The Unstable Adams Spectral Sequence for Two-Stage Towers

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Abstract

Let $KP$ denote a generalized mod 2 Eilenberg-MacLane space and let $Y$ be the fiber of a map $X \rightarrow KP$ to which the Massey-Peterson theorem applies. We study the relationship of the mod 2 unstable Adams spectral sequence (UASS) for $X$ and for $Y$. Given conditions on $X$, we split the $E_2$-term for $Y$, and we use a primary level calculation to compute $d_2$ for $Y$ up to an error term. If the UASS for $X$ collapses at $E_2$ (for example, if $X$ is an Eilenberg-MacLane space), the UASS for $Y$ collapses at $E_3$, and we have the entire UASS for $Y$. We also give examples and address a conjecture of Bousfield on the UASS for the Lie group SO.

Key words: unstable Adams spectral sequence, unstable Ext groups, SO

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1 Introduction

Let $KP$ denote a mod 2 generalized Eilenberg-MacLane space. In this paper we study the mod 2 unstable Adams spectral sequence (UASS) of a space $Y$ with polynomial cohomology which is obtained as the fiber of a map between simply connected spaces $X \rightarrow KP$ to which the Massey-Peterson theorem applies. By using specially constructed Adams resolutions, we study the relationship between the UASS for $X$ and for $Y$. Given certain conditions on $X$, we give a splitting of the $E_2$-term of the UASS for $Y$, and we show how to use a primary level calculation to compute almost complete information about the $d_2$ differentials in the UASS for $Y$. In the case that the UASS for $X$ collapses at $E_2$, the UASS for $Y$ collapses at $E_3$, and we have almost complete information about it.

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To state our results more precisely, we make some definitions. Throughout the paper, we take \( p = 2 \) and cohomology with mod 2 coefficients. Let \( \mathcal{U} \) denote the category of unstable modules over the mod 2 Steenrod algebra \( \mathcal{A} \). If \( M \) is an object in \( \mathcal{U} \), let \( U(M) \) denote the free unstable \( \mathcal{A} \)-algebra generated by \( M \), as described by [5]. If \( P \) is a free unstable \( \mathcal{A} \)-module, we write \( KP \) for the Eilenberg-MacLane space with \( H^*KP \cong U(P) \).

We call a map \( X \to KP \) Massey-Peterson if \( H^*X \cong U(M) \), \( H^*X \) is polynomial of finite type, \( X \) and \( KP \) are simply connected, and \( H^*KP \to H^*X \) is induced by a map \( f : P \to M \). We think of the topological map as realizing \( f \). If \( X \to KP \) is Massey-Peterson, with fiber \( Y \), then the Massey-Peterson theorem [4] tells us that \( H^*Y \cong U(N) \), where there is a short exact sequence

\[
0 \to \text{cok}(f) \to N \to \Omega\ker(f) \to 0.
\]

In general, this fundamental sequence does not split as \( \mathcal{A} \)-modules. However, the following theorem states conditions under which it does split at the level of \( \text{Ext} \).

**Theorem 1.1** Let \( X \) be a simply connected space with polynomial mod 2 cohomology \( H^*X \cong U(M) \), and let \( X \to KP \) be Massey-Peterson, inducing \( f : P \to M \) in cohomology. Let \( Y \) be the fiber of \( X \to KP \), and suppose \( H^*Y \cong U(N) \) is polynomial. If \( X \) has \( E_2 = E_3 \) in its unstable Adams spectral sequence, then there is a splitting

\[
\text{Ext}_\mathcal{U}^* (N, \Sigma'F_2) \cong \text{Ext}_\mathcal{U}^* (\text{cok}(f), \Sigma'F_2) \oplus \text{Ext}_\mathcal{U}^* (\Omega\ker(f), \Sigma'F_2).
\]

In general, the \( E_2 \)-term of an UASS is an \( \text{Ext} \) group taken over the category \( \mathcal{K} \) of unstable algebras over the Steenrod algebra. Because \( H^*Y \cong U(N) \),

\[
\text{Ext}_\mathcal{K}^* (H^*Y, H^*S') \cong \text{Ext}_\mathcal{U}^* (N, \Sigma'F_2),
\]

and hence Theorem 1.1 describes the \( E_2 \)-term of the UASS for \( Y \). Many differentials in the UASS for \( Y \) can be described in this case; we make a precise statement in Theorem 4.5. Roughly speaking, the \( d_2 \) differentials that go between the two summands of \( \text{Ext}_\mathcal{U}^* (N, \Sigma'F_2) \) can be computed by a primary level calculation involving an algebraic mapping cone. In fact, if \( X \) has \( E_2 = E_\infty \) in its UASS, then the UASS for \( Y \) can be computed almost completely by this calculation (Corollary 4.7).

The motivation for studying this situation was provided by a conjecture of Bousfield from the 1970s about the \( E_2 \)-term of the UASS for the Lie group \( SO \). We describe this conjecture in Section 5, along with examples of the application of Theorem 1.1 and Theorem 4.5.

Our strategy for studying the fibration \( Y \to X \to KP \) where \( H^*X \cong U(M) \)
and the fibration is induced by $f: P \to M$ is to build a resolution of $M$ using $P$ and resolutions of $\text{cok}(f)$ and $\text{ker}(f)$. When realized, this resolution gives an Adams tower for $X$ which can be manipulated to give an Adams tower for $Y$. Comparison of the two towers gives information about the differentials in the UASS for $Y$.

The organization of the paper is as follows. In Section 2 we construct a special chain complex resolving $M$ and use it to construct an Adams tower for $X$ and a tower with inverse limit $Y$ which are closely related. In Section 3, we construct an Adams tower for $Y$ by “rearranging” the tower for $X$. In Section 4, we study the homotopy spectral sequences of $X$ and $Y$ to prove Theorem 1.1, and we state and prove further results on differentials in the UASS for $Y$ (Theorem 4.5 and Corollary 4.7). In Section 5, we give examples of the application of Theorem 1.1 and Theorem 4.5.

The notation set up in the introduction will be carried through the paper.

2 Adams tower for $X$

In this section we construct an Adams tower for $X$ and use it to obtain a tower whose inverse limit is the 2-completion of $Y$. The tower for $Y$ will not be an Adams tower, but we show in Section 3 that its $k$-invariants can be “rearranged” in a controllable way to give an Adams tower for $Y$. We will retain enough information through the rearrangement that we can relate the homotopy spectral sequences of the two towers, thus obtaining information about the UASS for $Y$ from that of $X$.

We will need to make frequent use of the algebraic looping functor on $U$, and so we review the basic facts here. (See also [4].) The functor $\Omega: U \to U$ is right adjoint to the suspension functor $\Sigma: U \to U$. The module $\Omega M$ is the largest unstable quotient of the desuspension of $M$:

$$\Omega M \equiv (\Sigma^{-1} M)/(\Sigma^{-1} \text{Sq}_0 M),$$

where $\text{Sq}_0 x = \text{Sq}^{|x|} x$. We write $\Omega^k$ for the $k$-fold iterate of $\Omega$. The functor $\Omega$ is not exact, but only its first derived functor, which we denote $\Omega^1$, can be nonzero. The module $\Omega^1 M$ can be expressed as a regrading of the kernel of $\text{Sq}_0$ on $M$. In particular, if $\text{Sq}_0$ acts freely on $M$, then $\Omega^1 M = 0$. We write $\Omega^k_j$ for the $j$th derived functor of $\Omega^k$ and note that there is a composite functor spectral sequence (the Singer spectral sequence) $\Omega^k_j \Omega^k_l M \Rightarrow \Omega^{k+j}_{l+j} M$ which allows us to calculate derived functors of $\Omega^k$ inductively. For any unstable module $M$, $\Omega^k_j M = 0$ for $j > k$.

