

Operads and cosimplicial objects: an introduction.

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1 Introduction.

This paper is an introduction to the series of papers [26, 28, 29, 30], in which we develop a combinatorial theory of certain important operads and their actions.¹ The operads we consider are A_∞ operads, E_∞ operads, the little n -cubes operad and the framed little disks operad. Sections 2, 6 and 9, which can be read independently, are an introduction to the theory of operads.

The reader is also referred to the very interesting papers of Batanin ([3, 4]), which treat similar questions from a categorical point of view.

Here is an outline of the paper.

In Section 2 we motivate the concept of non-symmetric operad. We give the definition of A_∞ space and state the characterization (up to weak equivalence) of loop spaces: a space is weakly equivalent to a loop space if and only if it is a grouplike A_∞ space.

In Section 3 we introduce the total space construction Tot for cosimplicial spaces and the related conormalization construction for cosimplicial abelian groups.

In Section 4 we address the question: when is Tot of a cosimplicial space an A_∞ space? We obtain a useful sufficient condition for this to happen.

In Section 5 we reformulate the main result of Section 4 in a way which is convenient for generalization.

In Section 6 we motivate the concept of operad. We give the definition of E_∞ space and state the characterization (up to weak equivalence) of infinite loop spaces: a space is weakly equivalent to an infinite loop space if and only if it is a grouplike E_∞ space.

Section 7 contains motivation for the main result of Section 8.

In Section 8 we give a sufficient condition for Tot of a cosimplicial space to be an E_∞ space.

In Section 9 we introduce the little n -cubes operad \mathcal{C}_n . Operads weakly equivalent to \mathcal{C}_n are called E_n operads. We give the characterization (up to weak equivalence) of n -fold loop

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¹The relation between these papers and [27] is explained in Remarks 12.5 and 15.5.

spaces: a space is weakly equivalent to an n -fold loop space if and only if it has a grouplike action of an E_n operad.

In Section 10 we give a sufficient condition for Tot of a cosimplicial space to be an E_n space.

In Section 11 we describe some category theory which is used in the proof of the main theorem of Section 10.

In Section 12 we outline the proof of the main theorem of Section 10. As a byproduct we get a new, combinatorial, description (up to weak equivalence) of \mathcal{C}_n .

In Section 13 we describe applications of the main result in Section 10 to a certain space of knots and to topological Hochschild cohomology.

In Section 14 we develop a combinatorial description of the framed little disks operad.

In Section 15 we observe that the theory of Sections 4, 5, 8, 10, 12 and 14 remains valid with spaces replaced by chain complexes. In particular this leads to concrete and explicit chain models for \mathcal{C}_n and for the framed little disks operad.

In Section 16 we give some applications of the theory developed in Section 15; in particular we discuss Deligne's Hochschild cohomology conjecture.

2 Loop spaces and the little intervals operad.

Historically, the first use of operads was to give a precise meaning to the idea that loop spaces are monoids up to higher homotopy. In this section we recall how this works.

The first step is to reformulate the concept of monoid in a way that is amenable to generalization.

Proposition 2.1. *A monoid structure on a set S determines and is determined by a family of maps*

$$M(k) : S^k \rightarrow S$$

for $k \geq 0$ (where S^k denotes the k -fold Cartesian product) such that

- (a) $M(1)$ is the identity map, and
- (b) the set $\{M(k)\}_{k \geq 0}$ is closed under multivariable composition.

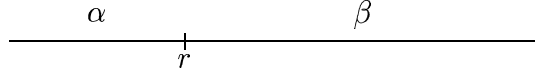
Proof. If S is a monoid with multiplication $M : S^2 \rightarrow S$ and unit $e : S^0 \rightarrow S$ we define $M(0)$ to be e , $M(1)$ to be the identity map, and $M(k)$ to be the iterated multiplication for $k \geq 2$. The monoid axioms show that the set $\{M(k)\}_{k \geq 0}$ is closed under multivariable composition.

Conversely, if S is a set with maps $M(k)$ satisfying (a) and (b) then S is a monoid with multiplication $M(2)$ and unit $M(0)$. \square

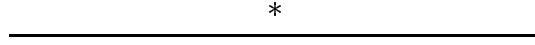
Next let Z be a based space with basepoint denoted by $*$. We consider the space ΩZ of based loops on Z . For each $r \in (0, 1)$ there is a multiplication

$$M_r : (\Omega Z)^2 \rightarrow \Omega Z$$

which takes a pair of loops (α, β) to the loop $M_r(\alpha, \beta)$ which is α (suitably rescaled) on the interval $[0, r]$ and β (suitably rescaled) on the interval $[r, 1]$. We represent the loop $M_r(\alpha, \beta)$ by the picture



We write $* \in \Omega Z$ for the constant loop at the basepoint, which we represent by the picture



and we write

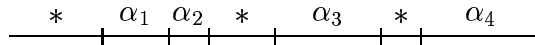
$$e : (\Omega Z)^0 \rightarrow \Omega Z$$

for the map whose image is $*$.

Motivated by Proposition 2.1, we consider the space $\mathcal{M}(k)$ of all maps

$$(\Omega Z)^k \rightarrow \Omega Z$$

that can be obtained by multivariable composition from the maps M_r and e . A typical example is the map in $\mathcal{M}(4)$ which takes a 4-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to the loop



In general, a map in $\mathcal{M}(k)$ is determined by k closed intervals in $[0, 1]$ with disjoint interiors (notice that, as in the example just given, the union of these intervals doesn't have to be all of $[0, 1]$). This motivates the following definition:

Definition 2.2. Let $\mathcal{A}(k)$ be the set in which an element is a set of k closed intervals in $[0, 1]$ with disjoint interiors (in particular, $\mathcal{A}(0)$ is a point). Give $\mathcal{A}(k)$ the topology induced by the following imbedding of $\mathcal{A}(k)$ in \mathbb{R}^{2k} : given k closed intervals in $[0, 1]$, list the $2k$ endpoints of the intervals in increasing order.

What we have shown so far is that $\mathcal{M}(k)$ is homeomorphic to $\mathcal{A}(k)$.² Moreover, it is easy to see that each $\mathcal{A}(k)$ is contractible, so to sum up we have

Proposition 2.3. *If $Y = \Omega Z$ for some Z then there is a family of subspaces $\mathcal{M}(k) \subset \text{Map}(Y^k, Y)$ such that*

- (a) $\mathcal{M}(1)$ contains the identity map,
- (b) the family $\mathcal{M} = \{\mathcal{M}_k\}_{k \geq 0}$ is closed under multivariable composition, and
- (c) each $\mathcal{M}(k)$ is contractible.

The crucial fact about this situation is that Proposition 2.3 has a converse up to weak equivalence: if Y is any connected space which has a family of contractible subspaces $\mathcal{M}(k) \subset \text{Map}(Y^k, Y)$ satisfying (a), (b) and (c) then Y is weakly equivalent to ΩZ for some Z (this

²In making this statement we must exclude the special case where the path-component of the basepoint in Z is a single point. On the other hand, the proposition which follows remains true in this case.

is a special case of Theorem 2.12 below). This gives us a way of recognizing that a space is a loop space (up to weak equivalence) without knowing in advance that it is a loop space.

Motivated by Propositions 2.1 and 2.3 we make a first attempt at the definition of non-symmetric³ operad.

Provisional Definition 2.4. A non-symmetric operad \mathcal{O} is a collection of subspaces

$$\mathcal{O}(k) \subset \text{Map}(Y^k, Y) \quad k \geq 0$$

(for some space Y) such that

- (a) $\mathcal{O}(1)$ contains the identity map and
- (b) the collection \mathcal{O} is closed under multivariable composition.

Critique of Provisional Definition 2.4. This definition is formally analogous to the nineteenth-century definition of a group as a family of bijections of a set S , closed under composition and inverses. The advantage of such a definition is its concreteness and the ease with which our minds assimilate it. The disadvantage (in the case of groups) is that the set S is really external to the group. The resolution of this difficulty (in the case of groups) was to split the original definition into two concepts: the concept of (abstract) group and the concept of group action.

Motivated by the Critique, we will split the Provisional Definition into two concepts: the concept of (abstract) non-symmetric operad and the concept of operad action.

First observe that (in the situation of 2.4) the multivariable composition operations in $\{\text{Map}(Y^k, Y)\}_{k \geq 0}$ restrict to give maps

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \rightarrow \mathcal{O}(j_1 + \cdots + j_k)$$

for each choice of $k, j_1, \dots, j_k \geq 0$. The associativity property of multivariable composition implies that the following diagram commutes for all choices of $k, j_1, \dots, j_k, \{i_{mn}\}_{m \leq k, n \leq j_m}$.

$$(2.1) \quad \begin{array}{ccc} \mathcal{O}(k) \times \prod_{m=1}^k \left(\mathcal{O}(j_m) \times \prod_{n=1}^{j_m} \mathcal{O}(i_{mn}) \right) & \xrightarrow{1 \times \gamma} & \mathcal{O}(k) \times \prod_{m=1}^k \mathcal{O}(i_{m1} + \cdots + i_{mj_m}) \\ \downarrow = & & \downarrow \gamma \\ \left(\mathcal{O}(k) \times \prod_{m=1}^k \mathcal{O}(j_m) \right) \times \prod_{m,n} \mathcal{O}(i_{mn}) & & \\ \downarrow \gamma \times 1 & & \downarrow \gamma \\ \mathcal{O}(j_1 + \cdots + j_k) \times \mathcal{O}(i_{11}) \times \cdots \times \mathcal{O}(i_{kj_k}) & \xrightarrow{\gamma} & \mathcal{O}(i_{11} + \cdots + i_{kj_k}) \end{array}$$

Definition 2.5. A non-symmetric operad \mathcal{O} is a collection of spaces $\{\mathcal{O}(k)\}_{k \geq 0}$ together with an element $1 \in \mathcal{O}(1)$ and maps

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \rightarrow \mathcal{O}(j_1 + \cdots + j_k)$$

³The word “non-symmetric” refers to the fact that we haven’t yet used, or needed, the action of the symmetric group on Y^k ; see Section 6.

(for each choice of $k, j_1, \dots, j_k \geq 0$) such that

- (a) for each k and each $s \in \mathcal{O}(k)$, $\gamma(1, s) = s$ and $\gamma(s, 1, \dots, 1) = s$, and
- (b) Diagram (2.1) commutes for all choices of $k, j_1, \dots, j_k, i_{11}, \dots, i_{kj_k}$.

Remark 2.6. Any collection \mathcal{O} which satisfies Provisional Definition 2.4 will also satisfy Definition 2.5.

Remark 2.7. Here are some examples which will be important later.

(a) The collection $\mathcal{A} = \{\mathcal{A}(k)\}_{k \geq 0}$ defined in Definition 2.2 is a non-symmetric operad (by Remark 2.6); it is called the *little intervals non-symmetric operad*. It is instructive to work out the explicit description of the maps γ in this case (cf. Section 9).

(b) If Y is any space the collection

$$\{\text{Map}(Y^k, Y)\}_{k \geq 0},$$

with its usual multivariable composition, is called the *endomorphism operad of Y* and denoted \mathcal{E}_Y .

