

# Survey on Classifying Spaces for Families of Subgroups

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## Abstract

We define for a topological group  $G$  and a family of subgroups  $\mathcal{F}$  two versions for the classifying space for the family  $\mathcal{F}$ , the  $G$ -CW-version  $E_{\mathcal{F}}(G)$  and the numerable  $G$ -space version  $J_{\mathcal{F}}(G)$ . They agree if  $G$  is discrete, or if  $G$  is a Lie group and each element in  $\mathcal{F}$  compact, or if  $G$  is totally disconnected and  $\mathcal{F}$  is the family of compact subgroups or of compact open subgroups. We discuss special geometric models for these spaces for the family of compact open groups in special cases such as almost connected groups  $G$  and word hyperbolic groups  $G$ . We deal with the question whether there are finite models, models of finite type, finite dimensional models. We also discuss the relevance of these spaces for the Baum-Connes Conjecture about the topological  $K$ -theory of the reduced group  $C^*$ -algebra, for the Farrell-Jones Conjecture about the algebraic  $K$ - and  $L$ -theory of group rings, for Completion Theorems and for classifying spaces for equivariant vector bundles and for other situations.

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## 0 Introduction

We define for a topological group  $G$  and a family of subgroups  $\mathcal{F}$  two versions for the classifying space for the family  $\mathcal{F}$ , the  $G$ -CW-version  $E_{\mathcal{F}}(G)$  and the numerable  $G$ -space version  $J_{\mathcal{F}}(G)$ . They agree, if  $G$  is discrete, or if  $G$  is a Lie group and each element in  $\mathcal{F}$  compact, or if  $G$  is totally disconnected and  $\mathcal{F}$

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is the family of compact subgroups or of compact open subgroups, but not in general

One motivation for the study of these classifying spaces comes from the fact that they appear in the Baum-Connes Conjecture about the topological  $K$ -theory of the reduced group  $C^*$ -algebra and in the Farrell-Jones Conjecture about the algebraic  $K$ - and  $L$ -theory of group rings and that they play a role in the formulations and constructions concerning Completion Theorems and classifying spaces for equivariant vector bundles and other situations. Because of the Baum-Connes Conjecture and the Farrell-Jones Conjecture the computation of the relevant  $K$ - and  $L$ -groups can be reduced to the computation of certain equivariant homology groups applied to these classifying spaces for the family of finite subgroups or the family of virtually cyclic subgroups. Therefore it is important to have nice geometric models for these spaces  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$  and in particular for the orbit space  $G \backslash E_{\mathcal{FIN}}(G)$ .

The space  $E_{\mathcal{F}}(G)$  has for the family of compact open subgroups or of finite subgroups nice geometric models for instance in the cases, where  $G$  is an almost connected group  $G$ , where  $G$  is a discrete subgroup of a connected Lie group, where  $G$  is a word hyperbolic group, arithmetic group, mapping class group, one-relator group and so on. Models are given by symmetric spaces, Teichmüller spaces, outer space, Rips complexes, buildings, trees and so on. On the other hand one can construct for any  $CW$ -complex  $X$  a discrete group  $G$  such that  $X$  and  $G \backslash E_{\mathcal{FIN}}(G)$  are homotopy equivalent.

We deal with the question whether there are finite models, models of finite type, finite dimensional models. In some sense the algebra of a discrete group  $G$  is reflected in the geometry of the spaces  $E_{\mathcal{FIN}}(G)$ . For torsionfree discrete groups  $E_{\mathcal{FIN}}(G)$  is the same as  $EG$ . For discrete groups with torsion the space  $E_{\mathcal{FIN}}(G)$  seems to carry relevant information which is not present in  $EG$ . For instance for a discrete group with torsion  $EG$  can never have a finite-dimensional model, whereas this is possible for  $E_{\mathcal{FIN}}(G)$  and the minimal dimension is related to the notion of virtual cohomological dimension.

The space  $J_{\mathcal{COM}}(G)$  associated to the family of compact subgroups is sometimes also called the classifying space for proper group actions. We will abbreviate it as  $\underline{J}G$ . For a discrete group  $G$  it agrees with  $E_{\mathcal{FIN}}(G)$  which we will abbreviate by  $\underline{E}G$ . Sometimes the abbreviation  $\underline{E}G$  is used in the literature, especially in connection with the Baum-Connes Conjecture, also for a topological group, but corresponds in the notion used in this article to  $\underline{J}G = J_{\mathcal{COM}}(G)$ . There is no difference between  $\underline{E}G$  and  $\underline{J}G$ , if  $G$  is discrete, a Lie group, or totally disconnected.

A reader, who is only interested in discrete groups, can skip Sections 2 and 3 completely.

Group means always locally compact Hausdorff topological group. Examples are discrete groups and Lie groups but we will also consider other groups. Space always means Hausdorff space. Subgroups are always assumed to be closed. Notice that isotropy groups of  $G$ -spaces are automatically closed. A map is always understood to be continuous.

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# 1 $G$ -CW-Complex-Version

In this section we explain the  $G$ -CW-complex version of the classifying space for a family  $\mathcal{F}$  of subgroups of a group  $G$ .

## 1.1 Basics about $G$ -CW-Complexes

**Definition 1.1 ( $G$ -CW-complex).** A  $G$ -CW-complex  $X$  is a  $G$ -space together with a  $G$ -invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that  $X$  carries the colimit topology with respect to this filtration (i.e. a set  $C \subseteq X$  is closed if and only if  $C \cap X_n$  is closed in  $X_n$  for all  $n \geq 0$ ) and  $X_n$  is obtained from  $X_{n-1}$  for each  $n \geq 0$  by attaching equivariant  $n$ -dimensional cells, i.e. there exists a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

The space  $X_n$  is called the  $n$ -skeleton of  $X$ . Notice that only the filtration by skeletons belongs to the  $G$ -CW-structure but not the  $G$ -pushouts, only their existence is required. An *equivariant open  $n$ -dimensional cell* is a  $G$ -component of  $X_n - X_{n-1}$ , i.e. the preimage of a path component of  $G \backslash (X_n - X_{n-1})$ . The closure of an equivariant open  $n$ -dimensional cell is called an *equivariant closed  $n$ -dimensional cell*. If one has chosen the  $G$ -pushouts in Definition 1.1, then the equivariant open  $n$ -dimensional cells are the  $G$ -subspaces  $Q_i(G/H_i \times (D^n - S^{n-1}))$  and the equivariant closed  $n$ -dimensional cells are the  $G$ -subspaces  $Q_i(G/H_i \times D^n)$ .

**Remark 1.2 (Proper  $G$ -CW-complexes).** A  $G$ -space  $X$  is called *proper* if for each pair of points  $x$  and  $y$  in  $X$  there are open neighborhoods  $V_x$  of  $x$  and  $W_y$  of  $y$  in  $X$  such that the closure of the subset  $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$  of  $G$  is compact. A  $G$ -CW-complex  $X$  is proper if and only if all its isotropy groups are compact [47, Theorem 1.23]. In particular a free  $G$ -CW-complex is always proper. However, not every free  $G$ -space is proper.

**Remark 1.3 ( $G$ -CW-complexes with open isotropy groups).** Let  $X$  be a  $G$ -space with  $G$ -invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \bigcup_{n \geq 0} X_n = X.$$

Then the following assertions are equivalent. i.) Every isotropy group of  $X$  is open and the filtration above yields a  $G$ - $CW$ -structure on  $X$ . ii.) The filtration above yields a (non-equivariant)  $CW$ -structure on  $X$  such that each open cell  $e \subseteq X$  and each  $g \in G$  with  $ge \cap e \neq \emptyset$  left multiplication with  $g$  induces the identity on  $e$ .

In particular we conclude for a discrete group  $G$  that a  $G$ - $CW$ -complex  $X$  is the same as a  $CW$ -complex  $X$  with  $G$ -action such that for each open cell  $e \subseteq X$  and each  $g \in G$  with  $ge \cap e \neq \emptyset$  left multiplication with  $g$  induces the identity on  $e$ .

**Example 1.4 (Lie groups acting properly and smoothly on manifolds).** If  $G$  is a Lie group and  $M$  is a (smooth) proper  $G$ -manifold, then an equivariant smooth triangulation induces a  $G$ - $CW$ -structure on  $M$ . For the proof and for *equivariant smooth triangulations* we refer to [36, Theorem I and II].

**Example 1.5 (Simplicial actions).** Let  $X$  be a simplicial complex on which the group  $G$  acts by simplicial automorphisms. Then all isotropy groups are closed and open. Moreover,  $G$  acts also on the barycentric subdivision  $X'$  by simplicial automorphisms. The filtration of the barycentric subdivision  $X'$  by the simplicial  $n$ -skeleton yields the structure of a  $G$ - $CW$ -complex what is not necessarily true for  $X$ .

A  $G$ -space is called *cocompact* if  $G \backslash X$  is compact. A  $G$ - $CW$ -complex  $X$  is *finite* if  $X$  has only finitely many equivariant cells. A  $G$ - $CW$ -complex is finite if and only if it is cocompact. A  $G$ - $CW$ -complex  $X$  is *of finite type* if each  $n$ -skeleton is finite. It is called *of dimension  $\leq n$*  if  $X = X_n$  and *finite dimensional* if it is of dimension  $\leq n$  for some integer  $n$ . A free  $G$ - $CW$ -complex  $X$  is the same as a  $G$ -principal bundle  $X \rightarrow Y$  over a  $CW$ -complex  $Y$  (see Remark 2.8).

**Theorem 1.6 (Whitehead Theorem for Families).** *Let  $f: Y \rightarrow Z$  be a  $G$ -map of  $G$ -spaces. Let  $\mathcal{F}$  be a set of (closed) subgroups of  $G$  which is closed under conjugation. Then the following assertions are equivalent:*

- (i) *For any  $G$ - $CW$ -complex  $X$ , whose isotropy groups belong to  $\mathcal{F}$ , the map induced by  $f$*

$$f_*: [X, Y]^G \rightarrow [X, Z]^G, \quad [g] \mapsto [g \circ f]$$

*between the set of  $G$ -homotopy classes of  $G$ -maps is bijective;*

- (ii) *For any  $H \in \mathcal{F}$  the map  $f^H: Y^H \rightarrow Z^H$  is a weak homotopy equivalence i.e. the map  $\pi_n(f^H, y): \pi_n(Y^H, y) \rightarrow \pi_n(Z^H, f^H(y))$  is bijective for any base point  $y \in Y^H$  and  $n \in \mathbb{Z}, n \geq 0$ .*

*Proof.* (i)  $\Rightarrow$  (ii) Evaluation at  $1H$  induces for any  $CW$ -complex  $A$  (equipped with the trivial  $G$ -action) a bijection  $[G/H \times A, Y]^G \xrightarrow{\cong} [A, Y^H]$ . Hence for any  $CW$ -complex  $A$  the map  $f^H$  induces a bijection

$$(f^H)_*: [A, Y^H] \rightarrow [A, Z^H], \quad [g] \mapsto [g \circ f^H].$$

This is equivalent to  $f^H$  being a weak homotopy equivalence by the classical non-equivariant Whitehead Theorem [79, Theorem 7.17 in Chapter IV.7 on page 182].

(ii)  $\Rightarrow$  (i) We only give the proof in the case, where  $Z$  is  $G/G$  since this is the main important case for us and the basic idea becomes already clear. The general case is treated for instance in [73, Proposition II.2.6 on page 107]. We have to show for any  $G$ -CW-complex  $X$  such that two  $G$ -maps  $f_0, f_1: X \rightarrow Y$  are  $G$ -homotopic provided that for any isotropy group  $H$  of  $X$  the  $H$ -fixed point set  $Y^H$  is *weakly contractible* i.e.  $\pi_n(Y^H, y)$  consists of one element for all base points  $y \in Y^H$ . Since  $X$  is  $\text{colim}_{n \rightarrow \infty} X_n$  it suffices to construct inductively over  $n$   $G$ -homotopies  $h[n]: X_n \times [0, 1] \rightarrow Z$  such that  $h[n]_i = f_i$  holds for  $i = 0, 1$  and  $h[n]|_{X_{n-1} \times [0, 1]} = h[n-1]$ . The induction beginning  $n = -1$  is trivial because of  $X_{-1} = \emptyset$ , the induction step from  $n - 1$  to  $n \geq 0$  done as follows. Fix a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

One easily checks that the desired  $G$ -homotopy  $h[n]$  exists if and only if we can find for each  $i \in I$  an extension of the  $G$ -map

$$\begin{aligned} & f_0 \circ Q_i^n \cup f_1 \circ Q_i^n \cup h[n-1] \circ (q_i^n \times \text{id}_{[0,1]}): \\ & G/H_i \times D^n \times \{0\} \cup G/H_i \times D^n \times \{1\} \cup G/H_i \times S^{n-1} \times [0, 1] \rightarrow Y \end{aligned}$$

to a  $G$ -map  $G/H_i \times D^n \times [0, 1] \rightarrow Y$ . This is the same problem as extending the (non-equivariant) map  $D^n \times \{0\} \cup D^n \times \{1\} \cup S^{n-1} \times [0, 1] \rightarrow Y$ , which is given by restricting the  $G$ -map above to  $1H_i$ , to a (non-equivariant) map  $D^n \times [0, 1] \rightarrow Y^{H_i}$ . Such an extension exists since  $Y^{H_i}$  is weakly contractible. This finishes the proof of Theorem 1.6.  $\square$

A  $G$ -map  $f: X \rightarrow Y$  of  $G$ -CW-complexes is a  $G$ -homotopy equivalence if and only if for any subgroup  $H \subseteq G$  which occurs as isotropy group of  $X$  or  $Y$  the induced map  $f^H: X^H \rightarrow Y^H$  is a weak homotopy equivalence. This follows from the Whitehead Theorem for Families 1.6 above.

A  $G$ -map of  $G$ -CW-complexes  $f: X \rightarrow Y$  is *cellular* if  $f(X_n) \subseteq Y_n$  holds for all  $n \geq 0$ . There is an equivariant version of the *Cellular Approximation Theorem*, namely, every  $G$ -map of  $G$ -CW-complexes is  $G$ -homotopic to a cellular one and each  $G$ -homotopy between cellular  $G$ -maps can be replaced by a cellular  $G$ -homotopy [73, Theorem II.2.1 on page 104].

## 1.2 The $G$ -CW-Version for the Classifying Space for a Family

**Definition 1.7 (Family of subgroups).** A family  $\mathcal{F}$  of subgroups of  $G$  is a set of (closed) subgroups of  $G$  which is closed under conjugation and finite intersections.

Examples for  $\mathcal{F}$  are

$$\begin{aligned} \mathcal{TR} &= \{\text{trivial subgroup}\}; \\ \mathcal{FIN} &= \{\text{finite subgroups}\}; \\ \mathcal{VCX} &= \{\text{virtually cyclic subgroups}\}; \\ \mathcal{COM} &= \{\text{compact subgroups}\}; \\ \mathcal{COMOP} &= \{\text{compact open subgroups}\}; \\ \mathcal{ALL} &= \{\text{all subgroups}\}. \end{aligned}$$

**Definition 1.8 (Classifying  $G$ -CW-complex for a family of subgroups).**

Let  $\mathcal{F}$  be a family of subgroups of  $G$ . A model  $E_{\mathcal{F}}(G)$  for the classifying  $G$ -CW-complex for the family  $\mathcal{F}$  of subgroups is a  $G$ -CW-complex  $E_{\mathcal{F}}(G)$  which has the following properties: i.) All isotropy groups of  $E_{\mathcal{F}}(G)$  belong to  $\mathcal{F}$ . ii.) For any  $G$ -CW-complex  $Y$ , whose isotropy groups belong to  $\mathcal{F}$ , there is up to  $G$ -homotopy precisely one  $G$ -map  $Y \rightarrow E_{\mathcal{F}}(G)$ .

We abbreviate  $\underline{E}G := E_{\mathcal{COM}}(G)$  and call it the universal  $G$ -CW-complex for proper  $G$ -actions.

In other words,  $E_{\mathcal{F}}(G)$  is a terminal object in the  $G$ -homotopy category of  $G$ -CW-complexes, whose isotropy groups belong to  $\mathcal{F}$ . In particular two models for  $E_{\mathcal{F}}(G)$  are  $G$ -homotopy equivalent and for two families  $\mathcal{F}_0 \subseteq \mathcal{F}_1$  there is up to  $G$ -homotopy precisely one  $G$ -map  $E_{\mathcal{F}_0}(G) \rightarrow E_{\mathcal{F}_1}(G)$ .

**Theorem 1.9 (Homotopy characterization of  $E_{\mathcal{F}}(G)$ ).** Let  $\mathcal{F}$  be a family of subgroups.

- (i) There exists a model for  $E_{\mathcal{F}}(G)$  for any family  $\mathcal{F}$ ;
- (ii) A  $G$ -CW-complex  $X$  is a model for  $E_{\mathcal{F}}(G)$  if and only if all its isotropy groups belong to  $\mathcal{F}$  and for each  $H \in \mathcal{F}$  the  $H$ -fixed point set  $X^H$  is weakly contractible.

*Proof.* (i) A model can be obtained by attaching equivariant cells  $G/H \times D^n$  for all  $H \in \mathcal{F}$  to make the  $H$ -fixed point sets weakly contractible. See for instance [47, Proposition 2.3 on page 35].

(ii) This follows from the Whitehead Theorem for Families 1.6 applied to  $f: X \rightarrow G/G$ .  $\square$

A model for  $E_{\mathcal{ALL}}(G)$  is  $G/G$ . In Section 4 we will give many interesting geometric models for classifying spaces  $E_{\mathcal{F}}(G)$ , in particular for the case, where  $G$  is discrete and  $\mathcal{F} = \mathcal{FIN}$  or, more generally, where  $G$  is a (locally compact topological Hausdorff) group and  $\mathcal{F} = \mathcal{COM}$ . In some sense  $\underline{E}G = E_{\mathcal{COM}}(G)$  is the most interesting case.



## 2 Numerable $G$ -Space-Version

In this section we explain the numerable  $G$ -space version of the classifying space for a family  $\mathcal{F}$  of subgroups of group  $G$ .

**Definition 2.1 ( $\mathcal{F}$ -numerable  $G$ -space).** A  $\mathcal{F}$ -numerable  $G$ -space is a  $G$ -space, for which there exists an open covering  $\{U_i \mid i \in I\}$  by  $G$ -subspaces such that there is for each  $i \in I$  a  $G$ -map  $U_i \rightarrow G/G_i$  for some  $G_i \in \mathcal{F}$  and there is a locally finite partition of unity  $\{e_i \mid i \in I\}$  subordinate to  $\{U_i \mid i \in I\}$  by  $G$ -invariant functions  $e_i: X \rightarrow [0, 1]$ .

Notice that we do not demand that the isotropy groups of a  $\mathcal{F}$ -numerable  $G$ -space belong to  $\mathcal{F}$ . If  $f: X \rightarrow Y$  is a  $G$ -map and  $Y$  is  $\mathcal{F}$ -numerable, then  $X$  is also  $\mathcal{F}$ -numerable.

**Lemma 2.2.** *Let  $\mathcal{F}$  be a family. Then a  $G$ -CW-complex is  $\mathcal{F}$ -numerable if all its isotropy groups belong to  $\mathcal{F}$ .*

*Proof.* This follows from the Slice Theorem for  $G$ -CW-complexes [47, Theorem 1.37] and the fact that  $G \backslash X$  is a CW-complex and hence paracompact [60].  $\square$

**Definition 2.3 (Classifying numerable  $G$ -space for a family of subgroups).** Let  $\mathcal{F}$  be a family of subgroups of  $G$ . A model  $J_{\mathcal{F}}(G)$  for the classifying numerable  $G$ -space for the family  $\mathcal{F}$  of subgroups is a  $G$ -space which has the following properties: i.)  $J_{\mathcal{F}}(G)$  is  $\mathcal{F}$ -numerable. ii.) For any  $\mathcal{F}$ -numerable  $G$ -space  $X$  there is up to  $G$ -homotopy precisely one  $G$ -map  $X \rightarrow J_{\mathcal{F}}(G)$ .

We abbreviate  $\underline{J}G := J_{\mathcal{CM}}(G)$  and call it the universal numerable  $G$ -space for proper  $G$ -actions, or briefly the universal space for proper  $G$ -actions.

In other words,  $J_{\mathcal{F}}(G)$  is a terminal object in the  $G$ -homotopy category of  $\mathcal{F}$ -numerable  $G$ -spaces. In particular two models for  $J_{\mathcal{F}}(G)$  are  $G$ -homotopy equivalent, and for two families  $\mathcal{F}_0 \subseteq \mathcal{F}_1$  there is up to  $G$ -homotopy precisely one  $G$ -map  $J_{\mathcal{F}_0}(G) \rightarrow J_{\mathcal{F}_1}(G)$ .

