The Type of the Classifying Space of a Topological Group for the Family of Compact Subgroups

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Abstract

Let $G$ be a locally compact topological group. We investigate the type of the classifying space of $G$ for the family of compact subgroups. We give criteria for this space to have a $d$-dimensional $G$-$CW$-model, a finite $G$-$CW$-model or a $G$-$CW$-model of finite type. Essentially we reduce these questions to discrete groups and to the homological algebra of the orbit category of discrete groups with respect to certain families of subgroups.

Key words: classifying space of a group for a family, topological group

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Introduction

Throughout this paper we denote by $G$ a locally compact topological group (where locally compact always includes Hausdorff). We denote by $G_0$ the component of the identity element and by $\overline{G} = G/G_0$ its component group. Notice that $G_0$ is locally compact and connected and $\overline{G}$ is locally compact and totally disconnected, i.e. each component consists of exactly one point. Subgroup will always mean closed subgroup. Sometimes we make the additional assumption

\[ \text{(S)} \]

For any closed subgroup $H \subset G$ the projection $p : G \longrightarrow G/H$ has a local cross section, i.e. there is a neighborhood $U$ of $eH$ together with a map $s : U \to G$ satisfying $p \circ s = \text{id}_U$.

Condition (S) is automatically satisfied if $G$ is discrete, if $G$ is a Lie group, or more generally, if $G$ is locally compact and second countable and has finite covering dimension [15]. The metric needed in [15] follows under our assumptions, since a locally compact Hausdorff space is regular and regularity in a second countable space implies metrizability.

A family $\mathcal{F}$ consists of a set of (closed) subgroups of $G$ with the property that for any $H, K \in \mathcal{F}$ and $g \in G$, the subgroups $g^{-1}Hg$ and $H \cap K$ belong to $\mathcal{F}$. Notice that we do not require that $\mathcal{F}$ is closed under taking subgroups. A classifying space of $G$ for the family $\mathcal{F}$

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is a $G$-$CW$-complex $E(G, \mathcal{F})$ such that the fixed point set $E(G, \mathcal{F})^H$ is weakly contractible for $H \in \mathcal{F}$ and all its isotropy groups belong to $\mathcal{F}$. Recall that a map $f : X \to Y$ of spaces is a \textit{weak homotopy equivalence} if and only if the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all base points $x \in X$ and $n \geq 0$ and that a space $X$ is \textit{weakly contractible} if and only if the projection $X \to \{ * \}$ to the space consisting of one point is a weak homotopy equivalence. The $G$-$CW$-complex $E(G, \mathcal{F})$ has the universal property that for any $G$-$CW$-complex $X$ whose isotropy groups belong to $\mathcal{F}$, there is up to $G$-homotopy precisely one $G$-map $X \to E(G, \mathcal{F})$ and in particular it is unique up to $G$-homotopy. The notion of a $G$-$CW$-complex can be found for instance in \cite[Definition 1.2]{10}, the existence of $E(G, \mathcal{F})$ follows for instance from \cite[2.2]{10}, and the universal property of $E(G, \mathcal{F})$ is a consequence of the Equivariant Whitehead Theorem \cite[Theorem 2.4]{10}. If $\mathcal{F}$ is the family of compact subgroups $\mathcal{COM}$, we will often abbreviate $E(G, \mathcal{COM})$ by $EG$. Notice that for a discrete group $\mathcal{COM}$ is the same as the family $\mathcal{FIN}$ of finite subgroups.

In Section 1 we will explain this notion and put it into context with the similar notion due to tom Dieck \cite[section I.6]{4}. We mention that these spaces $E(G, \mathcal{F})$ and in particular $EG$ play an important role in the formulation of the Baum-Connes Conjecture \cite[Conjecture 3.15 on p. 254]{3}, the Isomorphism Conjecture in algebraic $K$- and $L$-theory of Farrell and Jones \cite{5}, the generalization of the completion theorem of Atiyah and Segal for finite groups to infinite discrete groups \cite{12} and in the construction of classifying spaces for equivariant bundles \cite[Section I.8 and I.9]{4}. More information about models for $EG$ can be found for instance in \cite{3}.

We want to get information about the possible type of $E(G, \mathcal{F})$, i.e. whether there is an $m$-dimensional $G$-$CW$-model, a finite $G$-$CW$-model or a $G$-$CW$-model of finite type. A $G$-$CW$-complex $X$ is \textit{finite} if it is built by finitely many equivariant cells or, equivalently, $G \setminus X$ is compact. It is called \textit{of finite type} if each skeleton $X_n$ is finite. For discrete groups the type of $EG$ has been investigated in \cite{9}, \cite{11} and \cite{16}. We will give in Section 2 a necessary and sufficient algebraic criterion which not only applies to $\mathcal{FIN}$ but to any family $\mathcal{F}$. Namely, in Section 2 we will explain and prove

\textbf{Theorem 0.1} \textit{Let $G$ be a discrete group and let $d \geq 3$. Then we have:}

(a) \textit{There is a $d$-dimensional $G$-$CW$-model for $E(G, \mathcal{F})$ if and only if the constant $\mathbb{Z}Or(G, \mathcal{F})$-module $\mathbb{Z}$ has a $d$-dimensional projective resolution;}

(b) \textit{There is a $G$-$CW$-model for $E(G, \mathcal{F})$ of finite type if and only if $E(G, \mathcal{F})$ has a $G$-$CW$-model with finite 2-skeleton and the constant $\mathbb{Z}Or(G, \mathcal{F})$-module $\mathbb{Z}$ has a projective resolution of finite type;}

(c) \textit{There is a finite $G$-$CW$-model for $E(G, \mathcal{F})$ if and only if $E(G, \mathcal{F})$ has a $G$-$CW$-model with finite 2-skeleton and the constant $\mathbb{Z}Or(G, \mathcal{F})$-module $\mathbb{Z}$ has a finite free resolution over $Or(G, \mathcal{F});$}

(d) \textit{There is a $G$-$CW$-model with finite 2-skeleton for $EG = E(G, \mathcal{FIN})$ if and only if there are only finitely many conjugacy classes of finite subgroups $H \subset G$ and for any finite subgroup $H \subset G$ its Weyl group $WH := NH/H$ is finitely presented.}
In Section 3 we will reduce the case of a totally disconnected group to the one of a discrete group. Throughout the paper we will denote the discretization of a topological group $G$ by $G_d$, i.e. the same group but now with the discrete topology. Given a family $\mathcal{F}$ of (closed) subgroups of $G$, denote by $\mathcal{F}_d$ the same set of subgroups, but now in connection with $G_d$. Notice that $\mathcal{F}_d$ is again a family.