(c) More generally, if \mathcal{U} is any topological category with a monoidal product \square (see [22, Section VII.1]) and D is an object of \mathcal{U} then the collection of spaces

$$\{\text{Hom}_{\mathcal{U}}(D^{\square k}, D)\}_{k \geq 0}$$

with the evident multivariable composition is a non-symmetric operad. The proof is left as an exercise for the reader; see Section 11 for a hint.

(d) With \mathcal{U} as above, note that \square is also a monoidal product for \mathcal{U}^{op} . Applying part (c) to \mathcal{U}^{op} gives a non-symmetric operad whose k -th space is

$$\text{Hom}_{\mathcal{U}^{\text{op}}}(D^{\square k}, D) = \text{Hom}_{\mathcal{U}}(D, D^{\square k})$$

Next we formulate the concept of operad action. First observe that in the situation of Provisional Definition 2.4 the evaluation maps

$$\text{Map}(Y^k, Y) \times Y^k \rightarrow Y$$

restrict to maps

$$\theta : \mathcal{O}(k) \times Y^k \rightarrow Y$$

and that the diagram

$$(2.2) \quad \begin{array}{ccc} \mathcal{O}(k) \times \left(\prod_{m=1}^k \mathcal{O}(j_m) \right) \times Y^{j_1 + \dots + j_k} & \xrightarrow{\cong} & \mathcal{O}(k) \times \prod_{m=1}^k (\mathcal{O}(j_m) \times Y^{j_m}) \\ \downarrow \gamma \times 1 & & \downarrow 1 \times \prod \theta \\ \mathcal{O}(j_1 + \dots + j_k) \times Y^{j_1 + \dots + j_k} & \xrightarrow{\theta} & \mathcal{O}(k) \times Y^k \\ & & \downarrow \theta \\ & & Y \end{array}$$

commutes for all $k, j_1, \dots, j_k \geq 0$.

Definition 2.8. Let \mathcal{O} be a non-symmetric operad and let Y be a space. An action of \mathcal{O} on Y consists of a map

$$\theta : \mathcal{O}(k) \times Y^k \rightarrow Y$$

for each $k \geq 0$ such that

- (a) $\theta(1, x) = x$ for all $x \in Y$, and
- (b) Diagram (2.2) commutes for all $k, j_1, \dots, j_k \geq 0$.

Example 2.9. (a) The non-symmetric operad \mathcal{A} mentioned in Remark 2.7(a) acts on ΩZ for any space Z .

- (b) The endomorphism operad \mathcal{E}_Y acts on Y .

We conclude this section by stating the most general converse to Proposition 2.3. First we need two definitions.

Definition 2.10. An A_∞ operad is a non-symmetric operad \mathcal{O} for which each space $\mathcal{O}(k)$ is weakly equivalent to a point.

For example, the non-symmetric operad \mathcal{A} of Remark 2.7(a) is an A_∞ operad.

Notice the relationship between this definition and Proposition 2.1: the one-point sets $\{M(k)\}$ in Proposition 2.1 are replaced by contractible spaces in Definition 2.10.

Now let Y be a space with an action of an A_∞ operad \mathcal{O} . Because $\mathcal{O}(2)$ and $\mathcal{O}(0)$ are connected, the maps

$$\theta : \mathcal{O}(2) \times Y^2 \rightarrow Y$$

and

$$\theta : \mathcal{O}(0) \times Y^0 \rightarrow Y$$

induce a monoid structure on $\pi_0 Y$.

Definition 2.11. The action of \mathcal{O} on Y is *grouplike* if the monoid $\pi_0 Y$ is a group.

For example, the action in Example 2.9(a) is grouplike. Also, if Y is connected then all actions are grouplike.

Theorem 2.12. *Y is weakly equivalent to ΩZ for some space $Z \iff Y$ has a grouplike action of an A_∞ operad.*

Remark 2.13. This theorem developed gradually during the period from 1960 to 1974. In the \Leftarrow direction, the first version was proved by Stasheff [34], assuming that Y is connected and using a particular non-symmetric operad, now called the Stasheff operad (but the concept of operad hadn't yet been defined at that time). Boardman and Vogt proved the \Leftarrow direction for general A_∞ operads (except that they used PROP's instead of operads), but still assuming Y connected, in [5, 6]. May defined the concept of operad in [24] and proved the \Leftarrow direction for connected Y ; he proved the general version (for group-complete actions) in [25]. The \Rightarrow direction (for PROP's, which implies the result for operads) is due to Boardman and Vogt [5, 6].

3 Cosimplicial objects and totalization.

Theorem 2.12 leads to the question of how we can tell when a space Y has an action of an A_∞ operad. In the next section we will give an answer to this question in the important special case where Y is the total space of a cosimplicial space X^\bullet . In this section we pause for some background about cosimplicial objects.

Throughout this paper we will use the following conventions for cosimplicial objects.

Definition 3.1. (a) Define Δ to be the category of nonempty finite totally ordered sets (this is equivalent to the category usually called Δ). Define $[m]$ to be the finite totally ordered set $\{0, \dots, m\}$. Define

$$d^i : [m] \rightarrow [m+1] \quad 0 \leq i \leq m+1$$

to be the unique ordered injection whose image does not contain i , and define

$$s^i : [m] \rightarrow [m-1] \quad 0 \leq i \leq m-1$$

to be the unique ordered surjection for which the inverse image of i contains two points.

(b) Given a category \mathcal{C} , a cosimplicial object X^\bullet in \mathcal{C} is a functor from Δ to \mathcal{C} . If S is a nonempty finite totally ordered set then X^S will denote the value of X^\bullet at S , except that we write X^m instead of $X^{[m]}$. The maps

$$X^m \rightarrow X^{m+1}$$

induced by the d^i are called *coface maps* and the maps

$$X^m \rightarrow X^{m-1}$$

induced by the s^i are called *codegeneracy maps*.

Note that every object in Δ has a unique isomorphism to an object of the form $[m]$, so we can specify a cosimplicial object by giving its value on the objects $[m]$ (together with the coface and codegeneracy maps). For example:

Definition 3.2. Δ^\bullet is the cosimplicial space whose value at $[m]$ is the simplex Δ^m , with the usual coface and codegeneracy maps.

Next we define the cosimplicial analog of geometric realization. First recall (for example, from [15, Example 2.4(3)]) that the geometric realization of a simplicial space U_\bullet is a tensor product over Δ (also called a coend):

$$|U_\bullet| = U_\bullet \otimes_\Delta \Delta^\bullet$$

When we change the variance from simplicial to cosimplicial it is natural to replace \otimes_Δ by Hom_Δ , which leads us to the following definition.

Definition 3.3. Let X^\bullet be a cosimplicial space. The *total space* of X^\bullet , denoted $\text{Tot}(X^\bullet)$, is the space of cosimplicial maps $\text{Hom}_\Delta(\Delta^\bullet, X^\bullet)$.

Here's a more explicit description: a point in $\text{Tot}(X^\bullet)$ is a sequence

$$\alpha_0 : \Delta^0 \rightarrow X^0, \alpha_1 : \Delta^1 \rightarrow X^1, \alpha_2 : \Delta^2 \rightarrow X^2, \dots$$

which is *consistent*, i.e.,

$$d^i \circ \alpha_n = \alpha_{n+1} \circ d^i$$

and

$$s^i \circ \alpha_n = \alpha_{n-1} \circ s^i$$

for all i .

Example 3.4. Given a based space Z with basepoint $*$, we define a cosimplicial space $F^\bullet Z$ whose total space is ΩZ ($F^\bullet Z$ is called the geometric cobar construction on Z). The m -th space $F^m Z$ is the Cartesian product Z^m . The coface

$$d^i : F^m Z \rightarrow F^{m+1} Z$$

is defined by

$$d^i(z_1, \dots, z_m) = \begin{cases} (*, z_1, \dots, z_m) & \text{if } i = 0 \\ (z_1, \dots, z_i, z_i, \dots, z_m) & \text{if } 1 \leq i \leq m \\ (z_1, \dots, z_m, *) & \text{if } i = m + 1. \end{cases}$$

and the codegeneracy $s^i : F^m Z \rightarrow F^{m-1} Z$ deletes the $(i - 1)$ -st coordinate. The proof that $\text{Tot}(F^\bullet Z)$ is homeomorphic to ΩZ is left as an exercise for the reader. (Hint: if $m > 1$ then the map

$$\prod_{i=0}^{m-1} s_i : F^m Z \rightarrow \prod_{i=0}^{m-1} F^{m-1} Z$$

is a monomorphism).

We will also consider cosimplicial abelian groups.

Example 3.5. Let W be a space and define $S^\bullet W$ to be the cosimplicial abelian group $\text{Map}(S_\bullet W, \mathbb{Z})$, where $S_\bullet W$ is the usual simplicial set associated to W ([9, Example 1.28]) and Map means maps of sets.

Next we define the analog of Tot in this context. Let Δ_{simp}^m be the standard simplicial model of Δ^m ([9, Example 1.4]).

Definition 3.6. Let Δ_\bullet^* denote the cosimplicial chain complex which in degree m is the normalized chain complex ([38, pages 265–266]) of Δ_{simp}^m .

Definition 3.7. Let A^\bullet be a cosimplicial abelian group. The *conormalization*⁴ of A^\bullet , denoted $\mathbf{C}(A^\bullet)$, is the cochain complex

$$\mathrm{Hom}_\Delta(\Delta_\bullet, A^\bullet) \subset \prod_{m=0}^{\infty} \mathrm{Hom}(\Delta_\bullet^m, A^m).$$

Here Hom_Δ is Hom in the category of cosimplicial graded abelian groups (with A^m concentrated in dimension 0), and the differential is induced by the differentials of the Δ_\bullet^m .

Remark 3.8. Here are two concrete descriptions of $\mathbf{C}(A^\bullet)$; they are dual to the two standard ways of describing the normalization of a simplicial abelian group ([38, pages 265–266]). The proof that they agree with Definition 3.7 is left to the reader.

(a) Let $\mathbf{C}'(A^\bullet)$ be the cochain complex whose m -th group is the intersection of the kernels of the codegeneracies $s^i : A^m \rightarrow A^{m-1}$ and whose differential is $\sum (-1)^i d^i$. Then $\mathbf{C}(A^\bullet)$ is isomorphic to $\mathbf{C}'(A^\bullet)$.

(b) Let $\mathbf{C}''(A^\bullet)$ be the cochain complex whose m -th group is the cokernel of

$$\bigoplus_{i>0} d^i : \bigoplus_{i>0} A^{m-1} \rightarrow A^m$$

and whose differential is induced by d^0 . Then $\mathbf{C}(A^\bullet)$ is isomorphic to $\mathbf{C}''(A^\bullet)$

Example 3.9. The conormalization of $S^\bullet W$ (Example 3.5) is the complex of singular cochains that vanish on all degenerate singular chains. This is what is usually called the normalized singular cochain complex of W ; we will denote it by s^*W .

