Light Open Mappings On Compact 
n-Manifolds Do Not Raise Dimension

And A

Proof Of The Hilbert-Smith Conjecture

by

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Suppose that $M$ is a compact $n$-manifold and $\phi$ is a light open mapping of $M$ onto a metric space $Y$. It is shown that $\dim Y = n$.

The symbol $\rho$ is used for the metric on both $M$ and $Y$. Recall that $\phi$ is light iff for each $x \in M$, $\phi^{-1}\phi(x)$ is totally disconnected.

The following lemma is crucial to defining certain coverings of $M$ with distinguished families of open sets.

Lemma 1. Suppose that $\phi$ is a light open mapping of a compact connected $n$-manifold $M$ onto a metric space $Y$. For each $z \in Y$ and $\epsilon > 0$, there is a connected open set $U$ such that

1. $\text{diam } U < \epsilon$, 
2. $z \in U$, and 
3. $\phi^{-1}(U) = \bigcup_{i=1}^{s} U_i$ where $s$ is a natural number such that 
   (a) $U_i$ is a component of $\phi^{-1}(U)$ for each $i$, $1 \leq i \leq s$, 
   (b) $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, and 
   (c) $\phi(U_i) = U$ for each $i$.

Proof. It follows from Whyburn’s theory of light open mappings [5; p. 148] that for $\epsilon > 0$, there is a connected open set $U$ such that (1) $\text{diam } U < \epsilon$, (2) $z \in U$, and (3) $\phi^{-1}(U)$
consists of a finite number of components $U_1, U_2, \ldots, U_s$ such that $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ for $i \neq j$, and $\phi(U_i) = U$ for each $i$.

**Standing Hypothesis:** In the following, $M$ is a compact metric $n$-manifold. Suppose also that $Y$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that (1) for each $i$, $B_i$ is connected and uniformly locally connected and (b) if $H$ is any subcollection of $Q$ and $\bigcap_{h \in H} h \neq \emptyset$, then $\bigcap_{h \in H} h$ is connected and uniformly locally connected (a consequence of a theorem due to Bing and Floyd [1]). All of the open sets used in $Y$ below to construct coverings of $Y$ are in $Q$. The metric $\rho$ is chosen such that for each $\epsilon > 0$ and $x \in M$, $N_\epsilon(x)$, the $\epsilon$-neighborhood of $x$, is connected. Similarly, for $y \in Y$, $N_\epsilon(y)$ is connected.

One can use the alternative below and assume that $n > 1$.

**Alternative Standing Hypothesis:** In the following, $M$ is a compact metric $n$-manifold, $n > 1$. Suppose also that $Y$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that for each $i$, $B_i$ is connected and has a connected boundary (a consequence of a Theorem of Jones [3]). All of the open sets used in $Y$ below to construct coverings of $Y$ are in $Q$. The metric $\rho$ is chosen such that for each $\epsilon > 0$ and $x \in M$, $N_\epsilon(x)$, the $\epsilon$-neighborhood of $x$, is connected. Similarly, for $y \in Y$, $N_\epsilon(y)$ is connected [5].

If one uses *The Alternative Standing Hypothesis*, then make note of the following:

It is known that light open mappings on compact metric 1-manifolds are finite-to-one and do not raise dimension [5]. Furthermore, light open mappings on compact metric $n$-manifolds do not lower dimension [4]. Consequently, if $f$ is a light open mapping on a compact metric $n$-manifold $M$ onto a metric space $Y$ with $n > 1$, then $Y$ does not have any local separating points and, hence, by the theorem of Jones [3] has a basis of connected open sets with connected boundaries.

**Lemma 2.** Suppose that $\phi$ is a light open mapping of $M$ onto $Y$ and $G$ is an open covering of $Y$. Then there exists a finite open covering $R$ of $Y$ which refines $G$ such that
(1) if \( y \in Y \), then there is \( r \in R \) such that \( y \in r \), \( r \in Q \) where \( Q \) is the basis in The Standing Hypothesis, \( \phi^{-1}(r) = r_1 \cup r_2 \cup \cdots \cup r_q \), for some natural number \( q \), such that for each \( i = 1, 2, \ldots, q \), \( r_i \) is a component of \( \phi^{-1}(r) \), \( r_i \) maps onto \( r \) under \( \phi \), and \( \overline{r_i} \cap \overline{r_j} = \emptyset \) for \( i \neq j \), and

(2) \( R \) is irreducible.

Proof. Since \( Y \) is compact, use Lemma 1 to obtain a finite irreducible covering \( R \) of \( Y \) for the conditions of the lemma.

Let \( V^1 = \{ c \mid c \) is a component of \( \phi^{-1}(r) \) such that \( r \in R = R^1 \} \).

Definition. For each \( r^1_i \in R^1 \), \( \phi^{-1}(r^1_i) = \bigcup_{j=1}^{t_i} f^1_{ij} \) and \( \{ f^1_{ij} \}_{j=1}^{t_i} = F^1_i \) is called a distinguished family of open sets in \( V^1 \) where \( f^1_{ij} \) is a component of \( \phi^{-1}(r^1_i) \).

Construction Of \( U^1 \) Of Order \( n + 1 \) Which Star Refines \( V^1 \)

List the distinguished families of \( V^1 \) as \( F^1_1, F^1_2, \ldots, F^1_n \), where the degenerate families, if any, are listed last in the ordering.

Recall from the Standing Hypothesis that \( Y \) has a countable basis \( Q \) with certain properties. Use \( Q \) to obtain a finite open star refinement \( \hat{R} \subset Q \) of \( R^1 \) which covers \( Y \) such that each \( r \in \hat{R} \) is connected and uniformly locally connected (ulc) and inherits certain other properties from \( Q \).

Let \( \hat{U}^1 = \{ c \mid c \) is a component of \( \phi^{-1}(r) \) where \( r \in \hat{R} \} \). Clearly, \( \hat{U}^1 \) star refines \( V^1 \). Recall also that if \( c \) is a component of \( \phi^{-1}(r) \), then \( \phi(c) = r \).

Construction Of \( U^1 \) Which Refines \( \hat{U}^1 \) In A Special Way

For any collection \( S \) of sets, let \( \cup S \) be the union of sets in \( S \) and \( \cap S \) be their intersection.