Now we turn to the construction of an Adams tower for $X$. Recall that $H^* X \cong$
\( U(M) \) and that \( Y \) is the fiber of a Massey-Peterson map \( X \to KP \) which realizes \( f : P \to M \). We know that \( H^*Y \cong U(N) \) where \( N \) is given by an extension

\[
0 \to \text{cok}(f) \to N \to \Omega \ker(f) \to 0.
\]

We will construct an Adams tower for \( X \) that can be related to a tower for \( Y \) by using a particular resolution of \( M \). There is a short exact sequence

\[
0 \to P/\ker(f) \to M \to \text{cok}(f) \to 0,
\]

and so we note that both \( M \) and \( N \) are given by extensions which involve \( \text{cok}(f) \) and \( \ker(f) \). Let \( C_* \) and \( D_* \) be minimal projective resolutions of \( \text{cok}(f) \) and \( \ker(f) \), respectively. Later in the paper we will need the following lemma.

**Lemma 2.1** \( \Omega D_* \) is a projective resolution of \( \Omega \ker(f) \).

**Proof.** The \( k \)th homology group of the complex \( \Omega D_* \) is \( \Omega^1_k \ker(f) \). If \( k > 1 \), then the functor \( \Omega^1_k \) is identically zero. On the other hand, \( \text{Sq}_0 \) acts freely on \( \ker(f) \) because \( \ker(f) \subseteq P \), which is itself \( \text{Sq}_0 \)-free. Hence \( \Omega^1_1 \ker(f) = 0 \). □

We construct a projective resolution of \( M \) as follows. Since \( D_* \) is a projective resolution of \( \ker(f) \) and \( P \) is projective, we can make a resolution of \( P/\ker(f) \) by shifting \( D_i \) to homological dimension \( i + 1 \) and putting \( P \) into homological dimension 0. Let \( B_* \) be this chain complex,

\[
\cdots \to D_1 \to D_0 \to P,
\]

which resolves \( P/\ker(f) \). Augment \( B_* \) by \( f : P \to M \) to obtain \( H_iB_* = 0 \) for \( i \geq 0 \) and \( H_{-1}B_* = \text{cok}(f) \). Let \( \text{Shift}(C_*) \) be the augmented chain complex obtained by shifting \( C_* \) down one homological degree, that is, \( \text{Shift}(C_i)_i = C_{i+1,i} \), and let the augmentation be the differential \( C_1 \to C_0 \). Then \( H_i\text{Shift}(C_*) = 0 \) for \( i \geq 0 \) and \( H_{-1}\text{Shift}(C_*) = \text{cok}(f) \). It is easy to construct a chain map \( g : \text{Shift}(C_*) \to B_* \),

\[
\cdots C_4 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \\
\cdots D_1 \longrightarrow D_0 \longrightarrow P \longrightarrow M,
\]

which is an isomorphism on all homology groups. In dimension \(-1\), we define \( g \) by a lift \( C_0 \to M \) of \( C_0 \to \text{cok}(f) \). Then we note that \( C_1 \to C_0 \to M \) lifts to \( P \), since \( C_1 \to C_0 \to M \to \text{cok}(f) \) is zero. The rest of the map \( g \) follows from projectivity of \( C_* \) and acyclicity of \( D_* \).
Let $X_*$ be the algebraic mapping cone. Hence $X_0 = C_0 \oplus P$, $X_i = C_i \oplus D_{i-1}$ for $i \geq 1$, $X_*$ has an augmentation to $M$, and the differential is defined by $d = (d_C, g \oplus d_D)$:

$$
\begin{array}{cccccc}
D_2 & \rightarrow & D_1 & \rightarrow & D_0 & \rightarrow & P \\
\oplus & \oplus & \oplus & \oplus & \oplus & \rightarrow & \oplus \\
C_3 & \rightarrow & C_2 & \rightarrow & C_1 & \rightarrow & C_0 & \rightarrow & M
\end{array}
$$

The augmented complex $X_*$ is acyclic because $g$ is a homology isomorphism. Hence $X_* \rightarrow M$ is a projective resolution, and it can be realized to give an Adams tower $\{X_k\}$ for $X$. (See, for example, [2, Section 3].) We note the following facts about the tower $\{X_k\}$, which follow from its being a realization of the projective resolution $X_* \rightarrow M$ where $H^*X \cong U(M)$:

1. $\lim_{\rightarrow} X_k \simeq X_2$.
2. The $k$-invariants $X_k \rightarrow K\Omega^k(C_{k+1} \oplus D_k)$ are Massey-Peterson maps.
3. The composite of the inclusion of a fiber with a $k$-invariant

$$
K\Omega^k(C_k \oplus D_{k-1}) \rightarrow X_k \rightarrow K\Omega^k(C_{k+1} \oplus D_k)
$$

realizes the looped down differential of $X_*$.
4. $X_0 \simeq KC_0 \times KP$.

To construct a chain complex modeling $Y$, we define $E_*$ as the quotient of $X_*$ by $P$ in dimension 0: $E_i = X_i$ for $i \geq 1$ and $E_0 = X_0/P$. If we regard $P$ as a chain complex concentrated in degree 0, there is a short exact sequence of chain complexes

$$
0 \rightarrow P \rightarrow X_* \rightarrow E_* \rightarrow 0
$$

which can be regarded as modeling the fibration $Y \rightarrow X \rightarrow KP$. We realize this situation by defining a new tower as follows. Define a map of $\{X_k\}$ to the constant tower $\{KP\}$ by $X_k \rightarrow X_0 \rightarrow KP$, where the second map is projection, and let $\{E_k\}$ be the tower of fibers. The following lemma gives the essential facts about $\{E_k\}$.

**Lemma 2.2**

1. $\lim_{\rightarrow} E_k \simeq Y_2$.
2. The $k$-invariants $E_k \rightarrow K\Omega^k(C_{k+1} \oplus D_k)$ are Massey-Peterson maps.
3. The composites

$$
K\Omega^k(C_k \oplus D_{k-1}) \rightarrow E_k \rightarrow K\Omega^k(C_{k+1} \oplus D_k)
$$

realize the looped down differential of $E_*$.
The commutative diagram of $k$-invariants produced by the map of towers
$\{E_k\} \to \{X_k\}$ realizes the quotient map of chain complexes $X_* \to E_*$
appropriately looped down.