4 A sufficient condition for $\mathrm{Tot}(X^\bullet)$ to be an A_∞ space.

Definition 4.1. An A_∞ space is a space with an action of an A_∞ operad.

Let Z be a based space and let $F^\bullet Z$ be the cosimplicial space defined in Example 3.4. Then $\mathrm{Tot}(F^\bullet Z)$ is homeomorphic to ΩZ and in particular (as we have seen in Section 2) it is an A_∞ space. This leads us to the question:

Question 4.2. For what other cosimplicial spaces is Tot an A_∞ space?

As we have seen in Section 3, Tot is analogous to conormalization, so we can gain insight into Question 4.2 by examining a cosimplicial abelian group whose conormalization has a multiplicative structure, namely $S^\bullet W$ (see Example 3.5). The conormalization of $S^\bullet W$ is s^*W (see Example 3.9), and s^*W has an associative multiplication, the cup product, given by the usual Alexander-Whitney formula

$$(x \smile y)(\sigma) = x(\sigma(0, \dots, p)) \cdot y(\sigma(p, \dots, p+q));$$

⁴This functor is usually called normalization, but it seems desirable to have separate names for the cosimplicial and simplicial versions of normalization, analogous to the usual distinction between Tot and geometric realization.

here x has degree p , y has degree q , σ is in $S_{p+q}W$, \cdot is multiplication in \mathbb{Z} , and $\sigma(0, \dots, p)$ (resp., $\sigma(p, \dots, p+q)$) is the restriction of σ to the subsimplex of Δ^{p+q} spanned by the vertices $0, \dots, p$ (resp., $p, \dots, p+q$). The key point for our purpose is that the same formula defines a map

$$(4.1) \quad \smile: S^p W \times S^q W \rightarrow S^{p+q} W$$

and we can examine the relation between \smile and the coface and codegeneracy maps of $S^\bullet W$. This relation is given by the following formulas:

$$(4.2) \quad d^i(x \smile y) = \begin{cases} d^i x \smile y & \text{if } i \leq p \\ x \smile d^{i-p} y & \text{if } i > p \end{cases}$$

$$(4.3) \quad d^{p+1} x \smile y = x \smile d^0 y$$

$$(4.4) \quad s^i(x \smile y) = \begin{cases} s^i x \smile y & \text{if } i \leq p-1 \\ x \smile s^{i-p} y & \text{if } i \geq p \end{cases}$$

Next we observe that the cosimplicial space $F^\bullet Z$ has the same kind of structure as $S^\bullet W$: if we define

$$\smile: F^p Z \times F^q Z \rightarrow F^{p+q} Z$$

to be the obvious juxtaposition map

$$Z^p \times Z^q \rightarrow Z^{p+q}$$

then \smile satisfies (4.2), (4.3) and (4.4). Moreover, it is associative:

$$(x \smile y) \smile z = z \smile (y \smile z),$$

and unital: there is an element $e \in F^0 Z$ (namely the basepoint) such that

$$x \smile e = e \smile x = x$$

for all x . This suggests that, as a way of answering Question 4.2, we consider cosimplicial spaces having the same kind of structure as $S^\bullet W$ or $F^\bullet Z$:

Theorem 4.3. *If X^\bullet is a cosimplicial space with a cup product*

$$\smile: X^p \times X^q \rightarrow X^{p+q}$$

which is associative and unital and satisfies (4.2), (4.3) and (4.4) then $\text{Tot}(X^\bullet)$ is an A_∞ space.

Remark 4.4. (a) This result is due to Batanin [2, Theorems 5.1 and 5.2] with a simplified proof by us [28, Section 3].

(b) Theorem 4.3 gives a sufficient but not a necessary condition for $\text{Tot}(X^\bullet)$ to be an A_∞ space. However, we expect that any A_∞ space is weakly equivalent to one produced by Theorem 4.3 (in fact it is likely that Tot induces a Quillen equivalence between cosimplicial spaces satisfying the hypothesis of Theorem 4.3 and A_∞ spaces).

The remainder of this section gives an outline of the proof of Theorem 4.3; for details see [28, Section 3].

The first step in the proof is:

Proposition 4.5. *The category of cosimplicial spaces has a monoidal structure \square with the property that X^\bullet satisfies the hypothesis of Theorem 4.3 if and only if it is a \square -monoid.*

This is due to Batanin [1]. The definition of \square is modeled on equations (4.2), (4.3) and (4.4): $X^\bullet \square Y^\bullet$ is the cosimplicial space whose m -th space is

$$\left(\coprod_{p+q=m} X^p \times Y^q \right) / \sim$$

where \sim is the equivalence relation generated by $(x, d^0 y) \sim (d^{|x|+1} x, y)$. The coface maps are defined by

$$d^i(x, y) = \begin{cases} (d^i x, y) & \text{if } i \leq |x| \\ (x, d^{i-|x|} y) & \text{if } i > |x| \end{cases}$$

and the codegeneracy maps by

$$s^i(x, y) = \begin{cases} (s^i x, y) & \text{if } i \leq |x| - 1 \\ (x, s^{i-|x|} y) & \text{if } i \geq |x| \end{cases}$$

Next we apply Remark 2.7(d) with $D = \Delta^\bullet$ to get a non-symmetric operad \mathcal{B} . The space $\mathcal{B}(k)$ is

$$\text{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\square k})$$

The composition maps γ are defined as follows: if $f \in \mathcal{B}(k)$ and $g_i \in \mathcal{B}(j_i)$ for $1 \leq i \leq k$ then $\gamma(f, g_1, \dots, g_k) \in \mathcal{B}(j_1 + \dots + j_k)$ is the composite

$$\Delta^\bullet \xrightarrow{f} (\Delta^\bullet)^{\square k} \xrightarrow{g_1 \square \dots \square g_k} (\Delta^\bullet)^{\square(j_1 + \dots + j_k)}$$

Now let X^\bullet be a \square -monoid. We define an action of \mathcal{B} on $\text{Tot}(X^\bullet)$ by letting

$$\theta : \mathcal{B}(k) \times (\text{Tot}(X^\bullet))^k \rightarrow \text{Tot}(X^\bullet)$$

take $(f, \tau_1, \dots, \tau_k)$ to the composite

$$\Delta^\bullet \xrightarrow{f} (\Delta^\bullet)^{\square k} \xrightarrow{\tau_1 \square \dots \square \tau_k} (X^\bullet)^{\square k} \xrightarrow{\mu} X^\bullet$$

where μ is the monoidal structure map of X^\bullet .

To complete the proof of Theorem 4.3 it only remains to show that each $\mathcal{B}(k)$ is contractible. This is an easy consequence of the fact (due to Grayson [14]) that $(\Delta^\bullet)^{\square k}$ is isomorphic as a cosimplicial space to Δ^\bullet . See [28, Section 3] for details.

5 A reformulation.

Our next goal is to generalize Theorem 4.3. However, it turns out that the analogs of equations (4.2), (4.3) and (4.4) for the situations we will be considering are rather complicated and inconvenient, so we pause to reformulate Theorem 4.3 in a way that is more amenable to generalization.

Let us return to the motivating example $S^\bullet W$. Define

$$\sqcup : S^p W \otimes S^q W \rightarrow S^{p+q+1} W$$

by

$$(x \sqcup y)(\sigma) = x(\sigma(0, \dots, p)) \cdot y(\sigma(p+1, \dots, p+q+1))$$

for $\sigma \in S_{p+q+1} W$. Note that, in contrast to the cup product, the vertex p is not repeated in the formula for \sqcup .

This operation is related to the coface and codegeneracy operations in $S^\bullet W$ by the following equations:

$$(5.1) \quad d^i(x \sqcup y) = \begin{cases} d^i x \sqcup y & \text{if } i \leq p+1 \\ x \sqcup d^{i-p-2} y & \text{if } i > p+1 \end{cases}$$

$$(5.2) \quad s^i(x \sqcup y) = \begin{cases} s^i x \sqcup y & \text{if } i < p \\ x \sqcup s^{i-p-1} y & \text{if } i > p \end{cases}$$

Note that there is no analog for \sqcup of equation (4.3). \sqcup is associative:

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$$

and unital: there exists $e \in X^0$ with

$$(5.3) \quad s^p(x \sqcup e) = s^0(e \sqcup x) = x.$$

The operations \smile and \sqcup determine each other:

$$(5.4) \quad x \sqcup y = (d^{p+1} x) \smile y = x \smile d^0 y$$

$$(5.5) \quad x \smile y = s^p(x \sqcup y)$$

Now let X^\bullet be a cosimplicial space. If X^\bullet has an operation

$$\sqcup : X^p \times X^q \rightarrow X^{p+q+1}$$

which is associative and unital (in the sense of equation (5.3)) and satisfies (5.1) and (5.2) then the operation \smile defined by equation (5.5) satisfies the hypothesis of Theorem 4.3 (the verification is left to the reader). Conversely, if X^\bullet has a cup product satisfying the hypothesis of Theorem 4.3 then the \sqcup product defined by (5.4) is associative, unital and satisfies (5.1) and (5.2). To sum up:

Proposition 5.1. X^\bullet satisfies the hypothesis of Theorem 4.3 if and only if it has a product

$$\sqcup : X^p \times X^q \rightarrow X^{p+q+1}$$

which is associative and unital and satisfies (5.1) and (5.2).

Corollary 5.2. If X^\bullet is a cosimplicial space with a product

$$\sqcup : X^p \times X^q \rightarrow X^{p+q+1}$$

which is associative and unital and satisfies (5.1) and (5.2) then $\text{Tot}(X^\bullet)$ is an A_∞ space.

6 Operads.

As we have seen in Section 2, the “associativity up to higher homotopy” of the multiplication on ΩZ can be formulated rigorously as the action of an A_∞ operad on ΩZ . We would now like to give an analogous formulation of “commutativity up to higher homotopy.” A multiplication is commutative in the ordinary sense if it is invariant under permutations of the factors; this suggests that we add symmetric-group actions to Definition 2.5. We begin with a provisional form of the definition.

Provisional Definition 6.1. An operad \mathcal{O} is a collection of subspaces

$$\mathcal{O}(k) \subset \text{Map}(Y^k, Y) \quad k \geq 0$$

(for some space Y) such that

- (a) $\mathcal{O}(1)$ contains the identity map,
- (b) the collection \mathcal{O} is closed under multivariable composition, and
- (c) each $\mathcal{O}(k)$ is closed under the permutation action of the symmetric group Σ_k .

As in Section 2 we split the provisional definition into the concept of (abstract) operad and the concept of operad action.

To formulate the definition of operad, we need to know the relation between the action of Σ_k and the multivariable composition maps γ in Provisional Definition 6.1. This is left as an exercise for the reader; the answer is given in [24, Definition 1.1(c)].

Definition 6.2. An operad is a non-symmetric operad \mathcal{O} together with, for each k , a right action of Σ_k satisfying the formulas of [24, Definition 1.1(c)].

Remark 6.3. (a) Let Y be a space. The endomorphism operad \mathcal{E}_Y (Remark 2.7(b)) is an operad, where the k -th space $\text{Map}(Y^k, Y)$ is given the obvious right action of Σ_k .

