \mathcal{H})$ of ku-algebra spectra, where $\mathcal{K}$ denotes the "unitary deformation K-theory" functor. Additionally, we give spectral sequences for computing the homotopy groups of $\mathcal{KG}$ and $\mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{KG}$ in terms of $\mathrm{PU}(n)$-equivariant K-theory and homology of spaces of $\mathcal{G}$-representations.

1. Introduction

To a finitely generated group $\mathcal{G}$ one associates the category $\mathcal{C}$ of finite dimensional unitary representations of $\mathcal{G}$, with equivariant morphisms. Elementary methods of representation theory allow this category to be analyzed; explicitly, the category of unitary $\mathcal{G}$-representations is naturally equivalent to a direct sum of copies of the (topological) category of unitary vector spaces.

However, there is more structure to $\mathcal{C}$. First, there is a bilinear tensor product pairing. Second, $\mathcal{C}$ can be given the structure of an internal category $\mathcal{C}^{top}$ in **Top**. This means that there are *spaces* $\mathrm{Ob}(\mathcal{C}^{top})$ and $\mathrm{Mor}(\mathcal{C}^{top})$, together with continuous domain, range, identity, and composition maps, satisfying appropriate associativity and unity diagrams. The topology on $\mathrm{Ob}(\mathcal{C}^{top})$ reflects the possibility that homomorphisms from $\mathcal{G}$ to $\mathrm{U}(n)$ can continuously vary from one isomorphism class of representations to another. The identity gives a continuous functor $\mathcal{C} \to \mathcal{C}^{top}$ which is bijective on objects.

Both of these categories have notions of direct sums and so are suitable for application of an appropriate infinite loop space machine. This yields a map of (ring) spectra as follows:

$$\mathbb{K}\mathcal{C} \simeq \bigvee_{\mathrm{Irr}(\mathcal{G})} ku \to \mathbb{K}\mathcal{C}^{top} = \mathcal{KG}. \tag{1}$$

Here $\mathbb{K}$ is the algebraic K-theory functor, ku is the connective K-theory spectrum, and $\mathrm{Irr}(\mathcal{G})$ is the set of irreducible unitary representations of $\mathcal{G}$. Note that $\pi_*(\mathbb{K}\mathcal{C}) \cong R[\mathcal{G}] \otimes \mathbb{Z}[\beta]$ as a ring, where $R[\mathcal{G}]$ is the unitary representation ring of $\mathcal{G}$.

The spectrum $\mathcal{KG}$ is the *unitary deformation K-theory* of $\mathcal{G}$. It differs from the C^*-algebra K-theory of $\mathcal{G}$ - for example, in section 8, we find

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that the unitary deformation K-theory of the discrete Heisenberg group has infinitely generated π_0.

When $\mathcal{G}$ is free on n generators, one can directly verify the formula

$$\mathcal{K}\mathcal{G} \simeq ku \vee \left(\bigvee^{n} \Sigma ku \right).$$

A more functorial description in this case is that $\mathcal{K}\mathcal{G}$ is the connective cover of the function spectrum $F(B\mathcal{G}_+, ku)$. Even in simple cases $\mathcal{K}\mathcal{G}$ can be difficult to directly compute, such as when $\mathcal{G}$ is free abelian on multiple generators.

In this paper, we will prove the following product formula for unitary deformation K-theory.

THEOREM 1. *The tensor product map induces a map of ku-algebra spectra $\mathcal{K}\mathcal{G} \wedge_{ku} \mathcal{K}\mathcal{H} \to \mathcal{K}(\mathcal{G} \times \mathcal{H})$, and this map is a weak equivalence.*

The reader should compare the formula

$$R[\mathcal{G}] \otimes R[\mathcal{H}] \cong R[\mathcal{G} \times \mathcal{H}]$$

for unitary representation rings.

The proof of Theorem 1 proceeds by making use of a natural filtration of $\mathcal{K}\mathcal{G}$ by subspectra $\mathcal{K}\mathcal{G}_n$. These subspectra correspond to representations of $\mathcal{G}$ whose irreducible components have dimension less than or equal to n. Specifically, we show in sections 5 and 6 that there is a fibration sequence of spectra

$$\mathcal{K}\mathcal{G}_{n-1} \to \mathcal{K}\mathcal{G}_n \to ku^{\mathrm{PU}(n)}(\mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))/\mathrm{Sum}(\mathcal{G}, n)).$$

Here $\mathrm{Sum}(\mathcal{G}, n)$ is the subspace of $\mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))$ consisting of those representations $\mathcal{G} \to \mathrm{U}(n)$ which have a nontrivial invariant subspace. The spectrum $ku^{\mathrm{PU}(n)}(X)$ is a connective $\mathrm{PU}(n)$-equivariant K-homology spectrum for X, discussed in section 3.

As side benefits of the existence of this filtration, Theorems 22 and 24 give spectral sequences for computing the homotopy groups of $\mathcal{K}\mathcal{G}$ and the homotopy groups of $\mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{K}\mathcal{G}$ respectively.

When $\mathcal{G}$ is free on k generators, Theorem 24 gives a spectral sequence converging to $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^k$ in dimension 1, and 0 otherwise, but the terms in the spectral sequence are highly nontrivial - they are the homology groups of the spaces of k-tuples of elements of $\mathrm{U}(n)$, mod conjugation and relative to the subspace of k-tuples which admit a nontrivial invariant subspace. The method by which the terms in this spectral sequence are eliminated is a bit mysterious.

The motivation for studying these deformation K-theory spectra comes from algebraic K-theory.

The underlying goal of many programs in algebraic K-theory is to understand the algebraic K-groups of a field F as being built from the K-groups of the algebraic closure of the field, together with the action of the absolute Galois group. Specifically, Carlsson's program (see [2]) is to construct a model for the algebraic K-theory *spectrum* using the Galois group and the K-theory spectrum of the algebraic closure $\overline{F}$.

In some specific instances, the absolute Galois group of the field F is explicitly the profinite completion $\hat{\mathcal{G}}$ of a discrete group $\mathcal{G}$. (For example, the absolute Galois group of the field $k(z)$ of rational functions, where k is an algebraically closed of characteristic zero, is the profinite completion of a free group.) In the case where F contains an algebraically closed subfield, the profinite completion of the deformation K-theory spectrum $\mathcal{K}\mathcal{G}$ is conjecturally homotopy equivalent to the profinite completion of the algebraic K-theory spectrum $\mathbb{K}F$.

The layout of this paper is as follows. Section 2 gives the necessary background on Γ-spaces acted on by a compact Lie group G to identify equivariant smash products. The model theory of such functors was considered when G is a finite group in [4], using simplicial spaces. Our approach to the proofs of the results we need follows the approach of [3]. Section 3 gives explicit constructions of an equivariant version of connective K-theory, and in section 4 the unitary deformation K-theory of $\mathcal{G}$ is defined. Sections 5 and 6 are devoted to constructing the localization sequences for deformation K-theory, and in particular explicitly identifying the base as an equivariant smash product. In section 7 the algebra and module structures are made explicit by making use of results of Mandell and Elmendorf. The main theorems are proved in sections 8 and 9.

A proof of the product formula for representations in $\mathrm{GL}(n)$, rather than $\mathrm{U}(n)$, would also be desirable. This paper makes use of quite rigid constructions that make apparent the identification of the base in the localization sequence with a particular model for the equivariant smash product. In the case of $\mathrm{GL}(n)$, the definitions of both the cofiber in the localization sequence and the equivariant smash product need to be replaced by notions that are more well-behaved from the point of view of homotopy theory.

2. Preliminaries on G-equivariant Γ-spaces

In this section, G is a compact Lie group, and actions of G on based spaces are assumed to fix the basepoint; a *free* action will be one that is free away from the basepoint. We will now carry out constructions of Γ-spaces in a naïve equivariant context. When G is trivial, these are the standard definitions for Γ-spaces.

For any natural number k, denote the based space $\{*, 1, \dots, k\}$ by k_+.

Let Γ^o_G be the category of *right* G-spaces which are a finite wedge of the form $\vee G_+$ with morphisms being G-equivariant. (Strictly speaking, we take a small skeleton for this category.) The set $\Gamma^o_G(X,Y)$ can be given the mapping space topology, which gives this category an enrichment in spaces. Explicitly,

$$\Gamma^o_G(\vee^k G_+, Z) \cong \prod^k Z$$

as a space. We refer to the opposite category as Γ_G. If G is trivial we drop it from the notation.

Definition 2. A Γ_G-space is a base-point preserving continuous functor $\Gamma^o_G \to \mathbf{Top}_*$.