For each \( y \in Y \), let \( Q(y) = \{ r \mid r \in \hat{R} \) and \( y \in r \} \). There are at most a finite number of such sets distinct from each other. Order these sets as \( Q_1, Q_2, \ldots, Q_{m_1} \) such that for \( i < j \), \( Q_i \neq Q_j \) and card \( Q_i \geq \text{card} \ Q_j \). Let \( O_i = \cap_{j<i} \cup_{j<i} (\cap Q_j) \). For each \( i, 1 \leq i \leq m_1 \), let
\[ B_{ir} = (\partial O_i) \cap \partial r \] where \( r \in Q_i \) and \((\partial O_i) \cap \partial r \neq \emptyset \). There are at most a finite number of such non empty closed sets distinct from each other. Let \( B_1, B_2, \ldots, B_{m_2} \) denote all those sets distinct from each other. For each \( i, 1 \leq i \leq m_2 \), there is \( r \in \hat{R} \) such that \( \partial r \supset B_i \). Let \( B = \bigcup_{i=1}^{m_2} B_i \). Clearly, \( B = \bigcup_{r \in \hat{R}} \partial r \). For each \( y \in B \), let \( D(y) = \{ B_t \mid y \in B_t \} \). There are at most a finite number of such non empty sets distinct from each other. Order these as \( D_1, D_2, \ldots, D_{m_3} \) such that if \( i < j \), then \( D_i \neq D_j \) and \( \text{card } D_i \geq \text{card } D_j \).

It follows from the definition that \( \phi^{-1}(B) \) is closed and contains no open set. Hence, dimension \( \phi^{-1}(B) \leq n - 1 \).

Let \( e = (\frac{1}{4}) \{ \min \{ \rho(\cap D_i, \cap D_j) \mid (\cap D_i) \cap (\cap D_j) = \emptyset \} \) and \( \epsilon \), the Lebesque number of the covering \( \hat{R} \) of \( Y \).

Observe that if \( B_s \neq \bigcup_{i=1}^{k} D_i \), then \( B_s \cap (\cap D_i) = \emptyset \) for \( 1 \leq i \leq k \); otherwise, cardinality \((B_s \cup D_i) > \text{card } D_i \) and \( B_s \in D_j \) for some \( j, 1 \leq j < i \).

Note: A finite open covering \( C \) of a closed subset \( N \) of \( M \) such that \( \text{dim } N = n - 1 \) where the elements of \( C \) are open relative to \( N \) and order \( C = n \) can be extended to a collection \( C' \) of open sets in \( M \) such that \( \text{card } C = \text{card } C' \), \( C' \) covers \( M \) and order \( C' = n \). See [6]. The open coverings below of subsets \( N \) of \( M \) such that \( \text{dim } N \leq n - 1 \) consist of open subsets of \( M \).

Let \( \{ D_i \}_{i=1}^{q_1} \) be the collection of all \( D_i \) such that \( \text{card } D_i = c_1 = \text{card } D_1 \) (maximum cardinality). Observe that \((\cap D_{i_1}) \cap (\cap D_{i_j}) = \emptyset \) for \( i \neq j \).

Cover \( \phi^{-1}\left( \bigcup_{i=1}^{q_1}(\cap D_{i_1}) \right) \) with a finite irreducible open covering \( H_1 \) such that

1. if \( h \in H_1 \), then \( \text{diam } \phi(h) < \epsilon \),

2. \( H_1 \) star refines \( \hat{U}^1 \) and \( \phi(H_1) = \{ \phi(h) \mid h \in H_1 \} \) star refines \( \hat{R} \),

3. if \( H_{1_i} \) is the subcollection of \( H_1 \) which covers \( \phi^{-1}(\cap D_{1_i}) \) irreducibly, then \( \overline{\cup H_{1_i}} \cap \overline{H_{1_j}} = \emptyset \) for \( i \neq j \), and

4. if for each \( i, 1 \leq i \leq q_1, h \in H_{1_i}, w \in H_{1_i}, \) and \( u \in \hat{U}^1 \) such that
   
   a. \( \phi(h) \cap \phi(u) \neq \emptyset \),

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(b) $\phi(h) \cap \phi(w) \neq \emptyset$, and

c) $B_s \notin D_{1i}$ for all $B_s$ such that $\partial\phi(u) \supset B_s$ (alternatively, $\phi(u) \cap (\cap D_{1i}) \neq \emptyset$), then $\phi(u) \supset \phi(h) \cup \phi(w)$. 

Now, (1) and (2) should be clear.

To see (3), observe that if $i \neq j$, then $\phi^{-1}(\cap D_{1i})$ and $\phi^{-1}(\cap D_{1j})$ are disjoint compact sets.

To see (4), assume the hypothesis of (4). For each $h \in H_{1i}$ there exists $y \in \phi(h) \cap (\cap D_{1i})$. It follows from (1) that $N_{e}(y) \supset \phi(h)$. Since $B_s \notin D_{1i}$, $\rho(\cap D_{1i}, B_s) > 4e$. Thus, $\phi(u) \supset N_{4e}(y)$ since $N_{4e}(y)$ is connected. Also, $\text{diam } \phi(w) < e$ and $\phi(w) \cap \phi(h) \neq \emptyset$.

Hence, $\phi(u) \supset \phi(h) \cup \phi(w)$ since $\phi(u) \supset N_{4e}(y) \supset \phi(h) \cup \phi(w)$.

Let $c_{1}, c_{2}, \ldots, c_{m_{4}}$ denote the cardinal numbers (distinct from each other) of the cardinality of the collections $D_{t}$, $1 \leq t \leq m_{3}$, such that $c_{i} > c_{i+1}$ for $1 \leq i < m_{4}$.

