ON THE COHOMOLOGY OF GENERALIZED HOMOGENEOUS SPACES

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Abstract. We observe that work of Gugenheim and May on the cohomology of classical homogeneous spaces \(G/H\) of Lie groups applies verbatim to the calculation of the cohomology of generalized homogeneous spaces \(G/H\), where \(G\) is a finite loop space or a \(p\)-compact group and \(H\) is a “subgroup” in the homotopical sense.

We are interested in the cohomology \(H^*(G/H; R)\) of a generalized homogeneous space \(G/H\) with coefficients in a commutative Noetherian ring \(R\). Here \(G\) is a “finite loop space” and \(H\) is a “subgroup”. More precisely, \(G\) and \(H\) are homotopy equivalent to \(\Omega BG\) and \(\Omega BH\) for path connected spaces \(BG\) and \(BH\), and \(G/H\) is the homotopy fiber of a based map \(f : BH \rightarrow BG\). We always assume this much, and we add further hypotheses as needed. Such a framework of generalized homogeneous spaces was first introduced by Rector [10], and a more recent framework of \(p\)-compact groups has been introduced and studied extensively by Dwyer and Wilkerson [4] and others.

We ask the following question: How similar is the calculation of \(H^*(G/H; R)\) to the calculation of the cohomology of classical homogeneous spaces of compact Lie groups? When \(R = \mathbb{F}_p\) and \(H\) is of maximal rank in \(G\), in the sense that \(H^*(H; \mathbb{Q})\) and \(H^*(G; \mathbb{Q})\) are exterior algebras on the same number of generators, the second author has studied the question in [8, 9]. There, the fact that \(H^*(BG; R)\) need not be a polynomial algebra is confronted and results similar to the classical theorems of Borel and Bott [2, 3] are nevertheless proven. The purpose of this note is to begin to answer the general question without the maximal rank hypothesis, but under the hypothesis that \(H^*(BG; R)\) and \(H^*(BH; R)\) are polynomial algebras.

In fact, we shall not do any new mathematics. Rather, we shall merely point out that work of the first author [7] that was done before the general context was introduced goes far towards answering the question. Essentially the following theorem was announced in [7] and proven in [5]. We give a brief sketch of its proof and then return to a discussion of its applicability to the question on hand. Let \(BT^n\) be a classifying space of an \(n\)-torus \(T^n\).

Theorem 1. Assume the following hypotheses.

(i) \(\pi_1(BG)\) acts trivially on \(H^*(G/H; R)\).
(ii) \(R\) is a PID and \(H_*(BG; R)\) is of finite type over \(R\).
(iii) \(H^*(BG; R)\) is a polynomial algebra.
(iv) There is a map \(e : BT^n \rightarrow BH\) such that \(H^*(BT^n; R)\) is a free \(H^*(BH; R)\)-module via \(e^*\).

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Then \( H^*(G/H; R) \) is isomorphic as an \( R \)-module to \( \operatorname{Tor}_{\cdot}(BG; R)(R, H^*(BH; R)) \), regraded by total degree. Moreover, there is a filtration on \( H^*(G/H; R) \) such that its associated bigraded \( R \)-algebra is isomorphic to \( \operatorname{Tor}_{\cdot}(BG; R)(R, H^*(BH; R)) \).

**Proof.** The first two hypotheses ensure that \( H^*(G/H; R) \) is isomorphic to the differential torsion product \( \operatorname{Tor}_{\cdot}(BG; R)(R, C^*(BH; R)) \). See, for example, [5, p. 21-25]. The second hypothesis allows Lemma 3.2 there to be applied with \( Z \) replaced by \( R \), thus allowing the finite type over \( Z \) hypothesis assumed there to be replaced by the finite type over \( R \) hypothesis assumed here. Therefore there is an Eilenberg-Moore spectral sequence that converges from \( \operatorname{Tor}_{\cdot}(BG; R)(R, H^*(BH; R)) \) to \( H^*(G/H; R) \). The conclusion of the theorem is that this spectral sequence collapses at \( E_2 \) with trivial additive extensions, but not necessarily trivial multiplicative extensions. The last hypothesis and a comparison of spectral sequences argument essentially due to Baum [1] shows that the conclusion holds in general if it holds when \( BH = BT^n \). See [5, p. 37-38]. Here the strange result [5, 4.1] gives that there is a morphism

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g : C^*(BT^n; R) \rightarrow H^*(BT^n; R)
\]

of differential algebras such that \( g \) induces the identity map on cohomology and annihilates all \( \cup_1 \)-products.