Here $\mathbf{Top}_*$ is the category of compactly generated weak Hausdorff spaces with nondegenerate basepoint.

Any Γ_G-space M has an underlying Γ-space $M(G_+ \wedge -)$. This Γ-space inherits a continuous left G-action because the left action of G on the first factor of $G_+ \wedge Y$ is right G-equivariant. In other words, there is a continuous composite homomorphism

$$G \to \Gamma^o_G(G_+ \wedge Y, G_+ \wedge Y) \to F(M(G_+ \wedge Y), M(G_+ \wedge Y)).$$

Remark 3. More generally, if $H \to G$ is a map of groups, the formula $M \mapsto M(- \wedge_H G_+)$ defines a restriction map from Γ_G-spaces to Γ_H-spaces with a left action of the Weyl group NH/H.

Let $X, Z \in \Gamma^o_G$, Y a based set. The continuous G-equivariant left action of G on $G_+ \wedge Y$, acting on the left-hand factor alone, gives rise to a continuous right action of G on $\Gamma^o_G(G_+ \wedge Y, Z)$. There is a map $\phi : X \to \Gamma^o_G(G_+ \wedge Y, X \wedge Y)$ given by $x \mapsto \phi_x$, where $\phi_x(g \wedge y) = xg \wedge y$. The map ϕ is clearly G-equivariant. Composing this map with the functor M gives a continuous G-equivariant map $X \to F(M(G_+ \wedge Y), M(X \wedge Y))$, and the adjoint is a natural map $X \wedge_G M(G_+ \wedge Y) \to M(X \wedge Y)$.

For any based right G-space X and Γ_G-space M, we can functorially form a new Γ-space $X \otimes_G M$ as follows. First, X defines a functor $F^G(-, X) : \Gamma_G \to \mathbf{Top}_*$.

Definition 4. The Γ-space $X \otimes_G M$ is defined as follows:

$$X \otimes_G M(Z) = \left(\coprod_{Y \in \Gamma_G^o} M(Y) \wedge F^G(Y, X \wedge Z) \right) / \sim$$

Here the equivalence relation $\sim$ is generated by relations $(u, f^*v) \sim (f_*u, v)$ for $f : Y \to Y', u \in M(Y), v \in F^G(Y', X \wedge Z)$. More concisely, $X \otimes_G M(Z)$ can be expressed as the coend

$$\int^Y M(Y) \wedge F^G(Y, X \wedge Z).$$

Remark 5. If $X_.$ is a simplicial object in the category Γ_G^o, there is a natural homeomorphism $|X_.| \otimes_G M \to |M(X_.)|$. (A short proof can be given by expressing $|X_.|$ as a coend $\int^n X_n \wedge \Delta_+^n$ and applying "Fubini's theorem" - see [10], chapter IX.) The reason for allowing G-CW complexes rather than simply restricting to these simplicial objects is that some G-homotopy types cannot be realized by simplicial objects. For example, any such simplicial object of the form X_+ is a principal G-bundle over X/G_+, and is classified by an element in $H^1_{discrete}(X/G, G)$. A general X_+ is classified by an element in $H^1_{cont.}(X/G, G)$.

We will refer to the space $(X \otimes_G M)(1_+)$ as $M(X)$; this agrees with the notation already defined when $X \in \Gamma_G^o$. We will only apply this construction to cofibrant objects in a certain model category of based G-spaces; specifically, we will only apply this construction to based G-CW complexes with free action away from the basepoint. Such objects are formed by iterated cell attachment of $G_+ \wedge D_+^n$ along $G_+ \wedge S_+^{n-1}$.

It will be useful to have homotopy theoretic control on $X \otimes_G M$, for the purposes of which we introduce a less rigid tensor product.

Definition 6. For M a Γ_G-space, we can define a simplicial Γ_G-space $LM_.$ by setting $LM(Z)_p$ equal to

$$\bigvee_{Z_0, \ldots, Z_p} \Gamma_G^o(Z_p, Z) \wedge \Gamma_G^o(Z_{p-1}, Z_p) \wedge \cdots \wedge \Gamma_G^o(Z_0, Z_1) \wedge M(Z_0).$$

The face maps are given by:

$$\begin{aligned} d_i(f_p \wedge \ldots \wedge f_0 \wedge m) &= f_p \wedge \ldots \wedge f_i \circ f_{i-1} \wedge \ldots \wedge f_0 \wedge m \quad \text{if } i < p \\ d_p(f_p \wedge \ldots \wedge f_0 \wedge m) &= f_p \wedge \ldots \wedge f_1 \wedge (M f_0)(m) \end{aligned}$$

The degeneracy map s_i is insertion of an identity map after f_i for $0 \leq i \leq p$.

The simplicial Γ_G-space $LM_{\cdot}$ has a natural augmentation $LM_{\cdot} \to M$. The augmented object $LM_{\cdot} \to M$ has an extra degeneracy map s_{-1}, defined by

$$s_{-1}(f_p \wedge \ldots \wedge f_0 \wedge m) = id \wedge f_p \wedge \ldots \wedge f_0 \wedge m.$$

As a result, the map $|LM_{\cdot}(Z)| \to M(Z)$ is a homotopy equivalence for any $Z \in \Gamma_G^o$. (Note that $LM_{\cdot}$ is the bar construction $B(\Gamma_G^o, \Gamma_G^o, M)$.)

For X a right G-space, consider the simplicial Γ_G-space $X \otimes_G LM_{\cdot}$. We have

$$X \otimes_G LM_p = \bigvee_{Z_0,\ldots,Z_p} \left[X \otimes_G \Gamma_G^o(Z_p, -)\right] \wedge \Gamma_G^o(Z_{p-1}, Z_p) \wedge \cdots \wedge M(Z_0)$$

as a Γ-space; the tensor construction distributes over wedge products and commutes with smashing with spaces. However, a straightforward calculation yields the formula

$$\left[X \otimes_G \Gamma_G^o(Y, -)\right](Z) \cong F^G(Y, X \wedge Z),$$

the space of G-equivariant based functions from Y to $X \wedge Z$.

PROPOSITION 7. *For X a G-CW complex with free action away from the basepoint, the augmentation map $X \otimes_G LM_{\cdot} \to X \otimes_G M$ is a levelwise weak equivalence of Γ-spaces after realization.*

Proof. It suffices to prove that $|LM_{\cdot}(X)| \to M(X)$ is a weak equivalence for any G-CW complex X. We will prove this by showing that it is filtered by weak equivalences.

Suppose X is a G-CW complex. For any $n \in \mathbb{N}$, define a restricted mapping space $F(Y, X)^{(n)}$ to be the subspace of $F(Y, X)$ consisting of those functions whose image contains representatives of at most n distinct G-orbits of X. The space $M(X)$ is the limit of a natural sequence of subspaces $M(X)^{(n)}$, which are defined by

$$M(X)^{(n)} = \left(\coprod_Y M(Y) \wedge F^G(Y, X)^{(n)}\right) / \sim,$$

where the equivalence relation is the same as that defining $M(X)$.

For any $n > 0$, this yields the following natural pushout square:

$$\begin{array}{ccc}
F(n_+, X)^{(n-1)} \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+) & \longrightarrow & F(n_+, X) \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+) \\
\downarrow & & \downarrow \\
M(X)^{(n-1)} & \longrightarrow & M(X)^{(n)}
\end{array}$$

Here $\Sigma_n \int G$ is the wreath product $G^n \rtimes \Sigma_n$. X is a G-CW complex, so the horizontal arrows are cofibrations. This identifies the cofiber of the map $M(X)^{(n-1)} \to M(X)^{(n)}$ with the space

$$C(n,X) \underset{\Sigma_n \int G}{\wedge} M(G_+ \wedge n_+).$$

Here $C(n,X)$ is the quotient of the space X^n by the subspace consisting of elements $(x_1,\ldots,x_n)$ where $x_i = *$ for some i or $x_i = gx_j$ for some $g \in G, i \neq j$. Since X was a free G-CW complex, $C(n,X)$ admits a free $(\Sigma_n \int G)$-CW complex structure.