Let $\{D_{ij}\}_{j=1}^{q_{i}}$ denote the collection of all $D_{t}$ such that card $D_{t} = c_{i}$. If $H_{j-1}$ has been defined, let $\epsilon_{j} > 0$ be such that $\epsilon_{j} < (\frac{1}{t}) \min \{\epsilon, \epsilon_{j-1}, \delta_{t}, 1 \leq t < j\}$, where $\delta_{t} = (\frac{1}{t}) \left\{ \min \left\{ \rho \left( \cap D_{ti}, Y - \bigcup_{s=1}^{t-1} \phi(\cup H_{s}) \right) \mid 1 \leq i \leq q_{t} \right\} \right\}$ and $\min \left\{ \rho \left( \cap D_{ia} - \bigcup_{s=1}^{t-1} \phi(\cup H_{s}), \cap D_{j} - \bigcup_{s=1}^{j-1} \phi(\cup H_{s}) \right) = \emptyset \right\}$. Let $H_{j}$ be a finite irreducible open covering of $\phi^{-1} \left( \bigcup_{i=1}^{q_{j}} (\cap D_{ji}) \right) - \bigcup_{t=1}^{j-1} (\cup H_{t})$ such that

(1) if $h \in H_{j}$, then $\text{diam } \phi(h) < \epsilon_{j}$ and $\phi(h) \cap \left( \bigcup_{i=1}^{q_{j}} (\cap D_{(j-1)i}) \right) = \emptyset$,

(2) $H_{j}$ star refines $\hat{U}^{1}$ and $\phi(H_{j}) = \{\phi(h) \mid h \in H_{j}\}$ star refines $\phi(\hat{U}^{1}) = \hat{R}$,

(3) if $H_{ji}$ is the subcollection of $H_{j}$ which covers $\phi^{-1}(\cap D_{ji}) - \bigcup_{t=1}^{j-1} (\cup H_{t})$ irreducibly, then $\overline{\cup H_{ji}} \cap \overline{\cup H_{js}} = \emptyset$ for $i \neq s$, and

(4) if for each $i$, $1 \leq i \leq q_{i}$, $h \in H_{ji}$, $w_{1} \in H_{ji}$, $w_{2} \in H_{ji}$, and $u \in \hat{U}^{1}$ such that

(a) $\phi(h) \cap \phi(u) \neq \emptyset$;

(b) $\phi(h) \cap \phi(w_{1}) \neq \emptyset$. 

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(c) \( \phi(w_1) \cap \phi(w_2) \neq \emptyset \), and

(d) \( B_s \notin D_{j_s} \) for all \( B_s \) such that \( \partial \phi(u) \supset B_s \) (alternatively, \( \phi(u) \cap \left( \cap D_{j_s} - \bigcup_{t=1}^{j-1} \phi(\cup H_t) \right) \neq \emptyset \)); then \( \phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2) \).

To see (3), observe that if \( i \neq s \), then \( \phi^{-1}(\cap D_{j_s}) - \bigcup_{t=1}^{j-1} (\cup H_t) \) and \( \phi^{-1}(\cap D_{j_s}) - \bigcup_{t=1}^{j-1} (\cup H_t) \) are disjoint compact sets.

To see (4), assume the hypothesis of (4). Since \( h \in H_{j_s} \), it follows that there is \( z \in \phi(h) \cap \left( \cap D_{j_s} - \bigcup_{t=1}^{j-1} \phi(\cup H_t) \right) \). Now, \( \rho \left( \cap D_{j_s} - \bigcup_{t=1}^{j-1} \phi(\cup H_t), \partial \phi(u) \right) > 4\epsilon_j \). Either (a) \( \phi(u) \supset N_{4\epsilon}(z) \) or (b) \( Y - \bar{\phi(u)} \supset N_{4\epsilon_j}(z) \) since \( N_{4\epsilon_j}(z) \) is connected. Since \( \phi(u) \cap \phi(h) \neq \emptyset \) and \( \text{diam } \phi(h) < \epsilon_j \), it follows that \( \phi(u) \supset N_{4\epsilon_j}(z) \) and \( \phi(u) \supset \phi(h) \). By hypothesis, \( \text{diam } \phi(w_i) < \epsilon_j, i \in \{1, 2\} \). Hence, \( \phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2) \).

It follows by mathematical induction that \( H = \bigcup_{i=1}^{m_4} H_i \) exists which covers \( \phi^{-1}(B) \) irreducibly such that

1. if \( h \in H_j \), then \( \text{diam } \phi(h) < \epsilon_j \), \( \phi(h) \cap \left( \bigcup_{i=1}^{q_{j-1}} (D_{(j-1)_i}) \right) = \emptyset \),

2. for each \( i, 1 < i \leq m_4, H_i \) is an irreducible open covering of \( \phi^{-1} \left( \bigcup_{i=1}^{q_i} (\cap D_{i}) \right) - \bigcup_{t=1}^{i-1} (\cup H_t) \) and \( H_1 \) is an irreducible open covering of \( \phi^{-1} \left( \bigcup_{i=1}^{q_1} (\cap D_{1_i}) \right), \)

3. \( H_j \) star refines \( U^1 \) and \( \phi(H_j) \) star refines \( \phi(U^1) = \hat{R} \),

4. if \( H_{j_i} \) is the subcollection of \( H_j \) which covers \( \phi^{-1}(\cap D_{j_i}) - \bigcup_{t=1}^{j-1} (\cup H_t) \), then \( \cup H_{j_i} \cap \cup H_{j_s} = \emptyset \) for \( i \neq s \), and

5. if for each \( i, 1 \leq i \leq q_j, h \in H_{j_i}, w_1 \in H_t, j \leq t \leq m_4, w_2 \in H_s, j \leq s \leq m_4, \) and \( u \in U^1 \) such that

   (a) \( \phi(h) \cap \phi(u) \neq \emptyset \),

   (b) \( \phi(h) \cap \phi(w_1) \neq \emptyset \),

   (c) \( \phi(w_1) \cap \phi(w_2) \neq \emptyset \), and
(d) $B_s \notin D_{j_i}$ for all $B_s$ such that $\partial \phi(u) \supset B_s$ (alternatively, $\phi(u) \cap \left( \bigcap_{t=1}^{j-1} \phi(\cup H_t) \big) \neq \emptyset$; then $\phi(u) \supset \phi(h) \cup \phi(w_1) \cup \phi(w_2)$.

It should be clear from the arguments above how to obtain properties (1)-(5).

The next step is to shrink $H$ to $H'$ which has order $n$ and retains Properties (1)-(5).

