A NOTE ON THE HOMOTOPY TYPE OF $BSL_3(\mathbb{Z})^\wedge_2$

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Abstract. It is known that for $p$-perfect groups $G$ of finite virtual cohomological dimension and finite type mod-$p$ cohomology, the $p$-completed classifying space $BG^\wedge_p$ has the property that $\Omega BG^\wedge_p$ is a retract of the loop space on a simply-connected, $\mathbb{F}_p$-finite, $p$-complete space. In this note we consider a particular example where this theorem applies, namely we study the homotopy type of $BSL_3(\mathbb{Z})^\wedge_2$. In particular we analyze $\Omega BSt_3(\mathbb{Z})^\wedge_2$, a double cover of $\Omega BSL_3(\mathbb{Z})^\wedge_2$, and obtain a splitting theorem for it in terms of 2-primary Moore spaces and fibres of degree $2^r$ maps on spheres. We also give a formula for the Poincaré series of $H_*(\Omega B\Gamma^\wedge_p,\mathbb{F}_p)$ for a general group $\Gamma$, as above, in terms of possibly simpler components. This formula is used to calculate the mod-2 homology of $\Omega B\Gamma^\wedge_p$ for $\Gamma = SL_3(\mathbb{Z})$ or $St_3(\mathbb{Z})$ as modules over a certain tensor subalgebra.

0. Introduction

Let $p$ be a prime number and let $\Gamma$ be a $p$-perfect group of finite virtual cohomological dimension and mod-$p$ cohomology of finite type. Then it shown in [4] that in this case there exists a simply-connected, $\mathbb{F}_p$-finite, $p$-complete space $S$ and a map $f : S \to B\Gamma^\wedge_p$ such that $\Omega f$ has a right homotopy inverse. If one makes the additional hypothesis that the $p$-adic cohomology of $B\Gamma$ is of finite type then it follows from [1] that $S$ can be thought of as the $p$-completion of a simply connected finite complex. Thus one can expect a rich interplay between the the homotopy theory associated to classifying spaces of groups $\Gamma$, as above, and the homotopy theory of finite complexes. In this note we investigate this relationship for the first non-trivial example among a particularly interesting family of groups, namely the special linear groups over the integers $SL_n(\mathbb{Z})$. The case we study is $n = 3$, where the cohomology was computed by Soulé [8] in 1978. For $n \geq 1$ let $St_n(\mathbb{Z})$ denote the Steinberg group of rank $n$ over the integers. For any $n$ the group $St_n(\mathbb{Z})$ is a central extension of $SL_n(\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$.

Throughout this paper we denote the groups $SL_3(\mathbb{Z})$ and $St_3(\mathbb{Z})$ by $Sl$ and $St$ respectively. We show that the spaces $\Omega(BSl)^\wedge_2$ and $\Omega(BSt)^\wedge_2$ are retracts of the loop spaces on certain 4-dimensional torsion complexes. Moreover, these complexes are shown to be given by wedges of mod-$2^r$
Moore spaces. We then proceed with a calculation of the mod-2 homology of \( \Omega(BSt)^\wedge_2 \) and \( \Omega(BSl)^\wedge_2 \) as finitely generated free modules over a certain subalgebra. This calculation is an easy consequence of a general formula for the Poincaré series of \( H_*(\Omega B\Gamma^\wedge_p; F_p) \), which we derive, expressing it in terms, which for our examples are easily computable.

Our calculations are used to obtain a splitting result for \( \Omega(BSt)^\wedge_2 \) in terms of an infinite product, where one of the factors is the fibre of the degree 16 map on the sphere \( S^3 \) and the other factors are given in terms of mod-2 Moore spaces. This splitting gives a considerably large family of non-trivial elements in the 2-primary unstable algebraic \( K \) groups \( K_{3,*}(\mathbb{Z}) = \pi_*(BSL_3(\mathbb{Z})_2^\wedge) \).

The point of this calculation is to show how homotopy theoretic information can be derived in those examples out of some naive representation theory associated to the groups under consideration and cohomological information. For \( n > 3 \) there isn’t nearly as much known about the cohomology of \( SL_n(\mathbb{Z}) \) as is the case for \( n = 3 \). However, the reader will notice that we do not use any high dimensional cohomological information but rather only need to know the cohomology of \( St_3(\mathbb{Z}) \) up to dimension 4. Thus one may hope that our techniques might generalize to higher values of \( n \).

The paper is organized as follows. Section 1 contains a preliminary discussion of the main claims of [4], which are a prerequisite for our study here. A summary of results is in section 2. The rest of the paper contains proofs of our statements.