We can applying this same construction to $|LM_.(X)|$; the space $|LM_.(X)|^{(n)}$ is the realization of the simplicial space $LM_.(X)^{(n)}$. The cofiber of the corresponding map $|LM_.(X)|^{(n-1)} \to |LM_.(X)|^{(n)}$ is the geometric realization

$$\left| C(n,X) \underset{\Sigma_n \int G}{\wedge} LM_.(G_+ \wedge n_+) \right| \cong C(n,X) \underset{\Sigma_n \int G}{\wedge} |LM_.(G_+ \wedge n_+)|.$$

There is a map of cofibration sequences as follows:

$$\begin{array}{ccccc} |LM_.(X)|^{(n-1)} & \longrightarrow & |LM_.(X)|^{(n)} & \longrightarrow & C(n,X) \wedge_{\Sigma_n \int G} |LM_.(G_+ \wedge n_+)| \\ \downarrow & & \downarrow & & \downarrow \\ M(X)^{(n-1)} & \longrightarrow & M(X)^{(n)} & \longrightarrow & C(n,X) \wedge_{\Sigma_n \int G} M(G_+ \wedge n_+) \end{array}$$

To show inductively that $|LM_.(X)|^{(n)} \to M(X)^{(n)}$ is a weak equivalence, it therefore suffices to show that the right-hand vertical map is a weak equivalence for all n. $C(n,X)$ is a cofibrant $(\Sigma_n \int G)$-space, so smashing with it over $\Sigma_n \int G$ preserves weak equivalences. The result follows because $|LM_.(G_+ \wedge n_+)| \to M(G_+ \wedge n_+)$ is a weak equivalence for all n.

COROLLARY 8. *If a map $X \to Y$ of free G-CW complexes is k-connected, so is the map $M(X) \to M(Y)$.*

Proof. It suffices to show that the map $|LM_.(X)| \to |LM_.(Y)|$ is k-connected. However, this is a map of simplicial spaces which levelwise is of the form

$$\begin{array}{c} \bigvee_{Z_0,\ldots,Z_p} F^G(Z_p,X) \wedge \Gamma^o_G(Z_{p-1},Z_p) \wedge \cdots \wedge \Gamma^o_G(Z_0,Z_1) \wedge M(Z_0) \\ \downarrow \\ \bigvee_{Z_0,\ldots,Z_p} F^G(Z_p,Y) \wedge \Gamma^o_G(Z_{p-1},Z_p) \wedge \cdots \wedge \Gamma^o_G(Z_0,Z_1) \wedge M(Z_0). \end{array}$$

This map is k-connected because the map $F^G(Z_p,X) \to F^G(Z_p,Y)$ is. Since $LM_.(X) \to LM_.(Y)$ is k-connected levelwise, so is the map of geometric realizations.

For any Γ_G-space M, we have an associated (naïve pre-)spectrum $\{M(G_+ \wedge S^n)\}$, which is the same as the spectrum of the underlying Γ-space $M(G_+ \wedge -)$. A map of Γ_G-spaces $M \to M'$ is called a stable equivalence if the associated map of spectra is a weak equivalence.

PROPOSITION 9. *For any Γ_G-space M and free based G-CW complex X, the map $X \wedge_G M(G_+ \wedge -) \to X \otimes_G M$ is a stable equivalence.*

Proof. It suffices to show that $X \wedge_G M(G_+ \wedge S^n) \to M(X \wedge S^n)$ is highly connected for large n. Using the levelwise weak equivalence $|X \otimes_G LM_.| \to X \otimes_G M$, it suffices to show that this statement is true for Γ-spaces of the form $\Gamma^o_G(Y, -)$ for $Y \in \Gamma^o_G$.

In this case, we have the following diagram.

$$\begin{array}{ccc}
X \wedge_G \bigvee_Y (G_+ \wedge S^n) & \longrightarrow & \bigvee_Y X \wedge_G (G_+ \wedge S^n) \\
\downarrow & & \downarrow \\
X \wedge_G \prod_Y (G_+ \wedge S^n) & \longrightarrow & \prod_Y (X \wedge S^n) \\
\| & & \| \\
X \wedge_G \Gamma^o_G(Y, G_+ \wedge S^n) & \longrightarrow & \Gamma^o_G(Y, X \wedge S^n)
\end{array}$$

The top vertical arrows are isomorphisms on homotopy groups up to roughly dimension $2n$ since $G_+ \wedge S^n$ is $(n-1)$-connected. The uppermost horizontal arrow is an isomorphism. Therefore, the bottom map is an equivalence on homotopy groups up to roughly dimension $2n$, as desired.

3. Connective equivariant K-homology

In this section, we construct for each n a $\Gamma_{\mathrm{PU}(n)}$-space whose underlying spectrum is homotopy equivalent to ku, the connective K-theory spectrum.

Let W be a fixed n-dimensional inner product space. For any $d \in \mathbb{N}$, we have the Stiefel manifold $V(nd)$ of isometric embeddings of $W \otimes \mathbb{C}^d$ into $\mathbb{C}^\infty$, where $\mathbb{C}^d$ has the standard inner product. The space $V(nd)$ has a free right action by $I \otimes \mathrm{U}(d)$ by precomposition. Denote the quotient space by $H(d)$, and write $H = \coprod_d H(d)$. (Note $H \simeq \coprod_d \mathrm{BU}(d)$.) H has a partially defined direct sum operation: if $\{V_i\}$ is a family of elements of H such that $V_i \perp V_j$ for $i \neq j$, there is a sum element $\oplus V_i$ in H.

There is also an action of $\mathrm{U}(n) \otimes I$ on $V(nd)$ that commutes with the action of $I \otimes \mathrm{U}(d)$, hence passes to an action on the quotient $H(d)$. Since $\lambda I \otimes I = I \otimes \lambda I$, the scalars in $\mathrm{U}(n)$ act trivially on $H(d)$, so the action factors through $\mathrm{PU}(n)$. We therefore get a right action of $\mathrm{PU}(n)$ on H. The direct sum operation is $\mathrm{PU}(n)$-equivariant.

For any $Z \in \Gamma_G^o$, define

$$ku^{\mathrm{PU}(n)}(Z) = \left\{ f \in F^{\mathrm{PU}(n)}(Z, H) \;\middle|\; f(z) \perp f(z') \text{ if } [z] \neq [z'] \right\}.$$

A point of $ku^{\mathrm{PU}(n)}(Z)$ consists of a vector space isomorphic to $W^{\dim f(z)}$ associated to each non-basepoint $[z]$ of Z/G such that the vector spaces associated to $[z]$ and $[z']$ are orthogonal if $[z] \neq [z']$.

Given a map $\alpha \in \Gamma_G^o(Z, Z')$ and $f \in ku^{\mathrm{PU}(n)}(Z)$, we get an element $ku^{\mathrm{PU}(n)}(\alpha)(f) \in ku^{\mathrm{PU}(n)}(Z')$ as follows:

$$ku^{\mathrm{PU}(n)}(\alpha)(f)(z') = \bigoplus_{\alpha(z)=z'} f(z)$$

This is well-defined: if the preimage of z is the family z_i, then the z_i all lie in distinct orbits. The map $ku^{\mathrm{PU}(n)}(\alpha)(f)$ is also clearly $\mathrm{PU}(n)$-equivariant, and takes distinct orbits to orthogonal elements of H. The spectrum attached to the underlying Γ-space of $ku^{\mathrm{PU}(n)}$ is weakly equivalent to the connective K-theory spectrum ku - see [17].

We also define a second Γ_G-space ku/β as follows. For any $z \in \Gamma_G^o$,

$$ku/\beta(Z) = \tilde{\mathbb{N}}[Z/G].$$

More explicitly, $ku/\beta(Z)$ is the quotient of the free abelian monoid on Z/G by the submonoid $\mathbb{N}[*]$. For $\alpha \in \Gamma_G^o(Z, Z')$, $ku/\beta(\alpha)(\sum n_z[z]) = \sum n_z[\alpha(z)]$. (The reason for the notation is that the underlying spectrum is the cofiber of the Bott map - this will be made more explicit in section 7.)

For X a free right $\mathrm{PU}(n)$-space, $X \otimes_{\mathrm{PU}(n)} ku/\beta$ is the infinite symmetric product $\mathrm{Sym}^\infty(X/\mathrm{PU}(n))$.

There is a natural map $\epsilon : ku^{\mathrm{PU}(n)} \to ku/\beta$ of Γ_G-spaces: if $f \in ku^{\mathrm{PU}(n)}(Z)$, define $\epsilon(f) = \sum_{[z]} \left(\frac{\dim f(z)}{n}\right)[z]$. The map ϵ represents the augmentation $ku \to \mathrm{H}\mathbb{Z}$ on the underlying spectra, the first stage of the Postnikov tower for ku.