**Remark 2.4 (Proper  $G$ -spaces).** A  $\mathcal{CM}$ -numerable  $G$ -space  $X$  is proper. Not every proper  $G$ -space is  $\mathcal{CM}$ -numerable. But a  $G$ -CW-complex  $X$  is proper if and only if it is  $\mathcal{CM}$ -numerable (see Lemma 2.2).

**Theorem 2.5 (Homotopy characterization of  $J_{\mathcal{F}}(G)$ ).** *Let  $\mathcal{F}$  be a family of subgroups.*

- (i) *There exists a model for  $J_{\mathcal{F}}(G)$  for any family  $\mathcal{F}$ ;*
- (ii) *Let  $X$  be a  $\mathcal{F}$ -numerable  $G$ -space. Equip  $X \times X$  with the diagonal action and let  $\text{pr}_i: X \times X \rightarrow X$  be the projection onto the  $i$ -th factor for  $i = 1, 2$ . Then  $X$  is a model for  $J_{\mathcal{F}}(G)$  if and only if for each  $H \in \mathcal{F}$  there is  $x \in X$  with  $H \subseteq G_x$  and  $\text{pr}_1$  and  $\text{pr}_2$  are  $G$ -homotopic.*
- (iii) *For  $H \in \mathcal{F}$  the  $H$ -fixed point set  $J_{\mathcal{F}}(G)^H$  is contractible.*

*Proof.* (i) A model for  $J_{\mathcal{F}}(G)$  is constructed by tom Dieck [73, Theorem I.6.6. on page 47], namely, as the infinite join  $*_{n=1}^{\infty} L$  for  $L = \coprod_{H \in \mathcal{F}} G/H$ . There  $G$  is assumed to be compact but the proof goes through for locally compact topological Hausdorff groups.

(ii) Let  $X$  be a model for the classifying space  $J_{\mathcal{F}}(G)$  for  $\mathcal{F}$ . Then  $X \times X$  with the diagonal  $G$ -action is a  $\mathcal{F}$ -numerable  $G$ -space. Hence  $\text{pr}_1$  and  $\text{pr}_2$  are  $G$ -homotopic by the universal property. Since for any  $H \in \mathcal{F}$  the  $G$ -space  $G/H$  is  $\mathcal{F}$ -numerable, there must exist a  $G$ -map  $G/H \rightarrow X$  by the universal property of  $J_{\mathcal{F}}(G)$ . If  $x$  is the image under this map of  $1H$ , then  $H \subseteq G_x$ .

Suppose that  $X$  is a  $G$ -space such that for each  $H \in \mathcal{F}$  there is  $x \in X$  with  $H \subseteq G_x$  and  $\text{pr}_1$  and  $\text{pr}_2$  are  $G$ -homotopic. We want to show that then  $X$  is a model for  $J_{\mathcal{F}}(G)$ . Let  $f_0, f_1: Y \rightarrow X$  be two  $G$ -maps. Since  $\text{pr}_i \circ (f_0 \times f_1) = f_i$  holds for  $i = 0, 1$ ,  $f_0$  and  $f_1$  are  $G$ -homotopic. It remains to show for any  $\mathcal{F}$ -numerable  $G$ -space  $Y$  that there exists a  $G$ -map  $Y \rightarrow X$ . Because of the universal property of  $J_{\mathcal{F}}(G)$  it suffices to do this in the case, where  $Y = *_{n=1}^{\infty} L$  for  $L = \coprod_{H \in \mathcal{F}} G/H$ . By assumption there is a  $G$ -map  $L \rightarrow X$ . Analogous to the construction in [7, Appendix 2] one uses a  $G$ -homotopy from  $\text{pr}_1$  to  $\text{pr}_2$  to construct a  $G$ -map  $*_{n=1}^{\infty} L \rightarrow X$ .

(iii) Restricting to  $1H$  yields a bijection

$$[G/H \times J_{\mathcal{F}}(G)^H, J_{\mathcal{F}}(G)^H]^G \xrightarrow{\cong} [J_{\mathcal{F}}(G)^H, J_{\mathcal{F}}(G)^H],$$

where we consider  $X^H$  as a  $G$ -space with trivial  $G$  action. Since  $G/H \times X^H$  is a  $\mathcal{F}$ -numerable  $G$ -space,  $[J_{\mathcal{F}}(G)^H, J_{\mathcal{F}}(G)^H]$  consists of one element. Hence  $J_{\mathcal{F}}(G)^H$  is contractible.  $\square$

**Remark 2.6.** We do not know whether the converse of Theorem 2.5 (iii) is true, i.e. whether a  $\mathcal{F}$ -numerable  $G$ -space  $X$  is a model for  $J_{\mathcal{F}}(G)$  if  $X^H$  is contractible for each  $H \in \mathcal{F}$ .

**Example 2.7 (Numerable  $G$ -principal bundles).** A *numerable (locally trivial)  $G$ -principal bundle*  $p: E \rightarrow B$  consists by definition of a  $\mathcal{TR}$ -numerable  $G$ -space  $E$ , a space  $B$  with trivial action and a surjective  $G$ -map  $p: E \rightarrow B$  such that the induced map  $G \backslash E \rightarrow B$  is a homeomorphism. A numerable  $G$ -principal bundle  $p: EG \rightarrow BG$  is *universal* if and only if each numerable  $G$ -bundle admits a  $G$ -bundle map to  $p$  and two such  $G$ -bundle maps are  $G$ -bundle homotopic. A numerable  $G$ -principal bundle is universal if and only if  $E$  is contractible. This follows from [26, 7.5 and 7.7]. More information about numerable  $G$ -principal bundles can be found for instance in [35, Section 9 in Chapter 4] [73, Chapter I Section 8].

If  $p: E \rightarrow B$  is a universal numerable  $G$ -principal bundle, then  $E$  is a model for  $J_{\mathcal{TR}}(G)$ . Conversely,  $J_{\mathcal{TR}}(G) \rightarrow G \backslash J_{\mathcal{TR}}(G)$  is a model for the universal numerable  $G$ -principal bundle. We conclude that a  $\mathcal{TR}$ -numerable  $G$ -space  $X$  is a model for  $J_{\mathcal{TR}}(G)$  if and only if  $X$  is contractible (compare Remark 2.6).

**Remark 2.8 ( $G$ -Principal bundles over  $CW$ -complexes).** Let  $p: E \rightarrow B$  be a (locally trivial)  $G$ -principal bundle over a  $CW$ -complex. Since any  $CW$ -complex is paracompact [60], it is automatically a numerable  $G$ -principal bundle. The  $CW$ -complex structure on  $B$  pulls back to  $G$ - $CW$ -structure on  $E$  [47, 1.25 on page 18]. Conversely, if  $E$  is a free  $G$ - $CW$ -complex, then  $E \rightarrow G \backslash E$  is a numerable  $G$ -principal bundle over a  $CW$ -complex by Lemma 2.2

The classifying bundle map from  $p$  above to  $J_{\mathcal{TR}}(G) \rightarrow G \backslash J_{\mathcal{TR}}(G)$  lifts to a  $G$ -bundle map from  $p$  to  $E_{\mathcal{TR}}(G) \rightarrow G \backslash E_{\mathcal{TR}}(G)$  and two such  $G$ -bundle maps from  $p$  to  $E_{\mathcal{TR}}(G) \rightarrow G \backslash E_{\mathcal{TR}}(G)$  are  $G$ -bundle homotopic. Hence for  $G$ -principal bundles over  $CW$ -complexes one can use  $E_{\mathcal{TR}}(G) \rightarrow G \backslash E_{\mathcal{TR}}(G)$  as the universal object.

We will compare the spaces  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$  in Section 3. In Section 4 we will give many interesting geometric models for  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$  in particular in the case  $\mathcal{F} = \mathcal{COM}$ . In some sense  $\underline{J}G = J_{\mathcal{COM}}(G)$  is the most interesting case.

### 3 Comparison of the Two Versions

In this section we compare the two classifying spaces  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$ .

Since  $E_{\mathcal{F}}(G)$  is a  $\mathcal{F}$ -numerable space by Lemma 2.2, there is up to  $G$ -homotopy precisely one  $G$ -map

$$u: E_{\mathcal{F}}(G) \rightarrow J_{\mathcal{F}}(G). \quad (3.1)$$

**Lemma 3.2.** *The following assertions are equivalent for a family  $\mathcal{F}$  of subgroups of  $G$ :*

- (i) *The map  $u: E_{\mathcal{F}}(G) \rightarrow J_{\mathcal{F}}(G)$  defined in 3.1 is a  $G$ -homotopy equivalence;*
- (ii) *The  $G$ -spaces  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$  are  $G$ -homotopy equivalent;*
- (iii) *The  $G$ -space  $J_{\mathcal{F}}(G)$  is  $G$ -homotopy equivalent to a  $G$ -CW-complex, whose isotropy groups belong to  $\mathcal{F}$ ;*
- (iv) *There exists a  $G$ -map  $J_{\mathcal{F}}(G) \rightarrow Y$  to a  $G$ -CW-complex  $Y$ , whose isotropy groups belong to  $\mathcal{F}$ ;*

*Proof.* This follows from the universal properties of  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$ .  $\square$

**Lemma 3.3.** *Suppose either that every element  $H \in \mathcal{F}$  is an open (and closed) subgroup of  $G$  or that  $G$  is a Lie group and  $\mathcal{F} \subseteq \mathcal{COM}$ . Then the map  $u: E_{\mathcal{F}}(G) \rightarrow J_{\mathcal{F}}(G)$  defined in 3.1 is a  $G$ -homotopy equivalence.*

*Proof.* We have to inspect the construction in [73, Lemma 6.13 in Chapter II on page 49] and will use the same notation as in that paper. Let  $Z$  be a  $\mathcal{F}$ -numerable  $G$ -space. Let  $X = \coprod_{H \in \mathcal{F}} G/H$ . Then  $*_{n=1}^{\infty} X$  is a model for  $J_{\mathcal{F}}(G)$  by [73, Lemma 6.6 in Chapter II on page 47]. We inspect the construction of a  $G$ -map  $f: Z \rightarrow *_{n=1}^{\infty} X$ . One constructs a countable covering  $\{U_n \mid n = 1, 2, \dots\}$  of  $Z$  by  $G$ -invariant open subsets of  $Z$  together with a locally finite subordinate partition of unity  $\{v_n \mid n = 1, 2, \dots\}$  by  $G$ -invariant functions  $v_n: Z \rightarrow [0, 1]$  and  $G$ -maps  $\phi_n: U_n \rightarrow X$ . Then one obtains a  $G$ -map

$$f: Z \rightarrow *_{n=1}^{\infty} X, \quad z \mapsto (v_1(z)\phi_1(z), v_2(z)\phi_2(z), \dots),$$

where  $v_n(z)\phi_n(z)$  means  $0x$  for any  $x \in X$  if  $z \notin U_n$ . Let  $i_k: *_{n=1}^k X \rightarrow *_{n=1}^{\infty} X$  and  $j_k: *_{n=1}^k X \rightarrow *_{n=1}^{k+1} X$  be the obvious inclusions. Denote by  $\alpha_k: *_{n=1}^k X \rightarrow \text{colim}_{k \rightarrow \infty} *_{n=1}^k X$  the structure map and by  $i: \text{colim}_{k \rightarrow \infty} *_{n=1}^k X \rightarrow *_{n=1}^{\infty} X$  the map induced by the system  $\{i_k \mid k = 1, 2, \dots\}$ . This  $G$ -map is a (continuous) bijective  $G$ -map but not necessarily a  $G$ -homeomorphism. Since the partition  $\{v_n \mid n = 1, 2, \dots\}$  is locally finite, we can find for each  $z \in Z$  an open  $G$ -invariant neighborhood  $W_z$  of  $z$  in  $Z$  and a positive integer  $k_z$  such that  $v_n$  vanishes on  $W_z$  for  $n > k_z$ . Define a map

$$f'_z: W_z \rightarrow *_{n=1}^{k_z} X, \quad z \mapsto (v_1(z)\phi_1(z), v_2(z)\phi_2(z), \dots, v_{k_z}(z)\phi_{k_z}(z)).$$

Then  $\alpha_{k_z} \circ f'_z: W_z \rightarrow \operatorname{colim}_{k \rightarrow \infty} *_{n=1}^k X$  is a well-defined  $G$ -map whose composition with  $i: \operatorname{colim}_{k \rightarrow \infty} *_{n=1}^k X \rightarrow *_{n=1}^\infty X$  is  $f|_{W_z}$ . Hence the system of the maps  $\alpha_{k_z} \circ f'_z$  defines a  $G$ -map

$$f': Z \rightarrow \operatorname{colim}_{k \rightarrow \infty} *_{n=1}^k X$$

such that  $i \circ f' = f$  holds.

Let

$$\Delta_{n-1} = \{(t_1, t_2 \dots t_n) \mid t_i \in [0, 1], \sum_{i=1}^n t_i = 1\} \subseteq \prod_{n=1}^k [0, 1]$$

be the standard  $(n-1)$ -simplex. Let

$$p: \left( \prod_{n=1}^k X \right) \times \Delta_n \rightarrow *_{n=1}^k X, \quad (x_1, \dots, x_n), (t_1, \dots, t_n) \mapsto (t_1 x_1, \dots, t_n x_n)$$

be the obvious projection. It is a surjective continuous map but in general not an identification. Let  $\overline{*}_{n=1}^k X$  be the topological space whose underlying set is the same as for  $*_{n=1}^k X$  but whose topology is the quotient topology with respect to  $p$ . The identity induces a (continuous) map  $\overline{*}_{n=1}^k X \rightarrow *_{n=1}^k X$  which is not a homeomorphism in general. Choose for  $n \geq 1$  a (continuous) function  $\phi_n: [0, 1] \rightarrow [0, 1]$  which satisfies  $\phi_n^{-1}(0) = [0, 4^{-n}]$ . Define

$$u_k: *_{n=1}^k X \rightarrow \overline{*}_{n=1}^k X, \\ (t_n x_n \mid n = 1, \dots, k) \mapsto \left( \frac{\phi_n(t_n)}{\sum_{n=1}^k \phi_n(t_n)} x_n \mid n = 1, \dots, k \right).$$

It is not hard to check that this  $G$ -map is continuous. If  $\overline{j}_k: \overline{*}_{n=1}^k X \rightarrow \overline{*}_{n=1}^{k+1} X$  is the obvious inclusion, we have  $u_{k+1} \circ j_k = \overline{j}_k \circ u_k$  for all  $k \geq 1$ . Hence the system of the maps  $u_k$  induces a  $G$ -map

$$u: \operatorname{colim}_{k \rightarrow \infty} *_{n=1}^k X \rightarrow \operatorname{colim}_{k \rightarrow \infty} \overline{*}_{n=1}^k X.$$

Next we want to show that each  $G$ -space  $\overline{*}_{n=1}^k X$  has the  $G$ -homotopy type of a  $G$ - $CW$ -complex, whose isotropy groups belong to  $\mathcal{F}$ . We first show that  $\overline{*}_{n=1}^k X$  is a  $(\prod_{n=1}^k G)$ - $CW$ -complex. It suffices to treat the case  $k = 2$ , the general case follows by induction over  $k$ . We can rewrite  $X \overline{*} X$  as a  $G \times G$ -pushout

$$\begin{array}{ccc} X \times X & \xrightarrow{i_1} & CX \times X \\ i_2 \downarrow & & \downarrow \\ X \times CX & \longrightarrow & X \overline{*} X \end{array}$$

where  $CX$  is the cone over  $X$  and  $i_1$  and  $i_2$  are the obvious inclusions. Recall that we are working in the category of compactly generated spaces. Hence the

product of two  $G$ -CW-complexes is in a canonical way a  $(G \times G)$ -CW-complex, and, if  $(B, A)$  is a  $G$ -CW-pair,  $C$  a  $G$ -CW-complex and  $f: B \rightarrow C$  is a cellular  $G$ -map, then  $A \cup_f C$  inherits a  $G$ -CW-structure in a canonical way. Thus  $X \overline{\ast} X$  inherits a  $(G \times G)$ -CW-complex structure.

The problem is now to decide whether the  $(\prod_{n=1}^k G)$ -CW-complex  $\overline{\ast}_{n=1}^k X$  regarded as a  $G$ -space by the diagonal action has the  $G$ -homotopy type of a  $G$ -CW-complex. If each  $H \in \mathcal{F}$  is open, then each isotropy group of the  $G$ -space  $\overline{\ast}_{n=1}^k X$  is open and we conclude from Remark 1.3 that  $\overline{\ast}_{n=1}^k X$  with the diagonal  $G$ -action is a  $G$ -CW-complex. Suppose that  $G$  is a Lie group and each  $H \in \mathcal{F}$  is compact. Lemma 1.4 implies that for any compact subgroup  $K \subseteq \prod_{n=1}^k G$  the space  $(\prod_{n=1}^k G) / K$  regarded as  $G$ -space by the diagonal action has the  $G$ -homotopy type of a  $G$ -CW-complex. We conclude from [47, Lemma 7.4 on page 121] that  $\overline{\ast}_{n=1}^k X$  with the diagonal  $G$ -action has the  $G$ -homotopy type of a  $G$ -CW-complex. The isotropy groups  $\overline{\ast}_{n=1}^k X$  belong to  $\mathcal{F}$  since  $\mathcal{F}$  is closed under finite intersections and conjugation. It is not hard to check that each  $G$ -map  $\overline{j}_k$  is a  $G$ -cofibration. Hence  $\text{colim}_{k \rightarrow \infty} \overline{\ast}_{n=1}^k X$  has the  $G$ -homotopy type of a  $G$ -CW-complex, whose isotropy groups belong to  $\mathcal{F}$ .

Thus we have shown for every  $\mathcal{F}$ -numerable  $G$ -space  $Z$  that it admits a  $G$ -map to a  $G$ -CW-complex whose isotropy groups belong to  $\mathcal{F}$ . Now Lemma 3.3 follows from Lemma 3.2.  $\square$

**Definition 3.4 (Totally disconnected group).** *A (locally compact topological Hausdorff) group  $G$  is called totally disconnected if it satisfies one of the following equivalent conditions:*

- (T)  $G$  is totally disconnected as a topological space, i.e. each component consists of one point;
- (D) The covering dimension of the topological space  $G$  is zero;
- (FS) Any element of  $G$  has a fundamental system of compact open neighborhoods.

We have to explain why these three conditions are equivalent. The implication (T)  $\Rightarrow$  (D)  $\Rightarrow$  (FS) is shown in [33, Theorem 7.7 on page 62]. It remains to prove (FS)  $\Rightarrow$  (T). Let  $U$  be a subset of  $G$  containing two distinct points  $g$  and  $h$ . Let  $V$  be a compact open neighborhood of  $x$  which does not contain  $y$ . Then  $U$  is the disjoint union of the open non-empty sets  $V \cap U$  and  $V^c \cap U$  and hence disconnected.

**Lemma 3.5.** *Let  $G$  be a totally disconnected group and  $\mathcal{F}$  a family satisfying  $\mathcal{COMOP} \subseteq \mathcal{F} \subseteq \mathcal{COM}$ . Then the following square commutes up to  $G$ -homotopy and consists of  $G$ -homotopy equivalences*

$$\begin{array}{ccc} E_{\mathcal{COMOP}}(G) & \xrightarrow{u} & J_{\mathcal{COMOP}}(G) \\ \downarrow & & \downarrow \\ E_{\mathcal{F}}(G) & \xrightarrow{u} & J_{\mathcal{F}}(G) \end{array}$$

where all maps come from the universal properties.

*Proof.* We first show that any compact subgroup  $H \subseteq G$  is contained in a compact open subgroup. From [33, Theorem 7.7 on page 62] we get a compact open subgroup  $K \subseteq G$ . Since  $H$  is compact, we can find finitely many elements  $h_1, h_2, \dots, h_s$  in  $H$  such that  $H \subseteq \cup_{i=1}^s h_i K$ . Put  $L := \cap_{h \in H} h K h^{-1}$ . Then  $h L h^{-1} = L$  for all  $h \in H$ . Since  $L = \cap_{i=1}^s h_i K h_i^{-1}$  holds,  $L$  is compact open. Hence  $LH$  is a compact open subgroup containing  $H$ .

This implies that  $J_{\mathcal{F}}(G)$  is *COMOP*-numerable. Obviously  $J_{\text{COMOP}}(G)$  is  $\mathcal{F}$ -numerable. We conclude from the universal properties that  $J_{\text{COMOP}}(G) \rightarrow J_{\mathcal{F}}(G)$  is a  $G$ -homotopy equivalence.