**Proof.** For the first item, we note that since $E_k$ is the fiber of the map $X_k \to KP$, we have that $\lim_k E_k$ is the fiber of the inverse limit map $\lim_k X_k \to KP$. In view of the fact that $KP$ is its own 2-completion, since it is a mod 2 Eilenberg-MacLane space, the map between inverse limits is the 2-completion of $X \to KP$. Since all the spaces are simply connected, the fiber lemma for $p$-completion tells us that the fiber of the map between completions is the completion of the fiber, that is,

$$\lim_k E_k \simeq \text{Fiber}(\lim_k X_k \to KP)$$
$$\simeq \text{Fiber}(X^2 \to KP)$$
$$\simeq [\text{Fiber}(X \to KP)]_2$$
$$\simeq Y^2$$

To relate the tower $\{X_k\}$ and the tower $\{E_k\}$, we study the following commutative diagram:

$$
\begin{array}{ccc}
E_{k+1} & \longrightarrow & E_k \\
\downarrow & & \downarrow \\
X_{k+1} & \longrightarrow & X_k \\
\downarrow & & \downarrow \\
KP & \longrightarrow & KP
\end{array}
$$

The upper right hand square gives the $k$-invariants, and hence the third part of the lemma follows immediately from the corresponding fact for $\{X_k\}$ and the following commuting diagram of $k$-invariants and inclusions of fibers:

$$
\begin{array}{ccc}
K\Omega^k(C_k \oplus D_{k-1}) & \longrightarrow & E_k \\
\downarrow & & \downarrow \\
K\Omega^k(C_k \oplus D_{k-1}) & \longrightarrow & X_k \\
\downarrow & & \downarrow \\
& & \ast
\end{array}
$$

Likewise the fourth part of the lemma follows from noting that at the bottom
of the towers we have

\[
\begin{array}{ccc}
E_0 & \simeq & KC_0 \\
\downarrow & & \downarrow \\
X_0 & \simeq & K(P \oplus C_0)
\end{array}
\]

\[
\begin{array}{ccc}
\pi & & \\
\downarrow & & \downarrow \\
KC_0 & \longrightarrow & K(C_1 \oplus D_0)
\end{array}
\]

To prove the second part of the lemma, we proceed inductively. Consider the map of fibrations arising from the first k-invariant:

\[
\begin{array}{ccc}
E_1 & \longrightarrow & E_0 \simeq KC_0 \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & X_0 \simeq K(P \oplus C_0)
\end{array}
\]

Each of the k-invariants is clearly a Massey-Peterson map, and so the cohomologies of \( E_1 \) and \( X_1 \) are “very nice” in the sense of Massey-Peterson, say \( H^*E_1 \cong U(N_1) \) and \( H^*X_1 \cong U(M_1) \). The fundamental sequence is natural, giving us a ladder of short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{cok}(f) & \longrightarrow & N_1 & \longrightarrow & \Omega \ker(C_1 \oplus D_0 \to C_0) & \longrightarrow & 0 \\
0 & \longrightarrow & M & \longrightarrow & M_1 & \longrightarrow & \Omega \ker(C_1 \oplus D_0 \to C_0 \oplus P) & \longrightarrow & 0.
\end{array}
\]

In order to show that \( E_1 \to K\Omega(C_2 \oplus D_1) \) is Massey-Peterson, we need to know that \( H^*E_1 \) is polynomial. It suffices to check that \( \text{Sq}^0 \) acts freely on the two ends of the short exact sequence for \( N_1 \). Certainly \( \text{Sq}^0 \) acts freely on \( \Omega \ker(C_1 \oplus D_0 \to C_0) \), since this is a submodule of the projective \( \Omega(C_1 \oplus D_0) \). Further, by the assumption that \( H^*Y \cong U(N) \) is polynomial and the fact that \( \text{cok}(f) \subseteq N \), \( \text{Sq}^0 \) acts freely on \( \text{cok}(f) \). It follows easily that \( E_1 \to K\Omega(C_2 \oplus D_1) \) is Massey-Peterson.

We need to examine what happens in cohomology for the fibration \( E_2 \to E_1 \to K\Omega(C_2 \oplus D_1) \), and then the rest of the induction will follow. In particular, we need to know \( \text{cok}(\Omega(C_2 \oplus D_1) \to N_1) \) in order to understand the cokernel side of the fundamental sequence for \( H^*E_2 \). Consider the following commuting diagram, where the rows are Massey-Peterson fibrations:

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
E_1 & \longrightarrow & E_0 \simeq KC_0
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & K(C_1 \oplus D_0). \\
\downarrow & & \downarrow \\
\longrightarrow & K(C_1 \oplus D_0)
\end{array}
\]

(Commutativity of the right hand square follows from the fact that the com-
posite $C_1 \oplus D_0 \to C_0 \oplus P \to M$ is zero because it is the square of the differential in the chain complex $X_* \ldots$ We obtain a commuting ladder of fundamental sequences for $H^*Y$ and $H^*E_1$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{cok}(f) & \longrightarrow & N & \longrightarrow & \Omega \ker(f) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \text{cok}(f) & \longrightarrow & N_1 & \longrightarrow & \Omega \ker(C_1 \oplus D_0 \to C_0) & \longrightarrow & 0.
\end{array}
$$

Now the map $E_1 \to K\Omega(C_2 \oplus D_1)$ induces a map $\Omega(C_2 \oplus D_1) \to N_1$ which goes into the $\Omega \ker(C_1 \oplus D_0 \to C_0)$ part of $N_1$; specifically, it maps to $\Omega \ker(C_1 \oplus D_0 \to C_0 \oplus P)$. Therefore $\text{cok}(\Omega(C_2 \oplus D_1) \to N_1)$ is an extension of $\text{cok}(f)$ by

$$
\text{cok}[\Omega \ker(C_1 \oplus D_0 \to C_0 \oplus P) \to \Omega \ker(C_1 \oplus D_0 \to C_0)],
$$

which is $\Omega \ker(f)$, and the ladder of exact sequences above shows that this extension is $N$.

It is now easy to see that $\text{im}(H^*E_k \to H^*E_{k+1}) \cong U(N)$ for $k > 1$, and since the part of $H^*E_{k+1}$ coming from the fiber is also polynomial, we have that $H^*E_{k+1}$ is polynomial, as required. The induction now follows without difficulty.

\section{Adams tower for $Y$}

In this section, we take the tower $\{E_k\}$ constructed in Section 2 and “rearrange” it into an Adams tower for $Y$. We do this by taking the k-invariants, which are products, and separating them in such a way as to give a tower $\{E'_k\}$ which is interleaved with the tower $\{E_k\}$. In Section 4, we will study the homotopy spectral sequence of the tower $\{E'_k\}$ and show that this tower is an Adams tower for $Y$.

Recall that $E_{k+1}$ is the fiber of a Massey-Peterson map $E_k \to K\Omega^k(C_{k+1} \oplus D_k)$. The tower $\{E'_k\}$ is defined as follows.

**Definition 3.1** Let $E'_k$ be the homotopy fiber of $E_k \to K\Omega^kD_k$. Define a map $E'_{k+1} \to E'_k$ by the commutative ladder

$$
\begin{array}{cccccc}
E_{k+1} & \longrightarrow & E_k & \longrightarrow & K\Omega^k(C_{k+1} \oplus D_k) \\
\downarrow & & \downarrow & & \pi \\
E'_k & \longrightarrow & E_k & \longrightarrow & K\Omega^kD_k
\end{array}
$$

The map $E'_{k+1} \to E'_k$ is defined as the composite $E'_{k+1} \to E_{k+1} \to E'_k$. 

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Corollary 3.2 \( \lim_k E_k \simeq \lim_k E'_k \).

Our first task is to understand the k-invariants for the tower \( \{E'_k\} \), so that we will later be able to analyze the homotopy spectral sequence. We need the following lemma, which allows us to combine k-invariants of successive fibrations.

**Lemma 3.3** Suppose given a tower of principal fibrations

\[
\begin{array}{ccc}
W_3 & \rightarrow & W_2 \\
\downarrow & & \downarrow g \\
W_2 & \rightarrow & K_2 \\
p & \downarrow h & \rightarrow \\
W_1 & \rightarrow & K_1
\end{array}
\]

and suppose that \( g \) factors up to homotopy as \( g = hp \). Then the homotopy fiber of \( (f, h) : W_1 \rightarrow K_1 \times K_2 \) is homotopy equivalent to \( W_3 \).