(b) If \mathcal{U} is a topological category with a *symmetric* monoidal product \square (see [22, Section VII.7]) then the non-symmetric operads in Remarks 2.7(c) and (d) are operads, with the obvious Σ_k actions on $\text{Hom}_{\mathcal{U}}(D^{\square k}, D)$ and $\text{Hom}_{\mathcal{U}}(D, D^{\square k})$.

(c) If \mathcal{O} is a non-symmetric operad we can define an operad \mathcal{O}' by

$$\mathcal{O}'(k) = \mathcal{O}(k) \times \Sigma_k$$

with the obvious right Σ_k action; the definition of the composition maps γ for \mathcal{O}' is left as an exercise. We call \mathcal{O}' the operad generated by \mathcal{O} .

Definition 6.4. Let \mathcal{O} be an operad and let Y be a space. An action of \mathcal{O} on Y is an action of the underlying non-symmetric operad with the property that each map

$$\theta : \mathcal{O}(k) \times Y^k \rightarrow Y$$

factors through $\mathcal{O}(k) \times_{\Sigma_k} Y^k$.

Remark 6.5. If Y is a space then \mathcal{E}_Y acts on Y .

Definition 6.6. An E_∞ operad is an operad \mathcal{O} for which each space $\mathcal{O}(k)$ is weakly equivalent to a point.⁵

A space with an action of an E_∞ operad should be thought of as “commutative up to all higher homotopies.”

The analog of Theorem 2.12 in this setting is a statement about “infinite loop spaces.” Recall that an infinite loop space is a space X for which there exists a sequence X_1, X_2, \dots with X homeomorphic to ΩX_1 and X_i homeomorphic to ΩX_{i+1} for all i (thus an infinite loop space is the zeroth space of a spectrum).

Theorem 6.7. Y is weakly equivalent to an infinite loop space $\iff Y$ has a grouplike action of an E_∞ operad.

The \implies direction, and the \impliedby direction for connected Y , are due to Boardman and Vogt [5, 6]. May gave a simpler proof of the \impliedby direction for connected Y in [24] and proved the general case in [25].

7 A family of cochain operations.

We want to give an E_∞ analog of Corollary 5.2. In this section we prepare the way by returning to the motivating example, $S^\bullet W$, and defining a family of operations that generalizes \sqcup . The idea is that the definition of \sqcup is based on the partition of the set $\{0, \dots, p+q+1\}$ into $\{0, \dots, p\}$ and $\{p+1, \dots, p+q+1\}$; we can produce more operations by using other partitions.

First we need some notation. Recall that we have defined Δ to be the category of nonempty finite totally ordered sets T . For $T \in \Delta$ we define Δ^T to be the convex hull of T (in particular, $\Delta^{[m]}$ is the usual Δ^m). We define $S_T W$ to be the set of all continuous maps $\Delta^T \rightarrow W$ (in particular, $S_{[m]} W$ is what we have been calling $S_m W$) and $S^T W$ to be $\text{Map}(S_T W, \mathbb{Z})$ (so $S^{[m]} W$ is the same as $S^m W$).

Definition 7.1. Given a map $\sigma : \Delta^m \rightarrow W$ and a subset U of T , let $\sigma(U)$ be the restriction of σ to Δ^U .

Now observe that a partition of T into two pieces is the same thing as a surjective function $f : T \rightarrow \{1, 2\}$.

⁵For technical reasons, it is usual to require in addition that the action of each Σ_k should be free. The operad \mathcal{D} that we construct in Section 8 does have this property.

Definition 7.2. Given a surjection $f : T \rightarrow \{1, 2\}$, define a natural transformation

$$\langle f \rangle : S^{f^{-1}(1)}W \times S^{f^{-1}(2)}W \rightarrow S^T W$$

by the equation

$$\langle f \rangle(x, y)(\sigma) = x(\sigma(f^{-1}(1))) \cdot y(\sigma(f^{-1}(2)))$$

for $\sigma \in S_T W$; here \cdot is multiplication in \mathbb{Z} .

Remark 7.3. If f is the function $\{0, \dots, p+q+1\} \rightarrow \{1, 2\}$ that takes $\{0, \dots, p\}$ to 1 and $\{p+1, \dots, p+q+1\}$ to 2 then $\langle f \rangle$ is \sqcup .

Next we describe the relation between the operations $\langle f \rangle$ and the cosimplicial structure maps of $S^\bullet W$.

Proposition 7.4. *Let*

$$\begin{array}{ccc} T & \xrightarrow{\phi} & T' \\ & \searrow f & \swarrow g \\ & & \{1, 2\} \end{array}$$

be a commutative diagram, where ϕ is a map in Δ (i.e., an order-preserving map). For $i = 1, 2$ let

$$\phi_i : f^{-1}(i) \rightarrow g^{-1}(i)$$

be the restriction of ϕ .

Then the diagram

$$\begin{array}{ccc} S^{f^{-1}(1)}W \times S^{f^{-1}(2)}W & \xrightarrow{\langle f \rangle} & S^T W \\ (\phi_1)_* \times (\phi_2)_* \downarrow & & \downarrow \phi_* \\ S^{g^{-1}(1)}W \times S^{g^{-1}(2)}W & \xrightarrow{\langle g \rangle} & S^{T'} W \end{array}$$

commutes.

The proof is an immediate consequence of the definitions. In the special case of Remark 7.3 we recover equations (5.1) and (5.2).

Next we formulate the commutativity, associativity and unitality properties of the $\langle f \rangle$ operations. Commutativity is easy:

Proposition 7.5. *The diagram*

$$\begin{array}{ccc} S^{f^{-1}(1)}W \times S^{f^{-1}(2)}W & \xrightarrow{\langle f \rangle} & S^T W \\ \tau \downarrow & & \downarrow = \\ S^{f^{-1}(2)}W \times S^{f^{-1}(1)}W & \xrightarrow{\langle t \circ f \rangle} & S^T W \end{array}$$

commutes, where τ is the switch map and t is the transposition of $\{1, 2\}$.

For the associativity condition we need some notation. Define

$$\alpha : \{1, 2, 3\} \rightarrow \{1, 2\}$$

by $\alpha(1) = 1, \alpha(2) = 1, \alpha(3) = 2$ and

$$\beta : \{1, 2, 3\} \rightarrow \{1, 2\}$$

by $\beta(1) = 1, \beta(2) = 2, \beta(3) = 2$. Given a surjection $g : T \rightarrow \{1, 2, 3\}$ let g_1 be the restriction of g to $g^{-1}\{1, 2\}$ and let g_2 be the restriction of g to $g^{-1}\{2, 3\}$.

Proposition 7.6. *With the notation above, the diagram*

$$\begin{array}{ccc} S^{g^{-1}(1)}W \times S^{g^{-1}(2)}W \times S^{g^{-1}(3)}W & \xrightarrow{\langle g_1 \rangle \times 1} & S^{g^{-1}\{1,2\}}W \times S^{g^{-1}(3)}W \\ \downarrow 1 \times \langle g_2 \rangle & & \downarrow \langle \alpha \circ g \rangle \\ S^{g^{-1}(1)}W \times S^{g^{-1}\{2,3\}}W & \xrightarrow{\langle \beta \circ g \rangle} & S^T W \end{array}$$

commutes for every choice of T and g .

Again, the proof is immediate from the definitions.

For the unital property we need to extend $S^\bullet W$ to the category of all finite totally ordered sets, including the empty set.

Definition 7.7. (a) Define Δ_+ to be the category of finite totally ordered sets.

(b) Given a category \mathcal{C} , an *augmented cosimplicial object* in \mathcal{C} is a functor from Δ_+ to \mathcal{C} .

(c) Extend Δ^\bullet to a functor on Δ_+ by defining $\Delta^\emptyset = \emptyset$.

(d) Define $S_\bullet W$ as a functor from Δ_+^{op} to sets by $S_T W = \text{Map}(\Delta^T, W)$; in particular $S_\emptyset W$ is a point.

(e) Define $S^\bullet W$ as a functor from Δ_+ to abelian groups by $S^T W = \text{Map}(S_T W, \mathbb{Z})$; in particular $S^\emptyset W$ is isomorphic to \mathbb{Z} .

With these conventions, Definition 7.2 makes sense when f is not surjective, and Propositions 7.4 (with Δ replaced by Δ_+), 7.5 and 7.6 are still valid in this slightly more general context.

Now let $\varepsilon \in S^\emptyset W$ be the element corresponding to $1 \in \mathbb{Z}$.

Proposition 7.8. *If $f : T \rightarrow \{1, 2\}$ takes all of T to 1 then $\langle f \rangle(x, \varepsilon) = x$ for all x and if f takes all of T to 2 then $\langle f \rangle(\varepsilon, x) = x$ for all x .*

8 A sufficient condition for $\text{Tot}(X^\bullet)$ to be an E_∞ space.

Definition 8.1. An E_∞ space is a space with an action of an E_∞ operad.

In order to state the analog of Corollary 5.2 we need to use augmented cosimplicial spaces.

Definition 8.2. Let X^\bullet be an augmented cosimplicial space. Define $\text{Tot}(X^\bullet)$ to be

$$\text{Hom}_{\Delta_+}(\Delta^\bullet, X^\bullet).$$

This can be described more simply: $\text{Tot}(X^\bullet)$ is the total space (in the sense of Definition 3.3) of the restriction of X^\bullet to Δ .

Theorem 8.3. *Let X^\bullet be an augmented cosimplicial space with a map*

$$\langle f \rangle : X^{f^{-1}(1)} \times X^{f^{-1}(2)} \rightarrow X^T$$

for each $f : T \rightarrow \{1, 2\}$. Suppose that the maps $\langle f \rangle$ satisfy the analogs of Propositions 7.4, 7.5, and 7.6, and that there is an element $\varepsilon \in X^0$ satisfying the analog of Proposition 7.8. Then $\text{Tot}(X^\bullet)$ is an E_∞ space.

Remark 8.4. We expect that Tot induces a Quillen equivalence between augmented cosimplicial spaces satisfying the hypothesis of Theorem 8.3 and E_∞ spaces.

The remainder of this section gives an outline of the proof of Theorem 8.3; for details see [28].

The proof follows the same pattern as the proof of Theorem 4.3. The first step is

Proposition 8.5. *The category of augmented cosimplicial spaces has a symmetric monoidal structure \boxtimes with the property that X^\bullet satisfies the hypothesis of Theorem 8.3 if and only if it is a commutative \boxtimes -monoid.*

The basic idea in defining $X^\bullet \boxtimes Y^\bullet$ is that we build it from formal symbols $\langle f \rangle(x, y)$. In order to get a cosimplicial object we have to build in the cosimplicial operators, so we consider symbols of the form

$$\phi_*(\langle f \rangle(x, y))$$

where $f : T \rightarrow \{1, 2\}$ and $\phi : T \rightarrow S$ is an order-preserving map: such a symbol will represent a point in the S -th space $(X^\bullet \boxtimes Y^\bullet)^S$. We require these symbols to satisfy the relation in Proposition 7.4.

Our next two definitions make this precise.