The Γ_G-space $ku^{\mathrm{PU}(n)}$ determines a homology theory for $\mathrm{PU}(n)$-spaces. Specifically, we can define

$$ku_*^{\mathrm{PU}(n)}(X) = \pi_* \left(X \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \right).$$

Here π_* denotes the stable homotopy groups of the spectrum. In fact, since the underlying spectrum of $ku^{\mathrm{PU}(n)}$ is *special*, we can compute $ku_*^{\mathrm{PU}(n)}(X) = \pi_* \left(ku^{\mathrm{PU}(n)}(X) \right)$ for X connected. (See [16], 1.4.)

4. Unitary deformation K-theory

In this section we will have a fixed finitely generated *discrete* group $\mathcal{G}$. Carlsson, in [2], defined a notion of the "deformation K-theory" of $\mathcal{G}$ as a contravariant functor from groups to spectra, and in the introduction of this article an analogous notion of "unitary deformation K-theory" $\mathcal{K}\mathcal{G}$ was sketched. The following are weakly equivalent definitions of the corresponding notion of $\mathcal{K}\mathcal{G}$:

- The group completion of $\coprod_n \mathrm{EU}(n) \times_{\mathrm{U}(n)} \mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))$.
- The K-theory of a category of unitary representations of $\mathcal{G}$. This category is an internal category in **Top**: i.e., the objects and morphism sets are both given topologies.
- The simplicial object which is the K-theory of the singular complex of the category above. (This is essentially the definition given in [2].)

We will now describe another model for the unitary deformation K-theory of $\mathcal{G}$, equivalent to the first definition given above. The construction is based on the construction of connective topological K-homology of Segal in [17]. (Also see [18].)

Let $\mathcal{U} = \mathbb{C}^\infty$ be the infinite inner product space with orthonormal basis $\{e_i\}$. The group $\mathrm{U} = \operatorname{colim} \mathrm{U}(n)$ acts on $\mathcal{U}$. A $\mathcal{G}$-plane V of dimension k is a pair (V, ρ) where V is a k-plane in $\mathcal{U}$ and $\rho : \mathcal{G} \to \mathrm{U}(V)$ is an action of $\mathcal{G}$ on V.

We now describe a (non-equivariant) Γ-space $\mathcal{K}\mathcal{G}$. Define

$$\mathcal{K}\mathcal{G}(X) = \left\{ (V_x, \rho_x)_{x \in X} \;\middle|\; V_x \text{ a } \mathcal{G}\text{-plane, } V_x \perp V_{x'} \text{ if } x \neq x', V_* = 0 \right\}.$$

This is a special Γ-space. The underlying H-space is

$$\mathcal{K}\mathcal{G}(1_+) \simeq \coprod_n V(n) \times_{\mathrm{U}(n)} \mathrm{Hom}(\Gamma, \mathrm{U}(n)),$$

where $V(n)$ is the Stiefel manifold of n-frames in $\mathcal{U}$. We will now describe the simplicial space $X_. = \mathcal{K}\mathcal{G}(S^1)$. Since $\mathcal{K}\mathcal{G}$ is special, $\Omega|X_.| \simeq \Omega^\infty \mathcal{K}\mathcal{G}$.

For $p > 0$, X_p is the space

$$\left\{ (V_i, \rho_i)_{i=1}^p \;\middle|\; (V_i, \rho_i) \text{ a } \mathcal{G}\text{-plane, } V_i \perp V_j \text{ if } i \neq j \right\}.$$

(X_0 is a point.) Face maps are given by taking sums of orthogonal $\mathcal{G}$-planes or removing the first or last $\mathcal{G}$-plane. Degeneracy maps are given by insertion of 0-dimensional $\mathcal{G}$-planes.

The geometric realization of this simplicial space can be explicitly identified. Let Y be the space of pairs (A, ρ), where $A \in \mathrm{U}$ and ρ is a homomorphism $\mathcal{G} \to \mathrm{U}$ commuting with A. Call two such elements (A, ρ) and (A', ρ') equivalent if $A = A'$ and ρ, ρ' agree on all eigenspaces of A corresponding to eigenvalues $\lambda \neq 1$. Write the standard p-simplex Δ^p as the set of all $0 \leq t_1 \leq \ldots \leq t_p \leq 1$. Then there is a homeomorphism $|X_.| \to (Y/\sim)$ given by sending a point $((V_i, \rho_i)_{i=1}^p, 0 \leq t_1 \leq \ldots \leq t_p \leq 1)$ of $X_p \times \Delta^p$ to the pair (A, ρ), where A acts on V_i with eigenvalue $e^{2\pi i t_i}$ and by 1 on the orthogonal complement of ΣV_i, while ρ acts on V_i by ρ_i and acts by 1 on the orthogonal complement of ΣV_i. This map is a homeomorphism by the spectral theorem. (The essential details of this argument are from [8] and [12].)

We will refer to this space $|X.| \cong (Y/\sim)$ as E. It is space 1 of the Ω-spectrum associated to $\mathcal{KG}$, in the sense that $\Omega^\infty \mathcal{KG} \simeq \Omega E$.

This method is applicable to various other categories of representations of $\mathcal{G}$ that we will now examine in detail.

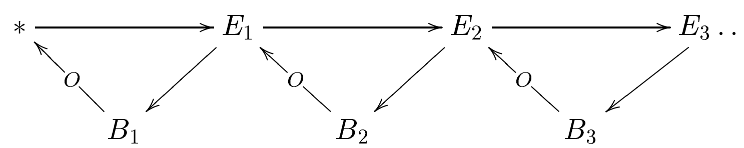
For any $i \geq 0$, there is a sub-Γ-space $\mathcal{KG}_i$ of $\mathcal{KG}$ such that $\mathcal{KG}_i(X)$ consists of those elements of $(V_x, \rho_x)_{x \in X}$ of $\mathcal{KG}(X)$ such that ρ_x breaks up into a direct sum of irreducible representations of dimension less than or equal to i. Each $\mathcal{KG}_i$ is a special Γ-space.

We have infinite loop spaces $E_i = |\mathcal{KG}_i(S^1)|$. For any $i \in \mathbb{N}$, E_i is the subspace of E which is the image of the space of pairs (A, ρ) such that ρ is a direct sum of irreducible representations of $\mathcal{G}$ of dimension less than or equal to i.

This gives a sequence of inclusions

$$* = E_0 \subset E_1 \subset E_2 \subset \ldots$$

of infinite loop spaces. Each of these inclusions is part of a quasifibration sequence $E_{i-1} \to E_i \to B_i$ where the base spaces will be explicitly identified. This gives rise to the following "exact couple" of spectra



Additionally, the inclusions of infinite loop spaces E_i are induced by maps of ku-modules, so the above is induced by an exact couple of ku-module spectra.

The intuition for the description of B_i is that the category of $\mathcal{G}$-representations whose irreducible summands have dimension less than i forms a Serre subcategory of the category of $\mathcal{G}$-representations whose irreducibles have dimension less than or equal to i, and the quotient category should be the category of sums of irreducible representations of dimension exactly i. The topology on the categories involved complicates the question of when a localization sequence of spectra exists in this situation, since the most obvious attempts to generalize of Quillen's Theorem B would not be applicable. We will construct the localization sequence explicitly.

There is a quotient Γ-space F_i of $\mathcal{KG}_i$ by the equivalence relation $(V_x, \rho_x)_{x \in X} \sim (V'_x, \rho'_x)_{x \in X}$ if for all $x \in X$:

- The subspace W_x of V_x generated by irreducible subrepresentations of ρ of dimension i coincides with the corresponding subspace for ρ'.
- ρ and ρ' agree on W_x.

Again, F_i is a special Γ-space.

Define B_i to be the space $|F_i(S^1)|$. B_i is the quotient of E_i by the following equivalence relation. We say $(A, \rho) \sim (A', \rho')$ if:

- The subspace W of $\mathcal{U}$ generated by irreducible subrepresentations of ρ of dimension i is the same as the subspace of $\mathcal{U}$ generated by irreducible subrepresentations of ρ' of dimension i.
- ρ and ρ' have the same action on W.
- A and A' have the same action on W.

Note that each equivalence class contains a unique pair (A, ρ) such that ρ acts trivially on the eigenspace of A for 1 and on the complementary subspace ρ is a direct sum of irreducible i-dimensional representations.