**Theorem 0.2** Let $G$ be a locally compact totally disconnected group and let $\mathcal{F}$ be a family of subgroups of $G$. Then there is a $G$-CW-model for $E(G, \mathcal{F})$ that is $d$-dimensional (resp. finite, resp. of finite type) if and only if there is a $G_d$-CW-model for $E(G_d, \mathcal{F}_d)$ that is $d$-dimensional (resp. finite, resp. of finite type).

The case of an almost connected group $G$, i.e. $\overline{G}$ is compact, has already been treated by Abels [2, Corollary 4.14]. Namely, for an almost connected (locally compact) group $G$ there is a model for $EG$ consisting of one equivariant cell $G/K$. Notice that $K$ is then necessarily a maximal compact subgroup of $G$ and uniquely determined by this property up to conjugation. In Section 4 we use this result to reduce the case of a locally compact group $G$ to a totally disconnected group. We show

**Theorem 0.3** Let $G$ be a locally compact group satisfying $(S)$ and let $\overline{G} := G/G_0$. Then there is a $G$-CW-model for $EG$ that is $d$-dimensional (resp. finite, resp. of finite type) if and only if $E\overline{G}$ has a $\overline{G}$-CW-model that is $d$-dimensional (resp. finite, resp. of finite type).

If we combine Theorem 0.1, Theorem 0.2 and Theorem 0.3 we get

**Theorem 0.4** Let $G$ be a locally compact group satisfying $(S)$. Denote by $\mathcal{COM}$ the family of compact subgroups of its component group $\overline{G}$ and let $d \geq 3$. Then

(a) There is a $d$-dimensional $G$-CW-model for $EG$ if and only if the constant $\mathbb{Z} \text{Or}(\overline{G}_d, \mathcal{COM}_d)$-module $\mathbb{Z}$ has a $d$-dimensional projective resolution;

(b) There is a $G$-CW-model for $EG$ of finite type if and only if $E(\overline{G}_d, \mathcal{COM}_d)$ has a $\overline{G}_d$-CW-model with finite 2-skeleton and the constant $\mathbb{Z} \text{Or}(\overline{G}_d, \mathcal{COM}_d)$-module $\mathbb{Z}$ has a projective resolution of finite type;

(c) There is a finite $G$-CW-model for $EG$ if and only if $E(\overline{G}_d, \mathcal{COM}_d)$ has a $\overline{G}_d$-CW-model with finite 2-skeleton and the constant $\mathbb{Z} \text{Or}(\overline{G}_d, \mathcal{COM}_d)$-module $\mathbb{Z}$ has a finite free resolution.

In particular we see from Theorem 0.3 that, for a Lie group $G$, type questions about $EG$ are equivalent to the corresponding type questions of $E\pi_0(G)$, since $\pi_0(G) = \overline{G}$ is discrete (cf. [11, Problem 7.1]). In this case the family $\mathcal{COM}_d$ appearing in Theorem 0.4 is just the family $\mathcal{FIN}$ of finite subgroups of $\pi_0(G)$. 

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1 Review of Classifying Spaces of a Group for a Family

Recall from the introduction the $G$-$CW$-complex $E(G,F)$. In particular, notice that we do not work with the stronger condition that $E(G,F)^H$ is contractible but only weakly contractible. If $G$ is discrete, then each fixed point set $E(G,F)^H$ has the homotopy type of a $CW$-complex and is contractible for $H \in F$. If $\mathcal{T}$ is discrete and $F = \mathcal{CM}$, then $E(G,F)^H$ is contractible for $H \in \mathcal{CM}$ by Proposition 4.3. In general $E(G,F)^H$ need not to be contractible as the following example shows.

Let $G$ be totally disconnected and let $F$ be the trivial family $\mathcal{T}R$ consisting of one element, namely the trivial group. We claim that then $E(G,\mathcal{T}R)$ is contractible if and only if $G$ is discrete. If $G$ is discrete, we already know that $E(G,\mathcal{T}R)$ is contractible. Suppose now that $E(G,\mathcal{T}R)$ is contractible. We obtain a numerable $G$-principal bundle $G \to E(G,\mathcal{T}R) \to G \backslash E(G,\mathcal{T}R)$ by the Slice Theorem [10, Theorem 1.37] and the fact that the quotient $G \backslash E(G,\mathcal{T}R)$ is a $CW$-complex and hence paracompact. This implies that it is a fibration by a result of Hurewicz [17, Theorem on p. 33]. Since $E(G,\mathcal{T}R)$ is contractible, $G$ and the loop space $\Omega(G \backslash E(G,\mathcal{T}R))$ are homotopy equivalent [17, 6.9 on p. 137, 6.10 on p. 138, Corollary 7.27 on p. 40]. Since $G \backslash E(G,\mathcal{T}R)$ is a $CW$-complex, $\Omega(G \backslash E(G,\mathcal{T}R))$ has the homotopy type of a $CW$-complex [13]. Let $f : G \to X$ be a homotopy equivalence from $G$ to a $CW$-complex $X$. Then the map induced between the set of path components $\pi_0(G) \to \pi_0(X)$ is bijective. Hence any preimage of a path component of $X$ is a point since $G$ is totally disconnected. Since $X$ is locally path-connected each path component of $X$ is open in $X$. We conclude that $G$ is the disjoint union of the preimages of the path components of $X$ and each of these preimages is open in $G$ and consists of one point. Hence $G$ is discrete.