Order the elements of $H$ as $\{h_{11}, h_{12}, \cdots, h_{1x_1}; h_{21}, h_{22}, \cdots, h_{2x_2}; \cdots, h_{m_41}, h_{m_42}, \cdots, h_{m_4x_{m_4}}\}$ where the elements of $H_i$ are ordered before the elements of $H_{i+1}$, $1 \leq i < m_4$. A corollary to a Theorem [10; p. 90] states that: A metric space $(X,d)$ has covering dimension $\leq n$ if and only if $X$ has a sequence $\{G_i\}$ such that (1) for each $i$, $G_i$ is an open covering of $X$, (2) for each $i$, $G_{i+1}$ refines $G_i$, and (3) $\{\text{mesh } G_i\} \to 0$. It follows that if $(X,d)$ is a compact metric space with $\text{dim } X \leq n$ and $Q$ is any finite irreducible covering of $X$, then there is a natural number $i$ such that $G_i$ refines $Q$ where $\{G_i\}$ is the sequence in the statement above from [10].

Now, $\text{dim } \phi^{-1}(B) \leq n - 1$. Using [10], it follows that there exists a finite irreducible open covering $G = \{g_1, g_2, \cdots, g_m\}$ of $\phi^{-1}(B)$ (the elements of $G$ are open in $M$) with sufficiently small mesh such that

1. the subcollection $G_1$ of $G$ which covers $\phi^{-1}(B) \cap \left( \cup H_1 - \bigcup_{j=1}^{m_4} (\cup H_j) \right)$ refines $H_1$,

2. if $i > 1$, then the subcollection $G_2$ of $G$ which covers $\phi^{-1} \left( \bigcup_{j=1}^{i} (\cap D_{ij}) \right) - \bigcup_{t=1}^{i-1} (\cup G_t) \big)$

3. for each $i$ and $k$, there is some $j$ such that $h_{ki} \supset g_j$ and $h_{st} \not\subset g_j$ for any ordered pair $(s,t) \neq (k,i)$ (this follows using the irreducibility of $H$), and

4. order $G \leq n$.

Let $h'_{i1}$ be the union of all $g_i$ such that $h_{i1} \supset g_i$ and let $h'_{i1}$ be the union of all $g_j$ such that $h_{i1} \supset g_j$ and $h_{it} \not\subset g_j$ for $1 \leq t < i$, that is, $h'_{i1} \not\subset g_j$. It follows that $H'_{i1} = \ldots$
\{h'_{11}, h'_{12}, \cdots, h'_{q_1}\} is an irreducible open cover of \(\phi^{-1}\left(\bigcup_{i=1}^{q_1} D_{1_i}\right)\). Continue. Let \(h'_{21}\) be the union of all \(g_i\) such that \(h_{21} \supset g_i\) and \(h_{1t} \not\supset g_i\), \(1 \leq t \leq q_1\). For each \(i, 1 < i \leq q_2\), let \(h'_{2i}\) be the union of all \(g_j\) such that \(h_{2i} \supset g_j, h_{1t} \not\supset g_j\), \(1 \leq t \leq q_1\), and \(h_{2t} \not\supset g_j\) for \(1 \leq t < i\). It follows that \(H'_2 = \{h'_{21}, h'_{22}, \cdots, h'_{2q_2}\}\) is an irreducible open cover of \(\phi^{-1}\left(\bigcup_{i=1}^{q_1} D_{2_i}\right) \cup H'_1\). To see that \(H'_2\) covers \(\phi^{-1}\left(\bigcup_{i=1}^{q_1} D_{2_i}\right) \cup H'_1\) suppose that there is some \(x\) in this set such that \(x \not\in h'_{2t}\) for any \(t, 1 \leq t \leq q_2\). By Property (2) above, \(G_2\) refines \(H_2\) and covers \(\left(\phi^{-1}\left(\bigcup_{i=1}^{q_1} D_{2_i}\right) \cup G_1\right) \cup \bigcup_{j=2}^{m_4}(\cup H_j)\). Now, if \(x \in g \in G_1\), then \(x \in h_{1s}\) for some \(s\) since \(G_1\) refines \(H_1\) and covers \(\phi^{-1}(B) \cap \left(\bigcup_{i=1}^{q_1} D_{2_i}\right) \cup H'_1\). Hence, \(x \not\in g\) for any \(g \in G_1\). Recall that \(\bigcup_{j=1}^{m_4}(\cup H_j) \cap \phi^{-1}\left(\bigcup_{j=1}^{q_2}(\cup D_{2j})\right) = \emptyset\). Thus, by definition of \(G_2\), \(x \in g\) for some \(g \in G_2\) and, consequently, \(x \in h_{2t}\) for some \(t\) since \(G_2\) refines \(H_2\). That is, there is a smallest \(t\) such that \(h_{2t} \supset g\). It can be shown in a similar way that for each \(i, 1 \leq i \leq m_4\), \(H'_i\) covers \(\phi^{-1}\left(\bigcup_{i=1}^{q_1} D_{i}^{(t)}\right) \cup H'_t\). Let \(H' = \bigcup_{i=1}^{m_4} H'_i\). Clearly, order \(H' \leq n\) since order \(G \leq n\) and no member of \(G\) is in two different elements of \(H'_i, 1 \leq t \leq m_4\). This is similar to a theorem of Nagata in [4].

It is not difficult to see that Properties (1)-(5) are true where \(H'_i\) replaces \(H_i, 1 \leq i \leq m_4\). For convenience of notation, suppose that Properties (1)-(5) are true for \(H_i\) as stated above.

Shrinking the elements of \(H_i\) as above does not violate any of the other properties.

Recall the definition of \(Q_i\) and \(O_i = \cap Q_i - \bigcup_{j<i}(\cap Q_j)\). Let \(Q_i = \{r_i\}_{t=1}^{x_i}\). For all possible selections of components \(c_{ij}^{(t)}\) of \(\phi^{-1}(r_i)\) such that \(\bigcap_{t=1}^{x_i} c_{ij}^{(t)} \neq \emptyset\), consider \(\phi^{-1}(O_i) \cap \bigcap_{t=1}^{x_i} c_{ij}^{(t)}\) to be an element of \(O\). The collection \(O\) of all such open sets distinct
from each other is a pairwise disjoint collection which covers $M - \phi^{-1}(B) = \bigcup_{i=1}^{m_1} \phi^{-1}(O_i)$.