Definition 8.6. Given $S \in \Delta_+$, let \mathcal{J}_S be the category whose objects are diagrams

$$(8.1) \quad \{1, 2\} \xleftarrow{f} T \xrightarrow{\phi} S$$

where T is a finite totally ordered set and ϕ is order-preserving, and whose morphisms are commutative diagrams

$$(8.2) \quad \begin{array}{ccccc} \{1, 2\} & \xleftarrow{f} & T & \xrightarrow{\phi} & S \\ = \downarrow & & \psi \downarrow & & = \downarrow \\ \{1, 2\} & \xleftarrow{f'} & T' & \xrightarrow{\phi'} & S \end{array}$$

with ψ order-preserving.

We will denote an object (8.1) of \mathcal{J}_S by (f, ϕ) . Given augmented cosimplicial spaces X^\bullet and Y^\bullet we consider the functor from \mathcal{J}_S to spaces which takes (f, ϕ) to

$$X^{f^{-1}(1)} \times Y^{f^{-1}(2)}$$

and a morphism (8.2) to the map

$$(\psi_1)_* \times (\psi_2)_*$$

where $\psi_i : f^{-1}(i) \rightarrow (f')^{-1}(i)$ is the restriction of ψ .

Definition 8.7. Define $X^\bullet \boxtimes Y^\bullet$ by

$$(X^\bullet \boxtimes Y^\bullet)^S = \operatorname{colim}_{(f, \phi) \in \mathcal{J}_S} X^{f^{-1}(1)} \times Y^{f^{-1}(2)}$$

for $S \in \Delta_+$.

The verification that \boxtimes is a symmetric monoidal product is given in [28, Section 6].

Remark 8.8. Readers familiar with Kan extensions will recognize that $X^\bullet \boxtimes Y^\bullet$ is one; see [28, Section 6].

Next we apply Remark 6.3(b) with $D = \Delta^\bullet$ to get an operad \mathcal{D} whose k -th space is

$$\operatorname{Hom}_\Delta(\Delta^\bullet, (\Delta^\bullet)^{\boxtimes k})$$

If X^\bullet is a commutative \boxtimes -monoid we define an action of \mathcal{D} on $\operatorname{Tot}(X^\bullet)$ by letting

$$\theta : \mathcal{D}(k) \times (\operatorname{Tot}(X^\bullet))^k \rightarrow \operatorname{Tot}(X^\bullet)$$

be the map that takes $(h, \tau_1, \dots, \tau_k)$ to the composite

$$\Delta^\bullet \xrightarrow{h} (\Delta^\bullet)^{\boxtimes k} \xrightarrow{\tau_1 \boxtimes \dots \boxtimes \tau_k} (X^\bullet)^{\boxtimes k} \xrightarrow{\mu} X^\bullet$$

where μ is the monoidal structure map of X^\bullet .

To complete the proof of Theorem 8.3 it only remains to show that each $\mathcal{D}(k)$ is contractible; see [28, Section 10] for the proof of this.

Remark 8.9. One can give a construction analogous to \boxtimes for the category of ordinary (unaugmented) cosimplicial spaces by requiring f to be a surjection in Definition 8.7. This gives a product which is coherently associative and commutative but not unital.

9 The little n -cubes operad.

We have seen in Section 2 that ΩZ is an A_∞ space. For $n \geq 2$ the space $\Omega^n Z$ has a commutativity property intermediate between A_∞ and E_∞ ; moreover, $\Omega^n Z$ has stronger commutativity than $\Omega^m Z$ if $n > m$. In this section we see how to make this precise.

Fix $n \geq 1$. Let I denote the interval $[0, 1]$.

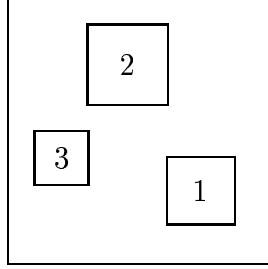
Definition 9.1. A TD-map $I^n \rightarrow I^n$ is a composite $T \circ D$, where T is a translation and D is a dilation (i.e., multiplication by a scalar).

A TD-map takes (t_1, \dots, t_n) to $(a_1 + bt_1, \dots, a_n + bt_n)$, where a_1, \dots, a_n and b are constants with $a_i \geq 0$, $b > 0$ and $a_i + b < 1$. The image of a TD-map is called a “little n -cube.” A TD-map is completely determined by its image.

Definition 9.2. (a) For $k \geq 0$, let $\mathcal{C}_n(k)$ be the space in which a point is a k -tuple $(\kappa_1, \dots, \kappa_k)$ of TD-maps $I^n \rightarrow I^n$ such that the images of the κ_i have disjoint interiors.

(b) Let \mathcal{C}_n be the collection of spaces $\{\mathcal{C}_n(k)\}_{k \geq 0}$.

In the special case $n = 2$, the elements of $\mathcal{C}_2(k)$ can be represented by pictures in the plane. For example, the picture



represents an element of $\mathcal{C}_2(3)$.

Next we define an operad structure for \mathcal{C}_n .

Definition 9.3. (a) Let $1 \in \mathcal{C}_n(1)$ be the identity map of I^n .

(b) Give $\mathcal{C}_n(k)$ the right Σ_k action that permutes the κ_i .

(c) Define

$$\gamma : \mathcal{C}_n(k) \times \mathcal{C}_n(j_1) \times \dots \times \mathcal{C}_n(j_k) \rightarrow \mathcal{C}_n(j_1 + \dots + j_k)$$

as follows: if

$$c = (\kappa_1, \dots, \kappa_k)$$

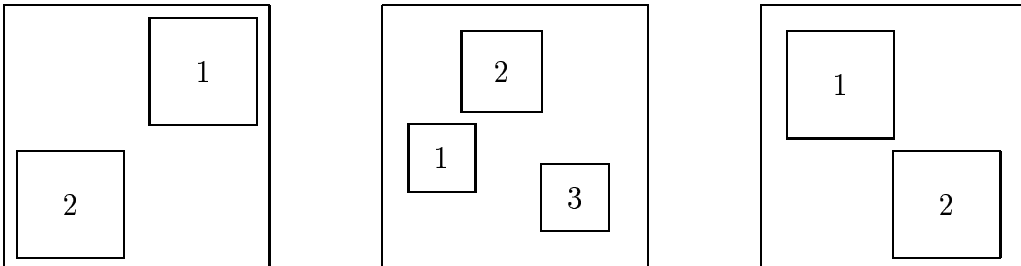
is a point of $\mathcal{C}_n(k)$, and

$$d_i = (\lambda_{i1}, \dots, \lambda_{ij_i})$$

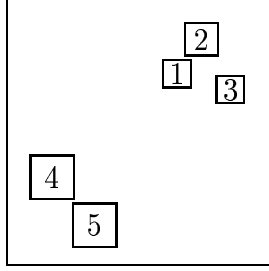
is a point of $\mathcal{C}_n(j_i)$ for $1 \leq i \leq k$, then $\gamma(c, d_1, \dots, d_k)$ is the point $(\nu_{11}, \dots, \nu_{kj_k})$, where

$$\nu_{il} = \kappa_i \circ \lambda_{il}.$$

For example, if $n = 2$ and $c \in \mathcal{C}_2(2)$, $d_1 \in \mathcal{C}_2(3)$ and $d_2 \in \mathcal{C}_2(2)$ are represented by



respectively, then $\gamma(c, d_1, d_2) \in \mathcal{C}_2(5)$ is represented by



Remark 9.4. (a) The definition of \mathcal{C}_n and its composition maps is due to Boardman and Vogt [5, 6]. They were working in a somewhat different context (PROP's instead of operads).

(b) \mathcal{C}_1 is the operad generated by the non-symmetric operad \mathcal{A} defined in Section 2 (see Remark 6.3(c)).

The reason for defining the operad \mathcal{C}_n is that it acts on $\Omega^n Z$. To describe this action we think of an element of $\Omega^n Z$ as a map $I^n \rightarrow Z$ which takes the boundary of I^n to the basepoint $*$ of Z . Then

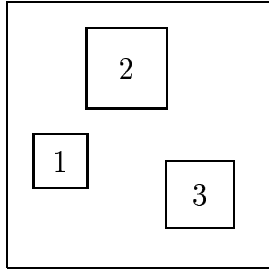
$$\theta : \mathcal{C}_n(k) \times (\Omega^n Z)^k \rightarrow \Omega^n Z$$

is defined as follows: if

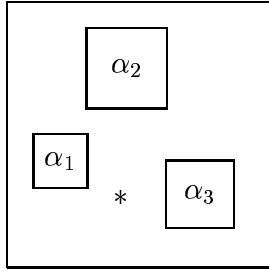
$$c = (\kappa_1, \dots, \kappa_k)$$

is a point of $\mathcal{C}_n(k)$ and $\alpha_i \in \Omega^n Z$ for $1 \leq i \leq k$ then $\theta(c, \alpha_1, \dots, \alpha_k)$ is the map $I^n \rightarrow Z$ which is $\alpha_i \circ (\kappa_i)^{-1}$ on the image of κ_i and $*$ for points which are not in the image of any κ_i .

For example, if c is the element of $\mathcal{C}_2(3)$ represented by



then $\theta(c, \alpha_1, \alpha_2, \alpha_3)$ is the map $I^2 \rightarrow Z$ represented by the picture



(where the α 's in the picture are appropriately scaled).

As one would expect, there is an analog of Theorems 2.12 and 6.7: if Y has a grouplike \mathcal{C}_n action then Y is weakly equivalent to $\Omega^n Z$ for some Z . In fact something a bit more

general is true; we pause to give the relevant definitions, which will also be used in Section 10.

Definition 9.5. Let $\mathcal{O}, \mathcal{O}'$ be operads. An *operad morphism* $\zeta : \mathcal{O} \rightarrow \mathcal{O}'$ is a sequence of maps

$$\zeta_k : \mathcal{O}(k) \rightarrow \mathcal{O}'(k)$$

such that

- (a) ζ_1 takes the unit element in $\mathcal{O}(1)$ to that in $\mathcal{O}'(1)$,
- (b) each ζ_k is Σ_k equivariant, and
- (c) the diagram

$$\begin{array}{ccc} \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) & \xrightarrow{\zeta_k \times \zeta_{j_1} \times \cdots} & \mathcal{O}'(k) \times \mathcal{O}'(j_1) \times \cdots \times \mathcal{O}'(j_k) \\ \gamma \downarrow & & \gamma \downarrow \\ \mathcal{O}(j_1 + \cdots + j_k) & \xrightarrow{\zeta_{j_1 + \cdots + j_k}} & \mathcal{O}'(j_1 + \cdots + j_k) \end{array}$$

commutes for all $k, j_1, \dots, j_k \geq 0$.

Definition 9.6. A morphism $\zeta : \mathcal{O} \rightarrow \mathcal{O}'$ is a weak equivalence if each ζ_k is a weak equivalence of spaces. Two operads \mathcal{O} and \mathcal{O}' are weakly equivalent if there is a diagram of operads and weak equivalences of operads

$$\mathcal{O} \leftarrow \cdots \rightarrow \mathcal{O}'$$

Definition 9.7. An operad is an E_n operad if it is weakly equivalent to \mathcal{C}_n .