5. Proof of the existence of the localization sequence

The proof that $p_i : E_i \to B_i$ is a quasifibration (and hence induces a long exact sequence on homotopy groups) proceeds inductively using the following result of Hardie [7]:

THEOREM 10. *Suppose that we have a diagram*

$$\begin{array}{ccccc} Q & \xleftarrow{h} & f^*(E) & \longrightarrow & E \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ Q' & \xleftarrow[g]{} & A & \xrightarrow[f]{} & B \end{array}$$

where f is a cofibration, p is a fibration, $f^*(E)$ is the pullback fibration, and λ is a quasifibration. If $h : s^{-1}(a) \to \lambda^{-1}(ga)$ is a *weak equivalence for all $a \in A$, then the induced map of pushouts $Q \coprod_{f^*(E)} E \to Q' \coprod_A B$ is a quasifibration.*

PROPOSITION 11. *The map $p_i : E_i \to B_i$ is a quasifibration with fiber E_{i-1}.*

Remark 12. This is what we might expect, as the map $E_i \to B_i$ is precisely the map which forgets the irreducible subrepresentations of dimension less than i. The fact that E_{i-1} is the honest fiber over any point is clear, but we need to show that E_{i-1} is also the homotopy fiber.

Proof. We will proceed by making use of a rank filtration. These rank filtrations were introduced in [12] and [14]. In particular, Mitchell explicitly describes this rank filtration for the connective K-theory spectrum.

For any j, let $B_{i,j}$ be the subspace of B_i generated by those pairs (A, ρ) such that ρ contains at most a sum of j irreducible representations. There is a sequence of inclusions $B_{i,j-1} \subset B_{i,j}$. Write $E_{i,j}$ for the subset of E_i lying over $B_{i,j}$.

The map $E_{i,0} \to B_{i,0}$ is a quasifibration, since $B_{i,0}$ is a point. Now suppose inductively that $E_{i,j-1} \to B_{i,j-1}$ is a quasifibration.

Let Y_j be the space of triples (A, ρ, W), where W is an ij-dimensional subspace of $\mathcal{U}$, A is an element of $\mathrm{U}(W)$, and ρ is a representation of $\mathcal{G}$ on W commuting with A and containing irreducible summands of dimension i or less. Let X_j be the subset of Y_j of triples (A, ρ, W) such that (A, ρ) represents a pair in $B_{i,j-1}$; in other words, ρ contains less than j distinct i-dimensional irreducible summands on the orthogonal complement of the eigenspace for 1 of A.

Next, we define a space Y'_j of triples (A, ρ, W), where $(A, \rho) \in E_{i,j}$ and W is an A- and ρ-invariant ij-dimensional subspace of $\mathcal{U}$ containing all the i-dimensional irreducible summands of ρ. There is a map $Y'_j \to Y_j$ given by forgetting the actions of A and ρ off W. Let X'_j be the fiber product of X_j and Y'_j over Y_j; it consists of triples (A, ρ, W) where ρ contains less than j distinct i-dimensional summands.

There is a map $X_j \to B_{j-1}$ given by sending (A, ρ, W) to (A, ρ), and a similar map $X'_j \to E_{j-1}$. These maps all assemble into the diagram below.

$$
\begin{array}{ccccc}
E_{i,j-1} & \longleftarrow & X'_j & \longrightarrow & Y'_j \\
\big\downarrow{\scriptstyle p_i} & & \big\downarrow & & \big\downarrow{\scriptstyle p} \\
B_{i,j-1} & \longleftarrow & X_j & \longrightarrow & Y_j
\end{array}
$$

There is an evident map from the pushout of the bottom row to $B_{i,j}$, and similarly a map from the pushout of the top row to $E_{i,j}$.

The map $X_j \to B_{i,j-1}$ is a quotient map; two points become identified by forgetting the "framing" subspace W, the non-i-dimensional summands of ρ, and the summands of ρ on the eigenspace for 1 of A. For points of Y_j not in X_j, the framing subspace W is determined by the image (A, ρ) in B_j since ρ must have j distinct i-dimensional irreducible summands covering all of W, and A can have no eigenspace for the eigenvalue 1. Therefore, the map from Y_j to the pushout of the bottom row is precisely the quotient map gotten by forgetting the framing W and any non-i-dimensional summands or summands lying on the eigenspace for 1 of A. However, this identifies the pushout with $B_{i,j}$. In exactly the same way, the pushout of the top row is $E_{i,j}$. The induced map of pushouts is the projection map $E_{i,j} \to B_{i,j}$.

The map $X_j \to Y_j$ is a cofibration because it is the colimit of geometric realizations of a closed inclusion of real points of algebraic varieties. (The subspace W is allowed to vary over the infinite Grassmannian. If we restrict its image to any finite subspace we get an inclusion of real algebraic varieties.)

The right-hand square is a pullback by construction, and the map p_i is assumed to be a quasifibration.

The map p is a fiber bundle with fiber E_{i-1}: An equivalence class of points $(A, \rho, W) \in Y'_j$ consists of a choice of ij-dimensional subspace W of $\mathcal{U}$, a choice of element in $(\bar{A}, \bar{\rho}, W)$ in Y_j to determine the action of A and ρ on W, and a choice of (A', ρ') acting on the orthogonal complement of W such that ρ' is made up of summands of dimension less than i. In other words, there is a pullback square:

$$\begin{array}{ccc} Y'_j & \longrightarrow & V \\ \downarrow & & \downarrow \\ Y_j & \longrightarrow & \mathrm{Gr(ij)} \end{array}$$

Here $\mathrm{Gr}(ij)$ is the Grassmannian of ij-dimensional planes in $\mathcal{U}$, and V is the bundle over the Grassmannian consisting of ij-dimensional planes in $\mathcal{U}$ and elements of E_{i-1} acting on their orthogonal complements.

Given any point (A, ρ, W) of X_j, the fiber in X'_j is E_{i-1} acting on the orthogonal complement of W. Suppose that (A, ρ) in $B_{i,j-1}$ is in canonical form: ρ acts by a sum of irreducible dimension i representations on some subspace $W' \subset W$ and trivial representations on the orthogonal complement, and A has eigenvalue 1 on the orthogonal complement of W'. Then the fiber over (A, ρ) in $E_{i,j-1}$ consists of all possible actions of E_{i-1} on the orthogonal complement of W'. The map from the fiber over (A, ρ, W) to the fiber over (A, ρ) is the inclusion of E_{i-1} acting on $W^\perp$ to E_{i-1} acting on $(W')^\perp$. This inclusion is a homotopy equivalence.

Therefore, $E_{i,j} \to B_{i,j}$ is a quasifibration with fiber E_{i-1}. Taking colimits in j, $E_i \to B_i$ is a quasifibration with fiber E_{i-1}.

COROLLARY 13. *The maps $\mathcal{KG}_{i-1} \to \mathcal{KG}_i \to F_i$ realize to a fibration sequence in the homotopy category of spectra.*

Proof. This follows because all three of these Γ-spaces are special.

6. Identification of the Γ-space F_n

Using the results of section 2, we will now identify the Γ-spaces F_n as equivariant smash products.

Let $\mathrm{Sum}(\mathcal{G}, n)$ be the subspace of $\mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))$ of reducible $\mathcal{G}$-representations of dimension n. Define $R_n = \mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))/\mathrm{Sum}(\mathcal{G}, n)$. There is a free action of $\mathrm{PU}(n)$ on R_n by conjugation. According to a result of Park and Suh ([13], Theorem 3.7), the algebraic variety $\mathrm{Hom}(\mathcal{G}, U(n))$ admits the structure of a $\mathrm{U}(n)$-CW complex. Since all isotropy groups of R_n contain the diagonal subgroup, this structure is actually the structure of a $\mathrm{PU}(n)$-CW complex, and R_n has an induced CW-structure.

PROPOSITION 14. *There is an isomorphism of Γ-spaces*

$$R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \to F_n.$$

Proof. By the universal property of the coend

$$R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(Z) = \int^Y ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z),$$

we can construct the map by exhibiting maps

$$ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z) \to F_n(Z),$$

natural in Z, that satisfy appropriate compatibility relations in Y.

Recall that a point of $ku^{\mathrm{PU}(n)}(Y)$ consists of an equivariant map $f : Y \to H = \coprod V(nd)/I \otimes \mathrm{U}(d)$ such that $f(y) \perp f(y')$ if $y \neq y'$.