Let $U^1 = O \cup H$. Observe that $U^1$ is a finite irreducible open covering of $M$ which star refines $V^1$ and order $U^1 = n + 1$. Also, $H$ has Properties (1)-(6) as listed above.

Construction of $V^2$ Which Star Refines $U^1$

Next, construct $V^2$. For each $y \in Y$, define $\hat{U}(y) = \{ u \mid u \in \hat{U}^1 \text{ and } y \in \phi(u) \}$. There is some $s$, $1 \leq s \leq z_1$, such that $r_s^1 \supset \bigcup_{u \in \hat{U}(y)} \phi(u)$ where $R^1 = \{ r_1^1, r_2^1, \ldots, r_z^1 \}$.

For each $y \in B$, choose $r_y^2 \in Q$ (the basis for $Y$ described above) such that

1. $y \in r_y^2$;

2. if $U^1(y) = \{ u \mid u \in U^1 \text{ and } u \cap \phi^{-1}(y) \neq \emptyset \}$ (that is, $U^1(y)$ covers $\phi^{-1}(y)$), then
   \[ \left( \bigcap_{u \in \hat{U}(y)} \phi(u) \right) \cap \left( \bigcap_{u \in \hat{U}(y)} \phi(u) \right) \supset r_y^2, \]

3. $\text{diam} r_y^2 < \left( \frac{1}{5} \right) \min \{ \rho(y, \partial \phi(v)) \mid v \in \hat{U}^1 \text{ and } y \notin \partial \phi(v) \}$,

4. $\phi^{-1}(r_y^2) = r_{y,1}^2 \cup r_{y,2}^2 \cup \cdots \cup r_{y,q}^2$, $r_{y,i}^2$ maps onto $r_y^2$ under $\phi$, and $r_{y,i}^2 \cap r_{y,j}^2 = \emptyset$ for $i \neq j$; and

5. if $u \in U^1(y)$, then there exists a component $c$ of $\phi^{-1}(r_y^2)$ such that $u \supset \overline{c}$ and $\text{diam} \overline{c} < \left( \frac{1}{4} \right) \rho(\overline{c}, \partial u)$ and for each component $k$ of $\phi^{-1}(r_y^2)$, there is some $v \in U^1(y)$ such that $v \supset \overline{k}$ and $\text{diam} \overline{k} < \left( \frac{1}{4} \right) \rho(\overline{k}, \partial v)$.

Observe that Property (4) follows from either Lemma 1 or [5; p. 148].

Note that Property (5) may be obtained by using [9; p. 78, Theorem (1.3)]. First, fix $x(u) \in u$ for each $u \in U^1(y)$. Choose $\delta > 0$ such that (a) $\delta < \left( \frac{1}{4} \right) \min \{ \rho(x(u), \partial u) \mid u \in U^1(y) \}$ and (b) if $x \in \phi^{-1}(y)$, then there is some $v \in U^1(y)$ such that $x \in v$ and $\rho(x, \partial v) < 4\delta$. By [9; p. 78, Theorem (1.3)], it follows that $r_y^2$ can be chosen with sufficiently small diameter such that for each component $c$ of $\phi^{-1}(r_y^2)$, $\text{diam} \overline{c} < \delta$. Hence, if $c_{x(u)}$ is the component of $\phi^{-1}(r_y^2)$ which contains $x(u)$, then (a) $u \supset \overline{c_{x(u)}}$, (b) $\text{diam} \overline{c_{x(u)}} < \left( \frac{1}{4} \right) \rho(\overline{c_{x(u)}}, \partial u)$, and (c) if $k$ is a component of $\phi^{-1}(r_y^2)$, then there is $v \in U^1(y)$ such that $v \supset \overline{k}$ and $\text{diam} \overline{k} < \left( \frac{1}{4} \right) \rho(\overline{k}, \partial v)$. 9
Let \( R_1^2 \) denote a finite irreducible collection of such sets \( r_y^2 \) which covers \( B \). If \( y \in Y \) and \( y \not\in \cup R_1^2 \), then choose \( r_y^2 \) satisfying (1)-(5) above such that \( \tilde{r}_y \cap B = \emptyset \) and let \( R_2^2 \) denote a finite irreducible cover of \( Y - (\cup R_1^2) \), consisting of such \( r_y^2 \). Let \( R^2 = R_1^2 \cup R_2^2 \) which is an irreducible cover of \( Y \).

Define \( V^2 = \{ c \mid c \text{ is a component of } \phi^{-1}(r_y^2) \text{ for some } i \text{, where } r_y^2 \in R_1^2 \cup R_2^2 \} \), which is an irreducible cover of \( M \). Observe that Property (5) implies that \( V^2 \) star refines \( U^1 \). Now \( U^1 \) is constructed so that \( U^1 \) refines \( \hat{U}^1 \) which star refines \( V^1 \). Hence, \( U^1 \) star refines \( V^1 \). The collection of components, \( \{ f_{ij}^2 \}_{j=1}^{t_i^2} \), of \( \phi^{-1}(r_y^2) \), \( r_y^2 \in R_1^2 \cup R_2^2 \), is a distinguished family in \( V^2 \).

**Definitions Of \( \alpha_1, \beta_1, \text{ And } \pi_1 = \beta_1 \alpha_1 \)**

**Case (1): \( y_i \in B \)**

Take any \( r_i^2 = r_{y_i}^2 \in R_1^2 \) chosen for \( y_i \in B \). Let \( F_i^2 = \{ f_{ij}^2 \}_{j=1}^{t_i^2} \) be the distinguished family in \( V^2 \) generated by \( r_{y_i}^2 \).

**Definition of \( e_{y_i} \) for \( y_i \in B \)**

Let \( e_{y_i} = \min \{ t \mid y_i \in \phi(\cup H_{e_q}) \text{ for some unique } q \} \).

Now, \( H(e_{y_i}) \subset H_{e_{y_i}} \) and \( H(e_{y_i}) \) covers \( \phi^{-1}(\cap D(e_{y_i})) - \bigcup_{s=1}^{e_{y_i} - 1} H_s \) irreducibly. Either (a) \( \phi^{-1}(y_i) \) is covered by \( H(e_{y_i}) \) or (b) \( \phi^{-1}(y_i) \) is not covered by \( H(e_{y_i}) \).