The map  $u: E_{\text{COMOP}}(G) \rightarrow J_{\text{COMOP}}(G)$  is a  $G$ -homotopy equivalence by Lemma 3.3.

This and Theorem 2.5 (iii) imply that  $E_{\text{COMOP}}(G)^H$  is contractible for all  $H \in \mathcal{F}$ . Hence  $E_{\text{COMOP}}(G) \rightarrow E_{\mathcal{F}}(G)$  is a  $G$ -homotopy equivalence by Theorem 1.9 (ii).  $\square$

**Definition 3.6 (Almost connected group).** *Given a group  $G$ , let  $G^0$  be the normal subgroup given by the component of the identity and  $\bar{G} = G/G^0$  be the component group. We call  $G$  almost connected if its component group  $\bar{G}$  is compact.*

A Lie group  $G$  is almost connected if and only if it has finitely many path components. In particular a discrete group is almost connected if it is finite.

**Theorem 3.7 (Comparison of  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$ ).** *The map  $u: E_{\mathcal{F}}(G) \rightarrow J_{\mathcal{F}}(G)$  defined in 3.1 is a  $G$ -homotopy equivalence if one of the following conditions is satisfied:*

- (i) *Each element in  $\mathcal{F}$  is an open subgroup of  $G$ ;*
- (ii) *The group  $G$  is discrete;*
- (iii) *The group  $G$  is a Lie group and every element  $H \in \mathcal{F}$  is compact;*
- (iv) *The group  $G$  is totally disconnected and  $\mathcal{F} = \text{COM}$  or  $\mathcal{F} = \text{COMOP}$ ;*
- (v) *The group  $G$  is almost connected and each element in  $\mathcal{F}$  is compact.*

*Proof.* Assertions (i), (ii), (iii) and (iv) have already been proved in Lemma 3.3 and Lemma 3.5. Assertion (v) follows from Lemma 3.2 (iii) and Theorem 4.3.  $\square$

The following example shows that the map  $u: E_{\mathcal{F}}(G) \rightarrow J_{\mathcal{F}}(G)$  defined in 3.1 is in general not a  $G$ -homotopy equivalence.

**Example 3.8 (Totally disconnected groups and  $\mathcal{TR}$ ).** Let  $G$  be totally disconnected. We claim that  $u: E_{\mathcal{TR}}(G) \rightarrow J_{\mathcal{TR}}(G)$  defined in 3.1 is a  $G$ -homotopy equivalence if and only if  $G$  is discrete. In view of Theorem 2.5 (iii)

and Lemma 3.3 this is equivalent to the statement that  $E_{\mathcal{TR}}(G)$  is contractible if and only if  $G$  is discrete. If  $G$  is discrete, we already know that  $E_{\mathcal{TR}}(G)$  is contractible. Suppose now that  $E_{\mathcal{TR}}(G)$  is contractible. We obtain a numerable  $G$ -principal bundle  $G \rightarrow E_{\mathcal{TR}}(G) \rightarrow G \backslash E_{\mathcal{TR}}(G)$  by Remark 2.8. This implies that it is a fibration by a result of Hurewicz [79, Theorem on p. 33]. Since  $E_{\mathcal{TR}}(G)$  is contractible,  $G$  and the loop space  $\Omega(G \backslash E_{\mathcal{TR}}(G))$  are homotopy equivalent [79, 6.9\* on p. 137, 6.10\* on p. 138, Corollary 7.27 on p. 40]. Since  $G \backslash E_{\mathcal{TR}}(G)$  is a  $CW$ -complex,  $\Omega(G \backslash E_{\mathcal{TR}}(G))$  has the homotopy type of a  $CW$ -complex [58]. Hence there exists a homotopy equivalence  $f: G \rightarrow X$  from  $G$  to a  $CW$ -complex  $X$ . Then the induced map  $\pi_0(G) \rightarrow \pi_0(X)$  between the set of path components is bijective. Hence the preimage of each path component of  $X$  is a path component of  $G$  and therefore a point since  $G$  is totally disconnected. Since  $X$  is locally path-connected each path component of  $X$  is open in  $X$ . We conclude that  $G$  is the disjoint union of the preimages of the path components of  $X$  and each of these preimages is open in  $G$  and consists of one point. Hence  $G$  is discrete.



## 4 Special Models

In this section we present some interesting geometric models for the space  $E_{\mathcal{F}}(G)$  and  $J_{\mathcal{F}}(G)$  focussing on  $\underline{E}G$  and  $\underline{J}G$ . In particular we are interested in cases, where these models satisfy finiteness conditions such as being finite, finite-dimensional or of finite type.

One extreme case is, where we take  $\mathcal{F}$  to be the family  $\mathcal{ACL}$  of all subgroups. Then a model for both  $E_{\mathcal{ACL}}(G)$  and  $J_{\mathcal{ACL}}(G)$  is  $G/G$ . The other extreme case is the family  $\mathcal{TR}$  consisting of the trivial subgroup. This case has already been treated in Example 2.7, Remark 2.8 and Example 3.8.

### 4.1 Operator Theoretic Model

Let  $G$  be a locally compact Hausdorff topological group. Let  $C_0(G)$  be the Banach space of complex valued functions of  $G$  vanishing at infinity with the supremum-norm. The group  $G$  acts isometrically on  $C_0(G)$  by  $(g \cdot f)(x) := f(g^{-1}x)$  for  $f \in C_0(G)$  and  $g, x \in G$ . Let  $PC_0(G)$  be the subspace of  $C_0(G)$  consisting of functions  $f$  such that  $f$  is not identically zero and has non-negative real numbers as values.

The next theorem is due to Abels [1, Theorem 2.4].

**Theorem 4.1 (Operator theoretic model).** *The  $G$ -space  $PC_0(G)$  is a model for  $\underline{J}G$ .*

**Remark 4.2.** Let  $G$  be discrete. Another model for  $\underline{J}G$  is the space

$$X_G = \{f: G \rightarrow [0, 1] \mid f \text{ has finite support, } \sum_{g \in G} f(g) = 1\}$$

with the topology coming from the supremum norm [7, page 248]. Let  $P_{\infty}(G)$  be the geometric realization of the simplicial set whose  $k$ -simplices consist of  $(k+1)$ -tuples  $(g_0, g_1, \dots, g_k)$  of elements  $g_i$  in  $G$ . This also a model for  $\underline{E}G$  [1, Example 2.6]. The spaces  $X_G$  and  $P_{\infty}(G)$  have the same underlying sets but in general they have different topologies. The identity map induces a (continuous)  $G$ -map  $P_{\infty}(G) \rightarrow X_G$  which is a  $G$ -homotopy equivalence, but in general not a  $G$ -homeomorphism (see also [75, A.2]).

### 4.2 Almost Connected Groups

The next result is due to Abels [1, Corollary 4.14].

**Theorem 4.3 (Almost connected groups).** *Let  $G$  be a (locally compact Hausdorff) topological group. Suppose that  $G$  is almost connected, i.e. the group  $G/G^0$  is compact for  $G^0$  the component of the identity element. Then  $G$  contains a maximal compact subgroup  $K$  which is unique up to conjugation. The  $G$ -space  $G/K$  is a model for  $\underline{J}G$ .*

The next result follows from Example 1.4, Theorem 3.7 (iii) and Theorem 4.3.

**Theorem 4.4 (Discrete subgroups of almost connected Lie groups).** *Let  $L$  be a Lie group with finitely many path components. Then  $L$  contains a maximal compact subgroup  $K$  which is unique up to conjugation. The  $L$ -space  $L/K$  is a model for  $\underline{E}L$ .*

*If  $G \subseteq L$  is a discrete subgroup of  $L$ , then  $L/K$  with the obvious left  $G$ -action is a finite-dimensional  $G$ -CW-model for  $\underline{E}G$ .*

### 4.3 Actions on Simply Connected Non-Positively Curved Manifolds

The next theorem is due to Abels [1, Theorem 4.15].

**Theorem 4.5 (Actions on simply connected non-positively curved manifolds).** *Let  $G$  be a (locally compact Hausdorff) topological group. Suppose that  $G$  acts properly and isometrically on the simply-connected complete Riemannian manifold  $M$  with non-positive sectional curvature. Then  $M$  is a model for  $\underline{J}G$ .*

### 4.4 Actions on CAT(0)-spaces

**Theorem 4.6 (Actions on CAT(0)-spaces).** *Let  $G$  be a (locally compact Hausdorff) topological group. Let  $X$  be a proper  $G$ -CW-complex. Suppose that  $X$  has the structure of a complete CAT(0)-space for which  $G$  acts by isometries. Then  $X$  is a model for  $\underline{E}G$ .*

*Proof.* By [13, Corollary II.2.8 on page 179] the  $K$ -fixed point set of  $X$  is non-empty convex subset of  $X$  and hence contractible for any compact subgroup  $K \subset G$ .  $\square$

### 4.5 Actions on Trees and Graphs of Groups

A *tree* is a 1-dimensional CW-complex which is contractible.

**Theorem 4.7 (Actions on trees).** *Suppose that  $G$  acts continuously on a tree  $T$  such that for each element  $g \in G$  and each open cell  $e$  with  $g \cdot e \cap e \neq \emptyset$  we have  $gx = x$  for any  $x \in e$ . Assume that the isotropy group of each  $x \in T$  is compact. Then  $T$  is a 1-dimensional model for  $E_{\text{COMOP}}(G) = J_{\text{COMOP}}(G)$ .*

*Proof.* We conclude from Remark 1.3 that  $T$  is a  $G$ -CW-complex and all isotropy groups are compact open. Let  $H \subseteq G$  be compact open. If  $e_0$  is a zero-cell in  $T$ , then  $H \cdot e_0$  is a compact discrete set and hence finite. Let  $T'$  be the union of all geodesics with extremities in  $H \cdot e_0$ . This is a  $H$ -invariant subtree of  $T$  of finite diameter. One shows now inductively over the diameter of  $T'$  that  $T'$  has a vertex which is fixed under the  $H$ -action (see [65, page 20] or [25, Proposition 4.7 on page 17]). Hence  $T^H$  is non-empty. If  $e$  and  $f$  are vertices

in  $T^H$ , the geodesic in  $T$  from  $e$  to  $f$  must be  $H$ -invariant. Hence  $T^H$  is a connected  $CW$ -subcomplex of the tree  $T$  and hence is itself a tree. This shows that  $T^H$  is contractible. Hence  $T$  is a model for  $E_{\mathcal{COMOP}}(G)$  which is the same as  $J_{\mathcal{COMOP}}(G)$  by Theorem 3.7 (i).  $\square$

Let  $G$  be a locally compact Hausdorff group. Suppose that  $G$  acts continuously on a tree  $T$  such that for each element  $g \in G$  and each open cell  $e$  with  $g \cdot e \cap e \neq \emptyset$  we have  $gx = x$  for any  $x \in e$ . If the  $G$ -action on a tree has possibly not compact isotropy groups, one can nevertheless get nice models for  $E_{\mathcal{COMOP}}(G)$  as follows. Let  $V$  be the set of equivariant 0-cells and  $E$  be the set of equivariant 1-cells of  $T$ . Then we can choose a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{e \in E} G/H_e \times \{-1, 1\} & \xrightarrow{q} & T_0 = \coprod_{v \in V} G/K_v \\ \downarrow & & \downarrow \\ \coprod_{e \in E} G/H_e \times [-1, 1] & \longrightarrow & T \end{array} \quad (4.8)$$

where the left vertical arrow is the obvious inclusion. Fix  $e \in E$  and  $\sigma \in \{-1, 1\}$ . Choose elements  $v(e, \sigma) \in V$  and  $g(e, \sigma) \in G$  such that  $q$  restricted to  $G/H_e \times \{\sigma\}$  is the  $G$ -map  $G/H_e \rightarrow G/K_{v(e, \sigma)}$  which sends  $1H_e$  to  $g(e, \sigma)K_{v(e, \sigma)}$ . Then conjugation with  $g(e, \sigma)$  induces a group homomorphism  $c_{g(e, \sigma)}: H_e \rightarrow K_{v(e, \pm 1)}$  and there is an up to equivariant homotopy unique  $c_{g(e, \sigma)}$ -equivariant cellular map  $f_{g(e, \sigma)}: E_{\mathcal{COMOP}}(H_e) \rightarrow E_{\mathcal{COMOP}}(K_{v(e, \sigma)})$ . Define a  $G$ -map

$$Q: \coprod_{e \in E} G \times_{H_e} E_{\mathcal{COMOP}}(H_e) \times \{-1, 1\} \rightarrow \coprod_{v \in V} G \times_{K_v} E_{\mathcal{COMOP}}(K_v)$$

by requiring that the restriction of  $Q$  to  $G \times_{H_e} E_{\mathcal{COMOP}}(H_e) \times \{\sigma\}$  is the  $G$ -map  $G \times_{H_e} E_{\mathcal{COMOP}}(H_e) \rightarrow G \times_{K_{v(e, \sigma)}} E_{\mathcal{COMOP}}(K_{v(e, \sigma)})$ ,  $(g, x) \mapsto (g, f_{g(e, \sigma)}(x))$ .

Let  $T_{\mathcal{COMOP}}$  be the  $G$ -pushout

$$\begin{array}{ccc} \coprod_{e \in E} G \times_{H_e} E_{\mathcal{COMOP}}(H_e) \times \{-1, 1\} & \xrightarrow{Q} & \coprod_{v \in V} G \times_{K_v} E_{\mathcal{COMOP}}(K_v) \\ \downarrow & & \downarrow \\ \coprod_{e \in E} G \times_{H_e} E_{\mathcal{COMOP}}(H_e) \times [-1, 1] & \longrightarrow & T_{\mathcal{COMOP}} \end{array}$$

The  $G$ -space  $T_{\mathcal{COMOP}}$  inherits a canonical  $G$ - $CW$ -structure with compact open isotropy groups. Notice that for any open subgroup  $L \subseteq G$  one can choose as model for  $E_{\mathcal{COMOP}}(L)$  the restriction  $\text{res}_G^L E_{\mathcal{COMOP}}(G)$  of  $E_{\mathcal{COMOP}}(G)$  to  $L$  and that there is a  $G$ -homeomorphism  $G \times_L \text{res}_G^L E_{\mathcal{COMOP}}(G) \xrightarrow{\cong} G/L \times E_{\mathcal{COMOP}}(G)$  which sends  $(g, x)$  to  $(gL, gx)$ . This implies that  $T_{\mathcal{COMOP}}$  is  $G$ -homotopy equivalent to  $T \times E_{\mathcal{COMOP}}(G)$  with the diagonal  $G$ -action. If  $H \subseteq G$  is compact open, then  $T^H$  is contractible. Hence  $(T \times E_{\mathcal{COMOP}}(G))^H$  is contractible for compact open subgroup  $H \subseteq G$ . Theorem 1.9 (ii) shows

**Theorem 4.9 (Models based on actions on trees).** *The  $G$ -CW-complex  $T_{\text{COMOP}}$  is a model for  $E_{\text{COMOP}}(G)$ .*

The point is that it may be possible to choose nice models for the various spaces  $E_{\text{COMOP}}(H_e)$  and  $E_{\text{COMOP}}(K_v)$  and thus get a nice model for  $E_{\text{COMOP}}(G)$ . If all isotropy groups of the  $G$ -action on  $T$  are compact, we can choose all spaces  $E_{\text{COMOP}}(H_e)$  and  $E_{\text{COMOP}}(K_v)$  to be {pt.} and we rediscover Theorem 4.7.

Next we recall which discrete groups  $G$  act on trees. Recall that an oriented graph  $X$  is a 1-dimensional CW-complex together with an orientation for each 1-cell. This can be codified by specifying a triple  $(V, E, s: E \times \{-1, 1\} \rightarrow V)$  consisting of two sets  $V$  and  $E$  and a map  $s$ . The associated oriented graph is the pushout

$$\begin{array}{ccc} E \times \{-1, 1\} & \xrightarrow{s} & V \\ \downarrow & & \downarrow \\ E \times [0, 1] & \longrightarrow & X \end{array}$$

So  $V$  is the set of vertices,  $E$  the set of edges, and for a edge  $e \in E$  its initial vertex is  $s(e, -1)$  and its terminal vertex is  $s(e, 1)$ . A *graph of groups*  $\mathcal{G}$  on a connected oriented graph  $X$  consists of two sets of groups  $\{K_v \mid v \in V\}$  and  $\{H_e \mid e \in E\}$  with  $V$  and  $E$  as index sets together with injective group homomorphisms  $\phi_{v,\sigma}: H_e \rightarrow K_{s(e,\sigma)}$  for each  $e \in E$ . Let  $X_0 \subseteq X$  be some maximal tree. We can associate to these data *the fundamental group*  $\pi = \pi(\mathcal{G}, X, X_0)$  as follows. Generators of  $\pi$  are the elements in  $K_v$  for each  $v \in V$  and the set  $\{t_e \mid e \in E\}$ . The relations are the relations in each group  $K_v$  for each  $v \in V$ , the relation  $t_e = 1$  for  $e \in V$  if  $e$  belongs to  $X_0$ , and for each  $e \in E$  and  $h \in H_e$  we require  $t_e^{-1}\phi_{e,-1}(h)t_e = \phi_{e,+1}(h)$ . It turns out that the obvious map  $K_v \rightarrow \pi$  is an injective group homomorphism for each  $v \in V$  and we will identify in the sequel  $K_v$  with its image in  $\pi$  [25, Corollary 7.5 on page 33], [65, Corollary 1 in 5.2 on page 45]. We can assign to these data a tree  $T = T(X, X_0, \mathcal{G})$  with  $\pi$ -action as follows. Define a  $\pi$ -map

$$q: \prod_{e \in E} \pi / \text{im}(\phi_{e,-1}) \times \{-1, 1\} \rightarrow \prod_{v \in V} \pi / K_v$$

by requiring that its restriction to  $\pi / \text{im}(\phi_{e,-1}) \times \{-1\}$  is the  $\pi$ -map given by the projection  $\pi / \text{im}(\phi_{e,-1}) \rightarrow \pi / K_{s(e,-1)}$  and its restriction to  $\pi / \text{im}(\phi_{e,-1}) \times \{1\}$  is the  $\pi$ -map  $\pi / \text{im}(\phi_{e,-1}) \rightarrow \pi / K_{s(e,1)}$  which sends  $g \text{im}(\phi_{e,-1})$  to  $gt_e \text{im}(\phi_{e,1})$ . Now define a 1-dimensional  $G$ -CW-complex  $T = T(\mathcal{G}, X, X_0)$  using this  $\pi$ -map  $q$  and the  $\pi$ -pushout analogous to (4.8). It turns out that  $T$  is contractible [25, Theorem 7.6 on page 33], [65, Theorem 12 in 5.3 on page 52].

On the other hand, suppose that  $T$  is a 1-dimensional  $G$ -CW-complex. Choose a  $G$ -pushout (4.8). Let  $X$  be the connected oriented graph  $G \setminus T$ . It has a set of vertices  $V$  and as set of edges the set  $E$ . The required map  $s: E \times \{-1, 1\} \rightarrow V$  sends  $s(e, \sigma)$  to the vertex for which  $q(G/H_e \times \{\sigma\})$  meets and hence is equal to  $G/K_{s(e,\sigma)}$ . Moreover, we get a graph of groups  $\mathcal{G}$  on  $X$  as follows. Let  $\{K_v \mid v \in V\}$  and  $\{H_e \mid e \in E\}$  be the set of groups given by (4.8).

Choose an element  $g(e, \sigma) \in G$  such that the  $G$ -map induced by  $q$  from  $G/H_e$  to  $G/K_{s(e, \sigma)}$  sends  $1H_e$  to  $g(e, \sigma)K_{s(e, \sigma)}$ . Then conjugation with  $g(e, \sigma)$  induces a group homomorphism  $\phi_{e, \sigma}: H_e \rightarrow K_{s(e, \sigma)}$ . After a choice of a maximal tree  $X_0$  in  $X$  one obtains an isomorphism  $G \cong \pi(\mathcal{G}, X, X_0)$ . (Up to isomorphism) we get a bijective correspondence between pairs  $(G, T)$  consisting of a group  $G$  acting on an oriented tree  $T$  and a graph of groups on connected oriented graphs. For details we refer for instance to [25, I.4 and I.7] and [65, §5].