Suppose $f \wedge g \in ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z)$. For every $y \in Y$ the element $g(y) = r(y) \wedge z(y)$ determines an irreducible action $r(y)$ of $\mathcal{G}$ on $\mathbb{C}^n$. The element $f(y) \in V(nd)/I \otimes \mathrm{U}(d)$ is the image of some element $\widetilde{f(y)} \in V(nd)$, which determines an isometric embedding $W \otimes \mathbb{C}^d \to \mathcal{U}$. Combining these two gives an action of $\mathcal{G}$ on an nd-plane of $\mathcal{U}$, together with a marking $z(y)$ of the plane by an element of Z. The action of $I \otimes \mathrm{U}(d)$ commutes with the $\mathcal{G}$-action on $W \otimes \mathbb{C}^d$, so the choice of lift $\widetilde{f(y)}$ does not change the resulting $\mathcal{G}$-plane. For $g \in \mathcal{G}$, $r(gy) = g \cdot r(y) = gr(y)g^{-1}$, and $f(gy) = (g \otimes I)f(y)(g^{-1} \otimes I)$, so the resulting plane only depends on the orbit Gy. The resulting $\mathcal{G}$-plane breaks up into irreducibles of dimension precisely n. Assembling these $\mathcal{G}$-planes over the distinct orbits gives a collection of orthogonal hyperplanes with $\mathcal{G}$-actions, marked by points of Z, which break up into a direct sum of n-dimensional irreducible representations. As $r(y)$ approaches the basepoint of R_n, the representation becomes reducible, so the map determines a well-defined element of $F_n(Z)$. The compatibility of this map with maps in Y is due to the fact that it preserves direct sums.

This map is bijective; associated to any point of $F_n(Z)$ there is a unique equivalence class of points which map to it. We leave it to the reader to verify that the inverse map is continuous.

7. E_∞-algebra and module structures

In this section we will make explicit the following. The tensor product of representations leads to the following multiplicative structures:

1. ku is an E_∞-ring spectrum.

2. $\mathcal{KG}$ is an E_∞-algebra over ku.

3. The sequence of maps $\mathcal{KG}_1 \to \mathcal{KG}_2 \to \cdots \to \mathcal{KG}$ is a sequence of E_∞-ku-module maps.

4. There are compatible E_∞-ku-linear pairings $\mathcal{KG}_n \wedge \mathcal{KG}_m \to \mathcal{KG}_{nm}$ for all n, m.

5. The map $\mathcal{KG}_n \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$ is a map of E_∞-ku-modules.

All of the above structures are natural in $\mathcal{G}$. From this point onward, we are only interested in derived categories of module spectra, and so all smash products are meant in the derived sense.

To begin, we will first recall the definition of a *multicategory*. A multicategory is an "operad with several objects", as follows. See [6].

Definition 15. A multicategory **M** consists of the following:

1. A class of objects Ob(**M**).

2. A set $\mathbf{M}_k(a_1, \ldots, a_k; b)$ for each $a_1, \ldots, a_k, b \in \mathrm{Ob}(\mathbf{M})$, $k \geq 0$ of "k-morphisms" from $(a_1, \ldots, a_k)$ to b.

3. A right action of the symmetric group Σ_k on the class of all k-morphisms such that σ^* maps the set $\mathbf{M}_k(a_1, \ldots, a_k; b))$ to the set $\mathbf{M}_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}; b)$.

4. An "identity" map $1_a \in M_1(a; a)$ for all $a \in \mathrm{Ob}(\mathbf{M})$.

5. A "composition" map

$$\mathbf{M}_n(b_1, \ldots, b_n; c) \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \times \cdots \\ \to \mathbf{M}_{k_1+\cdots+k_n}(a_{11}, \ldots, a_{nk_n}; c)$$

   which is associative, unital, and respects the symmetric group action.

We will not make precise these last definitions; they are essentially the same as the definitions for an operad. A map between multicategories that preserves the appropriate structure will be referred to as a multifunctor.

Example 16. Any symmetric monoidal category $(\mathbf{C}, \Box)$ is a multicategory, with

$$\mathbf{C}_k(a_1, \ldots, a_k; b) = \mathbf{C}(a_1 \Box \cdots \Box a_k, b).$$

For example, the categories of Γ-spaces or symmetric spectra under $\wedge$ are multicategories.

There is a (lax) symmetric monoidal functors $\mathbb{U}$ from Γ-spaces to symmetric spectra [11]. This naturally leads to a multifunctor from the multicategory of Γ-spaces to the multicategory of symmetric spectra.

Remark 17. The smash product of Γ-spaces of Γ-simplicial sets is defined using left Kan extension. As a result, we can equivalently define a multicategory structure on Γ-spaces without reference to the smash product by declaring the set of k-morphisms from $(M_1, \ldots, M_k)$ to N to be the set of collections of maps

$$M_1(Y_1) \wedge \cdots \wedge M_k(Y_k) \to N(Y_1 \wedge \cdots \wedge Y_k)$$

natural in $Y_1, \ldots, Y_k$.

We will now define two multicategories enriched over topological spaces. The first multicategory $\mathbf{F}$ is a parameter multicategory for an E_∞-filtered algebra. The second multicategory $\mathbf{M}^2$ is a parameter multicategory for maps of E_∞-modules. Let $\mathcal{E}(n)$ be the space of linear isometric embeddings of $\mathcal{U}^{\otimes n}$ in $\mathcal{U}$. Together the $\mathcal{E}(n)$ form an E_∞-operad.

Definition 18. The multicategory $\mathbf{F}$ has objects R (the ring), A (the algebra), and A_n for $n \geq 1$ (the algebra filtrations). Define a function filt on the objects by $R \mapsto 1, A_n \mapsto n, A \mapsto \infty$. The spaces of multicategory maps as defined as follows:

$$\mathbf{F}_k(B_1, \ldots, B_k; C) = \begin{cases} \emptyset & \text{if } \prod \text{filt}(B_i) > \text{filt}(C) \\ \mathcal{E}(n) & \text{if } \prod \text{filt}(B_i) \leq \text{filt}(C) \end{cases}$$

Composition is given by operad composition for $\mathcal{E}$.

Definition 19. The multicategory $\mathbf{M}^2$ has objects R (the ring), M_1, and M_2 (the modules). The mapping space $\mathbf{M}^2_k(B_1, \ldots, B_k; C)$ is equal to $\mathcal{E}(n)$ in the following cases:

- $C = B_i = R$
- $C = M_1$ and there is a unique i such that $B_i = M_1$; all other B_i are equal to R
- $C = M_2$ and there is a unique i such that $B_i = M_1$ or M_2; all other B_i are equal to R

Otherwise the mapping space is $\emptyset$. Composition is given by operad composition for $\mathcal{E}$.

PROPOSITION 20. *There are multifunctors $T : \mathbf{F} \to \Gamma$-spaces and $S_n : \mathbf{M}^2 \to \Gamma$-spaces, continuous with respect to the enrichment in spaces, such that:*

- $T(R) = ku$, $T(A) = \mathcal{KG}$, $T(A_n) = \mathcal{KG}_n$
- *The images under T of the identity maps in $\mathcal{E}(1)$ are the standard maps $ku \to \mathcal{KG}_1 \to \mathcal{KG}_2 \to \cdots \to \mathcal{KG}$*
- $S(R) = ku$, $S(M_1) = \mathcal{KG}_n$, $S(M_2) = F_n$
- *The image under S of the identity map in $\mathbf{M}^2(M_1, M_2) = \mathcal{E}(1)$ is the map $\mathcal{KG}_n \to F_n$*

Proof. Write $\mathcal{KG}^{\mathcal{U}}_n$ for the Γ-space $\mathcal{KG}_n$ indexed on the universe $\mathcal{U}$. For groups $\mathcal{G}_1, \ldots, \mathcal{G}_k$, there is a well-defined exterior tensor product of representations:

$$\mathcal{K}(\mathcal{G}_1)^{\mathcal{U}}_{n_1}(Z_1) \wedge \cdots \wedge \mathcal{K}(\mathcal{G}_k)^{\mathcal{U}}_{n_k}(Z_k) \to \mathcal{K}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_k)^{\mathcal{U}^{\otimes k}}_{n_1 \cdots n_k}(Z_1 \wedge \cdots \wedge Z_k)$$

Post-composition with linear isometric embeddings $\mathcal{U}^{\otimes k} \to \mathcal{U}$ then gives a map of Γ-spaces

$$\mathcal{E}(k)_+ \wedge \mathcal{K}(\mathcal{G}_1)_{n_1} \wedge \cdots \wedge \mathcal{K}(\mathcal{G}_k)_{n_k} \to \mathcal{K}(\mathcal{G}_1 \times \cdots \times \mathcal{G}_k).$$

If all $\mathcal{G}_i$ are equal to $\mathcal{G}$ or the trivial group, we can pull back along the diagonal map to get a map

$$\mathcal{E}(k)_+ \wedge B_1 \wedge \cdots \wedge B_k \to C,$$

where the B_i and C are either $\mathcal{KG}$ or ku. This map has a continuous adjoint which defines the multifunctor T. This map preserves composition and units.