There is another variant of the classifying space of a group $G$ for a family $F$ which we review next ([4, Theorem 6.6. on p. 47]. The assumption that $G$ is a compact Lie group is not needed). We denote the space considered there by $J(G,F)$ since it is constructed by a variant of Milnor’s infinite join construction. Namely, a model for $J(G,F)$ is $*_{n=1}^{\infty}Z$, where $Z$ is a disjoint union of homogeneous spaces $G/H_i$ such that each $G/H_i$ is $G$-isomorphic to $G/H$ for one $H \in F$ and each $H \in F$ occurs this way. This is an $F$-numerable $G$-space with the universal property that for any $F$-numerable $G$-space $X$ there is up to homotopy precisely one $G$-map $X \to J(G,F)$. Again $J(G,F)$ is unique up to $G$-homotopy.

In contrast to $E(G,F)$ the $H$-fixed point set $J(G,F)^H$ is always contractible for $H \in F$. Since $E(G,F)$ is a $G$-$CW$-complex and hence an $F$-numerable $G$-space, there is a $G$-map $f : E(G,F) \to J(G,F)$ unique up to $G$-homotopy. Obviously $f$ is a weak $G$-homotopy equivalence, i.e. $f^H : E(G,F)^H \to J(G,F)^H$ is a weak homotopy equivalence for each $H \subset G$. In other words, $E(G,F)$ is a $G$-$CW$-approximation of $J(G,F)$. We know that $f$ cannot be a $G$-homotopy equivalence in general since $E(G,F)^H$ is not contractible in general. Hence these concepts are different. However, for any $G$-$CW$-complex $X$ whose isotropy groups belong to $F$, any $G$-map $X \to J(G,F)$ lifts uniquely up to $G$-homotopy over the $G$-map $f : E(G,F) \to J(G,F)$. Moreover, if $G$ is discrete or if $G$ is a Lie group and $F$ contained in $\mathcal{CM}$, then $f : E(G,F) \to J(G,F)$ is a $G$-homotopy equivalence and these concepts agree. This can be seen as follows.

Under the assumptions on $G$ and $F$, $*_{n=0}^{\infty}Z$ has the $G$-homotopy type of a $G$-$CW$-
complex and hence $\ast_{n=1}^\infty Z_{\text{weak}}$ has the $G$-homotopy type of a $G$-$CW$-complex, where $\ast_{n=1}^\infty Z_{\text{weak}}$ is equipped with the weak topology with respect to the filtration by the subspaces $\ast_{n=0}^k Z$ for $k = 1, 2, \ldots$. This follows for instance from [10, section 7]. (See also [8].) One checks that for a $G$-space $X$ with a $G$-invariant covering and locally finite $G$-invariant subordinate partition of unity the $G$-map $X \to J(G, \mathcal{F})$ constructed in [4, Lemma 6.13 on p. 49 and Lemma 6.9 on p. 48] actually factorizes through $\ast_{n=1}^\infty Z_{\text{weak}}$, since locally this maps takes values in one of the subspaces $\ast_{n=0}^k Z$. In particular we obtain a $G$-map $J(G, \mathcal{F}) \to \ast_{n=1}^\infty Z_{\text{weak}}$. Since $\ast_{n=1}^\infty Z_{\text{weak}}$ is a $G$-$CW$-complex we obtain a $G$-map $h : J(G, \mathcal{F}) \to E(G, \mathcal{F})$. By the universal properties both compositions $h \circ f$ and $f \circ h$ are $G$-homotopic to the identity.

In the case $\mathcal{F} = \mathcal{COM}$ there is another model for the universal classifying space of $G$, well known from harmonic analysis, described in [2, §2]. Denote by $C_0(G)$ the space of complex-valued continuous functions on $G$ vanishing at infinity, endowed with the sup-norm-topology. By $g \cdot f(x) := f(g^{-1}x)$, $g \in G$, $f \in C_0(G)$, $G$ acts isometrically on $C_0(G)$. Denote by $PC_0(G)$ the subspace of real-valued functions $f \neq 0$ that only take non-negative values. Then $PC_0(G)$ is a final object in the homotopy category of numerably proper $G$-spaces. As [2] and [4] work over the same category ([2, Prop. 3.9]), the models $PC_0(G)$ and $J(G, \mathcal{COM})$ are $G$-homotopy equivalent.

2 Discrete Groups

Throughout this section $G$ denotes a discrete group. Finiteness conditions for $E(G, \mathcal{F})$ focussing on the family $\mathcal{F} = \mathcal{FIN}$ of finite subgroups have already been studied in [9], [11] and [16]. In this section we translate questions about $E(G, \mathcal{F})$ for $G$ and a family of subgroups $\mathcal{F}$ to homological algebra of modules over the associated orbit category $\text{Or}(G, \mathcal{F})$. We begin by recalling some basic definitions.

The orbit category $\text{Or}(G)$ of $G$ is the small category whose objects are homogeneous $G$-spaces $G/H$ and whose morphisms are $G$-maps. Let $\text{Or}(G, \mathcal{F})$ be the full subcategory of $\text{Or}(G)$ consisting of those objects $G/H$ for which $H$ belongs to $\mathcal{F}$. A $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module is a contravariant functor from $\text{Or}(G, \mathcal{F})$ to the category of $\mathbb{Z}$-modules. A morphism of such modules is a natural transformation. The category of $\mathbb{Z}\text{Or}(G, \mathcal{F})$-modules inherits the structure of an abelian category from the standard structure of an abelian category on the category of $\mathbb{Z}$-modules. In particular the notion of a projective $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module is defined. The free $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module $\mathbb{Z}\text{map}(G/?, G/K)$ based at the object $G/K$ is the $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module that assigns to an object $G/H$ the free $\mathbb{Z}$-module $\mathbb{Z}\text{map}_G(G/H, G/K)$ generated by the set $\text{map}_G(G/H, G/K)$. The key property of it is that for any $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module $N$ there is a natural bijection of $\mathbb{Z}$-modules

$$\text{hom}_{\mathbb{Z}\text{Or}(G, \mathcal{F})}(\mathbb{Z}\text{map}_G(G/?, G/K), N) \xrightarrow{\cong} N(G/K), \phi \mapsto \phi(G/K)(\text{id}_{G/K})$$