**Proof.** The commutative square

\[
\begin{array}{ccc}
W_2 & \rightarrow & W_1 \\
\downarrow & & \downarrow f \\
\ast & \rightarrow & K_1
\end{array}
\]

is a homotopy pullback square, hence remains so when we cross the bottom row with \( K_2 \) to obtain the square

\[
\begin{array}{ccc}
W_2 & \rightarrow & W_1 \\
g & & (f, h) \\
\downarrow & & \downarrow \\
K_2 & \rightarrow & K_1 \times K_2.
\end{array}
\]

Since the fibers of the vertical maps are homotopy equivalent and the fiber of \( g \) is \( W_3 \), the lemma follows.

We want to apply this lemma to find a k-invariant that will get us from \( E'_k \) to \( E'_{k+1} \), and as a preliminary, we find a k-invariant to go from \( E'_k \) to \( E_{k+1} \).

**Lemma 3.4** The homotopy fiber of the composite \( E'_k \rightarrow E_k \rightarrow K\Omega^k C_{k+1} \) is \( E_{k+1} \).
Proof. Apply the same reasoning as in the previous lemma, crossing the bottom row of the homotopy pullback square

\[
\begin{array}{ccc}
E'_k & \longrightarrow & E_k \\
\downarrow & & \downarrow \\
* & \longrightarrow & K\Omega^k D_k,
\end{array}
\]

with \( K\Omega^k C_{k+1} \). Then the fiber of the new vertical maps will be \( E_{k+1} \). \( \square \)

To show there is a k-invariant to go from \( E'_k \) to \( E'_{k+1} \) and to compute it, we look at the following tower of three spaces:

\[
\begin{array}{ccc}
E'_{k+1} & \\
\downarrow & \\
E_{k+1} & \longrightarrow & K\Omega^{k+1} D_{k+1} \\
\downarrow & \\
E'_k & \longrightarrow & K\Omega^k C_{k+1}.
\end{array}
\]

We next show that the second k-invariant factors through \( E'_k \) and apply Lemma 3.3.

Lemma 3.5 The map \( E_{k+1} \rightarrow K\Omega^{k+1} D_{k+1} \) factors through \( E_{k+1} \rightarrow E'_k \) up to homotopy.

Proof. Since we are mapping into an Eilenberg-MacLane space, the statement amounts to a factoring on the level of cohomology. To explain the algebraic fact that makes this work, consider a part of the tower \( \{E_k\} \):

\[
K\Omega^{k+1}(C_{k+1} \oplus D_k) \longrightarrow E_{k+1} \longrightarrow K\Omega^{k+1}(C_{k+2} \oplus D_{k+1}) \\
\downarrow \\
E_k \longrightarrow K\Omega^k (C_{k+1} \oplus D_k).
\]

The composite \( K\Omega^{k+1}(C_{k+1} \oplus D_k) \rightarrow E_{k+1} \rightarrow K\Omega^{k+1}(C_{k+2} \oplus D_{k+1}) \) is a realization of the looped down chain complex differential of the complex \( \mathcal{E}_* \). The critical fact we need for our proof is that the composite

\[
D_{k+1} \hookrightarrow C_{k+2} \oplus D_{k+1} \rightarrow C_{k+1} \oplus D_k \rightarrow C_{k+1}
\]

is zero. Heuristically, this means that putting in \( K\Omega^{k+1} C_{k+1} \) is not necessary in order to put in \( K\Omega^{k+1} D_{k+1} \); once we have put in \( K\Omega^{k+1} D_k \) (to obtain \( E'_k \)), we can then put in \( K\Omega^{k+1} D_{k+1} \).
We compare the fundamental sequences for the cohomology of $E_{k+1}$ and $E'_k$ arising from the fibrations

$$
E_{k+1} \longrightarrow E_k \longrightarrow K\Omega^k(C_{k+1} \oplus D_k)
$$

Let $H^*E_k \cong U(N_k)$, let $H^*E'_k \cong U(N'_k)$, and note that the cokernel of $\Omega^k(C_{k+1} \oplus D_k) \to N_k$ is $N$. The ladder of fundamental sequences corresponding to the above diagram is

$$
\begin{array}{cccc}
0 & \to & \text{cok}(\Omega^k D_k \to N_k) & \to & N_k' & \to & \Omega\ker(\Omega^k D_k \to N_k) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & N & \to & N_{k+1} & \to & \Omega\ker(\Omega^k(C_{k+1} \oplus D_k) \to N_k) & \to 0.
\end{array}
$$

The left vertical arrow is an epimorphism because $N \cong \text{cok}[\Omega^k(C_{k+1} \oplus D_k) \to N_k]$, and we are mapping to it from the cokernel of $\Omega^k D_k \to N_k$.

Now we want to consider the map $\Omega^{k+1} D_{k+1} \to N_{k+1}$ and show that it lifts to $N_k'$. The composite

$$
\Omega^{k+1} D_{k+1} \to N_{k+1} \to \Omega\ker(\Omega^k(C_{k+1} \oplus D_k) \to N_k)
$$

is zero into the factor $\Omega^k(C_{k+1})$, because of the critical algebraic fact mentioned above. Hence $\Omega^{k+1} D_{k+1}$ lifts to $\Omega\ker(\Omega^k D_k \to N_k)$. Then a standard diagram chase using the projectivity of $\Omega^{k+1} D_{k+1}$ and the fact that $\text{cok}(\Omega^k D_k \to N_k) \to N$ is an epimorphism shows that $\Omega^{k+1} D_{k+1} \to N_{k+1}$ lifts to $N_k'$.

**Corollary 3.6** $E'_{k+1}$ is the fiber of a Massey-Peterson map

$$
E'_k \to K\Omega^k C_{k+1} \times K\Omega^{k+1} D_{k+1}.
$$

Note that, in particular, $E'_0 \cong KC_0 \times K\Omega D_0$.

## 4 Homotopy Spectral Sequences

In this section we study the homotopy spectral sequence for the tower $\{E'_k\}$ constructed in Section 3. By examining $d_1$ we show that $\{E'_k\}$ is an Adams tower for $Y$. Because the homotopy spectral sequence of $\{E_k\}$ is almost the same as that of $\{X_k\}$, that is, the UASS for $X$, we can compare the UASS
for $X$ and for $Y$ using the two towers $\{E_k\}$ and $\{E'_k\}$. We deduce relationships between the differentials to establish Theorem 1.1 and the related results
Theorem 4.5 and Corollary 4.7.

To study $d_1$ of the homotopy spectral sequence of $\{E'_k\}$, we need to examine the composite

$$K\Omega^k C_k \times K\Omega^{k+1} D_k \to E'_k \to K\Omega^k C_{k+1} \times K\Omega^{k+1} D_{k+1}.$$ 

The following lemma relates these maps to the corresponding $k$-invariants and inclusions of fibers for $\{E_k\}$.

