THE 2-COMPACT GROUPS IN THE A-FAMILY ARE N-DETERMINED

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ABSTRACT. The 2-compact groups associated to central quotients of $\mathrm{SU}(n+1)$, $n \geq 1$, are shown to be determined up to isomorphism by their maximal torus normalizers.

1. INTRODUCTION

A 2-compact group is a 2-complete connected based space BX such that $H^*(\Omega BX; \mathbf{F}_2)$ is finite where ΩBX is the loop space [6]. It is customary, though sometimes confusing, to refer to BX by the symbol X.

Any 2-compact group BX comes equipped with a maximal torus normalizer $BN(X) \to BX$ where $BN(X)$ is the Borel construction

$$BT(X) \to BN(X) \to BW(X)$$

for the action of the Weyl group $W(X)$ on the maximal torus $T(X)$ [6, 9.8]. Does $BN(X)$ determine BX?

The answer to this question is "no" for the following reason. Let G be a Lie group and $N(G) \to G$ its Lie group maximal torus normalizer. Assuming that the component group $\pi_0(G)$ is a finite 2-group, $B\widehat{G}$ is a 2-compact group and $B\widehat{N}(G) \to B\widehat{G}$ its 2-compact group maximal torus normalizer. (For any Lie group H, $B\widehat{H}$ stands for the partial 2-completion of the classifying space BH for H.) Since there are distinct Lie groups, such as $\mathrm{O}(2n)$ and $\mathrm{SO}(2n+1)$, with isomorphic maximal torus normalizers, there are also distinct 2-compact groups, such as $\widehat{\mathrm{O}}(2n)$ and $\widehat{\mathrm{SO}}(2n+1)$, with isomorphic maximal torus normalizers. Thus we need to replace the maximal torus normalizer by a more delicate invariant which retains information about component groups. The maximal torus normalizer pair is a candidate for such a more delicate invariant.

For a 2-compact group BX let BX_0, the identity component of X, denote the universal covering space of BX. Since BX_0 is again a 2-compact group, it has a maximal torus normalizer $BN(X_0) \to BX_0$. The maximal torus normalizers of X and X_0 are related by a commutative diagram

$$\begin{tikzcd}
BN(X_0) \arrow[r] \arrow[d] & BX_0 \arrow[d] \\
BN(X) \arrow[r] \arrow[d] & BX \arrow[d] \\
B\pi_0(X) \arrow[r, equal] & B\pi_0(X)
\end{tikzcd}$$

where the columns are fibration sequences. The fibration $BN(X_0) \to BN(X) \to B\pi_0(X)$, called the *maximal torus normalizer pair* associated to BX, has the built-in property that it fully informs about the component group of X. Does the maximal torus normalizer pair determine the 2-compact group up to isomorphism?

Focusing on the following properties for a 2-compact group X,

(1) X is determined by $(N(X), N(X_0))$
(2) Automorphisms of X are determined by their restrictions to $N(X)$
(3) Automorphisms of X are determined by their restrictions to $T(X)$

we shall say that

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- X is totally N-determined if it satisfies (1) and (2)
- X is uniquely N-determined if it satisfies (1) and (3)

In this terminology, one might formulate the conjecture that all 2-compact groups are totally N-determined, all *connected* 2-compact groups even uniquely N-determined. Here is an infinite family of simple 2-compact groups corroborating the conjecture.

1.1. **Theorem.** *The simple* 2*-compact group* $\mathrm{PGL}(n+1,\mathbf{C})$*,* $n \geq 1$*, is uniquely* N*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{PGL}(n+1,\mathbf{C}))$ *equals* $\mathbf{Z}_2^\times$ *for* $n > 1$ *and* $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times$ *for* $n = 1$*.*

It immediately follows (3.2, 4.3) that the Lie group $\mathrm{PGL}(n+1,\mathbf{C}) = \mathrm{PSL}(n+1,\mathbf{C})$ occuring in Theorem 1.1 can be replaced by any central quotient of $\mathrm{SL}(n+1,\mathbf{C})$. Indeed, the methods used here are not confined to simple, or semi-simple, 2-compact groups.

1.2. **Corollary.** [22, 1.9] *The* 2*-compact group* $\mathrm{GL}(n,\mathbf{C})$ *is uniquely* N*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{GL}(n,\mathbf{C}))$ *equals* $\mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$ *for* $n > 2$ *and* $\mathbf{Z}^\times\backslash\,\mathrm{Aut}_{\mathbf{Z}_2\Sigma_2}(\mathbf{Z}_2^2)$ *for* $n = 2$*.*

The methods are not even confined to the connected cases. For instance, it follows from Lemma 4.1 that the 2-compact group $\mathrm{GL}(n,\mathbf{C}) \rtimes C_2$, where C_2 acts on $\mathrm{GL}(n,\mathbf{C})$ by complex conjugation, is totally N-determined.