which is an application of the Yoneda Lemma. A $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module is free if it is isomorphic to a direct sum $\bigoplus_{i \in I} \mathbb{Z}\text{map}(G/?, G/K_i)$ for appropriate choice of objects $G/K_i$ and index set $I$. A $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module is called finitely generated if it is a quotient of a $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module of the shape $\bigoplus_{i \in I} \mathbb{Z}\text{map}(G/?, G/K_i)$ with a finite index set $I$. Notice that a lot of standard facts for $\mathbb{Z}$-modules carry over to $\mathbb{Z}\text{Or}(G, \mathcal{F})$-modules. For instance, a $\mathbb{Z}\text{Or}(G, \mathcal{F})$-module is
projective or finitely generated projective respectively if and only if it is a direct summand
in a free $\mathbb{Z} \text{Or}(G, F)$-module or a finitely generated free $\mathbb{Z} \text{Or}(G, F)$-module respectively. The notion of a projective resolution $P_*$ of a $\mathbb{Z} \text{Or}(G, F)$-module is obvious. We call $P_*$ of finite type if each $P_n$ is finitely generated projective. We call $P_*$ finite if $P_*$ is both of finite type and finite-dimensional. Each $\mathbb{Z} \text{Or}(G, F)$-module has a projective resolution.

**Definition 2.1** Let $G$ be a discrete group and $(X, A)$ a relative $G$-CW-complex whose isotropy groups belong to the family $F$. The contravariant functor

$$C^c_r(X, A) : \text{Or}(G, F) \longrightarrow \mathbb{Z} - \text{Chain complexes}$$

$$G/H \mapsto C^c_r(X^H, A^H)$$

is called the cellular $\mathbb{Z} \text{Or}(G, F)$-chain complex of $(X, A)$.

Functoriality comes from the fact that $X^H = \text{map}_G(G/H, X)$. Notice that $(X^H, A^H)$ is canonically a CW-complex, hence we can speak of its cellular chain complex $C^c_r(X^H, A^H)$. As in the nonequivariant situation the chain modules are free with basis given by the (equivariant) cells. Namely, we have

**Lemma 2.2** For any $n \in \mathbb{Z}$ the $n$-th chain module $C^c_n(X) : \text{Or}(G, F) \longrightarrow \mathbb{Z} - \text{Modules}$ is a free $\mathbb{Z} \text{Or}(G, F)$-module.

**Proof:** Let the $n$-skeleton of $X$ be given by a pushout

$$\biguplus_{I_n} G/H_i \times S^{n-1} \longrightarrow X_{n-1}$$

$$\text{id} \times i$$

$$\biguplus_{I_n} G/H_i \times D^n \longrightarrow X_n.$$ 

Since $\text{id} \times i$ is a cofibration, we get by excision in $G/H$ natural isomorphisms

$$C^c_n(X, A)(G/H) \cong H_n(X^H, X^H_{n-1}) \cong H_n(\biguplus_{I_n} (G/H_i)^H \times (D^n, S^{n-1}))$$

$$\cong \bigoplus_{I_n} H_n((G/H_i)^H \times (D^n, S^{n-1})) \cong \bigoplus_{I_n} H_0((G/H_i)^H) \cong \bigoplus_{I_n} \mathbb{Z}[(G/H_i)^H]$$

$$\cong \bigoplus_{I_n} \mathbb{Z}[\text{map}_G(G/H, G/H_i)]. \quad (2.3)$$

From the explicit description (2.3) of the $n$-th chain module $C^c_n(X)$ we immediately get the following corollary, linking the type of the $G$-CW-complex to the type of its cellular chain complex.

**Corollary 2.4** Let $X$ be a $G$-CW-complex whose isotropy groups belong to the family $F$. Then $X$ is $d$-dimensional (resp. finite, resp. of finite type) if and only if its cellular $\mathbb{Z} \text{Or}(G, F)$-chain complex $C^c_r(X)$ is $d$-dimensional (resp. finite, resp. of finite type).
Proposition 2.5 Let $h : Z \longrightarrow Y$ be a $G$-map between $G$-CW-complexes such that both $Z^H$ and $Y^H$ are simply-connected for $H \in \mathcal{F}$ and all their isotropy groups belong to the family $\mathcal{F}$. Let $r \geq 2$, $r \geq \dim Z$ and a free $\mathbb{Z}Or(G, \mathcal{F})$-chain complex $(D_*, d_*)$ be given. Finally, suppose that there is a chain homotopy equivalence $f_*: D_* \longrightarrow C^c_*(Y, B)$ such that $D_*|_r = C^c_*(Z)|_r$ and $f_*|_r = C^c_*(h)|_r$.

Then there is a $G$-CW-complex $X$ with $X_r = Z$ and a cellular $G$-homotopy equivalence $g : X \longrightarrow Y$ such that:

i) $g|_Z = h$;

ii) $D_* = C^c_*(X)$;

iii) $C^c_*(g) = f_*$.

Proof: The proof is exactly the same as in [10, Theorem 13.19 on p. 268]. There only proper actions are considered but the same methods go through because here we are dealing with the easy case where all fixed point sets are simply connected, the isotropy groups belong to $\mathcal{F}$ and $G$ is discrete.

Lemma 2.6 If $G$ is a discrete group and $E$ is a $G$-CW-model for $E(G, \mathcal{F})$, then $C^c_*(E)$ is a free resolution over the orbit category $Or(G, \mathcal{F})$ of the constant $Or(G, \mathcal{F})$-module $\mathbb{Z}$ with value $\mathbb{Z}$.

Proof: The modules are free by Lemma 2.2. It remains to show that $C^c_*(E^H)$, $H \in \mathcal{F}$ has the homology of a point. But this follows from the fact that for discrete $H$ the space $E^H$ has a canonical CW-structure whose $n$-skeleton in exactly $E^H_n$ and $E^H$ is weakly contractible and hence contractible.

To shorten the next proof we start with the following lemma. Its proof is purely technical and hence left out. Details of the proof can be found in [10, p. 279-280]. All modules are supposed to be over the orbit category.