**Lemma 4.1** The following diagram of $k$-invariants and inclusions of fibers commutes:

$$\begin{array}{ccc}
K\Omega^{k+1} C_{k+1} \times K\Omega^{k+1} D_k & \longrightarrow & E_{k+1} \\
\downarrow & & \downarrow \\
K\Omega^k C_k \times K\Omega^{k+1} D_k & \longrightarrow & E'_k \\
(\pi,*) & & (*,\pi) \\
K\Omega^k C_k \times K\Omega^k D_{k-1} & \longrightarrow & E_k \\
\downarrow & & \downarrow \\
K\Omega^k C_k \times K\Omega D_k & \longrightarrow & K\Omega^k C_{k+1} \times K\Omega^k D_k
\end{array}$$

**Proof.** The upper left square commutes by the commutative ladder which defines the map $E_{k+1} \to E'_k$. (See Definition 3.1.) The upper right square commutes by Lemma 3.5 and the fact that $E_{k+1} \to E'_k \to K\Omega^k C_{k+1}$ is a fibration. The lower right square commutes because $E'_k \to K\Omega^k C_{k+1}$ is defined as the composite $E'_k \to E_k \to K\Omega^k C_{k+1}$ and because $E'_k \to E_k \to K\Omega^k D_k$ is a fibration.

For the lower left square, we expand the $2 \times 2$ square

$$\begin{array}{ccc}
E_k & \longrightarrow & E'_{k-1} \\
\downarrow & & \downarrow \\
K\Omega^k D_k & \longrightarrow & * 
\end{array}$$

to a $3 \times 3$ square:

$$\begin{array}{ccc}
K\Omega^k C_k \times K\Omega^{k+1} D_k & \longrightarrow & E'_k \longrightarrow E'_{k-1} \\
\downarrow & & \downarrow \\
K\Omega^k C_k & \longrightarrow & E_k \longrightarrow E'_{k-1} \\
* & & * \\
K\Omega^k D_k & \longrightarrow & K\Omega^k D_k \longrightarrow *.
\end{array}$$
Because the composite $K\Omega^k C_k \to E_k \to K\Omega^k D_k$ is known to be null, the map $K\Omega^k C_k \times K\Omega^{k+1} D_k \to K\Omega^k C_k$ is homotopic to projection. \hfill \Box

**Proposition 4.2** \{E'_k\} is an Adams tower for $Y$.

**Proof.** We already know that $\lim\limits_{\leftarrow} E'_k \simeq \lim\limits_{\leftarrow} E_k \simeq Y_2$. We must show that

1. $Y \to E'_k$ gives an epimorphism $H^* E'_k \to H^* Y$;
2. $\ker(H^* E'_k \to H^* E'_{k+1}) \cong \ker(H^* E'_k \to H^* Y)$.

Consider the commutative ladder of fundamental sequences at the end of the proof of Lemma 3.5. The map $E_{k+1} \to E'_k$ induces a map $N'_k \to N_{k+1}$ which is onto $N$. Since $Y \to E_{k+1}$ induces an isomorphism from $N \subseteq N_{k+1}$ to $N \subseteq H^* Y \cong U(N)$, the desired epimorphism $H^* E'_k \to H^* Y$ follows. To prove the second item, we note that $Y \to E'_k$ factors as $Y \to E'_{k+1} \to E_{k+1} \to E'_k$. Since $\ker(H^* E'_k \to H^* E'_{k+1}) \cong \ker(H^* E'_k \to H^* Y)$ the desired isomorphism follows. \hfill \Box

Now we wish to calculate the homotopy spectral sequence of \{E'_k\}, which is the UASS for $Y$. The $d_1$ differential comes from the composite

$$K\Omega^k C_k \times K\Omega^{k+1} D_k \to E'_k \to K\Omega^k C_{k+1} \times K\Omega^{k+1} D_{k+1}.$$ 

The components $\pi_* K\Omega^k C_k \to \pi_* K\Omega^k C_{k+1}$ and $\pi_* K\Omega^{k+1} D_k \to \pi_* K\Omega^{k+1} D_{k+1}$ are zero because $C_*$ and $D_*$ are minimal resolutions. Further, by Lemma 4.1 $K\Omega^{k+1} D_k \to E'_k \to K\Omega^k C_{k+1}$ is actually null because it factors through $E'_k \to E_k$ and $K\Omega^{k+1} D_k \to E'_k \to E_k \to E_k$ is a fibration. Hence the only possible nonzero part of $d_1$ takes $\pi_* K\Omega^k C_k \to \pi_* K\Omega^{k+1} D_{k+1}$.

**Remark 4.3** Note the contrast between the $d_1$ differentials in the homotopy spectral sequences for \{E_k\} and \{E'_k\}. In the homotopy spectral sequence for \{E_k\}, $d_1$ comes from the composite

$$K\Omega^k C_k \times K\Omega^k D_{k-1} \to E_k \to K\Omega^k C_{k+1} \times K\Omega^k D_k.$$ 

The only possible nonzero component of $d_1$ is $\pi_* K\Omega^k D_{k-1} \to \pi_* K\Omega^k C_{k+1}$, corresponding to the part of the chain complex differential of $E_*$ taking $C_{k+1} \to D_{k-1}$.

Before giving the proof of Theorem 1.1, we need the following preliminary lemma.

**Lemma 4.4** Let $x \in \pi_* K\Omega^k C_k$ be a boundary in the homotopy spectral sequence of \{E_k\}. Then $x$ is in the kernel of $d_1$ of the homotopy spectral sequence of \{E'_k\}.
Proof. We use the diagram of Lemma 4.1. Let $y$ be the image of $x$ in $\pi_*E'_k$. If $x$ is a boundary in the homotopy spectral sequence of \{E_k\}, then $x$ goes to 0 in $\pi_*E_k$, and so $y$ does also. The long exact sequence for the fibration $K\Omega^{k+1}E_k \to E_k' \to E_k$ gives a preimage for $y$, call it $z$, in $\pi_*K\Omega^{k+1}D_k$. However, $d_1(z)$ is certainly zero, and therefore since $x$ and $z$ have the same image in $\pi_*E'_k$, we have $d_1(x) = 0$ also. 

We are now ready to prove our Ext splitting result, which we reproduce here for the convenience of the reader.

**Theorem 1.1.** Let $X$ be a simply connected space with polynomial mod 2 cohomology $H^*X \cong U(M)$, and let $X \to KP$ be Massey-Peterson, inducing $f : P \to M$ in cohomology. Let $Y$ be the fiber of $X \to KP$, and suppose $H^*Y \cong U(N)$ is polynomial. If $X$ has $E_2 = E_3$ in its unstable Adams spectral sequence, then there is a splitting

$$\text{Ext}^*_{\Omega} (N, \Sigma^tF_2) \cong \text{Ext}^*_{\Omega} (\text{cok}(f), \Sigma^tF_2) \oplus \text{Ext}^*_{\Omega} (\Omega \ker(f), \Sigma^tF_2).$$

Proof. In view of the remarks preceding the statement of the theorem in the introduction, and the fact that $\Omega D_* \to \Omega \ker(f)$ and $C_* \to \ker(f)$ are resolutions, we have only to show that $d_1$ in the homotopy spectral sequence of \{E'_k\} is zero. Assuming that $C_*$ and $D_*$ are chosen to be minimal resolutions, this will follow from showing that $\pi_*K\Omega^kC_k \to \pi_*K\Omega^{k+1}D_{k+1}$ is zero.