**Example 4.10 (The graph associated to amalgamated products).** Consider the graph  $D$  with one edge  $e$  and two vertices  $v_{-1}$  and  $v_1$  and the map  $s: \{e\} \times \{-1, 1\} \rightarrow \{v_{-1}, v_1\}$  which sends  $(e, \sigma)$  to  $v_\sigma$ . Of course this is just the graph consisting of a single segment which is homeomorphic to  $[-1, 1]$ . Let  $\mathcal{G}$  be a graph of groups on  $D$ . This is the same as specifying a group  $H_e$  and groups  $K_{-1}$  and  $K_1$  together with injective group homomorphisms  $\phi_\sigma: H_e \rightarrow K_\sigma$  for  $\sigma \in \{-1, 1\}$ . There is only one choice of a maximal subtree in  $D$ , namely  $D$  itself. Then the fundamental group  $\pi$  of this graph of groups is the amalgamated product of  $K_{-1}$  and  $K_1$  over  $H_e$  with respect to  $\phi_{-1}$  and  $\phi_1$ , i.e. the pushout of groups

$$\begin{array}{ccc} H_e & \xrightarrow{\phi_{-1}} & K_{-1} \\ \phi_1 \downarrow & & \downarrow \\ K_1 & \longrightarrow & \pi \end{array}$$

Choose  $\phi_\sigma$ -equivariant maps  $f_\sigma: \underline{E}H_e \rightarrow \underline{E}K_\sigma$ . They induce  $\pi$ -maps

$$F_\sigma: \pi \times_{H_e} \underline{E}H_e \rightarrow \pi \times_{K_\sigma} \underline{E}K_\sigma, \quad (g, x) \mapsto (g, f_\sigma(x)).$$

We get a model for  $\underline{E}\pi$  as the  $\pi$ -pushout

$$\begin{array}{ccc} \pi \times_{H_e} \underline{E}H_e \times \{-1, 1\} & \xrightarrow{F_{-1} \amalg F_1} & \pi \times_{K_{-1}} \underline{E}K_{-1} \amalg \pi \times_{K_1} \underline{E}K_1 \\ \downarrow & & \downarrow \\ \pi \times_{H_e} \underline{E}H_e \times [-1, 1] & \longrightarrow & \underline{E}\pi \end{array}$$

**Example 4.11 (The graph associated to an  $HNN$ -extension).** Consider the graph  $S$  with one edge  $e$  and one vertex  $v$ . There is only one choice for the map  $s: \{e\} \times \{-1, 1\} \rightarrow \{v\}$ . Of course this graph is homeomorphic to  $S^1$ . Let  $\mathcal{G}$  be a graph of groups on  $S$ . It consists of two groups  $H_e$  and  $K_v$  and two injective group homomorphisms  $\phi_\sigma: H_e \rightarrow K_v$  for  $\sigma \in \{-1, 1\}$ . There is only one choice of a maximal subtree, namely  $\{v\}$ . The fundamental group  $\pi$  of  $\mathcal{G}$  is the so called  $HNN$ -extension associated to the data  $\phi_\sigma: H_e \rightarrow K_v$  for  $\sigma \in \{-1, 1\}$ , i.e. the group generated by the elements of  $K_v$  and a letter  $t_v$  whose relations are those of  $K_v$  and the relations  $t_v^{-1}\phi_{-1}(h)t_v = \phi_1(h)$  for all  $h \in H_e$ . Recall that the natural map  $K_v \rightarrow \pi$  is injective and we will identify  $K_v$  with its image in  $\pi$ . Choose  $\phi_\sigma$ -equivariant maps  $f_\sigma: \underline{E}H_e \rightarrow \underline{E}K_v$ . Let  $F_\sigma: \pi \times_{\phi_\sigma} \underline{E}H_e \rightarrow \pi \times \underline{E}K_v$  be the  $\pi$ -map which sends  $(g, x)$  to  $gf_{-1}(x)$  for

$\sigma = -1$  and to  $gt_e f_1(x)$  for  $\sigma = 1$ . Then a model for  $\underline{E}\pi$  is given by the  $\pi$ -pushout

$$\begin{array}{ccc} \pi \times_{\phi_{-1}} \underline{E}H_e \times \{-1, 1\} & \xrightarrow{F_{-1} \amalg F_1} & \pi \times_{K_v} \underline{E}K_v \\ \downarrow & & \downarrow \\ \pi \times_{\phi_{-1}} \underline{E}H_e \times [-1, 1] & \longrightarrow & \underline{E}\pi \end{array}$$

Notice that this looks like a telescope construction which is infinite to both sides. Consider the special case, where  $H_e = K_v$ ,  $\phi_{-1} = \text{id}$  and  $\phi_1$  is an automorphism. Then  $\pi$  is the semidirect product  $K_v \rtimes_{\phi_1} \mathbb{Z}$ . Choose a  $\phi_1$ -equivariant map  $f_1: \underline{E}K_v \rightarrow \underline{E}K_v$ . Then a model for  $\underline{E}\pi$  is given by the to both side infinite mapping telescope of  $f_1$  with the  $K_v \rtimes_{\phi_1} \mathbb{Z}$  action, for which  $\mathbb{Z}$  acts by shifting to the right and  $k \in K_v$  acts on the part belonging to  $n \in \mathbb{Z}$  by multiplication with  $\phi_1^n(k)$ . If we additionally assume that  $\phi_1 = \text{id}$ , then  $\pi = K_v \times \mathbb{Z}$  and we get  $\underline{E}K_v \times \mathbb{R}$  as model for  $\underline{E}\pi$ .

**Remark 4.12.** All these constructions yield also models for  $EG = E_{\mathcal{TR}}(G)$  if one replaces everywhere the spaces  $\underline{E}H_e$  and  $\underline{E}K_v$  by the spaces  $EH_e$  and  $EK_v$ .

## 4.6 Affine Buildings

Let  $\Sigma$  be an *affine building*, sometimes also called *Euclidean building*. This is a simplicial complex together with a system of subcomplexes called *apartments* satisfying the following axioms:

- (i) Each apartment is isomorphic to an affine Coxeter complex;
- (ii) Any two simplices of  $\Sigma$  are contained in some common apartment;
- (iii) If two apartments both contain two simplices  $A$  and  $B$  of  $\Sigma$ , then there is an isomorphism of one apartment onto the other which fixes the two simplices  $A$  and  $B$  pointwise.

The precise definition of an affine Coxeter complex, which is sometimes called also Euclidean Coxeter complex, can be found in [17, Section 2 in Chapter VI], where also more information about affine buildings is given. An affine building comes with metric  $d: \Sigma \times \Sigma \rightarrow [0, \infty)$  which is non-positively curved and complete. The building with this metric is a CAT(0)-space. A simplicial automorphism of  $\Sigma$  is always an isometry with respect to  $d$ . For two points  $x, y$  in the affine building there is a unique line segment  $[x, y]$  joining  $x$  and  $y$ . It is the set of points  $\{z \in \Sigma \mid d(x, y) = d(x, z) + d(z, y)\}$ . For  $x, y \in \Sigma$  and  $t \in [0, 1]$  let  $tx + (1 - t)y$  be the point  $z \in \Sigma$  uniquely determined by the property that  $d(x, z) = td(x, y)$  and  $d(z, y) = (1 - t)d(x, y)$ . Then the map

$$r: \Sigma \times \Sigma \times [0, 1] \rightarrow \Sigma, \quad (x, y, t) \mapsto tx + (1 - t)y$$

is continuous. This implies that  $\Sigma$  is contractible. All these facts are taken from [17, Section 3 in Chapter VI] and [13, Theorem 10A.4 on page 344].

Suppose that the group  $G$  acts on  $\Sigma$  by isometries. If  $G$  maps a non-empty bounded subset  $A$  of  $\Sigma$  to itself, then the  $G$ -action has a fixed point [17, Theorem 1 in Section 4 in Chapter VI on page 157]. Moreover the  $G$ -fixed point set must be contractible since for two points  $x, y \in \Sigma^G$  also the segment  $[x, y]$  must lie in  $\Sigma^G$  and hence the map  $r$  above induces a continuous map  $\Sigma^G \times \Sigma^G \times [0, 1] \rightarrow \Sigma^G$ . This implies together with Theorem 1.9 (ii), Example 1.5, Lemma 3.3 and Lemma 3.5

**Theorem 4.13 (Affine buildings).** *Let  $G$  be a topological (locally compact Hausdorff group). Suppose that  $G$  acts on the affine building by simplicial automorphisms such that each isotropy group is compact. Then each isotropy group is compact open,  $\Sigma$  is a model for  $J_{\text{COMOP}}(G)$  and the barycentric subdivision  $\Sigma'$  is a model for both  $J_{\text{COMOP}}(G)$  and  $E_{\text{COMOP}}(G)$ . If we additionally assume that  $G$  is totally disconnected or is a Lie group, then  $\Sigma$  is a model for both  $\underline{J}G$  and  $\underline{E}G$ .*

**Example 4.14 (Bruhat-Tits building).** An important example is the case of a reductive  $p$ -adic algebraic group  $G$  and its associated affine Bruhat-Tits building  $\beta(G)$  [70],[71]. Then  $\beta(G)$  is a model for  $\underline{J}G$  and  $\beta(G)'$  is a model for  $\underline{E}G$  by Theorem 4.13.

## 4.7 The Rips Complex of a Word-Hyperbolic Group

Let  $G$  be a finitely generated discrete group. Let  $S$  be a finite set of generators. We will always assume that  $S$  is *symmetric*, i.e. that the identity element  $1 \in G$  does not belong to  $S$  and  $s \in S$  implies  $s^{-1} \in S$ . For  $g_1, g_2 \in G$  let  $d_S(g_1, g_2)$  be the minimal natural number  $n$  such that  $g_1^{-1}g_2$  can be written as a word  $s_1s_2 \dots s_n$ . This defines a left  $G$ -invariant metric on  $G$ , the so called *word metric*.

A metric space  $X = (X, d)$  is called  $\delta$ -*hyperbolic* for a given real number  $\delta \geq 0$  if for any four points  $x, y, z, t$  the following inequality holds

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\} + 2\delta. \quad (4.15)$$

A group  $G$  with a finite symmetric set  $S$  of generators is called  $\delta$ -*hyperbolic* if the metric space  $(G, d_S)$  is  $\delta$ -hyperbolic.

The *Rips complex*  $P_d(G, S)$  of a group  $G$  with a symmetric finite set  $S$  of generators for a natural number  $d$  is the geometric realization of the simplicial set whose set of  $k$ -simplices consists of  $(k+1)$ -tuples  $(g_0, g_1, \dots, g_k)$  of pairwise distinct elements  $g_i \in G$  satisfying  $d_S(g_i, g_j) \leq d$  for all  $i, j \in \{0, 1, \dots, k\}$ . The obvious  $G$ -action by simplicial automorphisms on  $P_d(G, S)$  induces a  $G$ -action by simplicial automorphisms on the barycentric subdivision  $P_d(G, S)'$  (see Example 1.5). The following result is proved in [56], [57].

**Theorem 4.16 (Rips complex).** *Let  $G$  be a (discrete) group with a finite symmetric set of generators. Suppose that  $(G, S)$  is  $\delta$ -hyperbolic for the real number  $\delta \geq 0$ . Let  $d$  be a natural number with  $d \geq 16\delta + 8$ . Then the barycentric subdivision of the Rips complex  $P_d(G, S)'$  is a finite  $G$ -CW-model for  $\underline{E}G$ .*

A metric space is called *hyperbolic* if it is  $\delta$ -hyperbolic for some real number  $\delta \geq 0$ . A finitely generated group  $G$  is called *hyperbolic* if for one (and hence all) finite symmetric set  $S$  of generators the metric space  $(G, d_S)$  is a hyperbolic metric space. Since for metric spaces the property hyperbolic is invariant under quasiisometry and for two symmetric finite sets  $S_1$  and  $S_2$  of generators of  $G$  the metric spaces  $(G, d_{S_1})$  and  $(G, d_{S_2})$  are quasiisometric, the choice of  $S$  does not matter. Theorem 4.16 implies that for a hyperbolic group there is a finite  $G$ -CW-model for  $\underline{EG}$ .

The notion of a hyperbolic group is due to Gromov and has intensively been studied (see for example [13], [29], [30]). The prototype is the fundamental group of a closed hyperbolic manifold.

## 4.8 Arithmetic Groups

Arithmetic groups in a semisimple connected linear  $\mathbb{Q}$ -algebraic group possess finite models for  $\underline{EG}$ . Namely, let  $G(\mathbb{R})$  be the  $\mathbb{R}$ -points of a semisimple  $\mathbb{Q}$ -group  $G(\mathbb{Q})$  and let  $K \subseteq G(\mathbb{R})$  a maximal compact subgroup. If  $A \subseteq G(\mathbb{Q})$  is an arithmetic group, then  $G(\mathbb{R})/K$  with the left  $A$ -action is a model for  $E_{FLN}(A)$  as already explained in Theorem 4.4. The  $A$ -space  $G(\mathbb{R})/K$  is not necessarily cocompact. The Borel-Serre completion of  $G(\mathbb{R})/K$  (see [10], [64]) is a finite  $A$ -CW-model for  $E_{FLN}(A)$  as pointed out in [2, Remark 5.8], where a private communication with Borel and Prasad is mentioned.

## 4.9 Outer Automorphism Groups of Free groups

Let  $F_n$  be the free group of rank  $n$ . Denote by  $\text{Out}(F_n)$  the group of outer automorphisms of  $F_n$ , i.e. the quotient of the group of all automorphisms of  $F_n$  by the normal subgroup of inner automorphisms. Culler and Vogtmann [21], [76] have constructed a space  $X_n$  called *outer space* on which  $\text{Out}(F_n)$  acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Fix a graph  $R_n$  with one vertex  $v$  and  $n$ -edges and identify  $F_n$  with  $\pi_1(R_n, v)$ . A *marked metric graph*  $(g, \Gamma)$  consists of a graph  $\Gamma$  with all vertices of valence at least three, a homotopy equivalence  $g: R_n \rightarrow \Gamma$  called marking and to every edge of  $\Gamma$  there is assigned a positive length which makes  $\Gamma$  into a metric space by the path metric. We call two marked metric graphs  $(g, \Gamma)$  and  $(g', \Gamma')$  equivalent if there is a homothety  $h: \Gamma \rightarrow \Gamma'$  such that  $g \circ h$  and  $g'$  are homotopic. Homothety means that there is a constant  $\lambda > 0$  with  $d(h(x), h(y)) = \lambda \cdot d(x, y)$  for all  $x, y$ . Elements in outer space  $X_n$  are equivalence classes of marked graphs. The main result in [21] is that  $X$  is contractible. Actually, for each finite subgroup  $H \subseteq \text{Out}(F_n)$  the  $H$ -fixed point set  $X_n^H$  is contractible [43, Propostion 3.3 and Theorem 8.1], [78, Theorem 5.1].

The space  $X_n$  contains a *spine*  $K_n$  which is an  $\text{Out}(F_n)$ -equivariant deformation retraction. This space  $K_n$  is a simplicial complex of dimension  $(2n - 3)$  on which the  $\text{Out}(F_n)$ -action is by simplicial automorphisms and cocompact. Actually the group of simplicial automorphisms of  $K_n$  is  $\text{Out}(F_n)$  [14]. Hence the



barycentric subdivision  $K'_n$  is a finite  $(2n - 3)$ -dimensional model of  $\underline{E}\text{Out}(F_n)$ .

#### 4.10 Mapping Class groups

Let  $\Gamma_{g,r}^s$  be the *mapping class group* of an orientable compact surface  $F$  of genus  $g$  with  $s$  punctures and  $r$  boundary components. This is the group of isotopy classes of orientation preserving selfdiffeomorphisms  $F_g \rightarrow F_g$ , which preserve the punctures individually and restrict to the identity on the boundary. We require that the isotopies leave the boundary pointwise fixed. We will always assume that  $2g + s + r > 2$ , or, equivalently, that the Euler characteristic of the punctured surface  $F$  is negative. It is well-known that the associated *Teichmüller space*  $\mathcal{T}_{g,r}^s$  is a contractible space on which  $\Gamma_{g,r}^s$  acts properly. Actually  $\mathcal{T}_{g,r}^s$  is a model for  $E_{\mathcal{FIN}}(\Gamma_{g,r}^s)$  by the results of Kerckhoff [41].

We could not find a clear reference in the literature for the to experts known statement that there exist a finite  $\Gamma_{g,r}^s$ -*CW*-model for  $E_{\mathcal{FIN}}(\Gamma_{g,r}^s)$ . The work of Harer [32] on the existence of a spine and the construction of the spaces  $T_S(\epsilon)^H$  due to Ivanov [37, Theorem 5.4.A] seem to lead to such models.

#### 4.11 Groups with Appropriate Maximal Finite Subgroups

Let  $G$  be a discrete group. Let  $\mathcal{MFIN}$  be the subset of  $\mathcal{FIN}$  consisting of elements in  $\mathcal{FIN}$  which are maximal in  $\mathcal{FIN}$ . Consider the following assertions concerning  $G$ :

- (M) Every non-trivial finite subgroup of  $G$  is contained in a unique maximal finite subgroup;
- (NM)  $M \in \mathcal{MFIN}, M \neq \{1\} \Rightarrow N_G M = M$ ;

For such a group there is a nice model for  $\underline{E}G$  with as few non-free cells as possible. Let  $\{(M_i) \mid i \in I\}$  be the set of conjugacy classes of maximal finite subgroups of  $M_i \subseteq G$ . By attaching free  $G$ -cells we get an inclusion of  $G$ -*CW*-complexes  $j_1: \coprod_{i \in I} G \times_{M_i} EM_i \rightarrow EG$ , where  $EG$  is the same as  $E_{\mathcal{TR}}(G)$ , i.e. a contractible free  $G$ -*CW*-complex. Define  $\underline{E}G$  as the  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ u_1 \downarrow & & \downarrow f_1 \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & \underline{E}G \end{array} \quad (4.17)$$

where  $u_1$  is the obvious  $G$ -map obtained by collapsing each  $EM_i$  to a point.

We have to explain why  $\underline{E}G$  is a model for the classifying space for proper actions of  $G$ . Obviously it is a  $G$ -*CW*-complex. Its isotropy groups are all finite. We have to show for  $H \subseteq G$  finite that  $(\underline{E}G)^H$  contractible. We begin with the case  $H \neq \{1\}$ . Because of conditions (M) and (NM) there is precisely one index

$i_0 \in I$  such that  $H$  is subconjugated to  $M_{i_0}$  and is not subconjugated to  $M_i$  for  $i \neq i_0$  and we get

$$\left( \prod_{i \in I} G/M_i \right)^H = (G/M_{i_0})^H = \{\text{pt.}\}.$$

Hence  $\underline{E}G^H = \{\text{pt.}\}$ . It remains to treat  $H = \{1\}$ . Since  $u_1$  is a non-equivariant homotopy equivalence and  $j_1$  is a cofibration,  $f_1$  is a non-equivariant homotopy equivalence and hence  $\underline{E}G$  is contractible (after forgetting the group action).

Here are some examples of groups  $Q$  which satisfy conditions (M) and (NM):

- Extensions  $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow F \rightarrow 1$  for finite  $F$  such that the conjugation action of  $F$  on  $\mathbb{Z}^n$  is free outside  $0 \in \mathbb{Z}^n$ .

The conditions (M), (NM) are satisfied by [54, Lemma 6.3].

- Fuchsian groups  $F$

The conditions (M), (NM) are satisfied (see for instance [54, Lemma 4.5]).

In [54] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated.

- One-relator groups  $G$

Let  $G$  be a one-relator group. Let  $G = \langle (q_i)_{i \in I} \mid r \rangle$  be a presentation with one relation. We only have to consider the case, where  $G$  contains torsion. Let  $F$  be the free group with basis  $\{q_i \mid i \in I\}$ . Then  $r$  is an element in  $F$ . There exists an element  $s \in F$  and an integer  $m \geq 2$  such that  $r = s^m$ , the cyclic subgroup  $C$  generated by the class  $\bar{s} \in G$  represented by  $s$  has order  $m$ , any finite subgroup of  $G$  is subconjugated to  $C$  and for any  $g \in G$  the implication  $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$  holds. These claims follows from [55, Propositions 5.17, 5.18 and 5.19 in II.5 on pages 107 and 108]. Hence  $G$  satisfies (M) and (NM).