Lemma 2.7 Let $C_*$ be a free, 2-dimensional chain complex, $D_*$ a free chain complex and $f_* : C_* \longrightarrow D_*$ a chain map with $H_i(\text{cone}(f_*)) = 0, i \leq 2$. Then there is a free chain complex $C'_*$ and a chain homotopy equivalence $g_* : C'_* \longrightarrow D_*$ with $C'_*|_2 = C_*$ and $g_*|_2 = f_*$. If $C_*$ is finite and $D_*$ is homotopic to a finite free chain complex, resp. a free complex of finite type, then $C'_*$ can be chosen to be finite, resp. of finite type. If $D_*$ is homotopic to a finite-dimensional free complex, then $C'_*$ can be chosen to be finite-dimensional.

Proof of Theorem 0.1: The “only if” case is clear for the first three assertions by Lemma 2.6 and Corollary 2.4. In the “if” case for the first three assertions let $P_*$ be the given projective resolution of the constant $\mathbb{Z}Or(G, \mathcal{F})$-module $\mathbb{Z}$ and let $E = E(G, \mathcal{F})$ be a $G$-CW-model with finite 2-skeleton in the second and third case. By adding elementary chain complexes, i.e. complexes concentrated in two consecutive dimensions with the identity as
only non-trivial differential, we can get $P_*$ to be a free resolution. (In the $d$-dimensional case we use the Eilenberg trick for the last module. Notice that in the finite case $P_*$ is assumed to be free.) Since $C^*_s(E)$ also gives a free resolution of $\mathbb{Z}$ by Lemma 2.6, we have a homotopy equivalence $g_* : P_* \longrightarrow C^*_s(E)$. Using Lemma 2.7, we get a new free complex $Q_*$ with inherited finiteness property of $P_*$ and a chain homotopy equivalence $f_* : Q_* \longrightarrow C^c_s(E)$ which induces the identity in dimensions 0, 1 and 2. Therefore we can apply Proposition 2.5 to the inclusion $i : E_2 \longrightarrow E$ and $f_*$. The result is a $G$-CW-complex $X$ with $X_2 = E_2$ together with a homotopy equivalence $k : X \longrightarrow E$ and $C^*_s(X) = Q_*$. So $X$ is a $G$-CW-model for $E(G, \mathcal{F})$ and has the desired properties by Corollary 2.4.

The same proof as of [11, Theorem 4.2], replacing the words “of finite type” by “with finite 2-skeleton”, yields the last assertion of Theorem 0.1.

### 3 TotallyDisconnected Groups

Recall that a topological space $X$ is totally disconnected, if any component consists of exactly one point. In this section we want to show that there is a close relation between classifying spaces of a totally disconnected group $G$ and its discretization $G_d$. The reason for this is, that homotopy does not see the difference between a totally disconnected group $G$ and $G_d$, i.e. the canonical map $G_d/H_d \longrightarrow \text{res}G_d G/H$ is a weak $G_d$-homotopy equivalence. The different topologies will only appear in the family of subgroups that has to be considered. We start by collecting some elementary facts about totally disconnected spaces.

Let $X$ be a topological space. Consider the following 3 conditions.

- $(T) X$ is totally disconnected;
- $(D)$ The covering dimension of $X$ is 0;
- $(FS)$ Any element of $X$ has a fundamental system of open and compact neighborhoods.

**Lemma 3.1** For a locally compact group the conditions $(T)$, $(D)$ and $(FS)$ are equivalent.

**Proof:** The implications $(T) \Rightarrow (D) \Rightarrow (FS)$ are shown in [7, Theorem 7.7 on p. 62]. The implication $(FS) \Rightarrow (T)$ is done as follows: Let $U$ be a set containing two distinct points $x$ and $y$. We show that $U$ is disconnected. Let $V$ be an open and compact neighborhood of $x$, not containing $y$. Then $(V \cap U) \amalg (V^c \cap U) = U$ is a disjoint union of two nonempty open subsets of $U$.

The elementary proofs of the next two lemmas are left to the reader.

**Lemma 3.2** Let $f : X \longrightarrow Y$ be a surjective and open map. If $X$ is locally compact, then $Y$ is locally compact. If $X$ has property $(FS)$, then so has $Y$. In particular, if $G$ is a totally disconnected locally compact group, then $(G/H)^K$ is totally disconnected and locally compact for all (closed) subgroups $H, K \subset G$. 

Lemma 3.3 Let $f : X \to Y$ be a map. If $f^{-1}(y)$ is weakly contractible for all $y \in Y$ and $Y$ is totally disconnected, then $f$ is a weak homotopy equivalence. If $f^{-1}(y)$ is contractible for all $y \in Y$ and $Y$ is discrete, then $f$ is a homotopy equivalence.

Lemma 3.4 Let $G$ be a totally disconnected locally compact group and $X$ be a $G_d$-CW-complex whose isotropy groups are all closed when viewed as subgroups of $G$. Then the map

$$i_X : X \to \text{res}^{G_d}_G G \times_{G_d} X, \quad x \mapsto [e, x]$$

is a weak $G_d$-homotopy equivalence.

Proof: We begin with the case, where $X$ is a homogeneous space $G_d/H_d$ for a closed subgroup $H \subset G$. Then

$$G \times_{G_d} G_d/H_d \to G/H, \quad [g, g'H_d] \mapsto gg'H$$

is a $G$-homeomorphism. The obvious map $G_d/H_d \to \text{res}^{G_d}_G G/H$ is a weak $G_d$-homotopy equivalence by Lemma 3.2 and Lemma 3.3. Hence $i_{G_d/H_d} : G_d/H_d \to \text{res}^{G_d}_G G \times_{G_d} G_d/H_d$ is a weak $G_d$-homotopy equivalence.