We will relate this part of $d_1$ to the UASS for $X$ using the diagram of Lemma 4.1. To do this, we must discuss the relationship of the homotopy spectral sequences for \{E_k\} and \{X_k\}, the latter being the UASS for $X$. We claim that they are almost the same. To be more precise, note the map of towers \{E_k\} \to \{X_k\} is the identity on the $k$-invariants except at the very bottom, where $E_0 \to X_0$ is the inclusion $KC_0 \to KC_0 \times KP$. Hence any nonzero differential in \{E_k\} gives rise to a corresponding nonzero differential in \{X_k\}, and any zero differential in \{E_k\} gives rise to a corresponding zero differential in \{X_k\}. In fact, there are only two differences between the homotopy spectral sequences for \{E_k\} and \{X_k\}. The first is that classes in the homotopy spectral sequence of \{X_k\} which are boundaries of $\pi_*KP$ will survive to $E_\infty$ in the homotopy spectral sequence of \{E_k\}. These correspond to the kernel of $\pi_*Y \to \pi_*X$. The second is that infinite cycles in the homotopy spectral sequence of \{X_k\} which are in $\pi_*KP$ will not appear in the homotopy spectral sequence of \{E_k\}. These correspond to the cokernel of $\pi_*Y \to \pi_*X$.

To complete the proof of the theorem, we claim that the diagram of Lemma 4.1 shows that a nonzero map $\pi_*K\Omega^kC_k \to \pi_*K\Omega^{k+1}D_{k+1}$ would produce a nonzero $d_2$ in the homotopy spectral sequence of \{E_k\}, and hence in the UASS.
of $X$. To prove this, we use the interplay between the homotopy spectral sequences for the towers $\{E_k\}$ and $\{E'_k\}$. If $x \in \pi_* K\Omega^k C_k$ has a nonzero image in $\pi_* K\Omega^{k+1} D_{k+1}$, then $x$ is not in the kernel of $d_1$ for the homotopy spectral sequence of $\{E_k\}$. Hence by Lemma 4.4, we know that $x$ is not a boundary in the homotopy spectral sequence for $\{E_k\}$. However, this means that $x$ survives to $E_2$ in the homotopy spectral sequence for $\{E_k\}$, because all elements of $\pi_* K\Omega^k C_k$ are in the kernel of $d_1$. On the other hand, since $E'_k$ is the fiber of $E_{k+1} \rightarrow K\Omega^{k+1} D_{k+1}$, $x$ does not lift even to $E'_{k+1}$, let alone to $E_{k+2}$, and so $x$ must support a $d_2$ differential.

Therefore if $X$ has $E_2 = E_3$ in its unstable Adams spectral sequence, that is, there are no nonzero $d_2$'s, it must be that $\pi_* K\Omega^k C_k \rightarrow \pi_* K\Omega^{k+1} D_{k+1}$ is zero, completing the proof of the theorem. $\square$

Lastly, we prove a theorem about $d_2$ in the UASS for $Y$.

**Theorem 4.5** Given the hypotheses of Theorem 1.1, let $x \in \pi_* K\Omega^k D_{k-1}$ and let $\delta : \pi_* K\Omega^k D_{k-1} \rightarrow \pi_* K\Omega^k C_{k+1}$ be given by $d_1$ in the homotopy spectral sequence of $\{E_k\}$. In the unstable Adams spectral sequence of $Y$, $d_2(x) = \delta(x) \oplus y$ where $y \in \pi_* K\Omega^{k+1} D_{k+1}$. If $x \in \ker(\delta)$, then $y = 0$.

**Remark 4.6** If we write $\text{Hom}^l(A, B)$ for morphisms $A \rightarrow \Sigma^l B$ in the category of unstable $\mathcal{A}$-modules, then $\delta$ is the only nonzero part of the differential in the cochain complex $\text{Hom}^*(\mathcal{E}_*, \mathbb{F}_2)$, where $\mathcal{E}_*$ is the chain complex constructed in Section 2. Hence this theorem says that $d_2$'s in the unstable Adams spectral sequence for $Y$ can be determined up to an error term $y$ by the primary level algebraic calculation of $\mathcal{E}_*$. Furthermore, if an element $x$ looks like a cycle under $d_2 (\delta(x) = 0)$, then it is a cycle (the error term vanishes).

**Proof of Theorem 4.5** Again we examine the diagram in Lemma 4.1, this time extending it downward by a row:

\[
\begin{array}{cccc}
K\Omega^k C_k \times K\Omega^{k+1} D_k & \longrightarrow & E'_k & \longrightarrow & K\Omega^k C_{k+1} \times K\Omega^{k+1} D_{k+1} \\
\downarrow \text{(\pi,*)} & & \downarrow & & \downarrow \text{(\pi,*)} \\
K\Omega^k C_k \times K\Omega^k D_{k-1} & \longrightarrow & E_k & \longrightarrow & K\Omega^k C_{k+1} \times K\Omega^k D_k \\
\downarrow \text{(\ast,\pi)} & & \downarrow & & \downarrow \text{(\ast,\pi)} \\
K\Omega^{k-1} C_{k-1} \times K\Omega^k D_{k-1} & \longrightarrow & E'_{k-1} & \longrightarrow & K\Omega^{k-1} C_k \times K\Omega^k D_k.
\end{array}
\]

The first part of the theorem follows from a diagram chase starting in the lower left hand corner. The second part follows once we note that if $x \in \ker(\delta)$, then $x$ lifts to $E'_{k+1}$, and therefore also to $E_{k+2}$ (there are no nonzero $d_2$ differentials.
in the homotopy spectral sequence for \{E_k\}. Therefore \(x\) must go to zero in \(\pi_*K\Omega^{k+1}D_{k+1}\) or there would be an obstruction to lifting.

\[\]

**Corollary 4.7** If \(E_2 = E_\infty\) for \(X\), then \(E_3 = E_\infty\) for \(Y\).

Essentially, the corollary says that if \(E_2 = E_\infty\) for \(X\), then the UASS for \(Y\) is almost completely determined by the primary level calculation of the chain complex \(E_*\). The only piece of information missing is the potentially nonzero term \(y\) of Theorem 4.5.

**Proof.** If \(E_2 = E_\infty\) for \(X\), then once a class in homotopy lifts from \(E_k\) to \(E_{k+1}\), it lifts all the way up the tower to \(Y\). If a class lifts from \(E_{k-1}\) to \(E_{k+1}\), it has lifted from \(E_k\) to \(E_{k+1}\) in the process, hence lifts all the way to \(Y\).

5 Examples

In this section, we discuss two examples of Theorem 1.1 and Theorem 4.5. In the first example, we show how the theory works itself out in a very easy case, by considering the unstable Adams spectral sequence for \(Y = K(Z/4, n)\). In the second example, which motivated looking for a theorem along the lines of Theorem 1.1, we discuss a conjecture of Bousfield on the unstable Adams spectral sequence for the Lie group \(SO\), and we give a result related to the first stage of the conjecture.

5.1 The UASS for \(K(Z/4, n)\)

As an example where Theorems 1.1 and 4.5 can be easily and completely worked out, we let \(X = K(Z/2, n)\), \(KP = K(Z/2, n + 1)\), and \(f : X \to KP\) be \(\text{Sq}^1\). Hence the space \(Y\) whose UASS is being described by the theorems is \(Y = K(Z/4, n)\). Of course, we already know everything there is to know about the UASS in this case, so it is simply an exercise in following the definitions of the previous sections and seeing where they lead, particularly regarding the differentials.