**Case (a):** By the choice of \( r_{y_i}^2 \) which generates \( F_i^2 \), it follows that for each \( j, 1 \leq j \leq t_i^2 \), there is \( U_{ij} \in H(e_{y_i}) \) such that \( U_{ij} \supset \tilde{f}_{ij}^2 \). This follows from Property (5) above. In this case, there is no \( u \in O \) such that \( u \supset \tilde{f}_{ij}^2 \).

**Case (b):** For some \( j, 1 \leq j \leq t_i^2 \), choose \( U_{ij} \in H(e_{y_i}) \) if possible such that \( U_{ij} \supset \tilde{f}_{ij}^2 \); if not, then choose \( U_{ij} \in H_t \) for the smallest \( t \) (where, of course, \( t > e_{y_i} \)) such that \( U_{ij} \supset \tilde{f}_{ij}^2 \). By definition of \( e_{y_i} \), there is some \( u \in H(e_{y_i}) \) such that \( u \cap \phi^{-1}(y_i) \neq \emptyset \). By Property (5), there is some \( t, 1 \leq t \leq t_i^2 \), such that \( u = U_{it} \supset \tilde{f}_{it}^2 \). If there is no \( U_{ij} \in H(e_{y_i}) \) such that \( U_{ij} \supset \tilde{f}_{ij}^2 \), then clearly there is \( u = U_{ij} \in U_1(y_i) \subset H \) such that...
$U_{ij} \supset f_{ij}^{2}$ by Property (5) above. Now, $U_{ij} \in H_t \subset H$ for the smallest subscript $t$. By the definition of $e_{y_i}$, it follows that $t > e_{y_i}$.

Definition of $d_{y_i}$ for $y_i \in B$

Let $d_{y_i} = \min\{j \mid D_{j_i} \supset D(e_{y_i})\}$ for the smallest $t$.

If $y_i \notin B$, then neither $e_{y_i}$ nor $d_{y_i}$ is defined.

Definition of $\hat{V}(y_i)$

If $y_i \notin B$, then $\hat{V}(y_i) = \hat{U}(y_i)$. For $y_i \in B$, let $\hat{V}(y_i) = \{u \mid u \in \hat{U}^1$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}\}$.

Property P

If $y_i \in B$, then $\hat{V}(y_i) \neq \emptyset$ and if $u \in \hat{V}(y_i)$, then

1. $u \in \hat{U}(y_i)$, that is, $\hat{U}(y_i) \supset \hat{V}(y_i)$,

2. $\phi(u) \supset \bigcup_{s=1}^{t_i} \phi(U_{i,s})$,

3. $\phi(u) \supset \cap D_{(d_{y_i})_t}$,

4. $\overline{\phi(u)} \supset \cup D_{(e_{y_i})_q}$,

5. $\overline{\phi(u)} \supset \cup D_{(e_{y_i})_q}$, and

6. $\phi(u) \supset \cap D_{(e_{y_i})_q} - \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t)$.

Proof. Recall that $e_{y_i} = \min\{t \mid y_i \in \phi(\cup H_t)\}$ for some $q$, $d_{y_i} = \min\{j \mid D_{j_i} \supset D(e_{y_i})\}$ for the smallest $t$, and $\hat{V}(y_i) = \{u \mid u \in \hat{U}^1$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}\}$.

Take $u \in \hat{U}^1$ such that $\phi(u) \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. It follows that $B_a \notin D_{(d_{y_i})_t}$, for each $B_a$ such that $\partial \phi(u) \supset B_a$. To see this, suppose that $\partial \phi(u) \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. Then there exists $B_x$ such that $\partial \phi(u) \supset B_x$ and $B_x \cap (\cap D_{(d_{y_i})_t}) \neq \emptyset$. There exists $c < d_{y_i}$ such that $D_{c_s} \supset D_{(d_{y_i})_t}$, for some unique $s$ where $B_x \in D_{c_s}$. This contradicts the choice of $d_{y_i}$ since $D_{c_s} \supset D_{(d_{y_i})_t} \supset D_{(e_{y_i})_q}$. Hence, $B_a \notin D_{(d_{y_i})_t}$, as claimed.

For any $B_x \in D_{(d_{y_i})_t}$, $B_x = B_{ir} = (\partial O_j) \cap \partial r$ where $r \in Q_j$ for some $j$, $\phi(u) \supset O_j$, and $\phi(u) \in Q_j$. Thus, $\overline{\phi(u)} \supset \cup D_{(d_{y_i})_t}$ and $\phi(u) \supset \cap D_{(d_{y_i})_t}$, since $\partial \phi(u) \cap (\cap D_{(d_{y_i})_t}) = \emptyset$.
This establishes (3), (4), and (5) since $D(d_{u_i}) \supset D(e_{y_i})$.

Take any such $u$ as above. It follows that $\overline{\phi(u)} \supset \cup D(e_{y_i})$. Observe that if $B_s \cap (\cap D(e_{y_i})_q) \neq \emptyset$, $B_s \notin D_{e_{y_i}}$, and $\partial \phi(u) \supset B_s$, then $B_s \in D_{a_m}$ for some $a < e_{y_i}$ and the smallest $m$. Thus, $\bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t) \supset \cap D_{a_m}$. Note also that $B_s \in D(d_{u_i})$.

Now, $B_a \notin D(e_{y_i})$ for each $B_a$ such that $\partial \phi(u) \supset B_a$. Consequently, $\phi(u) \supset \cap D(e_{y_i}) - e_{y_i}-1 \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t)$. Hence, (6) is established.

There is $x$, $1 \leq x \leq t_i^2$, such that $U_{i_x} \in H(e_{y_i})$. Hence, $\phi(u) \cap \phi(U_{i_x}) \neq \emptyset$ since $\phi(U_{i_x}) \cap \left(\cap D(e_{y_i}) - \bigcup_{t=1}^{e_{y_i}-1} \phi(\cup H_t)\right) \neq \emptyset$. Take any $U_{i_z}, 1 \leq z \leq t_i^2$. Let $h = U_{i_z}$, $w_1 = w_2 = U_{i_z} \in H_s$, $e_{y_i} \leq s \leq m_4$. Also, $y_i \in \phi(U_{i_x}) \cap \phi(U_{i_z})$. By Property (5) of $H$, $\phi(u) \supset \phi(U_{i_x}) \cup \phi(U_{i_z})$ and $\phi(u) \supset \bigcup_{t=1}^{t_i^2} \phi(U_{i_t})$. It follows that $y_i \in \phi(u)$ and $u \notin \hat{U}(y_i)$.