Next we prove the claim for all skeleta $X_n$ by induction over $n$. The induction beginning $n \leq 0$ follows from the case of a homogeneous space since $X_0$ is a disjoint union of homogeneous spaces. In the induction step from $n - 1$ to $n$ one chooses a $G_d$-pushout

$$\bigsqcup_{I_n} G_d/H_i \times S^{n-1} \to X_{n-1}$$

and checks using the fact that $G$ is locally compact that the induced diagram is a $G_d$-pushout

$$\bigsqcup_{I_n} \text{res}_{G}^{G_d} G \times_{G_d} G_d/H_i \times S^{n-1} \to \text{res}_{G}^{G_d} G \times_{G_d} X_{n-1}$$

$$\bigsqcup_{I_n} \text{res}_{G}^{G_d} G \times_{G_d} G_d/H_i \times D^n \to \text{res}_{G}^{G_d} G \times_{G_d} X_n.$$

Notice that in both diagrams the left vertical arrows are $G_d$-cofibrations and the various maps $i_Y$ for $Y = \bigsqcup_{I_n} G_d/H_i \times S^{n-1}$, $Y = \bigsqcup_{I_n} G_d/H_i \times D^n$, $Y = X_{n-1}$ and $Y = X_n$ map the two diagrams to one another. By the induction hypothesis the first three are weak $G_d$-homotopy equivalences. Hence $i_{X_n}$ is a weak $G_d$-homotopy equivalence [10, Lemma 2.13 on p. 38].

Let $K \subset G_d$ be a subgroup. Since $X^K$ has the weak topology with respect to the filtration given by the subspaces $X^K_n$ and $G$ is locally compact, $\left(\text{res}_{G}^{G_d} G \times_{G_d} X^K\right)^K$ has the
weak topology with respect to the filtration given by the subspaces \( (\text{res}_G^{G_d} G \times_{G_d} X_n)^K \).

Since \((i_X)^K\) is a weak homotopy equivalence for \(n \geq 0\), the same follows for \((i_X)^K\).

**Corollary 3.5** Let \(G\) be a totally disconnected locally compact group and \(\mathcal{F}\) a family of subgroups of \(G\). If \(E(G_d, \mathcal{F}_d)\) is a \(G_d\)-CW-model for the classifying space of \(G_d\) for the family \(\mathcal{F}_d\), then \(G \times_{G_d} E(G_d, \mathcal{F}_d)\) is a \(G\)-CW-model for \(E(G, \mathcal{F})\).

**Proof:** We have for any \(K \in \mathcal{F}\) by Lemma 3.4:

\[
(G \times_{G_d} E(G_d, \mathcal{F}_d))^K = (\text{res}_G^{G_d} G \times_{G_d} E(G_d, \mathcal{F}_d))^K \cong_w E(G_d, \mathcal{F}_d)^K \cong_w \{\ast\}.
\]

**Proposition 3.6** Let \(G\) be totally disconnected and let \(X\) be a \(G\)-CW-complex that is \(d\)-dimensional (resp. finite, resp. of finite type). Then \(\text{res}_G^{G_d} X\) has a \(G_d\)-CW-approximation \(Y\) that is \(d\)-dimensional (resp. finite, resp. of finite type) and whose isotropy groups are the same as the one of \(X\). If \(X\) is a \(G\)-CW-model for \(E(G, \mathcal{F})\), then \(Y\) is a \(G_d\)-CW-model for \(E(G_d, \mathcal{F}_d)\).

**Proof:** For \(n \geq -1\) we construct by induction a \(G_d\)-CW-complex \(Y_n\) together with a \(G_d\)-approximation \(f_n : Y_n \longrightarrow \text{res}_G^{G_d} X_n\) such that \(Y_{n-1}\) is a subcomplex of \(Y_n\) and \(f_n|_{Y_{n-1}} = f_{n-1}\). The induction begin \(n = -1\) is given by the empty set. For the induction step from \(n - 1\) to \(n\) we proceed as follows. Choose a pushout

\[
\begin{array}{ccc}
\bigsqcup_{I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\bigsqcup_{I_n} G/H_i \times D^n & \longrightarrow & X_n
\end{array}
\]

for the \(n\)-skeleton of \(X_n\). Let \(i : \bigsqcup_{I_n} G_d/(H_i)_d \to \bigsqcup_{I_n} \text{res}_G^{G_d} G/H_i\) be the obvious weak \(G_d\)-homotopy equivalence (see Lemma 3.4). Since by the induction hypothesis \(f_{n-1}\) is a weak \(G_d\)-homotopy equivalence, we can find using \([10, \text{Proposition 2.3 on p. 35}]\) and a version of the Cellular Approximation Theorem (see for instance \([10, \text{Theorem 2.1 on p. 32}]\)) a cellular \(G_d\)-map \(g : \bigsqcup_{I_n} G_d/(H_i)_d \times S^{n-1} \longrightarrow Y_{n-1}\) and a \(G_d\)-homotopy \(h : \bigsqcup_{I_n} G_d/(H_i)_d \times S^{n-1} \times I \longrightarrow \text{res}_G^{G_d} X_{n-1}\) between \(f_{n-1} \circ g\) and \(r \circ (i \times \text{id}_{S^{n-1}})\). Let \(f_{n-1}' : \text{cyl}(g) \longrightarrow \text{res}_G^{G_d} X_{n-1}\) be the obvious map given by \(h\) and \(f_{n-1}\). Its restriction to \(Y_{n-1} \subset \text{cyl}(g)\) is \(f_{n-1}\) and to \(\bigsqcup_{I_n} G_d/(H_i)_d \times S^{n-1}\) is \(r \circ (i \times \text{id}_{S^{n-1}})\). Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
\bigsqcup_{I_n} G_d/(H_i)_d \times D^n & \longrightarrow & \bigsqcup_{I_n} G_d/(H_i)_d \times S^{n-1} \\
\downarrow & & \downarrow \\
\bigsqcup_{I_n} \text{res}_G^{G_d} G/H_i \times D^n & \longrightarrow & \bigsqcup_{I_n} \text{res}_G^{G_d} G/H_i \times S^{n-1} \quad r \quad \text{res}_G^{G_d} X_{n-1}.
\end{array}
\]


Taking the pushout of the upper row yields a $n$-dimensional $G_d$-CW-complex $Y_n$ which contains $Y_{n-1}$ as $G_d$-CW-subcomplex and is finite if $X_n$ is finite. Moreover, we get by the pushout property a $G_d$-map $f_n : Y_n \to \text{res}_{G_d}^n X_n$ which extends $f_{n-1} : Y_{n-1} \to \text{res}_{G_d}^{n-1} X_{n-1}$ and is a weak $G_d$-homotopy equivalence, since all vertical maps are weak $G_d$-homotopy equivalences ([10, Lemma 2.13 on p. 38]). Now put $Y := \text{colim}_{n \to \infty} Y_n$. Then $f := \text{colim}_{n \to \infty} f_n : Y \to \text{res}_{G_d}^\infty X$ is the $G_d$-CW-approximation we look for.