## 4.12 One-Relator Groups

Let  $G$  be a one-relator group. Let  $G = \langle (q_i)_{i \in I} \mid r \rangle$  be a presentation with one relation. There is up to conjugacy one maximal finite subgroup  $C$  which is cyclic. Let  $p: *_{i \in I} \mathbb{Z} \rightarrow G$  be the epimorphism from the free group generated by the set  $I$  to  $G$ , which sends the generator  $i \in I$  to  $q_i$ . Let  $Y \rightarrow \bigvee_{i \in I} S^1$  be the  $G$ -covering associated to the epimorphism  $p$ . There is a 1-dimensional unitary  $C$ -representation  $V$  and a  $C$ -map  $f: SV \rightarrow \text{res}_G^C Y$  such that the following is true. The induced action on the unit sphere  $SV$  is free. If we equip  $SV$  and  $DV$  with the obvious  $C$ -CW-complex structures, the  $C$ -map  $f$  can be chosen to be cellular and we obtain a  $G$ -CW-model for  $\underline{E}G$  by the  $G$ -pushout

$$\begin{array}{ccc} G \times_C SV & \xrightarrow{\bar{f}} & Y \\ \downarrow & & \downarrow \\ G \times_C DV & \longrightarrow & \underline{E}G \end{array}$$

where  $\bar{f}$  sends  $(g, x)$  to  $gf(x)$ . Thus we get a 2-dimensional  $G$ - $CW$ -model for  $\underline{E}G$  such that  $\underline{E}G$  is obtained from  $G/C$  for a maximal finite cyclic subgroup  $C \subseteq G$  by attaching free cells of dimensions  $\leq 2$  and the  $CW$ -complex structure on the quotient  $G \backslash \underline{E}G$  has precisely one 0-cell, precisely one 2-cell and as many 1-cells as there are elements in  $I$ . All these claims follow from [16, Exercise 2 (c) II. 5 on page 44].

If  $G$  is torsionfree, the 2-dimensional complex associated to a presentation with one relation is a model for  $BG$  (see also [55, Chapter III §§9 -11]).

### 4.13 Special Linear Groups of (2,2)-Matrices

In order to illustrate some of the general statements above we consider the special example  $SL_2(\mathbb{R})$  and  $SL_2(\mathbb{Z})$ .

Let  $\mathbb{H}^2$  be the 2-dimensional hyperbolic space. We will use either the upper half-plane model or the Poincaré disk model. The group  $SL_2(\mathbb{R})$  acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations, i.e. a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acts by sending a complex number  $z$  with positive imaginary part to  $\frac{az+b}{cz+d}$ . This action is proper and transitive. The isotropy group of  $z = i$  is  $SO(2)$ . Since  $\mathbb{H}^2$  is a simply-connected Riemannian manifold, whose sectional curvature is constant  $-1$ , the  $SL_2(\mathbb{R})$ -space  $\mathbb{H}^2$  is a model for  $\underline{E}SL_2(\mathbb{R})$  by Theorem 4.5.

One easily checks that  $SL_2(\mathbb{R})$  is a connected Lie group and  $SO(2) \subseteq SL_2(\mathbb{R})$  is a maximal compact subgroup. Hence  $SL_2(\mathbb{R})/SO(2)$  is a model for  $\underline{E}SL_2(\mathbb{R})$  by Theorem 4.3. Since the  $SL_2(\mathbb{R})$ -action on  $\mathbb{H}^2$  is transitive and  $SO(2)$  is the isotropy group at  $i \in \mathbb{H}^2$ , we see that the  $SL_2(\mathbb{R})$ -manifolds  $SL_2(\mathbb{R})/SO(2)$  and  $\mathbb{H}^2$  are  $SL_2(\mathbb{R})$ -diffeomorphic.

Since  $SL_2(\mathbb{Z})$  is a discrete subgroup of  $SL_2(\mathbb{R})$ , the space  $\mathbb{H}^2$  with the obvious  $SL_2(\mathbb{Z})$ -action is a model for  $\underline{E}SL_2(\mathbb{Z})$  (see Theorem 4.4).

The group  $SL_2(\mathbb{Z})$  is isomorphic to the amalgamated product  $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ . From Example 4.10 we conclude that a model for  $\underline{E}SL_2(\mathbb{Z})$  is given by the following  $SL_2(\mathbb{Z})$ -pushout

$$\begin{array}{ccc} SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times \{-1, 1\} & \xrightarrow{F_{-1} \amalg F_1} & SL_2(\mathbb{Z})/(\mathbb{Z}/4) \amalg SL_2(\mathbb{Z})/(\mathbb{Z}/6) \\ \downarrow & & \downarrow \\ SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times [-1, 1] & \longrightarrow & \underline{E}SL_2(\mathbb{Z}) \end{array}$$

where  $F_{-1}$  and  $F_1$  are the obvious projections. This model for  $\underline{E}SL_2(\mathbb{Z})$  is a tree, which has alternately two and three edges emanating from each vertex. The other model  $\mathbb{H}^2$  is a manifold. These two models must be  $SL_2(\mathbb{Z})$ -homotopy equivalent. They can explicitly be related by the following construction.

Divide the Poincaré disk into fundamental domains for the  $SL_2(\mathbb{Z})$ -action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e. a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk,

is a tree  $T$  with  $SL_2(\mathbb{Z})$ -action. This is the tree model above. The tree is a  $SL_2(\mathbb{Z})$ -equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point  $p$  in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing  $p$ , through  $p$  to the first intersection point of this geodesic with  $T$ .

The tree  $T$  above can be identified with the Bruhat-Tits building of  $SL_2(\widehat{\mathbb{Q}_p})$  and hence is a model for  $\underline{E}SL_2(\widehat{\mathbb{Q}_p})$  (see [17, page 134]). Since  $SL_2(\mathbb{Z})$  is a discrete subgroup of  $SL_2(\widehat{\mathbb{Q}_p})$ , we get another reason why this tree is a model for  $SL_2(\mathbb{Z})$ .

#### 4.14 Manifold Models

It is an interesting question, whether one can find a model for  $\underline{E}G$  which is a smooth  $G$ -manifold. One may also ask whether such a manifold model realizes the minimal dimension for  $\underline{E}G$  or whether the action is cocompact. Theorem 4.5 gives some information about these questions for simply connected non-positively curved Riemannian manifolds and Theorem 5.24 for discrete subgroups of Lie groups with finitely many path components. On the other hand there exists a virtually torsionfree group  $G$  such that  $G$  acts properly and cocompactly on a contractible manifold (without boundary), but there is no finite  $G$ -CW-model for  $\underline{E}G$  [23, Theorem 1.1].

## 5 Finiteness Conditions

In this section we investigate whether there are models for  $E_{\mathcal{F}}(G)$  which satisfy certain finiteness conditions such as being finite, being of finite type or being of finite dimension as a  $G$ -CW-complex.

### 5.1 Review of Finiteness Conditions on $BG$

As an illustration we review the corresponding question for  $EG$  for a discrete group  $G$ . This is equivalent to the question whether for a given discrete group  $G$  there is a CW-complex model for  $BG$  which is finite, of finite type or finite dimensional.

We introduce the following notation. Let  $R$  be a commutative associative ring with unit. The trivial  $RG$ -module is  $R$  viewed as  $RG$ -module by the trivial  $G$ -action. A *projective resolution* or *free resolution* respectively for an  $RG$ -module  $M$  is an  $RG$ -chain complex  $P_*$  of projective or free respectively  $RG$ -modules with  $P_i = 0$  for  $i \leq -1$  such that  $H_i(P_*) = 0$  for  $i \geq 1$  and  $H_0(P_*)$  is  $RG$ -isomorphic to  $M$ . If additionally each  $RG$ -module  $P_i$  is finitely generated and  $P_*$  is finite dimensional, we call  $P_*$  *finite*.

An  $RG$ -module  $M$  has *cohomological dimension*  $\text{cd}(M) \leq n$ , if there exists a projective resolution of dimension  $\leq n$  for  $M$ . This is equivalent to the condition that for any  $RG$ -module  $N$  we have  $\text{Ext}_{RG}^i(M, N) = 0$  for  $i \geq n + 1$ . A group  $G$  has *cohomological dimension*  $\text{cd}(G) \leq n$  over  $R$  if the trivial  $RG$ -module  $R$  has cohomological dimension  $\leq n$ . An  $RG$ -module  $M$  is of *type*  $FP_n$ , if it admits a projective  $RG$ -resolution  $P_*$  such that  $P_i$  is finitely generated for  $i \leq n$  and of type  $FP_\infty$  if it admits a projective  $RG$ -resolution  $P_*$  such that  $P_i$  is finitely generated for all  $i$ . A group  $G$  is of type  $FP_n$  or  $FP_\infty$  respectively if the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  is of type  $FP_n$  or  $FP_\infty$  respectively.

Here is a summary of well-known statements about finiteness conditions on  $BG$ . A key ingredient in the proof of the next result is the fact that the cellular  $RG$ -chain complex  $C_*(EG)$  is a free and in particular a projective  $RG$ -resolution of the trivial  $RG$ -module  $R$  since  $EG$  is a free  $G$ -CW-complex and contractible, and that  $C_*(EG)$  is  $n$ -dimensional or of type  $FP_n$  respectively if  $BG$  is  $n$ -dimensional or has finite  $n$ -skeleton respectively.

**Theorem 5.1 (Finiteness conditions for  $BG$ ).** *Let  $G$  be a discrete group.*

- (i) *If there exists a finite dimensional model for  $BG$ , then  $G$  is torsionfree;*
- (ii) (a) *There exists a CW-model for  $BG$  with finite 1-skeleton if and only if  $G$  is finitely generated;*  
 (b) *There exists a CW-model for  $BG$  with finite 2-skeleton if and only if  $G$  is finitely presented;*  
 (c) *For  $n \geq 3$  there exists a CW-model for  $BG$  with finite  $n$ -skeleton if and only if  $G$  is finitely presented and of type  $FP_n$ ;*  
 (d) *There exists a CW-model for  $BG$  of finite type, i.e. all skeleta are finite, if and only if  $G$  is finitely presented and of type  $FP_\infty$ ;*

- (e) *There exists groups  $G$  which are of type  $FP_2$  and which are not finitely presented;*
- (iii) *There is a finite CW-model for  $BG$  if and only if  $G$  is finitely presented and there is a finite free  $\mathbb{Z}G$ -resolution  $F_*$  for the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$ ;*
- (iv) *The following assertions are equivalent:*
  - (a) *The cohomological dimension of  $G$  is  $\leq 1$ ;*
  - (b) *There is a model for  $BG$  of dimension  $\leq 1$ ;*
  - (c)  *$G$  is free.*
- (v) *The following assertions are equivalent for  $d \geq 3$ :*
  - (a) *There exists a CW-model for  $BG$  of dimension  $\leq d$ ;*
  - (b)  *$G$  has cohomological dimension  $\leq d$  over  $\mathbb{Z}$ ;*
- (vi) *For Thompson's group  $F$  there is a CW-model of finite type for  $BG$  but no finite dimensional model for  $BG$ .*

*Proof.* (i) Suppose we can choose a finite dimensional model for  $BG$ . Let  $C \subseteq G$  be a finite cyclic subgroup. Then  $C \backslash \widetilde{BG} = C \backslash EG$  is a finite dimensional model for  $BC$ . Hence there is an integer  $d$  such that we have  $H_i(BC) = 0$  for  $i \geq d$ . This implies that  $C$  is trivial [16, (2.1) in II.3 on page 35]. Hence  $G$  is torsionfree.

(ii) See [9] and [16, Theorem 7.1 in VIII.7 on page 205].

(iii) See [16, Theorem 7.1 in VIII.7 on page 205].

(iv) See [67] and [69].

(v) See [16, Theorem 7.1 in VIII.7 on page 205].

(vi) See [18]. □

## 5.2 Modules over the Orbit Category

Let  $G$  be a discrete group and let  $\mathcal{F}$  be a family of subgroups. The *orbit category*  $\text{Or}(G)$  of  $G$  is the small category, whose objects are homogeneous  $G$ -spaces  $G/H$  and whose morphisms are  $G$ -maps. Let  $\text{Or}_{\mathcal{F}}(G)$  be the full subcategory of  $\text{Or}(G)$  consisting of those objects  $G/H$  for which  $H$  belongs to  $\mathcal{F}$ . A  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -*module* is a contravariant functor from  $\text{Or}_{\mathcal{F}}(G)$  to the category of  $\mathbb{Z}$ -modules. A morphism of such modules is a natural transformation. The category of  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -modules inherits the structure of an abelian category from the standard structure of an abelian category on the category of  $\mathbb{Z}$ -modules. In particular the notion of a projective  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module is defined. The *free*  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\mathbb{Z} \text{map}(G/?, G/K)$  based at the object  $G/K$  is the  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module that assigns to an object  $G/H$  the free  $\mathbb{Z}$ -module  $\mathbb{Z} \text{map}_G(G/H, G/K)$

generated by the set  $\text{map}_G(G/H, G/K)$ . The key property of it is that for any  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $N$  there is a natural bijection of  $\mathbb{Z}$ -modules

$$\text{hom}_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)}(\mathbb{Z}\text{map}_G(G/?, G/K), N) \xrightarrow{\cong} N(G/K), \quad \phi \mapsto \phi(G/K)(\text{id}_{G/K}).$$

This is a direct consequence of the Yoneda Lemma. A  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module is *free* if it is isomorphic to a direct sum  $\bigoplus_{i \in I} \mathbb{Z}\text{map}(G/?, G/K_i)$  for appropriate choice of objects  $G/K_i$  and index set  $I$ . A  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module is called *finitely generated* if it is a quotient of a  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module of the shape  $\bigoplus_{i \in I} \mathbb{Z}\text{map}(G/?, G/K_i)$  with a finite index set  $I$ . Notice that a lot of standard facts for  $\mathbb{Z}$ -modules carry over to  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -modules. For instance, a  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module is projective or finitely generated projective respectively if and only if it is a direct summand in a free  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module or a finitely generated free  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module respectively. The notion of a *projective resolution*  $P_*$  of a  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module is obvious and notions like of cohomological dimension  $\leq n$  or of type  $FP_\infty$  carry directly over. Each  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module has a projective resolution. The trivial  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\underline{\mathbb{Z}}$  is the constant functor from  $\text{Or}_{\mathcal{F}}(G)$  to the category of  $\mathbb{Z}$ -modules, which sends any morphism to  $\text{id}: \mathbb{Z} \rightarrow \mathbb{Z}$ . More information about modules over a category can be found for instance in [47, Section 9].

The next result is proved in [50, Theorem 0.1]. A key ingredient in the proof of the next result is the fact that the cellular  $R\text{Or}_{\mathcal{F}}(G)$ -chain complex  $C_*(E_{\mathcal{F}}(G))$  is a free and in particular a projective  $R\text{Or}_{\mathcal{F}}(G)$ -resolution of the trivial  $R\text{Or}_{\mathcal{F}}(G)$ -module  $R$ .

**Theorem 5.2 (Algebraic and geometric finiteness conditions).** *Let  $G$  be a discrete group and let  $d \geq 3$ . Then we have:*

- (i) *There is  $G$ -CW-model of dimension  $\leq d$  for  $E_{\mathcal{F}}(G)$  if and only if the trivial  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\underline{\mathbb{Z}}$  has cohomological dimension  $\leq d$ ;*
- (ii) *There is a  $G$ -CW-model for  $E_{\mathcal{F}}(G)$  of finite type if and only if  $E_{\mathcal{F}}(G)$  has a  $G$ -CW-model with finite 2-skeleton and the trivial  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\underline{\mathbb{Z}}$  is of type  $FP_\infty$ ;*
- (iii) *There is a finite  $G$ -CW-model for  $E_{\mathcal{F}}(G)$  if and only if  $E_{\mathcal{F}}(G)$  has a  $G$ -CW-model with finite 2-skeleton and the trivial  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\underline{\mathbb{Z}}$  has a finite free resolution over  $\text{Or}_{\mathcal{F}}(G)$ ;*
- (iv) *There is a  $G$ -CW-model with finite 2-skeleton for  $\underline{EG} = E_{\mathcal{F}\mathbb{I}\mathbb{N}}(G)$  if and only if there are only finitely many conjugacy classes of finite subgroups  $H \subset G$  and for any finite subgroup  $H \subset G$  its Weyl group  $W_G H := N_G H / H$  is finitely presented.*

In the case, where we take  $\mathcal{F}$  to be the trivial family, Theorem 5.2 (i) reduces to Theorem 5.1 (v), Theorem 5.2 (ii) to Theorem 5.1 (ii)d and Theorem 5.2 (iii) to Theorem 5.1 (iii), and one should compare Theorem 5.2 (iv) to Theorem 5.1 (ii)b.

**Remark 5.3.** Nucinkis [63] investigates the notion of  $\mathcal{FIN}$ -cohomological dimension and relates it to the question whether there are finite-dimensional modules for  $\underline{E}G$ . It gives another lower bound for the dimension of a model for  $\underline{E}G$  but is not sharp in general [11].

### 5.3 Reduction from Topological Groups to Discrete Groups

The *discretization*  $G_d$  of a topological group  $G$  is the same group but now with the discrete topology. Given a family  $\mathcal{F}$  of (closed) subgroups of  $G$ , denote by  $\mathcal{F}_d$  the same set of subgroups, but now in connection with  $G_d$ . Notice that  $\mathcal{F}_d$  is again a family. We will need the following condition

- (S) For any closed subgroup  $H \subset G$  the projection  $p: G \rightarrow G/H$  has a local cross section, i.e. there is a neighborhood  $U$  of  $eH$  together with a map  $s: U \rightarrow G$  satisfying  $p \circ s = \text{id}_U$ .

Condition (S) is automatically satisfied if  $G$  is discrete, if  $G$  is a Lie group, or more generally, if  $G$  is locally compact and second countable and has finite covering dimension [61]. The metric needed in [61] follows under our assumptions, since a locally compact Hausdorff space is regular and regularity in a second countable space implies metrizability.

The following two results are proved in [50, Theorem 0.2 and Theorem 0.3].

**Theorem 5.4 (Passage from totally disconnected groups to discrete groups).** *Let  $G$  be a locally compact totally disconnected Hausdorff group and let  $\mathcal{F}$  be a family of subgroups of  $G$ . Then there is a  $G$ -CW-model for  $E_{\mathcal{F}}(G)$  that is  $d$ -dimensional or finite or of finite type respectively if and only if there is a  $G_d$ -CW-model for  $E_{\mathcal{F}_d}(G_d)$  that is  $d$ -dimensional or finite or of finite type respectively.*

**Theorem 5.5 (Passage from topological groups to totally disconnected groups).** *Let  $G$  be a locally compact Hausdorff group satisfying condition (S). Put  $\overline{G} := G/G^0$ . Then there is a  $G$ -CW-model for  $\underline{E}G$  that is  $d$ -dimensional or finite or of finite type respectively if and only if  $\underline{E}\overline{G}$  has a  $\overline{G}$ -CW-model that is  $d$ -dimensional or finite or of finite type respectively.*

If we combine Theorem 5.2, Theorem 5.4 and Theorem 5.5 we get

**Theorem 5.6 (Passage from topological groups to discrete groups).** *Let  $G$  be a locally compact group satisfying (S). Denote by  $\overline{\mathcal{COM}}$  the family of compact subgroups of its component group  $\overline{G}$  and let  $d \geq 3$ . Then*

- (i) *There is a  $d$ -dimensional  $G$ -CW-model for  $\underline{E}G$  if and only if the trivial  $\mathbb{Z}\text{Or}_{\overline{\mathcal{COM}}_d}(\overline{G}_d)$ -module  $\underline{\mathbb{Z}}$  has cohomological dimension  $\leq d$ ;*
- (ii) *There is a  $G$ -CW-model for  $\underline{G}$  of finite type if and only if  $E_{\overline{\mathcal{COM}}_d}(\overline{G}_d)$  has a  $\overline{G}_d$ -CW-model with finite 2-skeleton and the trivial  $\mathbb{Z}\text{Or}_{\overline{\mathcal{COM}}_d}(\overline{G}_d)$ -module  $\underline{\mathbb{Z}}$  is of type  $FP_{\infty}$ ;*



(iii) *There is a finite  $G$ -CW-model for  $\underline{E}G$  if and only if  $E_{\overline{\mathcal{COM}}_d}(\overline{G}_d)$  has a  $\overline{G}_d$ -CW-model with finite 2-skeleton and the trivial  $\mathbb{Z}\text{Or}_{\overline{\mathcal{COM}}_d}(\overline{G}_d)$ -module  $\underline{\mathbb{Z}}$  has a finite free resolution.*

In particular we see from Theorem 5.5 that, for a Lie group  $G$ , type questions about  $\underline{E}G$  are equivalent to the corresponding type questions of  $\underline{E}\pi_0(G)$ , since  $\pi_0(G) = \overline{G}$  is discrete. In this case the family  $\overline{\mathcal{COM}}_d$  appearing in Theorem 5.6. is just the family  $\mathcal{FLN}$  of finite subgroups of  $\pi_0(G)$ .