We write \(F(n)\) for the free unstable \(A\)-module on a generator of dimension \(n\), and we write \(\overline{F}(n) \equiv F(n)/\text{Sq}^1\). Then \(M = F(n)\), \(P = F(n+1)\), \(\text{cok}(f) = \overline{F}(n)\), and \(\ker(f) \equiv \overline{F}(n+2)\). The resolution \(C_* \to \text{cok}(f)\) has \(C_i = F(n+i)\), and the resolution \(D_* \to \ker(f)\) has \(D_i = F(n+i+2)\), while in both cases all the differentials are \(\text{Sq}^1\).
If we lift $C_0 \rightarrow \text{cok}(f)$ to $M$, we get the identity map $F(n) \rightarrow F(n)$. Hence the chain map $g : \text{Shift}(C_*) \rightarrow B_*$,

$$\cdots C_3 \longrightarrow C_2 \longrightarrow C_1 \stackrel{\epsilon}{\longrightarrow} C_0$$

becomes

$$\cdots F(n + 3) \overset{\text{Sq}^1}{\longrightarrow} F(n + 2) \overset{\text{Sq}^1}{\longrightarrow} F(n + 1) \overset{\epsilon}{\longrightarrow} F(n)$$

The complex $E_*$, which we realize to obtain $Y$, is the mapping cone, without augmentation and factored out by $P$:

$$\begin{array}{ccc}
D_2 & \longrightarrow & D_1 \\
\oplus & \longrightarrow & \oplus \\
C_3 & \longrightarrow & C_2 \\
\end{array}$$

$$\begin{array}{ccc}
& & \oplus \\
& & \oplus \\
& & \oplus \\
\end{array}$$

$$\begin{array}{ccc}
D_0 & \longrightarrow & D_0 \\
\oplus & \longrightarrow & \oplus \\
C_0 & \longrightarrow & C_1 \\
\end{array}$$

In our example, the complex $E_*$ becomes

$$\begin{array}{ccc}
F(n + 4) & \longrightarrow & F(n + 3) \\
\oplus & \longrightarrow & \oplus \\
F(n + 3) & \longrightarrow & F(n + 2) \\
\oplus & \longrightarrow & \oplus \\
\end{array}$$

where the slant maps are the identities, since they come from the vertical maps in $g_*$. To obtain the Adams resolution for $Y$, we “pull” $D_*$ down one homological dimension (looping as we do so since we are beginning in a lower homological dimension and the topological dimension must remain the same):

$$\begin{array}{ccc}
\Omega D_3 & \longrightarrow & \Omega D_2 \\
\oplus & \longrightarrow & \oplus \\
C_3 & \longrightarrow & C_2 \\
\end{array}$$

$$\begin{array}{ccc}
& \longrightarrow & \oplus \\
& \longrightarrow & \oplus \\
& \longrightarrow & \oplus \\
\end{array}$$

$$\begin{array}{ccc}
\Omega D_1 & \longrightarrow & \Omega D_0 \\
\oplus & \longrightarrow & \oplus \\
C_1 & \longrightarrow & C_0 \\
\end{array}$$

The dashed slant arrows indicate that we are remembering where $d_1$ used to go, because it will become $d_2$ in the UASS. In our example, we get

$$\begin{array}{ccc}
\Omega F(n + 5) & \longrightarrow & \Omega F(n + 4) \\
\oplus & \longrightarrow & \oplus \\
F(n + 3) & \longrightarrow & F(n + 2) \\
\oplus & \longrightarrow & \oplus \\
F(n + 3) & \longrightarrow & F(n + 2) \\
\oplus & \longrightarrow & \oplus \\
F(n + 1) & \longrightarrow & F(n) \\
\oplus & \longrightarrow & \oplus \\
\end{array}$$

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Finally, to get the $E_2$-term of the UASS for $Y$, we compute $\text{Hom}^*(-, F_2)$:

$$\Sigma^{n+4}F_2 \hookleftarrow \Sigma^{n+3}F_2 \hookleftarrow \Sigma^{n+2}F_2 \hookleftarrow \Sigma^{n+1}F_2 \oplus \cdots \oplus \Sigma^nF_2.$$

Rewriting this in the usual $s$ versus $(t-s)$ format, and including the $d_2$'s indicated by the slant arrows, we obtain the UASS for $Y = K(Z/4, n)$:

![UASS Diagram](image)

The UASS for $K(Z/4, n)$.

5.2 The UASS for SO

Theorem 1.1 and Theorem 4.5 came up as part of an effort to prove a conjecture of Bousfield about the $E_2$-term of the unstable Adams spectral sequence for the Lie group SO. Let $P_\infty = H^*RP^\infty$, and recall that $H^*SO \cong U(P_\infty)$. Hence in the UASS for SO, we have $E_2^{s,t} \cong \text{Ext}^s_{U}(P_\infty, \Sigma^tF_2)$. Writing the elements of $P_\infty$ as $x^i$, we filter $P_\infty$ by letting $P_n$ be the elements $x^i$ with at most $n$ nonzero digits in the dyadic expansion of $i$. There is an associated spectral sequence converging to $\text{Ext}^s_{U}(P_\infty, \Sigma^tF_2)$, and Bousfield’s conjecture is that this spectral sequence collapses, giving

$$\text{Ext}^s_{U}(P_\infty, \Sigma^tF_2) \cong \oplus_n \text{Ext}^s_{U}(P_n/P_{n-1}, \Sigma^tF_2).$$

For an algebraic exploration of the conjecture and of the conjectured workings of the UASS of SO, see [1].

It is a suggestion of Mahowald that one try to relate Bousfield’s conjecture to the Postnikov tower for SO, which consists of spaces with polynomial cohomology of the form $U(-)$ and $k$-invariants which are Massey-Peterson maps. In Figure 1 we display the beginning of the tower. Note that the composites $k_n i_n$ run cyclically through the list $Sq^3$, $Sq^5$, $Sq^2$, $Sq^2$. If we let $X_n$ be the $n$th space in the Postnikov tower, then $\text{im}(H^*X_n \to H^*SO) \cong U(P_n)$. That is, the filtration of $H^*SO$ given by the Postnikov tower is the same as that
\[K(Z/2, 8) \xrightarrow{i_4} X_4 \xrightarrow{k_4} K(Z/2, 10)\]
\[K(Z, 7) \xrightarrow{i_3} X_3 \xrightarrow{k_3} K(Z/2, 9)\]
\[K(Z, 3) \xrightarrow{i_2} X_2 \xrightarrow{k_2} K(Z, 8)\]
\[K(Z/2, 1) \xrightarrow{i_1} X_1 \xrightarrow{k_1} K(Z, 4)\]

Fig. 1. The Postnikov tower for SO

given by dyadic expansion. Hence the Postnikov tower gives some geometric significance to the dyadic filtration of \(P_\infty\). The theorems in this paper are a first attempt to study the conjecture of Bousfield by comparing neighboring terms in the Postnikov tower of SO.

It is an easy matter to split off the first two filtrations.

**Proposition 5.1** \(\text{Ext}^s_\mathcal{U}(P_\infty, \Sigma^tF_2)\) is isomorphic to the direct sum

\[\text{Ext}^s_\mathcal{U}(P_1, \Sigma^tF_2) \oplus \text{Ext}^s_\mathcal{U}(P_2/P_1, \Sigma^tF_2) \oplus \text{Ext}^s_\mathcal{U}(P_\infty/P_2, \Sigma^tF_2).\]

**Proof.** First, we claim that the boundary map in the long exact sequence for Ext arising from the short exact sequence

\[0 \to P_1 \to P_\infty \to P_\infty/P_1 \to 0\]

is zero. To prove this, it is sufficient to show that the map

\[\text{Ext}^s_\mathcal{U}(P_\infty, \Sigma^tF_2) \to \text{Ext}^s_\mathcal{U}(P_1, \Sigma^tF_2)\]

is an epimorphism. Since \(P_1 \cong F(1)\), a free module in the category of unstable \(A\)-modules, we have \(\text{Ext}^s_\mathcal{U}(P_1, \Sigma^tF_2) = 0\) when \(s > 0\), so surjectivity is clear for \(s > 0\). For \(s = 0\), we need only note that \(P_1 \hookrightarrow P_\infty\) takes the generator of \(P_1\) to a generator of \(P_\infty\).