**Proof of Theorem 0.2:** Follows from Corollary 3.5 and Proposition 3.6.

### 4 Locally Compact Groups

The strategy of our study of locally compact groups is the following. Any locally compact group $G$ gives rise to a short sequence of the form $1 \longrightarrow G_0 \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1$ with $G_0$ locally compact and connected and with $\overline{G}$ locally compact and totally disconnected. This reduces the study of $G$ to the study of locally compact connected groups, that are very similar to Lie groups by the solution of Hilbert’s fifth problem (cp. [6]), and to locally compact totally disconnected groups, that are similar to their discrete underlying group, as we saw in the preceding section. We start with some remarks on locally compact groups $G$ which are almost connected, i.e. whose component group $G$ is compact.

**Theorem 4.1** Let $G$ be a almost connected locally compact group. Then $G$ has a maximal compact subgroup $K$ which is unique up to conjugacy and $G/K$ is a model for both $E_G$ and $J_G := J(G, \text{COM})$.

**Proof:** [1, Appendix, Theorem A.5], [2, Corollary 4.14].

We now turn our attention to locally compact groups that are not necessarily almost connected. From now on any locally compact group $G$ is supposed to satisfy condition $(S)$ defined in the introduction.

**Lemma 4.2** Let $L$ be an almost connected subgroup of $G$ and let $K$ be a maximal compact subgroup of $L$. If $G/L$ is totally disconnected, then for any compact $H \subset G$ the projection $\text{pr}_H^H : (G/K)^H \longrightarrow (G/L)^H$ is a weak homotopy equivalence. If $G/L$ is discrete, then for any compact $H \subset G$ the projection $\text{pr}_H^H : (G/K)^H \longrightarrow (G/L)^H$ is a homotopy equivalence.

**Proof:** If $H$ is not subconjugated to $L$, all spaces are empty. So, let $H$ be subconjugated to $L$. Since we assumed the existence of local cross sections we know that $G \longrightarrow G/L$ is a principal $L$-bundle. Let $Y$ be any $L$-space. Fix an element $w \in G$. Then in the associated fiber bundle $Y \longrightarrow G \times_L Y \longrightarrow G/L$, the typical fiber maps homeomorphically onto the preimage $p^{-1}(wL)$ of $wL \in G/L$ by sending $y$ to the class of $(w, y)$. If $wL$ is in $(G/L)^H$ then this implies $wHw^{-1} \subset L$ and we get an induced homeomorphism $Y^{wHw^{-1}} \longrightarrow (\text{pr}_H^H)^{-1}(wL)$ for $\text{pr}_H^H : (G \times_L Y)^H \longrightarrow (G/L)^H$. Now let $Y$ be $L/K$ which is a model for $J(L, \text{COM})$ by Theorem 4.1. Therefore $Y^{wHw^{-1}}$ is contractible. Hence by Lemma 3.3 the map $\text{pr}_H^H$ is a weak...
homotopy equivalence if $G/L$ and hence $(G/L)^H$ is totally disconnected, and a homotopy equivalence, if $G/L$ and hence $(G/L)^H$ is discrete. ■

**Proposition 4.3** Given a $G$-CW-model $EG$, there is a $G$-CW-model $EG$ and a $G$-map $f : EG \rightarrow p^*EG$ with the following properties (where $p^*EG$ is $EG$ viewed as a $G$-space by the projection $p : G \rightarrow \overline{G}$):

(a) If $EG$ is $d$-dimensional (resp. finite, resp. of finite type), then $EG$ is $d$-dimensional (resp. finite, resp. of finite type);

(b) If $G$ is discrete, then $EG^H$ is contractible for all compact $H \subset G$;

(c) $G_0 \setminus f : G_0 \setminus EG \rightarrow \overline{EG}$ is a $G$-homotopy equivalence.

**Proof:** We will construct for each $n \geq -1$ an $n$-dimensional $G$-CW-complex $X_n$ and a $G$-map $f_n : X_n \rightarrow p^*\overline{EG}_n$ to the $n$-skeleton of $p^*\overline{EG}$ such that $X_{n-1}$ is the $(n-1)$-skeleton of $X_n$ and $f_n|_{X_{n-1}} = f_{n-1}$ with the following properties:

(a) $f_n^H : X_n^H \rightarrow (p^*\overline{EG}_n)^H$ is a weak homotopy equivalence for all compact $H \subset G$. If $G$ is discrete, $f_n^H : X_n^H \rightarrow (p^*\overline{EG}_n)^H$ is a homotopy equivalence for all compact $H \subset G$;

(b) The isotropy groups of $X_n$ are all compact;

(c) $X_{n-1}$ is the $n$-skeleton of $X_n$ and $f_n|_{X_{n-1}} = f_{n-1}$. There is a bijective correspondence between the equivariant $n$-dimensional cells of $X_n$ and of $\overline{EG}_n$;

(d) $G_0 \setminus f_n : G_0 \setminus X_n \rightarrow \overline{EG}_n$ is a $G$-homotopy equivalence.

Notice that we then can define $EG := \text{colim}_{n \rightarrow \infty} X_n$ and $f := \text{colim}_{n \rightarrow \infty} f_n$, and check that $EG$ and $f$ have the desired properties.