Similarly, the exterior tensor product of $\mathcal{G}$-representations with trivial representations preserves the dimension of irreducible subrepresentations. In the same manner, we get maps

$$\mathcal{E}(k)_+ \wedge B_1 \wedge \cdots \wedge B_k \to F_n,$$

whenever all B_i are equal to ku except for at most one, and the adjoints of these maps define the multifunctor S.

These multifunctors are multifunctors of categories enriched in topological spaces. We can now apply Theorem 1.4 of [6] to find weakly equivalent models which have the structure of strict ring, module, and algebra spectra. (The singular complex functor must first be applied to move from the category of Γ-spaces to the category of Γ-simplicial sets, and then the Γ-spaces must be realized as symmetric spectra.) We will abuse notation and not change the names. Combining previous results with this, we have the following.

COROLLARY 21. *There exists a ring symmetric spectrum ku and contravariant functors $\mathcal{K}(-)_n$ and $\mathcal{K}(-)$ from finitely generated discrete groups to connective ku-module symmetric spectra with the following properties.*

- *There are ku-module maps $\mathcal{KG}_1 \to \mathcal{KG}_2 \to \cdots \to \mathcal{KG}$, and $\mathcal{KG}$ is weakly equivalent to the homotopy colimit.*
- *There are strictly commutative and associative ku-module pairings $\mathcal{KG}_n \wedge_{ku} \mathcal{KG}_m \to \mathcal{KG}_{nm}$ which commute with the above maps. (We allow the case when n or m are equal to ∞, using the convention $\mathcal{KG}_\infty = \mathcal{KG}$.)*
- *For any n, the homotopy cofiber of the map $\mathcal{KG}_{n-1} \to \mathcal{KG}_n$ is weakly equivalent as a ku-module to $ku^{\mathrm{PU}(n)} \wedge_{\mathrm{PU}(n)} R_n$.*

We also note that the cofiber sequence $\Sigma^2 ku \to ku \to \mathrm{H}\mathbb{Z}$ can be smashed over ku with $ku^{\mathrm{PU}(n)}$. The quotient $\mathrm{H}\mathbb{Z} \wedge_{ku} ku^{\mathrm{PU}(n)}$ is weakly equivalent as a $\mathrm{PU}(n)$-spectrum to the spectrum denoted ku/β in section 3.

8. The exact couple for $\mathcal{KG}$

There is the following chain of equivalences of spectra:

$$\mathrm{H}\mathbb{Z} \underset{ku}{\wedge} \left(ku^{\mathrm{PU}(n)} \underset{\mathrm{PU}(n)}{\wedge} R_n \right) \simeq ku/\beta \underset{\mathrm{PU}(n)}{\wedge} R_n \simeq \mathrm{H}\mathbb{Z} \wedge (R_n/\mathrm{PU}(n))$$

Define $\mathrm{QIrr}(\mathcal{G}, n) = R_n/\mathrm{PU}(n)$. $\mathrm{QIrr}(\mathcal{G}, n)$ is the quotient space of isomorphism classes of representations $\mathcal{G}$ of dimension n modulo decomposable representations. (The notation is to avoid confusion with the standard notation for the *subspace* of isomorphism classes of irreducible representations.)

Corollary 21 identifies the following cofiber sequences. The homotopy colimit of the top row is $\mathcal{KG}$.

$$\begin{array}{ccccccc}
* & \longrightarrow & \mathcal{KG}_1 & \longrightarrow & \mathcal{KG}_2 & \longrightarrow & \mathcal{KG}_3 \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & ku \wedge R_1 & & ku^{\mathrm{PU}(2)} \wedge_{\mathrm{PU}(2)} R_2 & & ku^{\mathrm{PU}(3)} \wedge_{\mathrm{PU}(3)} R_3
\end{array}$$

The following spectral sequence results.

THEOREM 22. *There exists a convergent right-half-plane spectral sequence of the form*

$$E_1^{p,q} = ku_{q-p+1}^{\mathrm{PU}(n)}(R_{p-1}) \Rightarrow \pi_{p+q}(\mathcal{KG}).$$

Remark 23. The grading convention is such that d_r maps $E_r^{p,q}$ to $E_r^{p-r,q+r-1}$.

Smashing the previous diagram over ku with $\mathrm{H}\mathbb{Z}$ yields the following. The homotopy colimit of the top row is $\mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{KG}$.

$$\begin{array}{ccccccc}
* & \longrightarrow & \mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{KG}_1 & \longrightarrow & \mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{KG}_2 & \longrightarrow & \mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{KG}_3 \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathcal{G}, 1) & & \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathcal{G}, 2) & & \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathcal{G}, 3)
\end{array}$$

Again, the diagram results in a spectral sequence.

THEOREM 24. *There exists a convergent right-half-plane spectral sequence of the form*

$$E_1^{p,q} = H_{q-p+1}(\mathrm{QIrr}(\mathcal{G}, p-1)) \Rightarrow \pi_{p+q}(\mathrm{H}\mathbb{Z} \underset{ku}{\wedge} \mathcal{KG}).$$

Example 25. When $\mathcal{G}$ is finite or nilpotent, the cofiber sequences are all split. When $\mathcal{G}$ is finite, this is clear. When $\mathcal{G}$ is nilpotent, results of [9] show that the space of irreducible representations of dimension n is closed in $\mathrm{Hom}(\mathcal{G}, \mathrm{U}(n))$, which provides the desired splitting $ku^{\mathrm{PU}(n)} \wedge_{\mathrm{PU}(n)} R_n \to \mathcal{K}\mathcal{G}_n$.

As a result, we have a weak equivalence

$$\mathcal{K}\mathcal{G} \simeq \bigvee \left(ku^{\mathrm{PU}(n)} \underset{\mathrm{PU}(n)}{\wedge} R_n \right).$$

In this case, the spectral sequence of Theorem 24 degenerates at the E_1 page. For example, consider the integer Heisenberg group:

$$\begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}$$

The $E_1 = E_\infty$ page of the spectral sequence of Theorem 24 is as follows:

$$\begin{array}{|cccccccc}
 & & & \vdots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & 0 & \dots \\
\hline
 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & \\
 & & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & \\
 & & & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \\
 & & & \vdots & & &
\end{array}$$

Example 26. When $\mathcal{G}$ is free on k generators, we can compute the deformation K-theory spectrum explicitly. In this case, $\mathcal{K}\mathcal{G} \simeq ku \vee (\vee^k \Sigma ku)$.

When $\mathcal{G}$ is free on two generators, explicit computations with the spectral sequence of Theorem 24 give the following picture of the E_1 page.

$$
\begin{array}{|ccccc}
 & & \vdots & & \\
0 & 0 & ? & ? & \\
0 & \mathbb{Z} & ? & ? & \\
0 & \mathbb{Z}^2 & ? & ? & \\
\mathbb{Z} & \mathbb{Z} & ? & ? & \\
\mathbb{Z}^2 & 0 & ? & ? & \\
\mathbb{Z} & 0 & ? & ? & \ldots \\
\hline
 & 0 & 0 & ? & \\
 & & 0 & 0 & \\
 & & \vdots & &
\end{array}
$$

The differential $d_1 : E_1^{1,2} \to E_1^{0,2}$ is an isomorphism. The terms $E_1^{p,q}$ are zero on the set $\{p > 0, p+q < 2\}$, and also on the set $\{q > p^2+p+2\}$. The terms where $q = p^2 + p + 2$ are all isomorphic to $\mathbb{Z}$.

This spectral sequence converges to $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^2$ in dimension 1, and 0 in all other dimensions. The classes in $E_1^{0,0}$ and $E_1^{0,1}$ are precisely those classes which survive to the E_∞ term.

9. Proof of the product formula

In this section we will prove Theorem 1, the product formula for deformation K-theory spectra.

The proof requires the following lemmas.

LEMMA 27. *A map $M' \to M$ of connective ku-module spectra is a weak equivalence if and only if the map $\mathrm{H}\mathbb{Z} \wedge_{ku} M' \to \mathrm{H}\mathbb{Z} \wedge_{ku} M$ is a weak equivalence.*

Proof. By taking cofibers, it suffices to prove the equivalent statement that a connective ku-module spectrum M'' is weakly contractible if and only if $\mathrm{H}\mathbb{Z} \wedge_{ku} M'' \simeq *$.

However, smashing M'' with the cofiber sequence $\Sigma^2 ku \to ku \to \mathrm{H}\mathbb{Z}$ of ku-module spectra shows that $\mathrm{H}\mathbb{Z} \wedge_{ku} M'' \simeq *$ if and only if the Bott map $\beta : \Sigma^2 M'' \to M''$ is a weak equivalence. This would imply that the homotopy groups of M'' are periodic; since M'' is connective, the result follows.