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1. Preliminaries

Recall that a group \( \Gamma \) is \( p \)-perfect if its first \( p \)-cohomology vanishes. Let \( \Gamma \) be a \( p \)-perfect group of finite virtual cohomological dimension and \( p \)-cohomology of finite type. It is shown in [4] that for some \( p \)-compact group \( X \), there exists a map \( f : BG_\wedge_p^\wedge \rightarrow BX \), such that the homotopy fibre \( X_\Gamma \) of \( f \) is \( F_p \)-finite, i.e. its \( p \)-cohomology is a finite dimensional \( p \)-vector space. One may always take \( BX \) to be the \( p \)-completion of the classifying space of a suitable compact Lie group. A map \( f \) as above is said to be a faithful homotopy representation of \( \Gamma \) in \( X \). A general reference for \( p \)-compact groups is [3].

Given a faithful homotopy representation of a group \( \Gamma \) as above in a connected \( p \)-compact group \( X \), one gets, upon \( p \)-completion, a fibre sequence

\[
(1) \quad X \overset{\iota}{\rightarrow} (X_\Gamma)_\wedge \overset{j}{\rightarrow} BG_\wedge \overset{f}{\rightarrow} BX.
\]

Let \( S_f(\Gamma) \) denote the \( p \)-completion of the mapping cone of \( \iota \). Notice that \( S_f(\Gamma) \) is \( F_p \)-finite. The universal property of cofibrations in conjunction
with the fact that $B\Gamma_p$ is $p$-complete gives a map $\rho : S_f(\Gamma) \to B\Gamma_p$ and the main result of [4] is that $\Omega \rho$ has a right homotopy inverse. In general one has the formula

$$\Omega S_f(\Gamma) \simeq \Omega B\Gamma_p \times \Omega \Sigma(X \wedge \Omega B\Gamma_p)_p.$$  

Notice that this theorem implies that, given any explicit CW structure on $B\Gamma$ and letting $B_n\Gamma$ denote $n$-skeleton (for instance one may take the $n$-th stage of Milnor’s classifying space construction for $\Gamma$), then for some $n > 0$ the $p$-completion of the inclusion $B_n\Gamma \to B\Gamma$ admits a right homotopy inverse after looping. The minimal $n$ for which this holds is called the $p$-essential dimension $ed_p(\Gamma)$ of the group $\Gamma$. It is of course independent of the choice of a CW structure. In [4] the author gives an upper bound on $ed_p(\Gamma)$. For groups of positive vcd, the bound is given by the sum of the minimal dimension of a faithful homotopy representation of $\Gamma$ and its vcd.

2. STATEMENT OF RESULTS

For any $n \geq 2$ and any prime power $p^r$, let $P^n(p^r)$ denote the mod-$p^r$ Moore space, i.e. the cofibre of the degree $p^r$ map on the sphere $S^{n-1}$.

**Theorem 2.1.** The 2-essential dimension of $Sl$ and $St$ is precisely 4. Moreover, let $S = P^3(2) \vee P^4(2) \vee P^4(4) \vee P^4(8)$ and let $\tilde{S} = P^3(2) \vee P^4(4) \vee P^4(16)$. Then there are maps $\alpha : S \to (BSl)_2$ and $\tilde{\alpha} : \tilde{S} \to (BSt)_2$ such that $\Omega \alpha$ and $\Omega \tilde{\alpha}$ have right homotopy inverses.

As we shall see, formula (2) gives that there exist homotopy equivalences

$$\Omega S \simeq \Omega (BSl)_2 \times \Omega \Sigma(SO(3) \wedge \Omega (BSl)_2)$$

and

$$\Omega \tilde{S} \simeq \Omega (BSt)_2 \times \Omega \Sigma^4 \Omega (BSt)_2.$$  

One has to be a bit careful here in general, as a statement as above will be true only after an extra completion on the right hand side. However the factors on the right hand side of both equations are purely 2-torsion spaces and thus are 2-complete to begin with.

These equivalences imply that 8 and 16 annihilate the reduced homology of $\Omega (BSl)_2$ and $\Omega (BSt)_2$ respectively and, furthermore, that this order of torsion is actually present in the respective homology groups. The second bound is a direct consequence of the more explicit description of $\Omega (BSt)_2$ given below. However the existence of torsion of order 8 in the homology of $\Omega (BSl)_2$ is mildly curious as the homology of $(BSl)_2$ has exponent 4. At any rate the fact that loop space homology has an exponent at all is rather unusual. The reader is referred to [5] for a general theorem in this fashion for finite groups. It is worth pointing out though, that for higher
values of $n$, the groups $SL_n(\mathbb{Z})$ have rational homology, so an exponent result should not be expected.

We also remark that the upper bound for the 2-essential dimension of these groups, as follows from [4], is 6. Thus the approximation there is not best possible in these examples.

Since $S$ and $\tilde{S}$ in Theorem 2.1 are finite suspensions, their loop space homology algebras are finitely generated free associative algebras and so an immediate corollary is the following

**Corollary 2.2.** The algebras $H_*(\Omega(\text{BS}l)_2^\wedge; \mathbb{F}_2)$ and $H_*(\Omega(\text{BS}t)_2^\wedge; \mathbb{F}_2)$ are finitely generated.