Thus, $\hat{V}(y_i)$ exists and $\hat{U}(y_i) \supset \hat{V}(y_i)$. Hence, (1) and (2) are true and Property P is established.

**Definition of $s_i$ for $y_i \in B$**

Let $s_i = \min\{s \mid r^1_s \supset \overline{\phi(u)} \text{ for all } u \in \hat{V}(y_i)\}$. Take $F^1_{s_i} = \{f^{1}_{s_i,j}\}_{j=1}^{t_i}$ for the given $F^2_i$. Each $f^2_{i,j} \in F^2_i$ is in one and only one member of $F^1_{s_i}$, say $f^1_{s_i z_{ij}}$. To see that there is a unique $z_{ij}, 1 \leq z_{ij} \leq t_i^{s_i}$, such that $f^1_{s_i z_{ij}} \supset U_{ij} \supset f^2_{i,j}$, recall that $U^1$ star refines $\hat{U}^1$, that $\hat{U}^1$ star refines $V^1$, the definition of $U(y_i)$, and Property (2) in the construction of $V^2$. Since $\bigcup_{u \in \hat{V}(y_i)} \overline{\phi(u)} \subset r^1_{s_i}, r^1_{s_i} \subset R^1$, $\bigcap_{u \in \hat{V}(y_i)} \phi(u) \supset \overline{r^2_{y_i}}, \bigcap_{u \in \hat{U}(y_i)} \phi(u) \supset \overline{r^2_{y_i}}, r^2_{y_i} \subset R^2$, and $U_{ij} \supset f^2_{i,j}$, we have $r^1_{s_i} \supset \phi(U_{ij})$ and $f^1_{s_i z_{ij}} \supset U_{ij}$ for some unique $z_{ij}$.

Observe that the chosen collection $\{U_{ij}\}_{j=1}^{t_i}$ has the property that $U_{ij} \in H_t$ where $t \geq e_{y_i}$.

Let $\alpha_1(f^2_{i,j}) = U_{ij}, \beta_1(U_{ij}) = f^1_{s_i z_{ij}}$, and $\pi_1(f^2_{i,j}) = \beta_1 \alpha_1(f^2_{i,j}) = f^1_{s_i z_{ij}}$.

**Case (2):** $y_i \notin B$.

Take $r^2_{y_i} \in R^2$, here $y_i \notin B$ and $r^2_{y_i}$ generates $F^1_i = \{f^2_{i,j}\}_{j=1}^{t_i}$ in $V^2$. Let $s_i = \min\{s |$
\( r^1_s \supset \bigcup_{u \in \hat{U}(y_i)} \phi(u) \). Take \( F^1_{s_i} = \{ f^1_{s_i,j} \}_{j=1}^{t^1_{s_i}} \) for the given \( F^2_t \), then \( \phi(f^1_{s_i,j}) \supset \bigcup_{u \in \hat{U}(y_i)} \phi(u) \).

For each \( j, 1 \leq j \leq t^2_t \), choose \( U_{ij} \in O \subset U^1 \) such that \( U_{ij} \supset f^2_{ij} \) (there is such a \( U_{ij} \) by Property (2) of \( r^2_{y_i} \) in the construction of \( V^2 \) above). Then \( U_{ij} \in U(y_i) \) and there is a unique \( z_{ij}, 1 \leq z_{ij} \leq t^1_{s_i} \), such that \( f^1_{s_i,z_{ij}} \supset U_{ij} \supset f^2_{ij} \). Let \( \alpha_1(f^2_{ij}) = U_{ij}, \beta_1(U_{ij}) = f^1_{s_i,z_{ij}}, \) and \( \pi_1(f^2_{ij}) = \beta_1 \alpha_1(f^2_{ij}) = f^1_{s_i,z_{ij}} \).

It will be shown now that the mappings \( \alpha_1 \) and \( \beta_1 \) are well defined.

We fix the choice of open sets in \( U^1 \), which are images of elements of \( V^2 \) under the mapping \( \alpha_1 \), and the question is: whether the definition of \( \beta_1 \) is correct (well defined)?

Suppose that \( \beta_1 \) is not well defined and there exist \( F^2_i \) and \( F^2_k \), two different distinguished families in \( V^2 \) such that (a) \( s_i \neq s_k \) (if \( s_i = s_k \), then \( \beta_1 \) is well defined), \( F^1_{s_i} \) is chosen for \( F^2_i \), \( F^1_{s_k} \) is chosen for \( F^2_k \), and (b) \( U_{ij} = U_{kt} \supset f^2_{ij} \cup f^2_{kt} \) where \( F^2_i = \{ f^2_{ij} \}_{j=1}^{t^2_t}, F^2_k = \{ f^2_{kj} \}_{j=1}^{t^2_t}, \) where for some \( j, 1 \leq j \leq t^2_t, U_{ij} \in U^1 \) is chosen such that \( U_{ij} \supset \overline{f^2_{ij}} \), and for some \( t, 1 \leq t \leq t^2_t, U_{kt} = U_{ij} \in U^1 \) is chosen such that \( U_{kt} \supset \overline{f^2_{kt}} \) as described above.

Case A: \( U_{ij} = U_{kt} \in O \subset U^1 \). Then \( y_i \notin B \) and \( y_k \notin B \). Indeed, \( y_i \in O_m \) and \( y_k \in O_m \) for some \( m, 1 \leq m \leq n_1 \). In this case, it follows from the definition of \( O \) that for each \( u \in \hat{U}(y_i), \phi(u) \supset O_m \) and for each \( v \in \hat{U}(y_k), \phi(v) \supset O_m \). Thus, \( \hat{U}(y_i) = \hat{U}(y_k) = \hat{V}(y_i) = \hat{V}(y_k) \), and \( s_i = s_k \) contrary to the assumption above.