LEMMA 28. *Irreducible unitary representations of $\mathcal{G} \times \mathcal{H}$ are precisely of the form $V \otimes W$ for V, W irreducible unitary representations of $\mathcal{G}$ and $\mathcal{H}$ respectively.*

Proof. The tensor product of two irreducible representations is irreducible: Suppose V is an irreducible $\mathcal{G}$-representation, and W is an irreducible $\mathcal{H}$-representation. Then

$$\mathrm{Hom}(V \otimes W, V \otimes W) \cong \mathrm{Hom}(V,V) \otimes_{\mathbb{C}} \mathrm{Hom}(W,W).$$

Since V is irreducible, $\mathrm{Hom}_{\mathcal{G}}(V,V) \cong \mathbb{C}$, given by scalars, and similarly for W. Let $\{e_i\}$ be a basis for the vector space $\mathrm{Hom}(W,W)$. Suppose ϕ is a $\mathcal{G} \times \mathcal{H}$-linear endomorphism of $V \otimes W$. Then

$$\phi = \sum \phi_i \otimes e_i \in \mathrm{Hom}_{\mathcal{G}}(V \otimes W, V \otimes W) \Rightarrow \phi_i \in \mathbb{C},$$

so $\phi = 1 \otimes \psi$ for some $\psi \in \mathrm{Hom}(W,W)$. Since ϕ is also $\mathcal{H}$-linear, we find that ψ must be $\mathcal{H}$-linear, so ψ is scalar. Therefore, $V \otimes W$ is irreducible.

Any irreducible unitary representation of $\mathcal{G} \times \mathcal{H}$ is a tensor product: Suppose U is an irreducible representation. Since the actions of $\mathcal{G}$ and $\mathcal{H}$ commute, any $\mathcal{H}$-isotypic component of U is $\mathcal{G}$-invariant, so there is an irreducible unitary representation W of $\mathcal{H}$ such that $U \cong W^d$ as an $\mathcal{H}$-representation.

Consider the $\mathcal{G}$-vector space $\mathrm{Hom}_{\mathcal{H}}(W,U)$. There is an irreducible $\mathcal{G}$-representation V (which we do not assume to have an inner product) and a nonzero $\mathcal{G}$-map $V \to \mathrm{Hom}_{\mathcal{H}}(W,U)$. The adjoint of this map is a nonzero $\mathcal{G} \times \mathcal{H}$-map $V \otimes W \to U$. Both sides are irreducible, and hence this map is an isomorphism of $\mathcal{G} \times \mathcal{H}$-representations.

The irreducible $\mathcal{G}$-summands of U all come from a unique isomorphism class of irreducible $\mathcal{G}$-representations. Since the above map is nonzero, this $\mathcal{G}$-representation must be isomorphic to V, and hence V admits an inner product. (Note that an irreducible representation admits at most one invariant inner product up to scaling.)

Remark 29. Lemma 28 is precisely the portion of the proof which fails when we consider representations of $\mathcal{G} \times \mathcal{H}$ in other groups such as orthogonal groups and symmetric groups.

Proof. [of Theorem 1] The proof consists of constructing a filtration of the spectrum $\mathcal{K}\mathcal{G} \wedge_{ku} \mathcal{K}\mathcal{H}$ that agrees with the filtration on $\mathcal{K}(\mathcal{G} \times \mathcal{H})$.

We apply the results of Corollary 21 to get a map of ku-algebras as follows.

$$\mathcal{K}\mathcal{G} \underset{ku}{\wedge} \mathcal{K}\mathcal{H} \to \mathcal{K}(\mathcal{G} \times \mathcal{H}) \underset{ku}{\wedge} \mathcal{K}(\mathcal{G} \times \mathcal{H}) \to \mathcal{K}(\mathcal{G} \times \mathcal{H})$$

Similarly, whenever $p \cdot q \leq n$ there is a corresponding map of ku-modules

$$\mathcal{K}\mathcal{G}_p \underset{ku}{\wedge} \mathcal{K}\mathcal{H}_q \to \mathcal{K}(\mathcal{G} \times \mathcal{H})_n.$$

This diagram is natural in p, q, and n. If we define new ku-module spectra $M_n = \operatorname{hocolim}_{p\cdot q\leq n} \mathcal{KG}_p \wedge_{ku} \mathcal{KH}_q$, then there are induced ku-module maps

$$f_n : M_n \to \mathcal{K}(\mathcal{G}\times\mathcal{H})_n.$$

Since the maps $(\operatorname{hocolim} \mathcal{KG}_p) \to \mathcal{KG}$ and $(\operatorname{hocolim} \mathcal{KH}_q) \to \mathcal{KH}$ are weak equivalences, there is a weak equivalence

$$\operatorname{hocolim} M_n \simeq \operatorname*{hocolim}_{p,q} \mathcal{KG}_p \underset{ku}{\wedge} \mathcal{KH}_q \simeq \mathcal{KG} \underset{ku}{\wedge} \mathcal{KH}.$$

Therefore, it suffices to show $M_n \to \mathcal{K}(\mathcal{G}\times\mathcal{H})_n$ is a weak equivalence for all n. We have an induced map of cofiber sequences:

$$\begin{array}{ccccc} M_{n-1} & \longrightarrow & M_n & \longrightarrow & M_n/M_{n-1} \\ \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} \\ \mathcal{K}(\mathcal{G}\times\mathcal{H})_{n-1} & \longrightarrow & \mathcal{K}(\mathcal{G}\times\mathcal{H})_n & \longrightarrow & ku^{\mathrm{PU}(n)} \wedge_{\mathrm{PU}(n)} R_n(\mathcal{G}\times\mathcal{H}) \end{array}$$

To prove the theorem inductively it suffices to show that the map g_n is a weak equivalence.

The spectra M_n/M_{n-1} and $ku^{\mathrm{PU}(n)} \wedge_{\mathrm{PU}(n)} R_n(\mathcal{G}\times\mathcal{H})$ are connective ku-module spectra. Applying Lemma 27, it suffices to prove that the map $\mathrm{H}\mathbb{Z} \wedge_{ku} g_n$ is a weak equivalence.

Because the map $M_{n-1} \to M_n$ is a map from the homotopy colimit of a subdiagram into the full diagram, we can explicitly compute the homotopy cofiber of this map. The homotopy cofiber is weakly equivalent to the wedge

$$\bigvee_{p\cdot q=n} (\mathcal{KG}_p/\mathcal{KG}_{p-1}) \underset{ku}{\wedge} (\mathcal{KH}_q/\mathcal{KH}_{q-1})\,.$$

The spectra $\mathcal{KG}_p/\mathcal{KG}_{p-1}$, and the corresponding spectra for $\mathcal{H}$, are those which were identified as equivariant smash product spectra in Corollary 13 and Proposition 14. Smashing over ku with $\mathrm{H}\mathbb{Z}$ gives us the following identity.

$$\begin{aligned} \mathrm{H}\mathbb{Z} \underset{ku}{\wedge} M_n/M_{n-1} &\simeq \bigvee_{p\cdot q=n} \Big(\mathrm{H}\mathbb{Z}\wedge \mathrm{QIrr}(\mathcal{G},p)\Big) \underset{\mathrm{H}\mathbb{Z}}{\wedge} \Big(\mathrm{H}\mathbb{Z}\wedge \mathrm{QIrr}(\mathcal{H},q)\Big) \\ &\simeq \mathrm{H}\mathbb{Z}\wedge \left(\bigvee_{p\cdot q=n} \mathrm{QIrr}(\mathcal{G},p)\wedge \mathrm{QIrr}(\mathcal{H},q)\right) \end{aligned}$$

The map $\mathrm{H}\mathbb{Z}\wedge_{ku} g_n$ can be identified with the map

$$\mathrm{H}\mathbb{Z}\wedge \Big(\vee_{p\cdot q=n} \mathrm{QIrr}(\mathcal{G},p)\wedge \mathrm{QIrr}(\mathcal{H},q)\Big) \to \mathrm{H}\mathbb{Z}\wedge \mathrm{QIrr}(\mathcal{G}\times\mathcal{H},n)$$

which is induced by the tensor product of representations. The tensor product map $\otimes : \vee_{p\cdot q=n} \mathrm{QIrr}(\mathcal{G},p)\wedge \mathrm{QIrr}(\mathcal{H},q) \to \mathrm{QIrr}(\mathcal{G}\times\mathcal{H},n)$ is a continuous map between compact Hausdorff spaces. It is bijective by Lemma 28. Therefore, it is a homeomorphism.

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