Next we get a more explicit hold on the homology of $\Omega(\text{BS}l)_2^\wedge$ and $\Omega(\text{BS}t)_2^\wedge$. To do that we first obtain a general formula for the Poincaré series of $\Omega B\Gamma^\wedge_p$ for a general group $\Gamma$, satisfying our hypotheses. For a space $Y$ let $P_Y$ denote the Poincaré series for $H_*(Y; \mathbb{F}_p)$.

**Theorem 2.3.** Let $\Gamma$ be a group of finite $vcd$ and finite type mod-$p$ cohomology. Let a faithful homotopy representation $f$ of $\Gamma$ in some $p$-compact group $X$ be given and let $S = S_f(\Gamma)$ be defined as above. Then

$$P_{\Omega B\Gamma^\wedge_p} = \frac{P_{\Omega S}P_X}{1 - P_{\Omega S} + P_{\Omega S}P_X}.$$

One could hope that it would be easier to calculate $P_{\Omega S}$ than $P_{\Omega B\Gamma^\wedge_p}$, since $S$ is $\mathbb{F}_p$-finite, where $B\Gamma^\wedge_p$ generally has infinite dimensional mod-$p$ cohomology. Indeed in our examples the respective spaces $S$ and $\tilde{S}$ of Theorem 2.1 are suspensions, a fact which makes the calculation very easy. More generally, one has the following

**Corollary 2.4.** With the notation of Theorem 2.3, suppose that $S$ is a co-$H$ space. Then

$$P_{\Omega B\Gamma^\wedge_p}(t) = \frac{tP_X}{1 - P_S + tP_X}.$$

It should be pointed out that finite or finite $vcd$ groups $\Gamma$ with the property that $\Omega B\Gamma^\wedge_p$ is a retract of the loop space on a finite co-$H$ space appear to be rather rare.

Having computed the Poincaré series of $\Omega B\Gamma^\wedge_2$ for our examples, we are now able to get most of the product structure in the respective homology algebras.

**Theorem 2.5.** Let $T$ denote the free associative algebra over $\mathbb{F}_2$ generated by elements $a_1, b_2, c_2, x_2$, and $d_3$, where degrees are given by subscripts. Let $H$ denote $H_*(\Omega(\text{BS}l)_2^\wedge; \mathbb{F}_2)$ and let $\tilde{H}$ denote $H_*(\Omega(\text{BS}t)_2^\wedge; \mathbb{F}_2)$. Then $H$ and $\tilde{H}$ are finitely generated free $T$-modules. In fact there are isomorphisms of $T$-modules

$$H \cong T \otimes H_*(SO(3); \mathbb{F}_2) \quad \text{and} \quad \tilde{H} \cong T \otimes H_*(S^3; \mathbb{F}_2).$$
Finally we show that, as a space, $\Omega(BSt)^\wedge_2$ can be expressed in terms of simpler building blocks.

**Theorem 2.6.** Let $Y$ denote the space $P^3(2) \vee P^4(4)$ and let $P$ denote $\Omega S^3 \wedge \Omega Y$. Then there is a homotopy equivalence

$$
\Omega(BSt)^\wedge_2 \simeq \Omega \Sigma P \times S^3\{16\} \times \Omega Y.
$$

As we observe below, $\Omega(BSt)^\wedge_2$ is a double cover of $\Omega(Bl)^\wedge_2$. Thus $(Bl)^\wedge_2$ and $(BSt)^\wedge_2$ have the same 2-connected cover. Notice that Theorem 2.6 gives the homotopy type of $\Omega BSt_3(Z)$ in terms of very familiar spaces. Elementary homotopy theory can be used to show that $P$ splits as an infinite wedge, where most of the summands are mod-2 and 4 Moore spaces, while the others are given by iterated smash products of mod-2 Moore spaces. The factor $\Omega Y$ also admits a splitting with similar summands. Thus the theorem yields a large family of non-zero 2-primary classes in the unstable algebraic $K$-theory of the integers $K_3(Z) = \pi_*(BSL_3(Z)^\wedge_2)$.

### 3. Finite complexes associated to $SL_n(Z)$

For any Euclidean domain $E$, the Steinberg group $St_n(E)$ is a central extension of $SL_n(E)$. Moreover, the extension is natural with respect to ring homomorphisms. Thus, consider the faithful representation of $SL_n(Z)$ in $SL_n(\mathbb{R})$, given by the natural inclusion. For $E = \mathbb{Z}$ or $\mathbb{R}$ the kernel of the extension above is given by $\mathbb{Z}/2\mathbb{Z}$ and one gets the corresponding commutative diagram of central extensions.