Now the general theory of differential torsion products of [5] kicks in. In modern language, implicit in the discussion of [6, p. 70], there is a model category structure on the category of \( A \)-modules for any \( DGA \) \( A \) over \( R \) such that every right \( A \)-module \( M \) admits a cofibrant approximation of a very precise sort. Namely, for any \( HA \)-free resolution \( X \otimes_R HA \rightarrow HM \) of \( HM \), there is a cofibrant approximation \( P = X \otimes_R A \rightarrow M \). Grading is made precise in the cited sources. The essential point is that \( P \) is not a bicomplex but rather has differential with many components. When \( HA \) is a polynomial algebra and \( M = R \), we can take \( X \) to be an exterior algebra with one generator for each polynomial generator of \( HA \). Here, assuming that \( A \) has a \( \cup_1 \)-product that satisfies the Hirsch formula (\( \cup_1 \) is a graded derivation), [5, 2.2] specifies the required differential explicitly in terms of \( \cup_1 \)-products. Using \( g \) to replace \( C^*(BT^n; R) \) by \( H^*(BT^n; R) \) in our differential torsion product, we see that the differential torsion product \( \operatorname{Tor}_{\cdot}(BG; R)(R, H^*(BT^n; R)) \) is computed by exactly the same chain complex as the ordinary torsion product \( \operatorname{Tor}_{\cdot}(BG; R)(R, H^*(BT^n; R)) \). See [5, 2.3]. The conclusion follows.

Hypotheses (i) and (ii) in the theorem are reasonable and not very restrictive. Hypothesis (iii) is intrinsic to the method at hand. Note that \( H^*(BG; R) \) can have infinitely many polynomial generators, so that \( G \) need not be finite. The key hypothesis is (iv). Here the following homotopical version of a theorem of Borel is relevant. It was first noticed by Rector [10, 2.2] that Baum’s proof [1] of Borel’s theorem is purely homotopical. A generalized variant of Baum’s proof is given in [5, p. 40-42]. That proof applies directly to give the following theorem. We state it for \( H \) and \( G \) as in the first paragraph. However, we are interested in its applicability to \( T^n \) and \( H \) in Theorem 1, and we restate it as a corollary in that special case.

**Theorem 2.** Let \( R \) be a field and assume the following hypotheses.

(i) \( \tau_1(BG) \) acts trivially on \( H^*(G/H; R) \).
(ii) \( H^*(BH; R) \) and \( H^*(BG; R) \) are polynomial algebras on the same finite number of generators.
(iii) \( H^*(G/H; R) \) is a finite dimensional \( R \)-module.
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Let\( H \) as a left\( H^{(BG; R)} \)-module. In particular,\( H^{(BH; R)} \) is\( H^{(BG; R)} \)-free.

**Corollary 3.** Let\( R \) be a field and assume given a map\( e : BT^n \rightarrow BH \) that satisfies the following properties, where\( H/T^n \) is the fiber of\( e \).

(i) \( \pi_1(BH) \) acts trivially on\( H^*(H/T^n; R) \).

(ii) \( H^*(BH; R) \) is a polynomial algebra on\( n \) generators.

(iii) \( H^*(H/T^n; R) \) is a finite dimensional\( R \)-module.

Then\( H^*(H/T^n; R) \cong R \otimes_{H^{(BG; R)}} H^*(BT^n; R) \) as an algebra and

\[
H^*(BT^n; R) \cong H^*(BH; R) \otimes_R H^*(H/T^n; R)
\]

as a left\( H^*(BH; R)-\)module. In particular,\( H^*(BT^n; R) \) is\( H^*(BH; R) \)-free.

When Corollary 3 applies, its conclusion gives hypothesis (iv) of Theorem 1. We comment briefly on applications to the integral and\( p \)-compact settings for the study of generalized homogeneous spaces.

**Remark 4.** A counterexample of Rector [10] shows that not all finite loop spaces\( H \) have (integral) maximal tori. When\( H \) does have a maximal torus, hypothesis (iii) of the Corollary holds by definition. Assuming that\( H \) is simply connected, [9, 3.11] describes for which primes\( p \)\( H^*(BH; \mathbb{Z}) \) is\( p \)-torsion free, so that\( H^*(BH; \mathbb{F}_p) \) is a polynomial algebra. If\( H \) is the localization of\( \mathbb{Z} \) at the primes\( p \) for which\( H^*(H; \mathbb{Z}) \) is\( p \)-torsion free, then\( H^*(BH; R) \) is also a polynomial algebra, and\( H^*(BT; R) \) is a free\( H^*(BH; R) \)-module. That is, hypothesis (iv) of Theorem 1 holds for the localization of\( \mathbb{Z} \) away from the finitely many “bad primes” for which\( H^*(BH; \mathbb{F}_p) \) is not a polynomial algebra on\( n \) generators.

**Remark 5.** In the\( p \)-compact setting, taking\( R = \mathbb{F}_p \), Dwyer and Wilkerson [4, 8.13, 9.7] prove that if\( H \) is connected,\( BH \) is\( \mathbb{F}_p \)-complete,\( H^*(H; \mathbb{F}_p) \) is finite dimensional, and\( H^*(H; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q} \) is an exterior algebra on\( n \) generators, then there is a map\( e : BT^n \rightarrow BH \) such that\( H^*(H/T^n; \mathbb{F}_p) \) is finite dimensional. Here Corollary 3 applies whenever\( H^*(BH; \mathbb{F}_p) \) is a polynomial algebra on\( n \) generators.

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