To finish the proof of the proposition, we will show that the boundary map in
Ext associated to the short exact sequence

\[ 0 \to P_2/P_1 \to P_\infty/P_1 \to P_\infty/P_2 \to 0 \]

is zero. Again, it is sufficient to show that

\[ \text{Ext}_U^s (P_\infty/P_1, \Sigma^r F_2) \to \text{Ext}_U^s (P_2/P_1, \Sigma^r F_2) \]

is an epimorphism. Since \( P_2/P_1 \cong F(3) \), we know that \( \text{Ext}_U^s (P_2/P_1, \Sigma^r F_2) = 0 \) unless \( t - s = 3 \), so we certainly have an epimorphism for \( t - s \neq 3 \). On the other hand, \( P_\infty/P_2 \) has its first class in dimension 7, so \( \text{Ext}_U^s (P_\infty/P_2, \Sigma^r F_2) = 0 \) for \( t - s < 7 \). Therefore we have the required epimorphism for \( t - s = 3 \) also.

Because of the preceding proposition, to prove Bousfield’s conjecture it is sufficient to prove a splitting for \( \text{Ext}_U^s (P_\infty/P_2, \Sigma^r F_2) \). Note that \( U(P_\infty/P_2) \cong H^*SO(7) \), where \( SO(7) \) denotes the 6-connected cover of \( SO \). The first stage of a splitting of \( \text{Ext}_U^s (P_\infty/P_2, \Sigma^r F_2) \) would be

\[ \text{Ext}_U^s (P_4/P_2, \Sigma^r F_2) \cong \text{Ext}_U^s (P_4/P_3, \Sigma^r F_2) \oplus \text{Ext}_U^s (P_3/P_2, \Sigma^r F_2). \]

We can use Theorems 1.1 and 4.5 to offer a step in this direction. The first two stages of the Postnikov tower for \( SO(7) \) are

\[
\begin{array}{ccc}
K(Z/2,8) & \longrightarrow & SO[7,8] \\
& | & \\
& \downarrow & \\
K(Z,7) & \xrightarrow{\text{Sq}^2} & K(Z/2,9),
\end{array}
\]

and this is a situation to which we can apply Theorems 1.1 and 4.5.

We note the following facts, which follow directly from [3], [6], and the Massey-Peterson Theorem:

1. Let \( f : F(9) \to \overline{F}(7) \) be \( \text{Sq}^2 \). Then \( \text{cok}(f) \cong P_3/P_2 \), and so \( H^*SO[7,8] \cong U(N) \) where there is a short exact sequence

\[ 0 \to P_3/P_2 \to N \to \Omega \ker(f) \to 0. \]

2. There is a short exact sequence

\[ 0 \to \langle \text{Sq}^2\rangle \to \Omega \ker(f) \to P_4/P_3 \to 0. \]

3. There is a short exact sequence

\[ 0 \to \langle \text{Sq}^2\rangle \to N \to P_4/P_2 \to 0. \]
As an immediate corollary of these facts and Theorem 1.1, we have the following.

**Proposition 5.2**

\[
\text{Ext}^\ast_u (N, \Sigma^t F_2) \cong \text{Ext}^\ast_u (P_3/P_2, \Sigma^t F_2) \oplus \text{Ext}^\ast_u (\Omega \ker(f), \Sigma^t F_2).
\]

It further follows from the facts enumerated above that there is a filtration

\[
0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 = N
\]

with the property that

\[
\begin{align*}
N_1/N_0 &\cong P_3/P_2 \\
N_2/N_1 &\cong \langle Sq^2 t_8 \rangle \\
N_3/N_2 &\cong P_4/P_3.
\end{align*}
\]

There is an associated spectral sequence converging to \(\text{Ext}^\ast_u (N, \Sigma^t F_2)\), and the preceding proposition tells us that all elements of \(\text{Ext}^\ast_u (P_3/P_2, \Sigma^t F_2)\) are infinite cycles. Another result necessary for the first level of a splitting result would be the following.

**Conjecture 5.3** \(d_1 : \text{Ext}^\ast_u (\langle Sq^2 t_8 \rangle, \Sigma^t F_2) \rightarrow \text{Ext}^{\ast+1}_u (P_4/P_3, \Sigma^t F_2)\) is zero.

**Corollary to Conjecture 5.3**

\[
\text{Ext}^\ast_u (P_4/P_2, \Sigma^t F_2) \cong \text{Ext}^\ast_u (P_4/P_3, \Sigma^t F_2) \oplus \text{Ext}^\ast_u (P_3/P_2, \Sigma^t F_2).
\]

**Proof.** Conjecture 5.3 would tell us that

\[
\begin{align*}
\text{Ext}^\ast_u (N, \Sigma^t F_2) &\cong \text{Ext}^\ast_u (P_3/P_2, \Sigma^t F_2) \oplus \text{Ext}^\ast_u (\langle Sq^2 t_8 \rangle, \Sigma^t F_2) \\
&\hspace{1em} \oplus \text{Ext}^\ast_u (P_4/P_3, \Sigma^t F_2).
\end{align*}
\]

However, it is also possible to filter \(N\) by

\[
0 = N'_0 \subseteq N'_1 \subseteq N'_2 \subseteq N'_3 = N
\]

with the property that

\[
\begin{align*}
N'_1/N'_0 &\cong \langle Sq^2 t_8 \rangle \\
N'_2/N'_1 &\cong P_3/P_2 \\
N'_3/N'_2 &\cong P_4/P_3
\end{align*}
\]
and the spectral sequence for Ext arising from this filtration would also collapse. Therefore the inclusion \( \langle \text{Sq}^2 t_8 \rangle \to N \) would give an epimorphism

\[
\text{Ext}^\ast_{U_1} (N, \Sigma' F_2) \to \text{Ext}^\ast_{U_1} (\langle \text{Sq}^2 t_8 \rangle, \Sigma' F_2),
\]

and hence the short exact sequence

\[
0 \to \langle \text{Sq}^2 t_8 \rangle \to N \to P_4/P_2 \to 0
\]

would split on Ext. The corollary would then follow.

Even in the absence of Conjecture 5.3, however, we can apply Theorem 4.5 and Corollary 4.7 to get the following chart for the UASS of SO[7, 8]. The plain dots come from the summand \( \text{Ext}^\ast_{U_1} (P_3/P_2, \Sigma' F_2) \). The circled dots come from the summand \( \text{Ext}^\ast_{U_1} (\Omega \text{ker}(f), \Sigma' F_2) \). No circled dot can survive to be an element of homotopy, which is found only in dimensions 7 and 8, so since \( E_3 = E_\infty \), all circled dots support \( d_2 \) differentials. Further, the error term \( y \) of Theorem 4.5 is always zero, because \( y \) must be a \( d_2 \) cycle and in this situation there are no such cycles, as we have just discussed.

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References