Applying the classifying space functor followed by $R$-completion to this diagram, where $R \subseteq \mathbb{Q}$ or $R = \mathbb{F}_p$, and replacing Lie groups by their maximal compact subgroups, we get a homotopy commutative diagram of fibrations:

$$
\begin{array}{ccc}
Z(n) & = & Z(n) \\
\downarrow & & \downarrow \\
(BZ/2\mathbb{Z})^\wedge_R & \longrightarrow & BSt_n(Z)^\wedge_R \\
\downarrow & & \downarrow \\
(BZ/2\mathbb{Z})^\wedge_R & \longrightarrow & BSL_n(Z)^\wedge_R \\
\downarrow & & \downarrow \\
(BZ/2\mathbb{Z})^\wedge_R & \longrightarrow & BSpin(n)^\wedge_R \\
\downarrow & & \downarrow \\
(BZ/2\mathbb{Z})^\wedge_R & \longrightarrow & BSO(n)^\wedge_R \\
\end{array}
$$

The theory presented in section 1 gives that the cofibres $S(n)$ and $\tilde{S}(n)$ of the maps from $SO(n)^\wedge_R$ and $Spin(n)^\wedge_R$ to $Z(n)$, respectively, have the property that

$$
\Omega S(n)^\wedge_R \simeq \Omega BSL_n(Z)^\wedge_R \times \Omega \Sigma(SO(n)^\wedge_R \wedge \Omega BSL_n(Z)^\wedge_R)^\wedge_R
$$

and

$$
\Omega \tilde{S}(n)^\wedge_R \simeq \Omega BSt_n(Z)^\wedge_R \times \Omega \Sigma(Spin(n)^\wedge_R \wedge \Omega BSt_n(Z)^\wedge_R)^\wedge_R.
$$
Notice that $S(n)$ is the $R$-completion of the cofibre of the map from $SL_n(\mathbb{R})$ to $Z'(n)$, where $Z'(n)$ denotes the orbit space $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. The corresponding description can be given for $\tilde{S}(n)$.

By [4], for $R = \mathbb{F}_p$, the mod-$p$ cohomological dimension of both $S(n)$ and $\tilde{S}(n)$ is bounded above by the sum of the cohomological dimension of $SO(n)$ and the vcd of $SL_n(\mathbb{Z})$. The vcd of $SL_n(\mathbb{Z})$ was computed by Borel and Serre [2] to be $\frac{n(n-1)}{2}$, which apparently is the same as the topological dimension of $SO(n)$. Thus $n(n-1)$ is an upper bound for the cohomological dimension of $S(n)$. This is also an upper bound for the $p$-essential dimension of these groups for every prime $p$ although an improvement, which we don’t discuss here, can be obtained for odd primes. Notice that a bound for the 2-essential dimension of $SL$ and $St$, given by this discussion is 6, whereas the actual 2-essential dimension is 4, as we show below.

4. The cohomology of $Sl$ and $St$ and the related homotopy types

In [8] Soulé presents a cohomology calculation for $Sl$ and $St$ with coefficients in the 2-local integers $\mathbb{Z}(2)$. His methods though are applicable to any ring of coefficients, a fact which will be of use in the sequel. The actual calculation will be of limited benefit to us, but for the sake of completeness we start by recording it.

**Theorem 4.1** (Soulé). There are isomorphisms

$$H^*(BSl; \mathbb{Z}(2)) \cong P[x_3, y_3, a_4, b_4, t_5, u_6, v_6]/I$$

and

$$H^*(BSt; \mathbb{Z}(2)) \cong P[\alpha_3, \eta_4, \nu_4]/J,$$

where degrees are given by subscripts. The ideal $I$ is given by

$$2x = 2y = 4a = 4b = 2t = 2u = 2v = 0$$

$$vx = vb = vt = vu = yx = yb = yt = yu = 0$$

$$v^2 + vy^2 = ab + xt = au + ax^2 = au + t^2 = 0$$

$$xu + bt = ab^2 + u^2 = tu + tx^2 = 0.$$  

The ideal $J$ is given by

$$2\alpha = 4\eta = 16\nu = \alpha^2 = \alpha\eta = \eta\nu = 0$$

Notice that the cohomology of $St$ is dramatically simpler than that of $Sl$. In fact the cohomology of $St$ is periodic, as one can see from Theorem 4.1 or observe as follows. Recall that $Spin(3) \simeq S^3$. Thus, by pulling back the central column of Diagram 2 we get a fibration

$$(S^3)^\wedge_2 \longrightarrow Z \longrightarrow (BST)^\wedge_2,$$

where $Z = Z(3)$. Since $Z$ is of finite mod-2 cohomological dimension and since the fibration is principal, it follows that there is a non-trivial transgression in the associated Serre spectral sequence, which grants periodicity.
in positive dimensions. It takes an easy exercise in undetermined coefficients to see that the periodicity class, given by the image of transgression, is $a\eta + bv$, where $a$ and $b$ are the classes of some odd integers modulo 4 and 16 respectively. The following is a straight forward consequence of the Serre spectral sequence.