## 5.4 Poset of Finite Subgroups

Throughout this Subsection 5.4 let  $G$  be a discrete group. Define the  $G$ -poset

$$\mathcal{P}(G) := \{K \mid K \subset G \text{ finite}, K \neq 1\}. \quad (5.7)$$

An element  $g \in G$  sends  $K$  to  $gKg^{-1}$  and the poset-structure comes from inclusion of subgroups. Denote by  $|\mathcal{P}(G)|$  the geometric realization of the category given by the poset  $\mathcal{P}(G)$ . This is a  $G$ -CW-complex but in general not proper, i.e. it can have points with infinite isotropy groups.

Let  $N_G H$  be the *normalizer* and let  $W_G H := N_G H/H$  be the *Weyl group* of  $H \subset G$ . Notice for a  $G$ -space  $X$  that  $X^H$  inherits a  $W_G H$ -action. Denote by  $CX$  the *cone* over  $X$ . Notice that  $C\emptyset$  is the one-point-space.

If  $H$  and  $K$  are subgroups of  $G$  and  $H$  is finite, then  $G/K^H$  is a finite union of  $W_G H$ -orbits of the shape  $W_G H/L$  for finite  $L \subset W_G H$ . Now one easily checks

**Lemma 5.8.** *The  $W_G H$ -space  $\underline{E}G^H$  is a  $W_G H$ -CW-model for  $\underline{E}W_G H$ . In particular, if  $\underline{E}G$  has a  $G$ -CW-model which is finite, of finite type or  $d$ -dimensional respectively, then there is a  $W_G H$ -model for  $\underline{E}W_G H$  which is finite, of finite type or  $d$ -dimensional respectively.*

**Notation 5.9 (The condition  $b(d)$  and  $B(d)$ ).** *Let  $d \geq 0$  be an integer. A group  $G$  satisfies the condition  $b(d)$  or  $b(< \infty)$  respectively if any  $\mathbb{Z}G$ -module  $M$  with the property that  $M$  restricted to  $\mathbb{Z}K$  is projective for all finite subgroups  $K \subset G$  has a projective  $\mathbb{Z}G$ -resolution of dimension  $d$  or of finite dimension respectively. A group  $G$  satisfies the condition  $B(d)$  if  $W_G H$  satisfies the condition  $b(d)$  for any finite subgroup  $H \subset G$ .*

*The length  $l(H) \in \{0, 1, \dots\}$  of a finite group  $H$  is the supremum over all  $p$  for which there is a nested sequence  $H_0 \subset H_1 \subset \dots \subset H_p$  of subgroups  $H_i$  of  $H$  with  $H_i \neq H_{i+1}$ .*

**Lemma 5.10.** *Suppose that there is a  $d$ -dimensional  $G$ -CW-complex  $X$  with finite isotropy groups such that  $H_p(X; \mathbb{Z}) = H_p(*, \mathbb{Z})$  for all  $p \geq 0$  holds. This assumption is for instance satisfied if there is a  $d$ -dimensional  $G$ -CW-model for  $\underline{E}G$ . Then  $G$  satisfies condition  $B(d)$ .*

*Proof.* Let  $H \subset G$  be finite. Then  $X/H$  satisfies  $H_p(X/H; \mathbb{Z}) = H_p(*, \mathbb{Z})$  for all  $p \geq 0$  [12, III.5.4 on page 131]. Let  $C_*$  be the cellular  $\mathbb{Z}W_G H$ -chain complex of  $X/H$ . This is a  $d$ -dimensional resolution of the trivial  $\mathbb{Z}W_G H$ -module  $\mathbb{Z}$  and each chain module is a sum of  $\mathbb{Z}W_G H$ -modules of the shape  $\mathbb{Z}[W_G H/K]$  for some finite subgroup  $K \subset W_G H$ . Let  $N$  be a  $\mathbb{Z}W_G H$ -module such that  $N$  is projective over  $\mathbb{Z}K$  for any finite subgroup  $K \subset W_G H$ . Then  $C_* \otimes_{\mathbb{Z}} N$  with the diagonal  $W_G H$ -operation is a  $d$ -dimensional projective  $\mathbb{Z}W_G H$ -resolution of  $N$ .  $\square$

**Theorem 5.11 (An algebraic criterion for finite-dimensionality).** *Let  $G$  be a discrete group. Suppose that we have for any finite subgroup  $H \subset G$  an integer  $d(H) \geq 3$  such that  $d(H) \geq d(K)$  for  $H \subset K$  and  $d(H) = d(K)$  if  $H$  and  $K$  are conjugate in  $G$ . Consider the following statements:*

(i) *There is a  $G$ -CW-model  $\underline{E}G$  such that for any finite subgroup  $H \subset G$*

$$\dim(\underline{E}G^H) = d(H);$$

(ii) *We have for any finite subgroup  $H \subset G$  and for any  $\mathbb{Z}W_G H$ -module  $M$*

$$H_{\mathbb{Z}W_G H}^{d(H)+1}(EW_G H \times (C|\mathcal{P}(W_G H)|, |\mathcal{P}(W_G H)|); M) = 0;$$

(iii) *We have for any finite subgroup  $H \subset G$  that its Weyl group  $W_G H$  satisfies  $b(< \infty)$  and that there is a subgroup  $\Delta(H) \subset W_G H$  of finite index such that for any  $\mathbb{Z}\Delta(H)$ -module  $M$*

$$H_{\mathbb{Z}\Delta(H)}^{d(H)+1}(E\Delta(H) \times (C|\mathcal{P}(W_G H)|, |\mathcal{P}(W_G H)|); M) = 0.$$

*Then (i) implies both (ii) and (iii). If there is an upper bound on the length  $l(H)$  of the finite subgroups  $H$  of  $G$ , then these statements (i), (ii) and (iii) are equivalent.*

The proof of Theorem 5.11 can be found in [48, Theorem 1.6]. In the case that  $G$  has finite virtual cohomological dimension a similar result is proved in [20, Theorem III].

**Example 5.12.** Suppose that  $G$  is torsionfree. Then Theorem 5.11 reduces to the well-known result [16, Theorem VIII.3.1 on page 190, Theorem VIII.7.1 on page 205] that the following assertions are equivalent for an integer  $d \geq 3$ :

- (i) There is a  $d$ -dimensional CW-model for  $BG$ ;
- (ii)  $G$  has cohomological dimension  $\leq d$ ;
- (iii)  $G$  has virtual cohomological dimension  $\leq d$ .

**Remark 5.13.** If  $W_G H$  contains a non-trivial normal finite subgroup  $L$ , then  $|\mathcal{P}(W_G H)|$  is contractible and

$$\begin{aligned} H_{\mathbb{Z}W_G H}^{d(H)+1}(EW_G H \times (C|\mathcal{P}(W_G H)|, |\mathcal{P}(W_G H)|); M) &= 0; \\ H_{\mathbb{Z}\Delta(H)}^{d(H)+1}(E\Delta(H) \times (C|\mathcal{P}(W_G H)|, |\mathcal{P}(W_G H)|); M) &= 0. \end{aligned}$$

The proof of this fact is given in [48, Example 1.8].

The next result is taken from [48, Theorem 1.10]. A weaker version of it for certain classes of groups and in  $l$  exponential dimension estimate can be found in [42, Theorem B] (see [48, Remark 1.12]).

**Theorem 5.14 (An upper bound on the dimension).** *Let  $G$  be a group and let  $l \geq 0$  and  $d \geq 0$  be integers such that the length  $l(H)$  of any finite subgroup  $H \subset G$  is bounded by  $l$  and  $G$  satisfies  $B(d)$ . Then there is a  $G$ -CW-model for  $\underline{EG}$  such that for any finite subgroup  $H \subset G$*

$$\dim(\underline{EG}^H) \leq \max\{3, d\} + (l - l(H))(d + 1)$$

*holds. In particular  $\underline{EG}$  has dimension at most  $\max\{3, d\} + l(d + 1)$ .*

## 5.5 Extensions of Groups

In this subsection we consider an exact sequence of discrete groups  $1 \rightarrow \Delta \rightarrow G \rightarrow \pi \rightarrow 1$ . We want to investigate whether finiteness conditions about the type of a classifying space for  $\mathcal{FLN}$  for  $\Delta$  and  $\pi$  carry over to the one of  $G$ . The proof of the next Theorem 5.15 is taken from [48, Theorem 3.1]), the proof of Theorem 5.16 is an easy variation.

**Theorem 5.15 (Dimension bounds and extensions).** *Suppose that there exists a positive integer  $d$  which is an upper bound on the orders of finite subgroups of  $\pi$ . Suppose that  $\underline{E}\Delta$  has a  $k$ -dimensional  $\Delta$ -CW-model and  $\underline{E}\pi$  has a  $m$ -dimensional  $\pi$ -CW-model. Then  $\underline{EG}$  has a  $(dk + m)$ -dimensional  $G$ -CW-model.*

**Theorem 5.16.** *Suppose that  $\Delta$  has the property that for any group  $\Gamma$  which contains  $\Delta$  as subgroup of finite index, there is a  $k$ -dimensional  $\Gamma$ -CW-model for  $\underline{E}\Gamma$ . Suppose that  $\underline{E}\pi$  has a  $m$ -dimensional  $\pi$ -CW-model. Then  $\underline{EG}$  has a  $(k + m)$ -dimensional  $G$ -CW-model.*

We will see in Example 5.26 that the condition about  $\Delta$  in Theorem 5.16 is automatically satisfied if  $\Delta$  is virtually poly-cyclic.

The next two results are taken from [48, Theorem 3.2 and Theorem 3.3]).

**Theorem 5.17.** *Suppose for any finite subgroup  $\pi' \subset \pi$  and any extension  $1 \rightarrow \Delta \rightarrow \Delta' \rightarrow \pi' \rightarrow 1$  that  $\underline{E}\Delta'$  has a finite  $\Delta'$ -CW-model or a  $\Delta'$ -CW-model of finite type respectively and suppose that  $\underline{E}\pi$  has a finite  $\pi$ -CW-model or a  $\pi$ -CW-model of finite type respectively. Then  $\underline{EG}$  has a finite  $G$ -CW-model or a  $G$ -CW-model of finite type respectively.*

**Theorem 5.18.** *Suppose that  $\Delta$  is word-hyperbolic or virtually poly-cyclic. Suppose that  $\underline{E}\pi$  has a finite  $\pi$ -CW-model or a  $\pi$ -CW-model of finite type respectively. Then  $\underline{E}G$  has a finite  $G$ -CW-model or a  $G$ -CW-model of finite type respectively.*

## 5.6 One-Dimensional Models for $\underline{E}G$

The following result follows from Dunwoody [27, Theorem 1.1].

**Theorem 5.19 (A criterion for 1-dimensional models).** *Let  $G$  be a discrete group. Then there exists a 1-dimensional model for  $\underline{E}G$  if and only if the cohomological dimension of  $G$  over the rationals  $\mathbb{Q}$  is less or equal to one.*

If  $G$  is finitely generated, then there is a 1-dimensional model for  $\underline{E}G$  if and only if  $G$  contains a finitely generated free subgroup of finite index [40, Theorem 1]. If  $G$  is torsionfree, we rediscover the results due to Swan and Stallings stated in Theorem 5.1 (iv) from Theorem 5.19.

## 5.7 Groups of Finite Virtual Dimension

In this section we investigate the condition  $b(d)$  and  $B(d)$  of Notation 5.9 for a discrete group  $G$  and explain how our results specialize in the case of a group of finite virtual cohomological dimension.

**Remark 5.20.** There exists groups  $G$  with a finite dimensional model for  $\underline{E}G$ , which do not admit a torsionfree subgroup of finite index. For instance, let  $G$  be a countable locally finite group which is not finite. Then its cohomological dimension over the rationals is  $\leq 1$  and hence it possesses a 1-dimensional model for  $\underline{E}G$  by Theorem 5.19. Obviously it contains no torsionfree subgroup of finite index. An example of a group  $G$  with a finite 2-dimensional model for  $\underline{E}G$ , which does not admit a torsionfree subgroup of finite index, is described in [11, page 493].

A discrete group  $G$  has *virtual cohomological dimension*  $\leq d$  if and only if it contains a torsionfree subgroup  $\Delta$  of finite index such that  $\Delta$  has cohomological dimension  $\leq d$ . This is independent of the choice of  $\Delta \subseteq G$  because for two torsionfree subgroups  $\Delta, \Delta' \subseteq G$  we have that  $\Delta$  has cohomological dimension  $\leq d$  if and only if  $\Delta'$  has cohomological dimension  $\leq d$ . The next two results are taken from [48, Lemma 6.1, Theorem 6.3, Theorem 6.4].

**Lemma 5.21.** *If  $G$  satisfies  $b(d)$  or  $B(d)$  respectively, then any subgroup  $\Delta$  of  $G$  satisfies  $b(d)$  or  $B(d)$  respectively.*

**Theorem 5.22 (Virtual cohomological dimension and the condition  $B(d)$ ).** *If  $G$  contains a torsionfree subgroup  $\Delta$  of finite index, then the following assertions are equivalent:*

- (i)  $G$  satisfies  $B(d)$ ;

(ii)  $G$  satisfies  $b(d)$ ;

(iii)  $G$  has virtual cohomological dimension  $\leq d$ .

Next we improve Theorem 5.14 in the case of groups with finite virtual cohomological dimension. Notice that for such a group there is an upper bound on the length  $l(H)$  of finite subgroups  $H \subset G$ .

**Theorem 5.23 (Virtual cohomological dimension and  $\dim(\underline{EG})$ ).** *Let  $G$  be a discrete group which contains a torsionfree subgroup of finite index and has virtual cohomological dimension  $\text{vcd}(G) \leq d$ . Let  $l \geq 0$  be an integer such that the length  $l(H)$  of any finite subgroup  $H \subset G$  is bounded by  $l$ .*

*Then we have  $\text{vcd}(G) \leq \dim(\underline{EG})$  for any model for  $\underline{EG}$  and there is a  $G$ -CW-model for  $\underline{EG}$  such that for any finite subgroup  $H \subset G$*

$$\dim(\underline{EG}^H) = \max\{3, d\} + l - l(H)$$

*holds. In particular there exists a model for  $\underline{EG}$  of dimension  $\max\{3, d\} + l$ .*

**Theorem 5.24 (Discrete subgroups of Lie groups).** *Let  $L$  be a Lie group with finitely many path components. Then  $L$  contains a maximal compact subgroup  $K$  which is unique up to conjugation. Let  $G \subseteq L$  be a discrete subgroup of  $L$ . Then  $L/K$  with the left  $G$ -action is a model for  $\underline{EG}$ .*

*Suppose additionally that  $G$  contains a torsionfree subgroup  $\Delta \subseteq G$  of finite index. Then we have*

$$\text{vcd}(G) \leq \dim(L/K)$$

*and equality holds if and only if  $G \backslash L$  is compact.*

*Proof.* We have already mentioned in Theorem 4.4 that  $L/K$  is a model for  $\underline{EG}$ . The restriction of  $\underline{EG}$  to  $\Delta$  is a  $\Delta$ -CW-model for  $\underline{E\Delta}$  and hence  $\Delta \backslash \underline{EG}$  is a CW-model for  $B\Delta$ . This implies  $\text{vcd}(G) := \text{cd}(\Delta) \leq \dim(L/K)$ . Obviously  $\Delta \backslash L/K$  is a manifold without boundary. Suppose that  $\Delta \backslash L/K$  is compact. Then  $\Delta \backslash L/K$  is a closed manifold and hence its homology with  $\mathbb{Z}/2$ -coefficients in the top dimension is non-trivial. This implies  $\text{cd}(\Delta) \geq \dim(\Delta \backslash L/K)$  and hence  $\text{vcd}(G) = \dim(L/K)$ . If  $\Delta \backslash L/K$  is not compact, it contains a CW-complex  $X \subseteq \Delta \backslash L/K$  of dimension smaller than  $\Delta \backslash L/K$  such that the inclusion of  $X$  into  $\Delta \backslash L/K$  is a homotopy equivalence. Hence  $X$  is another model for  $B\Delta$ . This implies  $\text{cd}(\Delta) < \dim(L/K)$  and hence  $\text{vcd}(G) < \dim(L/K)$ .  $\square$

**Remark 5.25.** An often useful strategy to find smaller models for  $E_{\mathcal{F}}(G)$  is to look for a  $G$ -CW-subcomplex  $X \subseteq E_{\mathcal{F}}(G)$  such that there exists a  $G$ -retraction  $r: E_{\mathcal{F}}(G) \rightarrow X$ , i.e. a  $G$ -map  $r$  with  $r|_X = \text{id}_X$ . Then  $X$  is automatically another model for  $E_{\mathcal{F}}(G)$ . We have seen this already in the case  $SL_2(\mathbb{Z})$ , where we found a tree inside  $\mathbb{H}^2 = SL_2(\mathbb{R})/SO(2)$  as explained in Subsection 4.13. This method can be used to construct a model for  $\underline{ESL}_n(\mathbb{Z})$  of dimension  $\frac{n(n-1)}{2}$  and to show that the virtual cohomological dimension of  $SL_n(\mathbb{Z})$  is  $\frac{n(n-1)}{2}$ . Notice that  $SL_n(\mathbb{R})/SO(n)$  is also a model for  $\underline{ESL}_n(\mathbb{Z})$  by Theorem 4.4 but has dimension  $\frac{n(n+1)}{2} - 1$ .

**Example 5.26 (Virtually poly-cyclic groups).** Let the group  $\Delta$  be *virtually poly-cyclic*, i.e.  $\Delta$  contains a subgroup  $\Delta'$  of finite index for which there is a finite sequence  $\{1\} = \Delta'_0 \subseteq \Delta'_1 \subseteq \dots \subseteq \Delta'_n = \Delta'$  of subgroups such that  $\Delta'_{i-1}$  is normal in  $\Delta'_i$  with cyclic quotient  $\Delta'_i/\Delta'_{i-1}$  for  $i = 1, 2, \dots, n$ . Denote by  $r$  the number of elements  $i \in \{1, 2, \dots, n\}$  with  $\Delta'_i/\Delta'_{i-1} \cong \mathbb{Z}$ . The number  $r$  is called the *Hirsch rank*. The group  $\Delta$  contains a torsionfree subgroup of finite index. We call  $\Delta'$  *poly- $\mathbb{Z}$*  if  $r = n$ , i.e. all quotients  $\Delta'_i/\Delta'_{i-1}$  are infinite cyclic. We want to show:

- (i)  $r = \text{vcd}(\Delta)$ ;
- (ii)  $r = \max\{i \mid H_i(\Delta'; \mathbb{Z}/2) \neq 0\}$  for one (and hence all) poly- $\mathbb{Z}$  subgroup  $\Delta' \subset \Delta$  of finite index;
- (iii) There exists a finite  $r$ -dimensional model for  $\underline{E}\Delta$  and for any model  $\underline{E}\Delta$  we have  $\dim(\underline{E}\Delta) \geq r$ .

We use induction over the number  $r$ . If  $r = 0$ , then  $\Delta$  is finite and all the claims are obviously true. Next we explain the induction step from  $(r - 1)$  to  $r \geq 1$ . We can choose an extension  $1 \rightarrow \Delta_0 \rightarrow \Delta \xrightarrow{p} V \rightarrow 1$  for some virtually poly-cyclic group  $\Delta_0$  with  $r(\Delta_0) = r(\Delta) - 1$  and some group  $V$  which contains  $\mathbb{Z}$  as subgroup of finite index. The induction hypothesis applies to any group  $\Gamma$  which contains  $\Delta_0$  as subgroup of finite index. Since  $V$  maps surjectively to  $\mathbb{Z}$  or the infinite dihedral group  $D_\infty$  with finite kernel and both  $\mathbb{Z}$  and  $D_\infty$  have 1-dimensional models for their classifying space for proper group actions, there is a 1-dimensional model for  $\underline{E}V$ . We conclude from Theorem 5.16 that there is a  $r$ -dimensional model for  $\underline{E}\Delta$ .

The existence of a  $r$ -dimensional model for  $\underline{E}\Delta$  implies  $\text{vcd}(\Delta) \leq r$ .