We proceed by induction over $n$. The induction begin $n = -1$ is given by $X_{-1} := \emptyset$. For the induction step from $n - 1$ to $n$ we choose a $G$-pushout

$$\begin{align*}
\bigsqcup_{I_n} \overline{G/H_i} \times S^{n-1} & \xrightarrow{\bigsqcup_{I_n} q_i} \overline{EG}_{n-1} \\
\bigsqcup_{I_n} \overline{G/H_i} \times D^n & \rightarrow \overline{EG}_n.
\end{align*}$$

Put $L_i := p^{-1}(H_i) \subset G$ for $i \in I_n$. Obviously $L_i$ is almost connected. Let $K_i$ be a maximal compact subgroup of $L_i$ for $i \in I_n$ (see Theorem 4.1). Since the projection $p : G \rightarrow \overline{G}$ induces a homeomorphism $G/L_i \cong \overline{G/H_i}$, $G/L_i$ is totally disconnected. Hence Lemma 4.2 implies that $pr_i^H : (G/K_i)^H \rightarrow (G/L_i)^H$ is a weak homotopy equivalence for all
compact subgroups \( H \) of \( G \) and is a homotopy equivalence for all compact subgroups \( H \) of \( G \), provided that \( \overline{G} \) is discrete. The same is true for \( f_{n-1} \) by induction hypothesis. Therefore we have a bijection induced by \( f_{n-1} \) ([10, Prop. 2.3 on p.35])

\[
[G/K_i \times S^{n-1}, X_{n-1}]_G \xrightarrow{(f_{n-1})^*} [G/K_i \times S^{n-1}, p^*\overline{E}G_{n-1}]_G.
\]

Using the Equivariant Cellular Approximation Theorem [10, Theorem 2.1 on p. 32] we get a cellular \( G \)-map \( r_i : G/K_i \times S^{n-1} \to X_{n-1} \) together with a \( G \)-homotopy \( h_i : G/K_i \times S^{n-1} \times [0, 1] \to p^*\overline{E}G_{n-1} \) from \( f_{n-1} \circ r_i \) to \( q_i \circ (pr_i \times id_{S^{n-1}}) \). Consider the following commutative \( G \)-diagram

\[
\begin{array}{cccc}
\coprod_{I_n} p^*(\overline{G}/H_i) \times D^n & \xrightarrow{\coprod_{I_n} p^*(\overline{G}/H_i) \times S^{n-1}} & \coprod_{I_n} q_i & \xrightarrow{\coprod_{I_n} r_i} \coprod_{I_n} \overline{E}G_{n-1} \\
\coprod_{I_n} pr_i \times id_{D^n} & \xrightarrow{\coprod_{I_n} pr_i \times id_{S^{n-1}}} & id & \coprod_{I_n} r_i \circ (pr_i \times id_{S^{n-1}}) \\
\coprod_{I_n} G/K_i \times D^n & \xrightarrow{\coprod_{I_n} G/K_i \times S^{n-1}} & \coprod_{I_n} h_i & \xrightarrow{\coprod_{I_n} h_i} \coprod_{I_n} \overline{E}G_{n-1} \\
\coprod_{I_n} G/K_i \times D^n \times [0, 1] & \xrightarrow{\coprod_{I_n} G/K_i \times S^{n-1} \times [0, 1]} & \coprod_{I_n} f_{n-1} & X_{n-1}.
\end{array}
\]

Notice that the pushout of the first row is \( p^*\overline{E}G_n \). Denote the pushout of the second, third and fourth row respectively by \( X'_n, X''_n \) and \( X_n \). The diagram above together with the pushout property induces \( G \)-maps \( f'_n : X'_n \longrightarrow p^*\overline{E}G_n, f''_n : X''_n \longrightarrow X_n \) and \( f''_n : X_n \longrightarrow X''_n \). The map \( f''_n \) is a \( G \)-homotopy equivalence and the maps \((f'_n)^H\) and \((f''_n)^H\) are weak homotopy equivalences (homotopy equivalence if \( \overline{G} \) is discrete) for each compact subgroup \( H \subset G \) ([10, Lemma 2.13 on p. 38]). We can choose a \( G \)-map \((f''_n)^{-1} : X''_n \to X'_n \) which induces the identity on \( p^*\overline{E}G_{n-1} \) and is a \( G \)-homotopy inverse of \( f''_n \). Now define \( f_n : X_n \longrightarrow p^*\overline{E}G_n \) by the composition \( f_n \circ (f''_n)^{-1} \circ f''_n \). By construction \( f''_n \) is a weak homotopy equivalence (homotopy equivalence if \( \overline{G} \) is discrete) for all compact subgroups \( H \subset G \) and \( X_n \) is a \( G \)-CW-complex with \( X_{n-1} \) as \((n-1)\)-skeleton and has only compact isotropy groups.

It remains to show that \( G_0 \backslash f_n : G_0 \backslash X_n \to \overline{E}G_n \) is a \( \overline{G} \)-homotopy equivalence. Since \( L_i \) inherits the property \((S)\) from \( G \), we get a locally trivial fiber bundle \( K_i \longrightarrow L_i \longrightarrow L_i/K_i \) which is automatically a Serre fibration and hence induces a long exact homotopy sequence [14, Theorem 2.11 on p. 60, Theorem 3.6 on p. 65 and Corollary 3.11 on p. 67]. Thus we
We conclude that \( p(K_i) = H_i \) holds for all \( i \in I_n \). Hence \( G_0 \backslash pr_i : G_0 \backslash G / K_i \to \overline{G} / H_i \) is a \( \overline{G} \)-homeomorphism. Therefore \( G_0 \backslash f''_n \) is a \( \overline{G} \)-homeomorphism. \( G_0 \backslash (f''_n)^{-1} \) is a \( \overline{G} \)-homotopy equivalence, since \( (f''_n)^{-1} \) is a \( G \)-homotopy equivalence, and \( G_0 \backslash (f'_n) \) is a \( \overline{G} \)-homotopy equivalence, since \( G_0 \backslash f_{n-1} \) is a \( \overline{G} \)-homotopy equivalence by the induction hypothesis. Hence \( G_0 \backslash f_n \) is a \( \overline{G} \)-homotopy equivalence.

**Proof of Theorem 0.3:** Is implied by Proposition 4.3.

### References


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