**Lemma 4.2.** There is an isomorphism of abelian groups

$$H^*(Z; R) \cong H^*(P^3(2) \vee P^4(4) \vee S^3; R),$$

For $R = \mathbb{Z}_{(2)}$ or $\mathbb{F}_2$.

Consider the map $(S^3)^\wedge_2 \rightarrow Z$. The Serre spectral sequence calculation easily implies that the induced map is degree 16 on integral cohomology. On the other hand notice that there is a commutative diagram of cofibrations

$$
\begin{array}{cccc}
S^3 & \rightarrow & Z & \rightarrow & \tilde{S} \\
\pi & & \downarrow & & \\
SO(3) & \rightarrow & Z & \rightarrow & S
\end{array}
$$

and that $\pi$ is degree 2 on the top class. Here of course $S$ and $\tilde{S}$ are $S(3)$ and $\tilde{S}(3)$ of the preceding section respectively. It is now easy to conclude the following

**Lemma 4.3.** There are isomorphisms of abelian groups

$$H^*(S; R) \cong H^*(P^3(2) \vee P^3(2) \vee P^4(4) \vee P^4(8); R),$$

and

$$H^*(\tilde{S}; R) \cong H^*(P^3(2) \vee P^4(4) \vee P^4(16); R),$$

For $R = \mathbb{Z}_{(2)}$ or $\mathbb{F}_2$.

Notice that the map $\tilde{\rho} : \tilde{S} \rightarrow (BSl)_2^\wedge$ can be thought of as the inclusion of the 4-skeleton, but that $S$ actually has torsion of order 8 whereas the integral cohomology of $BSl$ is annihilated by 4. Furthermore, by the discussion in the preceding section, $\Omega\rho$ and $\Omega\tilde{\rho}$ have right homotopy inverses. We proceed by analyzing the homotopy structure of $S$ and $\tilde{S}$.

Throughout the end of this section we omit the 2-completion symbol at our convenience. All statement thus hold up to 2-completion. The homotopy type of $\tilde{S}$ can be determined rather easily if one knows its cohomology with $\mathbb{Z}/16\mathbb{Z}$ coefficients. Indeed consider the additive cohomological structure, given in Lemma 4.3. Clearly the 3-skeleton $\tilde{S}(3)$ is given by $P^3(2) \vee S^3 \vee S^3$. Analyzing $\pi_3$ of this space is elementary; there is a free summand of rank 2 given by $\pi_3(S^3 \vee S^3)$ with generators $\lambda_1, \lambda_2$ and a class of order 4 given by the generator of $\pi_3(P^3(2))$. The last is the class of the composite $S^3 \xrightarrow{\eta} S^2 \xrightarrow{\pi} P^3(2)$, where $\eta$ is the Hopf map.
Thus each of the two attaching maps for 4-cells in \( \tilde{S} \) is given as a sum \( a\lambda_1 + b\lambda_2 + c\pi \eta \), where \( a \) and \( b \) are the appropriate powers of 2 and \( c \in \mathbb{Z}/4\mathbb{Z} \). Clearly \( \tilde{S} \) is a wedge of Moore spaces if and only if \( c = 0 \) for each one of the attaching maps. Notice that if \( c \) is odd, the corresponding cohomology class will be the cup square of the bottom class. Thus such a summand can be detected by mod-2 cohomology for \( r \geq 1 \). On the other hand 2\( \pi \eta \) can be detected in mod-2 cohomology for \( r \geq 2 \). Thus, in particular, by computing the cup product structure in \( H^*(\tilde{S}; \mathbb{Z}/16\mathbb{Z}) \), we can complete the analysis of its homotopy type.

**Proposition 4.4.** All cup products in \( H^*(\tilde{S}; \mathbb{Z}/16\mathbb{Z}) \) are trivial. Thus there is a homotopy equivalence

\[
\tilde{S} \simeq P^3(2) \lor P^4(4) \lor P^4(16).
\]

**Proof.** The map \( \tilde{\rho} : \tilde{S} \longrightarrow (BSt)^\wedge_2 \) produced above induces an isomorphism on mod-2\( r \) cohomology for \( r \geq 1 \) through dimension 4. Thus our claim follows by using Soulé’s method to calculate the mod-16 cohomology of \( St \) in low dimensions.

Indeed Soulé shows that the 2-completed homotopy types of \( BSl \) and \( BST \) can be obtained as the 2-completions of certain homotopy colimits, which we describe below (compare to [6]). Thus the respective cohomology can be computed in terms of the spectral sequence associates to a homotopy colimit. In both cases the spectral sequence degenerates into a long exact, Mayer-Vietoris type, long exact sequence which allows an easy calculation of the cohomology. We proceed by describing this long exact sequence for \( St \).

Consider the diagram

\[
\begin{array}{ccc}
\Sigma_4 & \longrightarrow & \Sigma_4 \\
\downarrow j_2 & & \downarrow j_1 \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & D_8, \\
\end{array}
\]

where the morphisms \( j_1 \) factors through the alternating group \( A_4 \) and \( j_2 \) does not; the morphisms \( i_k, k = 1, 2 \), are specified by the requirement that they are not conjugate in \( D_8 \).