Now the analog of Theorems 2.12 and 6.7 is

Theorem 9.8. Y is weakly equivalent to $\Omega^n Z$ for some space $Z \iff Y$ has a grouplike action of an E_n operad.

The \implies direction, and the \impliedby direction for connected Y , are due to Boardman and Vogt [5, 6]. A simpler proof of the \impliedby direction for connected Y was given by May in [24]. The general case of the \impliedby direction is due to May [25].

Theorem 9.8 is aesthetically pleasing but has not often been applied because it is usually hard to show that a space is an E_n space (for $1 < n < \infty$) without knowing in advance that it is an n -fold loop space. In Section 10 we will address this difficulty by giving a sufficient condition for Tot of a cosimplicial space to be an E_n space.

Since this section is intended as an introduction to \mathcal{C}_n , we should mention that the most important uses of \mathcal{C}_n in algebraic topology come from the ‘‘approximation theorem’’ [24, Theorem 2.7]. This theorem gives a model for $\Omega^n \Sigma^n Z$, built from Z and \mathcal{C}_n . Using this model, Fred Cohen has given a complete description of the homology of $\Omega^n \Sigma^n Z$ [7]. Another basic fact is that the model splits stably as a wedge of pieces of the form

$$\mathcal{C}_n(k)_+ \wedge_{\Sigma_k} Z^{\wedge k}$$

(where $+$ means add a disjoint basepoint and \wedge is the smash product); this is called the Snaith splitting [33, 8].

10 A sufficient condition for $\text{Tot}(X^\bullet)$ to be an E_n space.

In this section we give the analog of Theorem 8.3 for E_n actions. The hypothesis of Theorem 8.3 refers to $\langle f \rangle$ operations where f ranges through functions $T \rightarrow \{1, 2\}$. For our current purpose we need to consider functions $f : T \rightarrow \{1, \dots, k\}$ for all $k \geq 2$ but we will only use those of “complexity $\leq n$ ” (see Definitions 10.3 and 10.4).

Let us return to the motivating example $S^\bullet W$. The extension of the definition of $\langle f \rangle$ to functions $f : T \rightarrow \{1, \dots, k\}$ is routine:

Definition 10.1. Given $f : T \rightarrow \{1, \dots, k\}$ define a natural transformation

$$\langle f \rangle : S^{f^{-1}(1)}W \times \dots \times S^{f^{-1}(k)}W \rightarrow S^T W$$

by the equation

$$\langle f \rangle(x_1, \dots, x_k)(\sigma) = x_1(\sigma(f^{-1}(1))) \cdot \dots \cdot x_k(\sigma(f^{-1}(k)))$$

for $\sigma \in S_T W$; here \cdot is multiplication in \mathbb{Z} .

These operations satisfy properties analogous to Propositions 7.4, 7.5, 7.6 and 7.8; the precise formulation is left to the reader (see [28, Definitions 9.3–9.7] for a hint).

Remark 10.2. $\langle f \rangle$ operations with $k > 2$ are composites of those with $k = 2$; that is why we were able to restrict to $k = 2$ in Sections 7 and 8. However, it is not true that an operation $\langle f \rangle$ with complexity $\leq n$ (see Definitions 10.3 and 10.4) can be decomposed into operations with $k = 2$ and complexity $\leq n$, which is why we cannot restrict to $k = 2$ in this section.

As we have seen in Section 9, an E_n operad encodes commutativity which is intermediate between A_∞ (no commutativity) and E_∞ (full commutativity). We therefore want, for each n , a family of operations which interpolates between \sqcup and the family of all $\langle f \rangle$ operations. Notice that, in general, the ordered sets $f^{-1}(1), \dots, f^{-1}(k)$ are mixed together in T , but when f corresponds to an iterate of \sqcup they are not mixed. We therefore introduce a way of measuring the amount of mixing.

We begin with the case $k = 2$. First observe that a function f from a finite totally ordered set to $\{1, 2\}$ is the same thing as a finite sequence of 1’s and 2’s.

Definition 10.3. (a) The complexity of a finite sequence of 1’s and 2’s is the number of times the sequence changes from 1 to 2 or from 2 to 1.

(b) The complexity of a function $f : T \rightarrow \{1, 2\}$ is the complexity of the corresponding sequence.

For example, if f corresponds to the sequence 11222122112 then the complexity of f is 5.

Next let $k > 2$. A function $f : T \rightarrow \{1, \dots, k\}$ corresponds to a finite sequence with values in $\{1, \dots, k\}$. We consider the subsequences that have only two values: for example in the sequence 12313212 we consider the subsequences 121212, 23322 and 13131. As in Definition 10.3, the complexity of such a subsequence is the number of times it changes its value.

Definition 10.4. (a) The complexity of a sequence with values in $\{1, \dots, k\}$ is the maximum of the complexities of the subsequences with only two values.

(b) The complexity of $f : T \rightarrow \{1, \dots, k\}$ is the complexity of the sequence corresponding to f .

In the example just given, the complexity of 121212 is 5, the complexity of 23322 is 2, and the complexity of 13131 is 4, so the complexity of 12313212 is 5.

Remark 10.5. The definition of complexity is suggested by [32]; the reason we use it here is that it is well-adapted to the proof of Theorem 10.6 below. There may be other ways of defining complexity that would also lead to Theorem 10.6, although this seems unlikely.

We can now state the analog of Theorem 8.3.

Theorem 10.6. *Fix n . Let X^\bullet be an augmented cosimplicial space with a map*

$$\langle f \rangle : X^{f^{-1}(1)} \times \dots \times X^{f^{-1}(k)} \rightarrow X^T$$

for each $f : T \rightarrow \{1, \dots, k\}$ with complexity $\leq n$. Suppose that the maps $\langle f \rangle$ are consistent with the cosimplicial operators (in the sense of [28, Definition 9.4]) and are commutative, associative and unital (in the sense of [28, Definitions 9.5, 9.6 and 9.7]). Then $\text{Tot}(X^\bullet)$ is an E_n space.

We expect that Tot induces a Quillen equivalence between augmented cosimplicial spaces satisfying the hypothesis of Theorem 10.6 and E_n spaces.

The proof of Theorem 10.6 is similar in outline to the proofs of Theorems 4.3 and 8.3. However, just as E_n interpolates between A_∞ and E_∞ , we need a way to interpolate between the concepts of monoidal product and symmetric monoidal product. The next section is devoted to this.

11 An extension of Remark 6.3(b).

Remark 6.3(b) says that a symmetric monoidal product \boxtimes on a topological category \mathcal{U} , together with a choice of an object $D \in \mathcal{U}$, leads to an operad \mathcal{O} . We begin with an outline of the proof of this fact.

\boxtimes is a binary operation, so the first step is to choose, for each $k > 2$, a specific way of inserting parentheses⁶ to get a k -fold iterate of \boxtimes which we denote by

$$\boxtimes^k : \mathcal{U}^k \rightarrow \mathcal{U}.$$

We also define \boxtimes^1 to be the identity functor and \boxtimes^0 to be the unit object of \boxtimes .

Next we define the spaces of the operad \mathcal{O} by

$$\mathcal{O}(k) = \text{Hom}_{\mathcal{U}}(D, \boxtimes^k(D, \dots, D))$$

for $k \geq 0$.

⁶A different choice gives a naturally isomorphic functor, by MacLane's coherence theorem [22, Section VII.7].

To define the action of Σ_k on $\mathcal{O}(k)$, we use MacLane’s coherence theorem. This gives, for each $\sigma \in \Sigma_k$, a natural isomorphism

$$\sigma_* : \boxtimes^k(X_1, \dots, X_k) \rightarrow \boxtimes^k(X_{\sigma(1)}, \dots, X_{\sigma(k)})$$

and in particular a self-map of $\boxtimes^k(D, \dots, D)$.

We use the coherence theorem again to get a natural isomorphism

$$\Gamma : \boxtimes^k(\boxtimes^{j_1}, \dots, \boxtimes^{j_k}) \rightarrow \boxtimes^{j_1 + \dots + j_k}$$

which induces the structure map γ of \mathcal{O} .

It remains to check that γ has the associativity, unitality and equivariance properties required by the definition of operad; for this we apply the coherence theorem one more time to see that Γ has associativity, unitality and equivariance properties (see [28, Section 4] for the explicit statements) from which those for γ can be deduced. This completes the proof of Remark 6.3(b).

The same proof proves something more general. Assume that for each k we are given a subfunctor of \boxtimes^k , that is, a functor \mathcal{F}_k with a natural monomorphism to \boxtimes^k . Assume further that \mathcal{F}_k is closed under σ_* (that is, σ_* takes $\mathcal{F}_k(X_1, \dots, X_k)$ to $\mathcal{F}_k(X_{\sigma(1)}, \dots, X_{\sigma(k)})$) and that the collection $\{\mathcal{F}_k\}_{k \geq 0}$ is closed under Γ . The argument given above shows:

Proposition 11.1. *Under these assumptions the collection $\{\text{Hom}_{\mathcal{U}}(D, \mathcal{F}_k(D, \dots, D))\}_{k \geq 0}$ is an operad, with Σ_k action induced by the maps σ_* and γ induced by Γ .*

Remark 11.2. In [28, Section 4] we give a more general version of 11.1, using the concept of “functor-operad.” A functor-operad is a collection of functors

$$\mathcal{F}_k : \mathcal{U}^k \rightarrow \mathcal{U}$$

with just enough structure to satisfy the conclusion of Proposition 11.1. This concept has been discovered independently, in a different context and in a more general form, by Batanin [3], who calls them “internal operads.”

12 Proof of Theorem 10.6.

Recall that in the proof of Theorem 8.3 we constructed a symmetric monoidal product \boxtimes on the category of augmented cosimplicial spaces. Our first task is to give a formula for the iterate \boxtimes^k .

Given $S \in \Delta_+$ and $k \geq 0$ let $\mathcal{J}_S(k)$ be the category whose objects are diagrams

$$(12.1) \quad \{1, \dots, k\} \xleftarrow{f} T \xrightarrow{\phi} S$$

where T is a finite totally ordered set and ϕ is order-preserving, and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} \{1, \dots, k\} & \xleftarrow{f} & T & \xrightarrow{\phi} & S \\ = \downarrow & & \psi \downarrow & & = \downarrow \\ \{1, \dots, k\} & \xleftarrow{f'} & T' & \xrightarrow{\phi'} & S \end{array}$$

with ψ order-preserving.

We will denote an object (12.1) of \mathcal{J}_S by (f, ϕ) .

Definition 12.1. Let $X_1^\bullet, \dots, X_k^\bullet$ be augmented cosimplicial spaces. Define $\Xi_k(X_1^\bullet, \dots, X_k^\bullet)$ to be the cosimplicial space whose value at $S \in \Delta_+$ is

$$\operatorname{colim}_{(f, \phi) \in \mathcal{J}_S(k)} X_1^{f^{-1}(1)} \times \dots \times X_k^{f^{-1}(k)}$$

When $k = 2$ this is the same as the definition of $X_1^\bullet \boxtimes X_2^\bullet$. In general we have

Proposition 12.2. \boxtimes^k is naturally isomorphic to Ξ_k .