For any torsionfree subgroup  $\Delta' \subset \Delta$  of finite index we have  $\max\{i \mid H_i(\Delta'; \mathbb{Z}/2) \neq 0\} \leq \text{vcd}(\Delta)$ ,

It is not hard to check by induction over  $r$  that we can find a sequence of torsionfree subgroups  $\{1\} \subseteq \Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_r \subseteq \Delta$  such that  $\Delta_{i-1}$  is normal in  $\Delta_i$  with  $\Delta_i/\Delta_{i-1} \cong \mathbb{Z}$  for  $i \in \{1, 2, \dots, r\}$  and  $\Delta_r$  has finite index in  $\Delta$ . We show by induction over  $i$  that  $H_i(\Delta_i; \mathbb{Z}/2) = \mathbb{Z}/2$  for  $i = 0, 1, \dots, r$ . The induction beginning  $i = 0$  is trivial. The induction step from  $(i - 1)$  to  $i$  follows from the part of the long exact Wang sequence

$$\begin{aligned} H_i(\Delta_{i-1}; \mathbb{Z}/2) = 0 \rightarrow H_i(\Delta_i; \mathbb{Z}/2) \rightarrow H_{i-1}(\Delta_{i-1}; \mathbb{Z}/2) = \mathbb{Z}/2 \\ \xrightarrow{\text{id} - H_{i-1}(f; \mathbb{Z}/2) = 0} H_{i-1}(\Delta_{i-1}; \mathbb{Z}/2) \end{aligned}$$

which comes from the Hochschild-Serre spectral sequence associated to the extension  $1 \rightarrow \Delta_{i-1} \rightarrow \Delta_i \rightarrow \mathbb{Z} \rightarrow 1$  for  $f: \Delta_{i-1} \rightarrow \Delta_{i-1}$  the automorphism induced by conjugation with some preimage in  $\Delta_i$  of the generator of  $\mathbb{Z}$ . This implies

$$r = \max\{i \mid H_i(\Delta_r; \mathbb{Z}/2) \neq 0\} = \text{cd}(\Delta_r) = \text{vcd}(\Delta).$$

Now the claim follows.

The existence of a  $r$ -dimensional model for  $\underline{E}G$  is proved for finitely generated nilpotent groups with  $\text{vcd}(G) \leq r$  for  $r \neq 2$  in [62], where also not necessarily finitely generated nilpotent groups are studied.

The work of Dekimpe-Igout [24] or Wilking [80, Theorem 3] implies that there is a model for  $E_{\mathcal{FIN}}(\Delta)$  whose underlying space is  $\mathbb{R}^r$ .

## 5.8 Counterexamples

The following problem is stated by Brown [15, page 32]. It created a lot of activities and many of the results stated above were motivated by it.

**Problem 5.27.** *For which discrete groups  $G$ , which contain a torsionfree subgroup of finite index and has virtual cohomological dimension  $\leq d$ , does there exist a  $d$ -dimensional  $G$ -CW-model for  $\underline{E}G$ ?*

The following four problems for discrete groups  $G$  are stated in the problem lists appearing in [48] and [77].

**Problem 5.28.** *Let  $H \subseteq G$  be a subgroup of finite index. Suppose that  $\underline{E}H$  has a  $H$ -CW-model of finite type or a finite  $H$ -CW-model respectively. Does then  $\underline{E}G$  have a  $G$ -CW-model of finite type or a finite  $G$ -CW-model respectively?*

**Problem 5.29.** *If the group  $G$  contains a subgroup of finite index  $H$  which has a  $H$ -CW-model of finite type for  $\underline{E}H$ , does then  $G$  contain only finitely many conjugacy classes of finite subgroups?*

**Problem 5.30.** *Let  $G$  be a group such that  $BG$  has a model of finite type. Is then  $BW_G H$  of finite type for any finite subgroup  $H \subset G$ ?*

**Problem 5.31.** *Let  $1 \rightarrow \Delta \xrightarrow{i} G \xrightarrow{p} \pi \rightarrow 1$  be an exact sequence of groups. Suppose that there is a  $\Delta$ -CW-model of finite type for  $\underline{E}\Delta$  and a  $G$ -CW-model of finite type for  $\underline{E}G$ . Is then there a  $\pi$ -CW-model of finite type for  $\underline{E}\pi$ ?*

Leary and Nucinkis [46] have constructed many very interesting examples of discrete groups some of which are listed below. Their main technical input is an equivariant version of the constructions due to Bestvina and Brady [9]. These examples show that the answer to the Problems 5.27, 5.28, 5.29, 5.30 and 5.31 above is *not* positive in general. A group  $G$  is *of type VF* if it contains a subgroup  $H \subseteq G$  of finite index for which there is a finite model for  $BH$ .

- (i) For any positive integer  $d$  there exist a group  $G$  of type *VF* which has virtually cohomological dimension  $\leq 3d$ , but for which any model for  $\underline{E}G$  has dimension  $\geq 4d$ ;
- (ii) There exists a group  $G$  with a finite cyclic subgroup  $H \subseteq G$  such that  $G$  is of type *VF* but the centralizer  $C_G H$  of  $H$  in  $G$  is not of type *FP* $_{\infty}$ ;

- (iii) There exists a group  $G$  of type  $VF$  which contains infinitely many conjugacy classes of finite subgroups;
- (iv) There exists an extension  $1 \rightarrow \Delta \rightarrow G \rightarrow \pi \rightarrow 1$  such that  $\underline{E}\Delta$  and  $\underline{E}G$  have finite  $G$ -CW-models but there is no  $G$ -CW-model for  $\underline{E}\pi$  of finite type.



## 6 The Orbit Space of $\underline{EG}$

We will see that in many computations of the group (co-)homology, of the algebraic  $K$ - and  $L$ -theory of the group ring or the topological  $K$ -theory of the reduced  $C^*$ -algebra of a discrete group  $G$  a key problem is to determine the homotopy type of the quotient space  $G \backslash \underline{EG}$  of  $\underline{EG}$ . The following result shows that this is a difficult problem in general and can only be solved in special cases. It was proved by Leary and Nucinkis [45] based on ideas due to Baumslag-Dyer-Heller [8] and Kan and Thurston [39].

**Theorem 6.1 (The homotopy type of  $G \backslash \underline{EG}$ ).** *Let  $X$  be a connected CW-complex. Then there exists a group  $G$  such that  $G \backslash \underline{EG}$  is homotopy equivalent to  $X$ .*

There are some cases, where the quotient  $G \backslash \underline{EG}$  has been determined explicitly using geometric input. We mention a few examples.

- (i) Let  $G$  be a planar group (sometimes also called NEC) group, i.e. a discontinuous group of isometries of the two-sphere  $S^2$ , the Euclidean plane  $\mathbb{R}^2$ , or the hyperbolic plane  $\mathbb{H}^2$ . Examples are Fuchsian groups and two-dimensional crystallographic groups. If  $G$  acts on  $\mathbb{R}^2$  or  $\mathbb{H}^2$  and the action is cocompact, then  $\mathbb{R}^2$  or  $\mathbb{H}^2$  is a model for  $\underline{EG}$  and the quotient space  $G \backslash \underline{EG}$  is a compact 2-dimensional surface. The number of boundary components, its genus and the answer to the question, whether  $G \backslash \underline{EG}$  is orientable, can be read off from an explicit presentation of  $G$ . A summary of these details can be found in [54, Section 4], where further references to papers containing proofs of the stated facts are given;
- (ii) Let  $G = \langle (q_i)_{i \in I} \mid r \rangle$  be a one-relator group. Let  $F$  be the free group on the letters  $\{q_i \mid i \in I\}$ . Then  $r$  is an element in  $F$ . There exists an element  $s \in F$  and an integer  $m \geq 1$  such that  $r = s^m$ , the cyclic subgroup  $C$  generated by the class  $\bar{s} \in G$  represented by  $s$  has order  $m$ , any finite subgroup of  $G$  is subconjugated to  $C$  and for any  $g \in G$  the implication  $g^{-1}Cg \cap C \neq \{1\} \Rightarrow g \in C$  holds (see [55, Propositions 5.17, 5.18 and 5.19 in II.5 on pages 107 and 108]).

In the sequel we use the two-dimensional model for  $\underline{EG}$  described in Subsection 4.12. Let us compute the integral homology of  $BG$  and  $G \backslash \underline{EG}$ . Since  $G \backslash \underline{EG}$  has precisely one 2-cell and is two-dimensional,  $H_2(G \backslash \underline{EG})$  is either trivial or infinite cyclic and  $H_k(G \backslash \underline{EG}) = 0$  for  $k \geq 3$ . We obtain the short exact sequence

$$0 \rightarrow H_2(BG) \xrightarrow{H_2(q)} H_2(G \backslash \underline{EG}) \xrightarrow{\partial_2} H_1(BC) \xrightarrow{H_1(Bi)} H_1(BG) \xrightarrow{H_1(q)} H_1(G \backslash \underline{EG}) \rightarrow 0$$

and for  $k \geq 3$  isomorphisms

$$H_k(Bi): H_k(BC) \xrightarrow{\cong} H_k(BG)$$

from the pushout coming from (4.17)

$$\begin{array}{ccc} BC & \xrightarrow{i} & BG \\ \downarrow & & \downarrow \\ \{\text{pt.}\} & \longrightarrow & G \backslash \underline{EG} \end{array}$$

Hence  $H_2(G \backslash \underline{EG}) = 0$  and the sequence

$$0 \rightarrow H_1(BC) \xrightarrow{H_1(Bi)} H_1(BG) \xrightarrow{H_1(q)} H_1(G \backslash \underline{EG}) \rightarrow 0$$

is exact, provided that  $H_2(BG) = 0$ . Suppose that  $H_2(BG) \neq 0$ . Hopf's Theorem says that  $H_2(BG) \cong R \cap [F, F]/[F, R]$  if  $R$  is the subgroup of  $G$  normally generated by  $r \in F$  (see [16, Theorem 5.3 in II.5 on page 42]). For every element in  $R \cap [F, F]/[F, R]$  there exists  $n \in \mathbb{Z}$  such that  $r^n$  belongs to  $[F, F]$  and the element is represented by  $r^n$ . Hence there is  $n \geq 1$  such that  $r^n$  does belong to  $[F, F]$ . Since  $F/[F, F]$  is torsion-free, also  $s$  and  $r$  belong to  $[F, F]$ . We conclude that both  $H_2(BG)$  and  $H_2(G \backslash \underline{EG})$  are infinite cyclic groups,  $H_1(BC) \rightarrow H_1(BG)$  is trivial and  $H_1(q): H_1(BG) \xrightarrow{\cong} H_1(G \backslash \underline{EG})$  is bijective. We also see that  $H_2(BG) = 0$  if and only if  $r$  does not belong to  $[F, F]$ .

- (iii) Let  $\text{Hei}$  be the three-dimensional discrete Heisenberg group which is the subgroup of  $GL_3(\mathbb{Z})$  consisting of upper triangular matrices with 1 on the diagonals. Consider the  $\mathbb{Z}/4$ -action given by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -z & y - xz \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Then a key result in [49] is that  $G \backslash \underline{EG}$  is homeomorphic to  $S^3$  for  $G = \text{Hei} \rtimes \mathbb{Z}/4$ ;

- (iv) A key result in [66, Corollary on page 8] implies that for  $G = SL_3(\mathbb{Z})$  the quotient space  $G \backslash \underline{EG}$  is contractible.

## 7 Relevance and Applications of Classifying Spaces for Families

In this section we discuss some theoretical aspects which involve and rely on the notion of a classifying space for a family of subgroups.

### 7.1 Baum-Connes Conjecture

Let  $G$  be a locally compact second countable Hausdorff group. Using the equivariant  $KK$ -theory due to Kasparov one can assign to a  $\mathcal{COM}$ -numerable  $G$ -space  $X$  its equivariant  $K$ -theory  $K_n^G(X)$ . Let  $C_r^*(G)$  be the reduced group  $C^*$ -algebra associated to  $G$ . The goal of the Baum-Connes Conjecture is to compute the topological  $K$ -theory  $K_p(C_r^*(G))$ . The following formulation is taken from [7, Conjecture 3.15].

**Conjecture 7.1 (Baum-Connes Conjecture).** *The assembly map defined by taking the equivariant index*

$$\text{asmb}: K_n^G(\underline{J}G) \xrightarrow{\cong} K_n(C_r^*(G))$$

*is bijective for all  $n \in \mathbb{Z}$ .*

More information about this conjecture and its relation and application to other conjectures and problems can be found for instance in [7], [34], [53], [59], [75].

### 7.2 Farrell-Jones Conjecture

Let  $G$  be a discrete group. Let  $R$  be a associative ring with unit. One can construct a  $G$ -homology theory  $\mathcal{H}_*^G(X; \mathbf{K})$  graded over the integers and defined for  $G$ - $CW$ -complexes  $X$  such that for any subgroup  $H \subseteq G$  the abelian group  $\mathcal{H}_n^G(G/H; \mathbf{K})$  is isomorphic to the algebraic  $K$ -groups  $K_n(RH)$  for  $n \in \mathbb{Z}$ . If  $R$  comes with an involution of rings, one can also construct a  $G$ -homology theory  $\mathcal{H}_*^G(X; \mathbf{L}^{(-\infty)})$  graded over the integers and defined for  $G$ - $CW$ -complexes  $X$  such that for any subgroup  $H \subseteq G$  the abelian group  $\mathcal{H}_n^G(G/H; \mathbf{L}^{(-\infty)})$  is isomorphic to the algebraic  $L$ -groups  $L_n^{-\infty}(RH)$  for  $n \in \mathbb{Z}$ . Let  $\mathcal{VCX}$  be the family of virtually cyclic subgroups of  $G$ . The goal of the Farrell-Jones Conjecture is to compute the algebraic  $K$ -groups  $K_n(RH)$  and the algebraic  $L$ -groups. The following formulation is equivalent to the original one appearing in [28, 1.6 on page 257].

**Conjecture 7.2 (Farrell-Jones Conjecture).** *The assembly maps induced by the projection  $E_{\mathcal{VCX}}(G) \rightarrow G/G$*

$$\text{asmb}: \mathcal{H}_n^G(E_{\mathcal{VCX}}(G), \mathbf{K}) \rightarrow \mathcal{H}_n^G(G/G, \mathbf{K}) = K_n(RG); \quad (7.3)$$

$$\text{asmb}: \mathcal{H}_n^G(E_{\mathcal{VCX}}(G), \mathbf{L}^{-\infty}) \rightarrow \mathcal{H}_n^G(G/G, \mathbf{L}^{-\infty}) = L_n^{-\infty}(RG), \quad (7.4)$$

*are bijective for all  $n \in \mathbb{Z}$ .*

More information about this conjecture and its relation and application to other conjectures and problems can be found for instance in [28] and [53].

We mention that for a discrete group  $G$  one can formulate the Baum-Connes Conjecture in a similar fashion. Namely, one can also construct a  $G$ -homology theory  $\mathcal{H}_*^G(X; \mathbf{K}^{\text{top}})$  graded over the integers and defined for  $G$ - $CW$ -complexes  $X$  such that for any subgroup  $H \subseteq G$  the abelian group  $\mathcal{H}_n^G(G/H; \mathbf{K}^{\text{top}})$  is isomorphic to the topological  $K$ -groups  $K_n(C_r^*(H))$  for  $n \in \mathbb{Z}$  and the assembly map appearing in the Baum-Connes Conjecture can be identified with the map induced by the projection  $\underline{J}G = \underline{E}G \rightarrow G/G$  (see [22], [31]). If the ring  $R$  is regular and contains  $\mathbb{Q}$  as subring, then one can replace in the Farrell-Jones Conjecture 7.2  $E_{\mathcal{V}\mathcal{X}\mathcal{Y}}(G)$  by  $\underline{E}G$  but this is not possible for arbitrary rings such as  $R = \mathbb{Z}$ . This comes from the appearance of Nil-terms in the Bass-Heller-Swan decomposition which do not occur in the context of the topological  $K$ -theory of reduced  $C^*$ -algebras.

Both the Baum-Connes Conjecture 7.1 and the Farrell-Jones Conjecture 7.2 allow to reduce the computation of certain  $K$ - and  $L$ -groups of the group ring or the reduced  $C^*$ -algebra of a group  $G$  to the computation of certain  $G$ -homology theories applied to  $\underline{J}G$ ,  $\underline{E}G$  or  $E_{\mathcal{V}\mathcal{X}\mathcal{Y}}(G)$ . Hence it is important to find good models for these spaces or to make predictions about their dimension or whether they are finite or of finite type.

### 7.3 Completion Theorem

Let  $G$  be a discrete group. For a proper finite  $G$ - $CW$ -complex let  $K_G^*(X)$  be its equivariant  $K$ -theory defined in terms of equivariant finite-dimensional complex vector bundles over  $X$  (see [52, Theorem 3.2]). It is a  $G$ -cohomology theory with a multiplicative structure. Assume that  $\underline{E}G$  has a finite  $G$ - $CW$ -model. Let  $I \subseteq K_G^0(\underline{E}G)$  be the augmentation ideal, i.e. the kernel of the map  $K^0(\underline{E}G) \rightarrow \mathbb{Z}$  sending the class of an equivariant complex vector bundle to its complex dimension. Let  $K_G^*(\underline{E}G)\hat{I}$  be the  $I$ -adic completion of  $K_G^*(\underline{E}G)$  and let  $K^*(BG)$  be the topological  $K$ -theory of  $BG$ .

**Theorem 7.5 (Completion Theorem for discrete groups).** *Let  $G$  be a discrete group such that there exists a finite model for  $\underline{E}G$ . Then there is a canonical isomorphism*

$$K^*(BG) \xrightarrow{\cong} K_G^*(\underline{E}G)\hat{I}.$$

This result is proved in [52, Theorem 4.4], where a more general statement is given provided that there is a finite-dimensional model for  $\underline{E}G$  and an upper bound on the orders of finite subgroups of  $G$ . In the case where  $G$  is finite, Theorem 7.5 reduces to the Completion Theorem due to Atiyah and Segal [3], [4].

### 7.4 Classifying Spaces for Equivariant Bundles

In [51] the equivariant  $K$ -theory for finite proper  $G$ - $CW$ -complexes appearing in Subsection 7.3 above is extended to arbitrary proper  $G$ - $CW$ -complexes (in-

cluding the multiplicative structure) using  $\Gamma$ -spaces in the sense of Segal and involving classifying spaces for equivariant vector bundles. These classifying spaces for equivariant vector bundles are again classifying spaces of certain Lie groups and certain families (see [73, Section 8 and 9 in Chapter I], [52, Lemma 2.4]).

## 7.5 Equivariant Homology and Cohomology

Classifying spaces for families play a role in computations of equivariant homology and cohomology for compact Lie groups such as equivariant bordism as explained in [72, Chapter 7], [73, Chapter III].

## 8 Computations using Classifying Spaces for Families

In this section we discuss some computations which involve and rely on the notion of a classifying space for a family of subgroups. These computations are possible since one understands in the cases of interest the geometry of  $\underline{EG}$  and  $G \backslash \underline{EG}$ . We focus on the case described in Subsection 4.11, namely of a discrete group  $G$  satisfying the conditions (M) and (NM). Let  $s: EG \rightarrow \underline{EG}$  be the up to  $G$ -homotopy unique  $G$ -map. Denote by  $j_i: M_i \rightarrow G$  the inclusion.

### 8.1 Group Homology

We begin with the group homology  $H_n(BG)$  (with integer coefficients). Let  $\tilde{H}_p(X)$  be the reduced homology, i.e. the kernel of the map  $H_n(X) \rightarrow H_n(\{\text{pt.}\})$  induced by the projection  $X \rightarrow \{\text{pt.}\}$ . The Mayer-Vietoris sequence applied to the pushout, which is obtained from the  $G$ -pushout (4.17) by dividing out the  $G$ -action, yields the long Mayer-Vietoris sequence

$$\begin{aligned} \dots \rightarrow H_{p+1}(G \backslash \underline{EG}) \xrightarrow{\partial_{p+1}} \bigoplus_{i \in I} \tilde{H}_p(BM_i) \xrightarrow{\bigoplus_{i \in I} H_p(Bj_i)} H_p(BG) \\ \xrightarrow{H_p(G \backslash s)} H_p(G \backslash \underline{EG}) \xrightarrow{\partial_p} \dots \end{aligned} \quad (8.1)$$

In particular we obtain an isomorphism for  $p \geq \dim(\underline{EG}) + 2$

$$\bigoplus_{i \in I} H_p(Bj_i): \bigoplus_{i \in I} \tilde{H}_p(BM_i) \xrightarrow{\cong} H_p(BG). \quad (8.2)$$

This example and the forthcoming ones show why it is important to get upper bounds on the dimension of  $\underline{EG}$  and to understand the quotient space  $G \backslash \underline{EG}$ . For Fuchsian groups and for one-relator groups we have  $\dim(G \backslash \underline{EG}) \leq 2$  and it is easy to compute the homology of  $G \backslash \underline{EG}$  in this case as explained in Section 6.