Applying the classifying space functor \( B(\cdot) \) to this diagram one obtains a map \( \phi \) from the homotopy colimit of the resulting diagram of spaces to the classifying space \( BSl \), which induces a mod-2 cohomology isomorphism and thus a homotopy equivalence after 2-completion. In particular \( \phi \) induces a cohomology isomorphism with respect to any 2-local ring of coefficients.

Now, consider the group \( \Sigma_4 \) given as the unique central extension of \( \Sigma_4 \) by \( \mathbb{Z}/2\mathbb{Z} \) which covers the projection \( Q_{16} \longrightarrow D_8 \), the target being the Sylow 2-subgroup of \( \Sigma_4 \). Then the diagram above can be pulled back along
the projection of $\tilde{\Sigma}_4$ onto $\Sigma_4$ to obtain a new diagram

$$\begin{array}{ccc}
\tilde{\Sigma}_4 & \xrightarrow{\tilde{i}_2} & \tilde{\Sigma}_4 \\
\xrightarrow{\tilde{j}_2} & & \xrightarrow{\tilde{i}_1} \\
\mathbb{Z}/2\mathbb{Z} & & \mathbb{Q}_{16}.
\end{array}$$

The process described above for $Sl$ applies here to give a homology decomposition of $BSt$ at 2, namely, applying $B(-)$ and taking the homotopy colimit, one obtains a map $\tilde{\phi}$ to $BSt$, which induces a cohomology isomorphism with respect to any 2-local ring of coefficients.

Notice that the nature of these diagrams implies that in the associated spectral sequence for the cohomology of the homotopy colimit, the non-trivial terms appear on the 0 and 1 vertical lines. In other word the functor $\lim^i(H^*(-))$ applied to these diagrams obtains trivial values for $i > 1$. Thus for the diagram corresponding to the Steinberg group one obtains each cohomology groups as an extension

$$0 \rightarrow \lim^1_\mathcal{D} H^{n-1}(-) \rightarrow H^n(BSt) \rightarrow \lim_\mathcal{D} H^n(-) \rightarrow 0,$$

where $\mathcal{D}$ is the diagram for $St$. Splicing these sequences dimensionwise with the exact sequence defining the inverse limit and its first derived functor and canceling redundant terms, one gets, as in [8], a long exact sequence with respect to any 2-local ring of coefficients:

$$(4) \cdots \rightarrow H^i(Sl) \xrightarrow{\alpha} H^i(\tilde{\Sigma}_4)^{\times 3} \xrightarrow{d} H^i(Q_{16}) \times H^i(\mathbb{Z}/2\mathbb{Z}) \rightarrow \cdots$$

where the map $d$ is given by the formula

$$d(x, y, z) = (\tilde{i}_2^*(x) - \tilde{i}_1^*(y), \tilde{j}_1^*(y) - \tilde{j}_2^*(z))$$

and $\alpha$ is the ring homomorphism induced by the natural map from the 3-fold free product of $\tilde{\Sigma}_4$ into $St$.

The cohomology of all groups involved is well known. In particular an explicit description of the cohomology of $\Sigma_4$ is in [8]. We write an additive set of generators for these groups in low dimensions. Throughout this section mod-16 coefficients are assumed. For $1 \leq k \leq 4$ the groups $H^k(\tilde{\Sigma}_4)$ are generated by elements $\sigma_1$, $\eta_2$, $\epsilon_3$ and $\tau_4$, where dimensions are given by subscripts and orders are 2, 2, 16 and 16 respectively. Similarly, the groups $H^k(Q_{16})$ are generated by $\alpha_1$, $\beta_1$, $\gamma_2$, $\delta_2$, $\epsilon_3$ and $\tau_4'$ of orders 2, 2, 2, 2, 16 and 16 respectively. Finally the groups $H^k(\mathbb{Z}/2\mathbb{Z})$ are generated by $a_1$, $b_2$, $c_3$ and $d_4$ all of order 4. Using Soulé’s integral cohomology calculation and the universal coefficients theorem, it is an elementary exercise to compute the following table, describing the effect of the maps induced on cohomology by the homomorphisms in the diagram above.
From this table one easily calculates the homomorphism \( d \) in (4) up to \( i = 4 \). In particular one has from the data above that the map \( \alpha \) is injective in dimension 4 and zero in dimension 2. By naturality of the cup product one concludes that the cup square of the generator of \( H^2(St) \) vanishes. This completes the proof.

The homotopy type of \( S \) is calculated similarly. However, a direct calculation of the cohomology of \( Sl \) is not necessary.