Case B: \( U_{ij} = U_{kt} \in H \subset U^1, y_i \in B, \) and \( y_k \in B \).

There is \( x, 1 \leq x \leq t^2_t \), such that \( U_{ix} \in H(e_{y_i})_q \) and there is \( z, 1 \leq z \leq t^2_t \), such that \( U_{kz} \in H(e_{y_k})_s \). Since \( U_{ij} = U_{kt}, y_i \in \phi(U_{ij}) \cap \phi(U_{ix}) \) and \( y_k \in \phi(U_{kt}) \cap \phi(U_{kz}) \).

Now, \( \rho(\cap D(e_{y_i})_q, \cap D(e_{y_k})_s) < e \) since \( \text{diam } \phi(U_{ix}) < \frac{1}{4e}, \text{diam } \phi(U_{ij}) < \frac{1}{4e}, \) and \( \text{diam } \phi(U_{kz}) < \frac{1}{4e} \). Consequently, \( \cap D(e_{y_i})_q \cap \cap D(e_{y_k})_s \neq \emptyset \) by the definition of \( e \). Observe that \( D(d_{y_i})_t \) is the collection of maximal cardinality \( d_{y_i} \) that contains \( D(e_{y_i})_q \) with the smallest subscript \( t \) (as all those collections of cardinality \( d_{y_i} \) are ordered) and \( D(d_{y_k})_r \) is the collection of maximal cardinality \( d_{y_k} \) that contains \( D(e_{y_k})_s \) with smallest subscript \( r \).
Since \( (\cap D(e_{y_k}) \cap (D(e_{y_k})_r) \neq \emptyset \), \( D(d_{y_k}) \supset D(e_{y_k})_r \)), and \( D(d_{y_k}) \supset D(e_{y_k})_r \), it follows that \( d_{y_i} = d_{y_k} \) and \( t = r \).

Clearly, \( \hat{V}(y_i) = \hat{V}(y_k) \) by the definitions of \( \hat{V}(y_i) \) and \( \hat{V}(y_k) \). Consequently, \( s_i = s_k \) contrary to the assumption above.

Case C: \( y_i \notin B \) and \( y_k \in B \).

Note that if \( y_i \notin B \), then \( U_{ij} \notin H \) and if \( y_k \in B \), then \( U_{kt} \in H \). Thus, \( U_{ij} \neq U_{kt} \).

It should be clear that \( \alpha_i \) and \( \beta_i \) are well defined.

Clearly, \( \beta_i \) is defined on \( U^1 \) since \( U^1 \) is irreducible and \( V^2 \) refines \( U^1 \). Observe that \( \pi_1 \) maps distinguished families onto distinguished families.

The Dimension of \( Y \) is \( n \)

For each \( i \), let \( Z_{s_i} \) denote the union of all \( F^2_{s_i} \), distinguished families in \( V^2 \), such that \( F^1_{s_i} \) is chosen for \( F^2_{s_i} \) as described above. That is, \( \pi_1(F^2_i) = F^1_{s_i} \). Observe that \( \phi^{-1}(\phi(Z_{s_i})) = Z_{s_i} \).

Claim: The order \( \{ Z_{s_i} \}_{i=1}^{\text{card } R^2} \leq n + 1 \). Let \( x \in \bigcap_{j=1}^{m} Z_{s_{ij}} \) where \( s_{ij} \neq s_{ik}, k \neq j \). Now, for each \( i_j, x \in f_{ijt_j}^2 \subset U_{ijt_j} = \alpha_1(f_{ijt_j}^2) \). By the well definedness of \( \beta_1 \), \( U_{ijt_j} \neq U_{ikt_k} \) for \( j \neq k \).

Since \( U_{ijt_j} \in U^1 \) and order \( U^1 \leq n + 1 \), it follows that \( m \leq n + 1 \) and order \( \{ Z_{s_i} \}_{i=1}^{\text{card } R^2} \leq n + 1 \). Thus, \( Z = \{ \phi(Z_{s_i}) \}_{i=1}^{\text{card } R^2} \) is a finite open covering of \( Y \) of order \( \leq n + 1 \). Clearly, if \( G \) is any finite open covering of \( Y \), then there is \( R = R^1 \) as defined above which covers \( Y \) and refines \( G \). Hence, \( Z \) refines \( G \) and order \( Z \leq n + 1 \). Thus, \( \dim Y \leq n \).

It is not difficult to show that if a \( p \)-adic group \( A_p \) acts effectively on an \( n \)-manifold \( M \), then the orbit mapping \( \phi : M \rightarrow M/A_p \) is light open and closed. Use the fact that there is a sequence \( A_p = H_0 \supset H_1 \supset H_2 \supset \cdots \) of open (and closed) subgroups of \( A_p \) which closes down on the identity \( e \) of \( A_p \) such that when \( j > i \), \( H_i/H_j \) is a cyclic group of order \( p^{j-i} \) and \( A_p/H_i \) is a cyclic group of order \( p^i \). The cyclic group \( A_p/H_i \) acts effectively on \( M/H_i \) with orbit space \( M/A_p \).
The following theorem is a consequence of the argument above.

**Theorem.** If a \( p \)-adic group \( A_p \) acts effectively on a compact connected \( n \)-manifold, then the orbit mapping \( \phi : M \to M/A_p = Y \) is a light open mapping and \( \dim Y = n \).

The Hilbert-Smith Conjecture

The Hilbert-Smith Conjecture states that if \( G \) is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then \( G \) is a Lie group.

It is well known that if a locally compact group \( G \) acts effectively on a connected \( n \)-manifold \( M \) and \( G \) is not a Lie group [6], then there is a subgroup \( H \) of \( G \) isomorphic to a \( p \)-adic group \( A_p \) which acts effectively on \( M \). Thus, the Hilbert-Smith Conjecture can be established by proving that there is no effective action by a \( p \)-adic group \( A_p \) on a connected \( n \)-manifold \( M \).

C.T. Yang [7] has shown that if a \( p \)-adic group \( A_p \) acts effectively on a compact \( n \)-manifold \( M \), then the dimension of the orbit space \( M/A_p = Y \) is \( n + 2 \) or infinity. This contradicts the work in this paper. Hence, there is no such action and the Hilbert-Smith Conjecture is true.

**References**