For the proof see [28, Section 6] (also cf. Remark 10.2).

Now fix n , and let $\mathcal{J}_S^n(k)$ be the full subcategory of $\mathcal{J}_S(k)$ whose objects are the (f, ϕ) for which the complexity of f is $\leq n$.

Definition 12.3. Let $X_1^\bullet, \dots, X_k^\bullet$ be augmented cosimplicial spaces. Define $\Xi_k^n(X_1^\bullet, \dots, X_k^\bullet)$ to be the cosimplicial space whose value at $S \in \Delta_+$ is

$$\operatorname{colim}_{(f, \phi) \in \mathcal{J}_S^n(k)} X^{f^{-1}(1)} \times \dots \times X^{f^{-1}(k)}$$

Proposition 12.4. For each n , the sequence of functors $\{\Xi_k^n\}_{k \geq 0}$ satisfies the hypothesis of Proposition 11.1.

For the proof see [28, Section 8].

Applying Proposition 11.1 we obtain an operad with k -th space

$$(12.2) \quad \operatorname{Hom}_{\Delta_+}(\Delta^\bullet, \Xi^n(\Delta^\bullet, \dots, \Delta^\bullet))$$

We will denote this operad by \mathcal{D}_n .

If X^\bullet satisfies the hypothesis of Theorem 10.6 then there are maps

$$\xi_k : \Xi_k^n(X^\bullet, \dots, X^\bullet) \rightarrow X^\bullet$$

for each $k \geq 0$. We define an action of \mathcal{D}_n on $\operatorname{Tot}(X^\bullet)$ by letting

$$\theta : \mathcal{D}_n(k) \times (\operatorname{Tot}(X^\bullet))^k \rightarrow \operatorname{Tot}(X^\bullet)$$

be the map that takes $(h, \tau_1, \dots, \tau_k)$ to the composite

$$\Delta^\bullet \xrightarrow{h} \Xi_k^n(\Delta^\bullet, \dots, \Delta^\bullet) \xrightarrow{\Xi_k^n(\tau_1, \dots, \tau_k)} \Xi_k^n(X^\bullet, \dots, X^\bullet) \xrightarrow{\xi_k} X^\bullet$$

To complete the proof of Theorem 10.6 it remains to show that \mathcal{D}_n is weakly equivalent to \mathcal{C}_n . This is more difficult than the corresponding step in the proofs of Theorems 4.3 and 8.3 because in those cases it was only necessary to show that certain spaces were contractible, whereas here we need to show not just that the spaces $\mathcal{D}_n(k)$ and $\mathcal{C}_n(k)$ are weakly equivalent but that the operad structures are compatible. The proof is given in [28, Section 12]; the basic idea is to show that the operads \mathcal{C}_n and \mathcal{D}_n can be written as homotopy colimits, over the same indexing category, of contractible sub-operads.

Remark 12.5. The operad \mathcal{D}_n is of interest in its own right, as an E_n operad whose structure is in some ways simpler than that of \mathcal{C}_n . In [28, Section 11] it is shown that $\mathcal{D}_n(k)$ is homeomorphic to

$$Z_k^n \times \text{Tot}(\Delta^\bullet)$$

where Z_k^n is the zeroth space of $\Xi_k^n(\Delta^\bullet, \dots, \Delta^\bullet)$. Moreover, the space Z_k^n has an explicit cell decomposition which is well-related to the operad structure of \mathcal{D}_n . The cellular chain complexes of the Z_k^n form a chain operad which is studied in [27] (where it is called \mathcal{S}_n).

13 Applications.

13.1 The topology of a space of knots.

The space of imbeddings of S^1 in \mathbb{R}^k is of considerable interest in topology. It turns out that a closely related space satisfies the hypothesis of Theorem 10.6 with $n = 2$, and is therefore a two-fold loop space.

To be specific, let us consider the manifold-with-boundary $\mathbb{R}^{k-1} \times I$. Fix points $x_0 \in \mathbb{R}^{k-1} \times \{0\}$ and $x_1 \in \mathbb{R}^{k-1} \times \{1\}$, and also fix tangent vectors v_0 at x_0 and v_1 at x_1 . Let $\text{Emb}(I, \mathbb{R}^{k-1} \times I)$ be the space of embeddings of I in $\mathbb{R}^{k-1} \times I$ which take 0 and 1 to x_0 and x_1 respectively, with tangent vectors v_0 at 0 and v_1 at 1. Let $\text{Imm}(I, \mathbb{R}^{k-1} \times I)$ be the analogous space with immersions instead of imbeddings. Finally, let $\text{Fib}(I, \mathbb{R}^{k-1} \times I)$ be the fiber of the forgetful map

$$\text{Emb}(I, \mathbb{R}^{k-1} \times I) \rightarrow \text{Imm}(I, \mathbb{R}^{k-1} \times I)$$

It follows from a theorem of Hirsch and Smale that $\text{Imm}(I, \mathbb{R}^{k-1} \times I)$ is homotopy equivalent to ΩS^{k-1} , so $\text{Fib}(I, \mathbb{R}^{k-1} \times I)$ contains most of the information in $\text{Emb}(I, \mathbb{R}^{k-1} \times I)$.

Now assume $k \geq 4$.

Dev Sinha [31] (building on earlier work of Goodwillie and Weiss) has given a cosimplicial space X^\bullet with $\text{Tot}(X^\bullet)$ weakly equivalent to $\text{Fib}(I, \mathbb{R}^{k-1} \times I)$. He has also shown that X^\bullet satisfies the hypothesis of Theorem 10.6 with $n = 2$. It follows that $\text{Fib}(I, \mathbb{R}^{k-1} \times I)$ is a two-fold loop space.

Remark 13.1. When a cosimplicial space X^\bullet satisfies the hypothesis of Theorem 10.6, the spectral sequence converging to the homology of $\text{Tot}(X^\bullet)$ will have extra structure coming from the $\langle f \rangle$ operations. This should be useful for analyzing the spectral sequence that converges to the homology of $\text{Fib}(I, \mathbb{R}^{k-1} \times I)$.

13.2 Topological Hochschild Cohomology.

Theorems 4.3, 8.3 and 10.6 are still true, with essentially the same proofs, for cosimplicial spectra (except that Cartesian products in the category of spaces are replaced by smash products in the category of spectra).

The definition of Hochschild cohomology for associative rings (which will be recalled in Section 16) has an analog for associative ring spectra in the sense of [11] or [16]. If R is an

associative ring spectrum there is a cosimplicial spectrum $TH^\bullet(R)$ (see [26, Example 3.4] for the definition) whose total spectrum $\text{Tot}(TH^\bullet(R))$ is called the topological Hochschild cohomology spectrum of R .

In [26] it is shown that $TH^\bullet(R)$ satisfies the hypothesis of Theorem 10.6 with $n = 2$, and therefore the topological Hochschild cohomology spectrum of R is an E_2 spectrum. This is a spectrum analog of Deligne's Hochschild cohomology conjecture (see Section 16).

14 The framed little disks operad.

The framed little disks operad was defined by Getzler in [13]; it is a variant of the little 2-cubes operad.

Let B denote the closed unit disk in \mathbb{R}^2 .

Definition 14.1. A TDR-map $B \rightarrow B$ is a composite $T \circ D \circ R$, where T is a translation, D is a dilation and R is a rotation.

Definition 14.2. (a) For $k \geq 0$, let $\mathcal{F}(k)$ be the space in which a point is a k -tuple $(\kappa_1, \dots, \kappa_k)$ of TDR-maps $B \rightarrow B$ such that the images of the κ_i have disjoint interiors.

(b) Let \mathcal{F} be the collection of spaces $\{\mathcal{F}(k)\}_{k \geq 0}$.

\mathcal{F} is an operad, where the Σ_k action on $\mathcal{F}(k)$ permutes the κ_i and the definition of γ is analogous to Definition 9.3(c).

Remark 14.3. It is instructive to consider the relation between \mathcal{F} and \mathcal{C}_2 .

(a) If we require the κ_i in Definition 14.2 to be TD maps (that is, composites of translations and dilations), we get a suboperad \mathcal{F}_0 of \mathcal{F} . By restricting TD maps $B \rightarrow B$ to the square inscribed in B we get an equivalence of operads $\mathcal{F}_0 \rightarrow \mathcal{C}_2$.

(b) The k -th space of \mathcal{F} is the Cartesian product $\mathcal{F}_0(k) \times (S^1)^k$ (but note that the projections $\mathcal{F}(k) \rightarrow \mathcal{F}_0(k)$ do not give a map of operads).

(c) An action of \mathcal{F} on a space X is the same thing as an \mathcal{F}_0 action together with a suitably compatible S^1 action.

Remark 14.4. One reason that \mathcal{F} is important is that an \mathcal{F} action on a space X induces a Batalin-Vilkovisky structure on H_*X (see [13]).

The analog of Theorem 10.6 for \mathcal{F} actions has a surprisingly simple form. As motivation we consider the following situation: let V_\bullet be a cyclic set, that is, a simplicial set together with maps

$$t : V_m \rightarrow V_m$$

for each $m \geq 0$ satisfying certain relations with the simplicial operators (see [38, Definition 9.6.1]). Define A^\bullet to be $\text{Map}(V_\bullet, \mathbb{Z})$. Then A^\bullet is a cocyclic abelian group, that is, it is a cosimplicial abelian group together with maps

$$\tau : A^m \rightarrow A^m$$

for each $m \geq 0$ which satisfy appropriate relations with the cosimplicial operators. We can define \sqcup and the other $\langle f \rangle$ operations on A^\bullet in analogy with Definition 7.2 (the precise

definition is left as an exercise for the reader; the basic idea is to use iterated face maps to interpret the symbol $\sigma(U)$ in this context). The relations between the maps t and the simplicial operators imply that all $\langle f \rangle$ operations of complexity ≤ 2 are generated by \sqcup and τ , subject to the relation

$$(14.1) \quad \tau^{p+1}(x \sqcup y) = y \sqcup x,$$

where x is in A^p and τ^{p+1} denotes the $(p+1)$ -st iterate of τ .

Theorem 14.5. *If X^\bullet is a cocyclic space with a product*

$$\sqcup : X^p \times X^q \rightarrow X^{p+q+1}$$

which is associative and unital and satisfies (5.1), (5.2) and (14.1) then $\text{Tot}(X^\bullet)$ has an action of \mathcal{F} .

The proof is similar to that of Theorem 10.6; see [30].

15 Cosimplicial chain complexes.

The theory developed in Sections 4, 5, 8, 10, 12 and 14 has a precise analog with spaces replaced by chain complexes; see [29]. In this section we give a brief discussion.

First we need the appropriate concept of operad. In fact one can define nonsymmetric operads in any monoidal category by replacing the Cartesian products in Definition 2.5 by the monoidal product, and one can define operads in any symmetric monoidal category by analogy with Definition 6.2. The category of chain complexes is a symmetric monoidal category (the monoidal product is the usual tensor product of chain complexes) and operads in this category are called *chain operads*.