**Proposition 4.5.** There is a homotopy equivalence
\[
S \simeq P^3(2) \vee P^4(2) \vee P^4(4) \vee P^4(8).
\]

**Proof.** Consider Diagram (3) above. It follows immediately that the map
\[
H^4(S; \mathbb{Z}/16\mathbb{Z}) \longrightarrow H^4(\tilde{S}; \mathbb{Z}/16\mathbb{Z})
\]
is injective. By naturality of the cup product, it follows at once that all cup products of elements in \( H^2(S; \mathbb{Z}/16\mathbb{Z}) \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) vanish. Thus a simple argument as above, involving the possible attaching maps for 4-cells in \( S \), gives that the two 4-cells are attached only to the respective 3-cells, as detected by cohomology, and the result follows.

This completes the proof of Theorem 2.1 except for the claim that the 2-essential dimension of the groups under consideration is precisely 4. It is obviously bounded above by 4. Furthermore, a simply-connected 2-dimensional space of finite type must be equivalent to a wedge of 2-spheres. Thus for any group \( \Gamma \) whose homology contains \( p \)-torsion, the \( p \)-essential dimension is larger than 2. Thus it remains to show that \( ed_2(\Gamma) > 3 \) in our examples.

Indeed, independently of the precise value of the 2-essential dimension, we compute the homology of \( \Omega(BSl)^{\wedge}_p \) and \( \Omega(BSt)^{\wedge}_p \). It follows immediately from the calculation that in both cases the loop space homology has indecomposable elements in dimension 3. On the other hand, every simply-connected 3-dimensional space \( T \) is a suspension and thus its loop space homology is a free associative algebra on generators in dimensions 1 and 2. It follows at once that if \( \Gamma \) is any group of \( p \)-essential dimension 3 then the homology of \( \Omega BT^p_\wedge \) is generated by elements in degrees 1 and 2. Since this is not the case in our examples, the proof is complete.
5. A formula for the Poincaré series

Let \( P_X \) denote the Poincaré series for the mod-\( p \) homology of a space \( X \).

Let \( \Gamma \) be a \( p \)-perfect finite vcd group satisfying the extra hypothesis of having a finite type \( p \)-cohomology module. Let a faithful homotopy representation \( f \) of \( \Gamma \) in some \( p \)-compact group \( X \) be given and construct the space \( S = S_f(\Gamma) \), as explained in section 1 above. Then there is a homotopy equivalence

\[
\Omega S_f(\Gamma) \simeq \Omega \Sigma (X \wedge \Omega B\Gamma_p^\wedge) \times \Omega B\Gamma_p^\wedge.
\]

Now since \( H_* (\Omega \Sigma A) \cong T[\tilde{H}_*(X)] \), where \( T \) denotes the tensor algebra, one has

\[
P_{\Omega \Sigma (X \wedge \Omega B\Gamma_p^\wedge)} = \frac{1}{1 - \tilde{P}_X \tilde{P}_{\Omega B\Gamma_p^\wedge}} = \frac{1}{1 - (P_X - 1)(P_{\Omega B\Gamma_p^\wedge} - 1)},
\]

where \( \tilde{P} \) denotes the Poincaré series for the reduced homology. Thus we get the formula

\[
P_{\Omega S} = \frac{P_{\Omega B\Gamma_p^\wedge}}{1 - (P_X - 1)(P_{\Omega B\Gamma_p^\wedge} - 1)}
\]

It is now easy to solve for \( P_{\Omega B\Gamma_p^\wedge} \) and get the formula of Theorem 2.3.

Having established this, Corollary 2.4 follows from the observation that if \( S \) is a co-H space then

\[
P_{\Omega S}(t) = \frac{t}{1 + t - P_S}.
\]

6. The loop space homology

Let \( H \) and \( \tilde{H} \) denote the homology algebras of \( \Omega (BSl)_2^\wedge \) and \( \Omega (BSt)_2^\wedge \) respectively. Let \( P_H \) and \( P_{\tilde{H}} \) denote the respective Poincaré series.

**Proposition 6.1.** With the notation above

\[
P_H(t) = \frac{1 + t^3}{1 - t - 3t^2 - t^3}
\]

and

\[
P_{\tilde{H}} = \frac{1 + t + t^2 + t^3}{1 - t - 3t^2 - t^3}
\]

**Proof.** Since both \( S \) and \( \tilde{S} \) are suspensions, we can apply Corollary 2.4, which gives

\[
P_H(t) = \frac{tP_{SO(3)}}{1 - P_S + tP_{SO(3)}} \quad \text{and} \quad P_{\tilde{H}} = \frac{tP_{S^3}}{1 - P_S + tP_{S^3}}.
\]

Using the information obtained above about \( S \) and \( \tilde{S} \), the reader immediately verifies the proposition. \( \square \)
Recall that we have fibrations
\[\Omega Z \longrightarrow \Omega(BSl)_2^\wedge \longrightarrow (S^3)_2^\wedge\]
and
\[\Omega Z \longrightarrow \Omega(BSt)_2^\wedge \longrightarrow SO(3)_2^\wedge.\]
But notice that \(Z\) is a suspension with the Poincaré series for the mod-2 homology of \(\Omega Z\) given by \(1 - t - 3t^2 - t^3\). On the other hand the numerators of the fractions above give the Poincaré polynomials of \(SO(3)\) and \(S^3\) respectively. Thus the Serre spectral sequences for the fibrations above both collapse at the \(E^2\) page and we get the statement of Theorem 2.5.