### 8.2 Topological $K$ -Theory of Group $C^*$ -Algebras

Analogously one can compute the source of the assembly map appearing in the Baum-Connes Conjecture 7.1. Namely, the Mayer-Vietoris sequence associated to the  $G$ -pushout (4.17) and the one associated to its quotient under the  $G$ -action look like

$$\begin{aligned} \dots \rightarrow K_{p+1}^G(\underline{EG}) \rightarrow \bigoplus_{i \in I} K_p^G(G \times_{M_i} EM_i) \\ \rightarrow \left( \bigoplus_{i \in I} K_p^G(G/M_i) \right) \bigoplus K_p^G(EG) \rightarrow K_p^G(\underline{EG}) \rightarrow \dots \end{aligned} \quad (8.3)$$

and

$$\begin{aligned} \dots \rightarrow K_{p+1}(G \backslash \underline{EG}) &\rightarrow \bigoplus_{i \in I} K_p(BM_i) \\ &\rightarrow \left( \bigoplus_{i \in I} K_p(\{\text{pt.}\}) \right) \bigoplus K_p(BG) \rightarrow \bigoplus_{i \in I} K_p(G \backslash \underline{EG}) \rightarrow \dots \end{aligned} \quad (8.4)$$

Notice that for a free  $G$ - $CW$ -complex  $X$  there is a canonical isomorphism  $K_p^G(X) \cong K_p(G \backslash X)$ . We can splice these sequences together and obtain the long exact sequence

$$\begin{aligned} \dots \rightarrow K_{p+1}(G \backslash \underline{EG}) &\rightarrow \bigoplus_{i \in I} K_p^G(G/M_i) \rightarrow \bigoplus_{i \in I} K_p(\{\text{pt.}\}) \bigoplus K_p^G(\underline{EG}) \\ &\rightarrow K_p(G \backslash \underline{EG}) \rightarrow \dots \end{aligned} \quad (8.5)$$

There are identification of  $K_0^G(G/M_i)$  with the complex representation ring  $R_{\mathbb{C}}(M_i)$  of the finite group  $M_i$  and of  $K_0(\{\text{pt.}\})$  with  $\mathbb{Z}$ . Under these identification the map  $K_0^G(G/M_i) \rightarrow K_0(\{\text{pt.}\})$  becomes the split surjective map  $\epsilon: R_{\mathbb{C}}(M_i) \rightarrow \mathbb{Z}$  which sends the class of a complex  $M_i$ -representation  $V$  to the complex dimension of  $\mathbb{C} \otimes_{\mathbb{C}[M_i]} V$ . The kernel of this map is denoted by  $\tilde{R}_{\mathbb{C}}(M_i)$ . The groups  $K_1^G(G/M_i)$  and  $K_1(\{\text{pt.}\})$  vanish. The abelian group  $R_{\mathbb{C}}(M_i)$  and hence also  $\tilde{R}_{\mathbb{C}}(M_i)$  are finitely generated free abelian groups. If  $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$  is ring such that the order of any finite subgroup of  $G$  is invertible in  $\Lambda$ , then the map

$$\Lambda \otimes_{\mathbb{Z}} K_p^G(s): \Lambda \otimes_{\mathbb{Z}} K_p^G(EG) \rightarrow \Lambda \otimes_{\mathbb{Z}} K_p^G(G \backslash \underline{EG})$$

is an isomorphism for all  $p \in \mathbb{Z}$  [54, Lemma 2.8 (a)]. Hence we conclude from the long exact sequence (8.5)

**Theorem 8.6.** *Let  $G$  be a discrete group which satisfies the conditions (M) and (NM) appearing in Subsection 4.11. Suppose that the Baum-Connes Conjecture 7.1 is true for  $G$ . Let  $\{(M_i) \mid i \in I\}$  be the set of conjugacy classes of maximal finite subgroups of  $G$ . Then there is an isomorphism*

$$K_1(C_r^*(G)) \xrightarrow{\cong} K_1(G \backslash \underline{EG})$$

and a short exact sequence

$$0 \rightarrow \bigoplus_{i \in I} \tilde{R}_{\mathbb{C}}(M_i) \rightarrow K_1(C_r^*(G)) \rightarrow K_1(G \backslash \underline{EG}) \rightarrow 0,$$

which splits if we invert the orders of all finite subgroups of  $G$ .

### 8.3 Algebraic $K$ -and $L$ -Theory of Group Rings

Suppose that  $G$  satisfies the Farrell-Jones Conjecture 7.2. Then the computation of the relevant groups  $K_n(RG)$  or  $L_n^{<-\infty}(RG)$  respectively is equivalent to the computation of  $\mathcal{H}_n^G(E_{\mathcal{X}\mathcal{Y}\mathcal{X}}(G), \mathbf{K})$  or  $\mathcal{H}_n^G(E_{\mathcal{X}\mathcal{Y}\mathcal{X}}(G), \mathbf{L}^{-\infty})$  respectively. The following result is due to Bartels [5]. Recall that  $\underline{EG}$  is the same as  $E_{\mathcal{FIN}}(G)$ .

**Theorem 8.7.** (i) For every group  $G$ , every ring  $R$  and every  $n \in \mathbb{Z}$  the up to  $G$ -homotopy unique  $G$ -map  $f: E_{\mathcal{F}LN}(G) \rightarrow E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G)$  induces a split injection

$$H_n^G(f; \mathbf{K}_R): H_n^G(E_{\mathcal{F}LN}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G); \mathbf{K}_R);$$

(ii) Suppose  $R$  is such that  $K_{-i}(RV) = 0$  for all virtually cyclic subgroups  $V$  of  $G$  and for sufficiently large  $i$  (for example  $R = \mathbb{Z}$  will do). Then we get a split injection

$$H_n^G(f; \mathbf{L}_R^{\langle -\infty \rangle}): H_n^G(E_{\mathcal{F}LN}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G); \mathbf{L}_R^{\langle -\infty \rangle}).$$

It remains to compute  $H_n^G(E_{\mathcal{F}LN}(G); \mathbf{K})$  and  $H_n^G(E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G), E_{\mathcal{F}LN}(G); \mathbf{K})$ , if we arrange  $f$  to be a  $G$ -cofibration and think of  $E_{\mathcal{F}LN}(G)$  as a  $G$ -CW-subcomplex of  $E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G)$ . Namely, we get from the Farrell-Jones Conjecture 7.2 and Theorem 8.7 an isomorphism

$$H_n^G(E_{\mathcal{F}LN}(G); \mathbf{K}) \bigoplus H_n^G(E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G), E_{\mathcal{F}LN}(G); \mathbf{K}) \xrightarrow{\cong} K_n(RG).$$

The analogous statement holds for  $\mathbf{L}_R^{\langle -\infty \rangle}$ , provided  $R$  satisfies the conditions appearing in Theorem 8.7 (ii).

Analogously to Theorem 8.6 one obtains

**Theorem 8.8.** Let  $G$  be a discrete group which satisfies the conditions (M) and (NM) appearing in Subsection 4.11. Let  $\{(M_i) \mid i \in I\}$  be the set of conjugacy classes of maximal finite subgroups of  $G$ . Then

(i) There is a long exact sequence

$$\begin{aligned} \dots \rightarrow H_{p+1}(G \setminus E_{\mathcal{F}LN}(G); \mathbf{K}(R)) &\rightarrow \bigoplus_{i \in I} K_p(R[M_i]) \\ \rightarrow \bigoplus_{i \in I} K_p(R) \bigoplus H_p^G(E_{\mathcal{F}LN}(G); \mathbf{K}_R) &\rightarrow \bigoplus H_p(G \setminus E_{\mathcal{F}LN}(G); \mathbf{K}(R)) \rightarrow \dots \end{aligned}$$

and analogously for  $\mathbf{L}_R^{\langle -\infty \rangle}$ .

(ii) For  $R = \mathbb{Z}$  there are isomorphisms

$$\bigoplus_{i \in I} \text{Wh}_n(M_i) \bigoplus H_n^G(E_{\mathcal{F}LN}(G), E_{\mathcal{V}\mathcal{X}\mathcal{X}}(G); \mathbf{K}_{\mathbb{Z}}) \xrightarrow{\cong} \text{Wh}_n(G).$$

**Remark 8.9.** These results about groups satisfying conditions (M) and (NM) are extended in [49] to groups which map surjectively to groups satisfying conditions (M) and (NM) with special focus on the semi-direct product of the discrete three-dimensional Heisenberg group with  $\mathbb{Z}/4$ .



**Remark 8.10.** In [66] a special model for  $\underline{E}SL_3(\mathbb{Z})$  is presented which allows to compute the integral group homology. Information about the algebraic  $K$ -theory of  $SL_3(\mathbb{Z})$  can be found in [68, Chapter 7], [74].

The analysis of the other term  $H_n^G(E_{\mathcal{W}\mathcal{X}\mathcal{X}}(G), E_{\mathcal{F}LN}(G); \mathbf{K})$  simplifies considerably under certain assumptions on  $G$ .

**Theorem 8.11 (On the structure of  $E_{\mathcal{W}\mathcal{X}\mathcal{X}}(G)$ ).** *Suppose that  $G$  satisfies the following conditions:*

- *Every infinite cyclic subgroup  $C \subseteq G$  has finite index in its centralizer  $C_G C$ ;*
- *There is an upper bound on the orders of finite subgroups.*

*(Each word-hyperbolic group satisfies these two conditions.) Then*

- (i) *For an infinite virtually cyclic subgroup  $V \subseteq G$  define*

$$V_{\max} = \bigcup \{N_G C \mid C \subset V \text{ infinite cyclic normal}\}.$$

*Then*

- (a)  $V_{\max}$  *is an infinite virtually cyclic subgroup of  $G$  and contains  $V$ ;*  
(b) *If  $V \subseteq W \subseteq G$  are infinite virtually cyclic subgroups of  $G$ , then  $V_{\max} = W_{\max}$ ;*  
(c) *Each infinite virtually cyclic subgroup  $V$  is contained in a unique maximal infinite virtually cyclic subgroup, namely  $V_{\max}$ , and  $N_G V_{\max} = V_{\max}$ ;*
- (ii) *Let  $\{V_i \mid i \in I\}$  be a complete system of representatives of conjugacy classes of maximal infinite virtually cyclic subgroups. Then there exists a  $G$ -pushout*

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{V_i} E_{\mathcal{F}LN}(V_i) & \longrightarrow & E_{\mathcal{F}LN}(G) \\ \text{pr} \downarrow & & \downarrow \\ \coprod_{i \in I} G/V_i & \longrightarrow & E_{\mathcal{W}\mathcal{X}\mathcal{X}}(G) \end{array}$$

*whose upper horizontal arrow is an inclusion of  $G$ -CW-complexes.*

- (iii) *There are natural isomorphisms*

$$\begin{aligned} \bigoplus_{i \in I} H_n^{V_i}(E_{\mathcal{W}\mathcal{X}\mathcal{X}}(V_i), E_{\mathcal{F}LN}(V_i); \mathbf{K}_R) &\xrightarrow{\cong} H_n^G(E_{\mathcal{W}\mathcal{X}\mathcal{X}}(G), E_{\mathcal{F}LN}(G); \mathbf{K}_R) \\ \bigoplus_{i \in I} H_n^{V_i}(E_{\mathcal{W}\mathcal{X}\mathcal{X}}(V_i), E_{\mathcal{F}LN}(V_i); \mathbf{L}_R^{\langle -\infty \rangle}) &\xrightarrow{\cong} H_n^G(E_{\mathcal{W}\mathcal{X}\mathcal{X}}(G), E_{\mathcal{F}LN}(G); \mathbf{L}_R^{\langle -\infty \rangle}). \end{aligned}$$

*Proof.* Each word-hyperbolic group  $G$  satisfies these two conditions by [13, Theorem 3.2 in III.Γ.3 on page 459 and Corollary 3.10 in III.Γ.3 on page 462].

(i) Let  $V$  be an infinite virtually cyclic subgroup  $V \subseteq G$ . Fix a normal infinite cyclic subgroup  $C \subseteq V$ . Let  $b$  be a common multiple of the orders of finite subgroups of  $G$ . Put  $d := b \cdot b!$ . Let  $e$  be the index of the infinite cyclic group  $dC = \{d \cdot x \mid x \in C\}$  in its centralizer  $C_G dC$ . Let  $D \subset dC$  be any non-trivial subgroup. Obviously  $dC \subseteq C_G D$ . We want to show

$$[C_G D : dC] \leq b \cdot e^2. \quad (8.12)$$

Since  $D$  is central in  $C_G D$  and  $C_G D$  is virtually cyclic and hence  $|C_G D/D| < \infty$ , the spectral sequence associated to the extension  $1 \rightarrow D \rightarrow C_G D \rightarrow C_G D/D \rightarrow 1$  implies that the map  $D = H_1(D) \rightarrow H_1(C_G D)$  is injective and has finite cokernel. In particular the quotient of  $H_1(C_G D)$  by its torsion subgroup  $H_1(C_G D)/\text{tors}$  is an infinite cyclic group. Let  $p_{C_G D}: C_G D \rightarrow H_1(C_G D)/\text{tors}$  be the canonical epimorphism. Its kernel is a finite normal subgroup. The following diagram commutes and has exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker(p_C) & \longrightarrow & C_G C & \xrightarrow{p_C} & H_1(C_G C)/\text{tors} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \ker(p_D) & \longrightarrow & C_G D & \xrightarrow{p_D} & H_1(C_G D)/\text{tors} & \longrightarrow & 1 \end{array}$$

where the vertical maps are induced by the inclusions  $C_G C \subseteq C_G D$ . All vertical maps are injections with finite cokernel. Fix elements  $z_C \in C_G C$  and  $z_D \in C_G D$  such that  $p_C(z_C)$  and  $p_D(z_D)$  are generators. Choose  $l \in \mathbb{Z}$  such that  $p_C(z_C)$  is sent to  $l \cdot p_D(z_D)$ . Then there is  $k \in \ker(p_D)$  with  $z_C = k \cdot z_D^l$ . The order of  $\ker(p_D)$  divides  $b$  by assumption. If  $\phi: \ker(p_D) \rightarrow \ker(p_D)$  is any automorphism, then  $\phi^{b!} = \text{id}$ . This implies for any element  $k \in \ker(p_D)$  that

$$\prod_{i=0}^{d-1} \phi^i(k) = \left( \prod_{i=0}^{b!-1} \phi^i(k) \right)^b = 1.$$

Hence we get in  $C_G D$  if  $\phi$  is conjugation with  $z_D^l$

$$z_C^d = (k \cdot z_D^l)^d = \prod_{i=0}^{d-1} \phi^i(k) \cdot z_D^{dl} = z_D^{dl}.$$

Obviously  $z_D \in C_G dC$  since  $z_C^d = z_D^{dl}$  generates  $dC$ . Hence  $z_D^e$  lies in  $dC$  and we get  $z_D^e = z_C^{df}$  for some integer  $f$ . This implies  $z_D^e = z_D^{ldf}$  and hence that  $l$  divides  $e$ . We conclude that the cokernel of the map  $H_1(C_G C)/\text{tors} \rightarrow H_1(C_G D)/\text{tors}$  is bounded by  $e$ . Hence the index  $[C_G D : C_G C]$  is bounded by  $b \cdot e$  since the order of  $\ker(p_D)$  divides  $b$ . Since  $dC \subseteq C_G C \subseteq C_G dC \subseteq C_G D$  holds, equation 8.12 follows.

Next we show that there is a normal infinite cyclic subgroup  $C_0 \subseteq V$  such that  $V_{\max} = N_G C_0$  holds. If  $C'$  and  $C''$  are infinite cyclic normal subgroups of

$V$ , then both  $C_G C'$  and  $C_G C''$  are contained in  $C_G(C' \cap C'')$  and  $C' \cap C''$  is again an infinite cyclic normal subgroup. Hence there is a sequence of normal infinite cyclic subgroups of  $V$

$$dC \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

which yields a sequence  $C_G dC \subseteq C_G C_1 \subseteq C_G C_2 \subseteq \dots$  satisfying

$$\bigcup \{C_G C_n \mid n \geq 1\} = \bigcup \{C_G C \mid C \subset V \text{ infinite cyclic normal}\}.$$

Because of 8.12 there is an upper bound on  $[C_G C_n : C_G dC]$  which is independent of  $n$ . Hence there is an index  $n_0$  with

$$C_G C_{n_0} = \bigcup \{C_G C \mid C \subset V \text{ infinite cyclic normal}\}.$$

For any infinite cyclic subgroup  $C \subseteq G$  the index of  $C_G C$  in  $N_G C$  is 1 or 2. Hence there is an index  $n_1$  with

$$N_G C_{n_1} = \bigcup \{N_G C \mid C \subset V \text{ infinite cyclic normal}\}.$$

Thus we have shown the existence of a normal infinite cyclic subgroup  $C \subseteq V$  with  $V_{\max} = C$ . Now assertion (i)a follows.

We conclude assertion (i)b from the fact that for an inclusion of infinite virtually cyclic group  $V \subseteq W$  there exists a normal infinite cyclic subgroup  $C \subseteq W$  such that  $C \subseteq V$  holds. Assertion (i)c is now obviously true. This finishes the proof of assertion (i).

(ii) Construct a  $G$ -pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{V_i} E_{\mathcal{F}IN}(V_i) & \xrightarrow{j} & E_{\mathcal{F}IN}(G) \\ \text{pr} \downarrow & & \downarrow \\ \coprod_{i \in I} G/V_i & \longrightarrow & X \end{array}$$

with  $j$  an inclusion of  $G$ -CW-complexes. Obviously  $X$  is a  $G$ -CW-complex whose isotropy groups are virtually cyclic. It remains to prove for virtually cyclic  $H \subseteq G$  that  $X^H$  is contractible.

Given a  $V_i$ -space  $Y$  and a subgroup  $H \subseteq G$ , there is after a choice of a map of sets  $s: G/V_i \rightarrow G$ , whose composition with the projection  $G \rightarrow G/V_i$  is the identity, a  $G$ -homeomorphism

$$\coprod_{\substack{w \in G/V_i \\ s(w)^{-1} H s(w) \subseteq V_i}} Y^{s(w)^{-1} H s(w)} \xrightarrow{\cong} (G \times_{V_i} Y)^H, \quad (8.13)$$

which sends  $y \in Y^{s(w)^{-1} H s(w)}$  to  $(s(w), y)$ .

If  $H$  is infinite, the  $H$ -fixed point set of the upper right and upper left corner is empty and of the lower left corner is the one-point space because of assertion

(i)c and equation 8.13. Hence  $X^H$  is a point for an infinite virtually cyclic subgroup  $H \subseteq G$ .

If  $H$  is finite, one checks using equation 8.13 that the left vertical map induces a homotopy equivalence on the  $H$ -fixed point set. Since the upper horizontal arrow induces a cofibration on the  $H$ -fixed point set, the right vertical arrow induces a homotopy equivalence on the  $H$ -fixed point sets. Hence  $X^H$  is contractible for finite  $H \subseteq G$ . This shows that  $X$  is a model for  $E_{\mathcal{VC}\mathcal{C}}(G)$ .

(iii) follows from excision and the induction structure. This finishes the proof of Theorem 8.11.  $\square$

Theorem 8.11 has also been proved by Daniel Juan-Pineda and Ian Leary [38] under the stronger condition that every infinite subgroup of  $G$ , which is not virtually cyclic, contains a non-abelian free subgroup. The case, where  $G$  is the fundamental group of a closed Riemannian manifold with negative sectional curvature is treated in [6].

**Remark 8.14.** In Theorem 8.11 the terms  $H_n^{V_i}(E_{\mathcal{VC}\mathcal{C}}(V_i), E_{\mathcal{FLN}}(V_i); \mathbf{K}_R)$  and  $H_n^{V_i}(E_{\mathcal{VC}\mathcal{C}}(V_i), E_{\mathcal{FLN}}(V_i); \mathbf{L}_R^{\langle -\infty \rangle})$  occur. They also appear in the direct sum decomposition

$$\begin{aligned} K_n(RV_i) &\cong H_n^{V_i}(E_{\mathcal{FLN}}(V_i); \mathbf{K}_R) \bigoplus H_n^{V_i}(E_{\mathcal{VC}\mathcal{C}}(V_i), E_{\mathcal{FLN}}(V_i); \mathbf{K}_R); \\ L_n(RV_i) &\cong H_n^{V_i}(E_{\mathcal{FLN}}(V_i); \mathbf{L}_R^{\langle -\infty \rangle}) \bigoplus H_n^{V_i}(E_{\mathcal{VC}\mathcal{C}}(V_i), E_{\mathcal{FLN}}(V_i); \mathbf{L}_R^{\langle -\infty \rangle}). \end{aligned}$$

They can be analysed further and contain information about and are build from the Nil and UNIL-terms in algebraic  $K$ -theory and  $L$ -theory of the infinite virtually cyclic group  $V_i$ . They vanish for  $L$ -theory after inverting 2 by results of [19]. For  $R = \mathbb{Z}$  they vanishes rationally for algebraic  $K$ -theory by results of [44].

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