Definition 15.1. (a) A chain complex is *weakly contractible* if its homology is \mathbb{Z} in dimension 0 and zero in all other dimensions.

(b) An A_∞ chain operad is a nonsymmetric chain operad \mathcal{O} for which each $\mathcal{O}(k)$ is a weakly contractible chain complex.

(c) An E_∞ chain operad is a chain operad \mathcal{O} for which each $\mathcal{O}(k)$ is a weakly contractible chain complex.⁷

Next we need the analog of Tot. We have already defined the conormalization of a cosimplicial abelian group (Definition 3.7). We now extend this definition to cosimplicial chain complexes. Recall the cosimplicial chain complex Δ_*^\bullet (Definition 3.6).

Definition 15.2. Let B_*^\bullet be a cosimplicial chain complex. The *conormalization* of B_*^\bullet , denoted $\mathbf{C}(B_*^\bullet)$, is the cochain complex

$$\text{Hom}_\Delta(\Delta_*^\bullet, B_*^\bullet) \subset \prod_{m=0}^{\infty} \text{Hom}(\Delta_*^m, B_*^m).$$

Here Hom_Δ is Hom in the category of cosimplicial graded abelian groups and the differential is induced by the differentials of Δ_*^\bullet and B_*^\bullet .

⁷For technical reasons, it is usual to require in addition that the action of each Σ_k should be free. The operad \mathcal{J} defined below has this property.

In practice it's useful to have an elementary description of $\mathbf{C}(B_*^\bullet)$. First note that by fixing the internal degree m we get a cosimplicial abelian group B_m^\bullet and hence a cochain complex $\mathbf{C}(B_m^\bullet)$ (see Remark 3.8 for elementary descriptions of this cochain complex). The differential in B_*^\bullet induces a differential

$$\mathbf{C}(B_m^\bullet) \rightarrow \mathbf{C}(B_{m-1}^\bullet)$$

so the cochain complexes $\mathbf{C}(B_m^\bullet)$ assemble into a bicomplex (with differentials lowering degree in one direction and raising degree in the other). The conormalization of B_*^\bullet is the totalization of this bicomplex:

$$\mathbf{C}(B_*^\bullet)^p = \prod_m \mathbf{C}(B_m^\bullet)^{p+m};$$

in general this is an infinite product.

Next we observe that the definition of \boxtimes in Section 8 has an analog for augmented cosimplicial chain complexes, with \times replaced by \otimes . As a consequence we get a chain operad \mathcal{T} with

$$\mathcal{T}(k) = \mathbf{C}((\Delta_*^\bullet)^{\boxtimes k})$$

Theorem 15.3. (a) \mathcal{T} is an E_∞ chain operad.

(b) If B_*^\bullet is a cosimplicial chain complex satisfying the hypothesis of Theorem 8.3 then \mathcal{T} acts on $\mathbf{C}(B_*^\bullet)$.

Remark 15.4. The definition of \mathcal{T} looks complicated, but in fact \mathcal{T} has a simple explicit description. For each fixed $k \geq 1$ and $q, r \geq 0$ let $\mathcal{U}_{q,r}(k)$ be the free abelian group generated by the symbols

$$\{1, \dots, k\} \xleftarrow{f} [q] \xrightarrow{\phi} [r]$$

where

- (a) f is onto,
- (b) the image of ϕ contains all of $\{1, \dots, r\}$ (but is allowed to not contain 0),
- (c) ϕ is order-preserving,
- (d) $\phi(i) = \phi(i+1) \Rightarrow f(i) \neq f(i+1)$.

Then the p -th group of the chain complex $\mathcal{T}(k)$ is

$$\prod_q \mathcal{U}_{q, q+1-p-k}(k)$$

It is not hard to show this from the definitions; see [29]. [29] also gives explicit formulas for the differential of $\mathcal{T}(k)$ and for the operad structure maps of \mathcal{T} .

Remark 15.5. \mathcal{T} is not the same as the E_∞ chain operad \mathcal{S} defined in [27], but they are related: \mathcal{S} can be obtained from \mathcal{T} by the condensation process described in [26, Section 7]. We will show in [29] that the structural formulas for \mathcal{S} can be deduced from those for \mathcal{T} ; this is less elementary than the treatment of the structural formulas in [27] but avoids the eight pages of sign verifications in that paper. Each of \mathcal{S} and \mathcal{T} has advantages: \mathcal{S} is much smaller but \mathcal{T} has useful formal properties (see Section 16.3).

Next we need the definition of weak equivalence for chain operads.

Definition 15.6. A morphism $\zeta : \mathcal{O} \rightarrow \mathcal{O}'$ of chain operads is a weak equivalence if each ζ_k is a homology isomorphism. Two chain operads \mathcal{O} and \mathcal{O}' are weakly equivalent if there is a diagram of operads and weak equivalences of operads

$$\mathcal{O} \leftarrow \cdots \rightarrow \mathcal{O}'$$

Now fix $n \geq 1$. Applying the normalized singular chain functor to the little n -cubes operad \mathcal{C}_n we obtain a chain operad $S_*\mathcal{C}_n$.

Definition 15.7. An E_n chain operad is a chain operad weakly equivalent to $S_*\mathcal{C}_n$.

The definition of Ξ_k^n in Section 12 has an analog for augmented cosimplicial chain complexes. As a consequence we get a chain operad \mathcal{T}_n with

$$\mathcal{T}_n(k) = \mathbf{C}(\Xi_k^n(\Delta_*^\bullet, \dots, \Delta_*^\bullet))$$

Theorem 15.8. (a) \mathcal{T}_n is an E_n chain operad.

(b) If B_*^\bullet is a cosimplicial chain complex satisfying the hypothesis of Theorem 10.6 then \mathcal{T}_n acts on $\mathbf{C}(B_*^\bullet)$.

Remark 15.9. \mathcal{T}_n has an explicit description similar to that in Remark 15.4, except that f is required to have complexity $\leq n$; see [29]. Also in [29], we show that the chain operad \mathcal{S}_n defined in [27] can be obtained from \mathcal{T}_n by condensation.

Remark 15.10. The theory described in Section 14 also has a chain analog. In [30] we construct a chain operad \mathcal{G} which is weakly equivalent to $S_*\mathcal{F}$, and we show that if B_*^\bullet is a cosimplicial chain complex satisfying the hypothesis of Theorem 14.5 then \mathcal{G} acts on $\mathbf{C}(B_*^\bullet)$.

16 Applications.

16.1 Deligne's Hochschild cohomology conjecture.

Let R be an associative ring. The *Hochschild cochain complex* $C^*(R)$ is the cochain complex which in degree p is

$$\mathrm{Hom}_{\mathbb{Z}}(R^{\otimes p}, R);$$

the differential is determined by the formula

$$\begin{aligned} (d\rho)(r_1 \otimes \cdots \otimes r_{p+1}) \\ = r_1\rho(r_2 \otimes \cdots) + \sum_{i=1}^p (-1)^i \rho(\cdots \otimes r_i r_{i+1} \otimes \cdots) + (-1)^{p+1} \rho(\cdots \otimes r_p) r_{p+1} \end{aligned}$$

where $\rho \in C^p(R)$. The *Hochschild cohomology* $H^*(R)$ is the cohomology of this complex. Hochschild defined a cup product on $C^*(R)$ by

$$(\rho_1 \smile \rho_2)(r_1 \otimes \cdots \otimes r_{p+q}) = \rho_1(\cdots \otimes r_p) \cdot \rho_2(r_{p+1} \otimes \cdots)$$

where $\rho_1 \in C^p(R)$ and $\rho_2 \in C^q(R)$. This induces a product, also denoted by \smile , on $H^*(R)$.

Gerstenhaber showed in 1963 (see [12]) that $H^*(R)$ is what is now known as a *Gerstenhaber algebra*. That is, he showed that the cup product on $H^*(R)$ is commutative and that there is a Lie bracket

$$[,] : H^p(R) \otimes H^q(R) \rightarrow H^{p+q-1}(R)$$

such that $[x, \]$ is a derivation with respect to \smile for each $x \in H^*(R)$.

About 10 years later, Fred Cohen showed that if X has a \mathcal{C}_2 action then H_*X is a Gerstenhaber algebra (but he didn't use this terminology); see [7].

In 1993, Deligne asked in a letter [10] whether these two examples of Gerstenhaber algebras were related: specifically he asked whether the cup product and Lie bracket on $H^*(R)$ are induced by an action of an E_2 chain operad on $C^*(R)$.

One reason this conjecture is important is because of its connection with Kontsevich's deformation quantization theorem; see [19].

The conjecture has been proved by several authors using quite different methods (see [35, 36, 26, 37, 19, 20, 17, 18]). In [27, Section 2] we gave an especially simple proof by showing that the E_2 operad \mathcal{S}_2 defined in that paper acts on $C^*(R)$ by explicit formulas. In [29] we show that the E_2 operad \mathcal{T}_2 (see Section 15) acts on $C^*(R)$, also by explicit formulas; this argument has the advantage that it avoids the complicated sign verifications needed in [27].

16.2 Strong Frobenius algebras.

By a *strong Frobenius algebra* we mean an algebra A over a field such that A is isomorphic to A^* as an A -bimodule. In [30] we show that if A is a strong Frobenius algebra then the chain operad \mathcal{G} (see Remark 15.10) acts on $C^*(A)$. This is a strong form of Deligne's Hochschild cohomology conjecture for these algebras.

16.3 A theorem of Kriz and May.

Let Ab^Δ denote the category of cosimplicial abelian groups, and let $\text{Ch}_{\geq 0}^*$ denote the category of non-negatively graded cochain complexes.

The conormalization functor gives an equivalence of categories

$$(16.1) \quad \mathbf{C} : \text{Ab}^\Delta \rightarrow \text{Ch}_{\geq 0}^*$$

(cf. [38, Section 8.4]).

We mentioned in Remark 8.9 that the category of cosimplicial spaces has a non-unital symmetric monoidal product \boxtimes ; essentially the same construction (with \times replaced by \otimes) gives a non-unital symmetric monoidal product \boxtimes for Ab^Δ . It is natural to ask how \boxtimes is related to the equivalence (16.1), and this question has a simple answer:

Theorem 16.1. $\mathbf{C}(A^\bullet \boxtimes B^\bullet) \cong \mathcal{T}(2) \otimes_{\mathcal{T}(1) \otimes \mathcal{T}(1)} (\mathbf{C}(A^\bullet) \otimes \mathbf{C}(B^\bullet))$, where \mathcal{T} is the chain operad defined in Section 15.

See [29] for the proof.

The formula in Theorem 16.1 is precisely analogous to Definition V.1.1 of [21]. As a corollary to Theorem 16.1 we recover the results of [21, Section V.3], but with the “linear isometries operad” (actually the singular chains of the usual linear isometries operad) replaced by \mathcal{T} . The operad \mathcal{T} has the advantage that it is much smaller than the linear isometries operad and its structure can be described explicitly.

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