7. SPLITTING IN TERMS OF MOORE SPACES

Let \(Z\) be, as before, the fibre of the map \((BSt)_2^\wedge \longrightarrow B(S^3)_2^\wedge\). Then the restriction map \(H^*((BSt)_2^\wedge) \longrightarrow H^*(Z)\) is an epimorphism with respect to any 2-local ring of coefficients. Thus the same argument as made for \(\tilde{S}\) in Proposition 4.4 gives that \(Z \simeq \tilde{Y} \vee S^3\), where \(\tilde{Y} = P^3(2) \vee P^4(4)\). Thus there is a diagram of fibrations

\[
\begin{array}{ccc}
\tilde{F} & \longrightarrow & S^3 \\
\downarrow & & \downarrow \\
\Omega(BSt)_2^\wedge & \longrightarrow & S^3 \\
\downarrow q & & \downarrow p \\
\Omega Y & \longrightarrow & Y
\end{array}
\]  

(5)

Notice that \(q\) has a right homotopy inverse, since \(p\) does. Hence, \(\Omega(BSt)_2^\wedge\) being a loop space, splits as a product
\[\Omega(BSt)_2^\wedge \simeq \tilde{F} \times \Omega Y.\]

The following two diagrams give a hold on \(\tilde{F}\). Let \(P\) denote \(\Omega S^3 \wedge \Omega Y\).

\[
\begin{array}{ccc}
\Omega\Sigma P & \longrightarrow & * \\
\downarrow & & \downarrow \\
\tilde{F} & \longrightarrow & S^3 \\
\downarrow \tilde{q} & & \downarrow = \\
S^3\{16\} & \longrightarrow & S^3 \times \Omega Y
\end{array} \quad \begin{array}{ccc}
\tilde{F} & \longrightarrow & S^3\{16\} \\
\downarrow \tilde{q} & & \downarrow = \\
\tilde{F} & \longrightarrow & S^3 \\
\downarrow \tilde{p}_1 & & \downarrow [16] \tilde{p}_1 \\
S^3 \times \Omega Y & \longrightarrow & S^3 \\
\end{array}
\]  

(6)

The left diagram implies that \(\tilde{q}\) has a right homotopy inverse, since \(\tilde{p}_1\) has one and since the 3-skeleton of \(S^3 \times \Omega Y\) is given by \(S^3\). The right diagram, on the other hand, gives that the fibration
\[\Omega\Sigma P \longrightarrow \tilde{F} \longrightarrow S^3\{16\}\]
is principal and hence that there is a homotopy equivalence
$$\tilde{F} \simeq \Omega \Sigma P \times S^3\{16\}. $$
Thus we get the statement of Theorem 2.6,
$$\Omega (BSt)^2_2 \simeq \Omega \Sigma (\Omega S^3 \wedge \Omega Y) \times S^3\{16\} \times \Omega Y. $$
To end this section, we point out that $\Sigma (\Omega S^3 \wedge \Omega Y)$ splits into an infinite wedge, where the factors are mod-2 and 4 Moore spaces and iterated smash products of mod-2 Moore spaces. These splittings are elementary but we describe them in some detail for the benefit of the reader who might not be a homotopy theorist. Good references are [9] and [7].

First, if $X$ is a space then the James splitting theorem gives that
$$\Sigma \Omega \Sigma X \simeq \Sigma \bigvee_{k \geq 1} X^{(k)}, $$
where $X^{(k)}$ denotes the $k$-fold smash product of $X$. It is also easy to see that for $r > s \geq 1$, there is the identity
$$P^n(2^r) \wedge P^k(2^s) \simeq P^{n+k-1}(2^s) \vee P^{n+k}(2^s). $$
However the cohomology of iterated smash products of mod-2 Moore spaces contains Steenrod squares of excess larger than 1. Thus these smash products do not split as a wedge.

In view of this one has
$$\Sigma (\Omega S^3 \wedge \Omega Y) \simeq \bigvee_{i=0}^{\infty} S^{3+2i} \wedge \Omega Y \simeq \bigvee_{i=0}^{\infty} \Sigma^{3+2i} \Omega \Sigma (P^2(2) \vee P^3(4)). $$
For each factor $\Sigma^{3+2i} \Omega \Sigma (P^2(2) \vee P^3(4))$ there is a decomposition
$$\Sigma^{3+2i} \Omega \Sigma (P^2(2) \vee P^3(4)) \simeq \Sigma^{3+2i} \bigvee_{j=1}^{\infty} (P^2(2) \vee P^3(4))^{(j)}. $$
The claim follows.

References


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