See [31, 33, 34] for classification results for other 2-compact groups (with polynomial $\mathbf{F}_2$-cohomology). The results for the automorphism groups are not new [18] but reproved here.

2. GENERALITIES

This sections contains the fundamental definitions and the first general results. Whereas p-compact groups are determined by their maximal torus normalizers [29, 1] when $p > 2$, a finer invariant is needed for 2-compact groups as there are examples (2.2) of distinct 2-compact groups with identical maximal torus normalizers.

2.1. **Maximal torus normalizer pairs.** Let $N_0 \to N$ be a maximal rank normal monomorphism between two extended 2-compact tori, meaning simply that there exists a short exact sequence of loop spaces $N_0 \to N \to \pi$ for some finite group π. We say that (N, N_0) is a maximal torus normalizer pair for the 2-compact group X, and we write $N(X, X_0) = (N, N_0)$, if there exists a morphism of loop space short exact sequences

$$\begin{tikzcd}
N_0 \arrow[r] \arrow[d, "j_0"] & N \arrow[r] \arrow[d, "j"] & \pi \arrow[d, "\cong"] \\
X_0 \arrow[r] & X \arrow[r] & \pi_0(X)
\end{tikzcd}$$

where j and j_0 are maximal torus normalizers for X and its identity component X_0. A maximal torus normalizer pair for X determines the maximal torus $T(X)$, the Weyl groups, $W(X)$ and $W(X_0)$, of X and X_0, the component group $\pi_0(X) = N(X)/N(X_0) = W(X)/W(X_0)$, and [7, 7.5] the center $Z(X_0) \to X_0$ of X_0.

2.2. **Example.** 1. Since $N(\mathrm{SO}(2n+1)) \subseteq \mathrm{O}(2n) \subsetneq \mathrm{SO}(2n+1)$, $\mathrm{O}(2n)$ and $\mathrm{SO}(2n+1)$ have the same maximal torus normalizer. Their maximal torus normalizer pairs are distinct, however, for $\mathrm{SO}(2n+1)$ is connected and $\mathrm{O}(2n)$ disconnected.
2. More generally [14], let G be any compact connected Lie group and $N(G)$ its maximal torus normalizer. If $N(G)$ is not maximal, there exists a compact Lie group H such that $N(G) \subseteq H \subsetneq G$. The two compact Lie groups, G and H, have isomorphic maximal torus normalizers but distinct maximal torus normalizer pairs as H is non-connected.
3. The Weyl groups for $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(n)$, $n \geq 3$, are isomorphic as reflection groups but $N(\mathrm{SO}(2n+1))$ is a split and $N(\mathrm{Sp}(n))$ a non-split extension [3]. Thus connected 2-compact groups can not be classified by their Weyl group alone.

2.3. **The Adams–Mahmud homomorphism.** For a 2-compact group (or extended 2-compact torus) X, we let $\mathrm{End}(X) = [BX, *; BX]$ denote the monoid of homotopy classes of endomorphism of X. The *automorphism group* $\mathrm{Aut}(X) \subseteq [BX, *; BX]$ of X is the group of invertible elements in $\mathrm{End}(X)$ and the *outer automorphism group* $\mathrm{Out}(X) = \mathrm{Aut}(X)/\pi_0(X) \subseteq [BX; BX]$ is the group of conjugacy classes of automorphisms of X.

Let X be a 2-compact group with maximal torus normalizer pair (N, N_0). Turn the maximal torus normalizer $Bj\colon BN \to BX$ into a fibration. Any automorphism $f\colon X \to X$ of the 2-compact group X restricts to an automorphism $\mathrm{AM}(f)\colon N \to N$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N)$ of the identity component X_0 of X, such that the diagram

$$\begin{tikzcd} BN \arrow[r, "B(\mathrm{AM}(f))"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r, "Bf"'] & BX \end{tikzcd}$$

commutes up to based homotopy [26, §3]. The Adams–Mahmud homomorphism is the resulting homomorphism

$$\mathrm{AM}\colon \operatorname{Aut}(X) \to W(X_0)\backslash \operatorname{Aut}(N) \tag{2.4}$$

of automorphism groups.

The automorphism group of N sits [24, 5.2] in a short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to \operatorname{Aut}(N) \xrightarrow{\pi_*} \operatorname{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.5}$$

where the normal subgroup to the left consists of all automorphisms of N that induce the identity on homotopy groups and the group to the right consists of all pairs $(\alpha, \theta) \in \operatorname{Aut}(W(X)) \times \operatorname{Aut}(\check{T}(X))$ such that θ is α-linear and the induced automorphism $H^2(\alpha^{-1}, \theta)$ [35, 6.7.6] preserves the extension class $e(X) \in H^2(W(X); \check{T}(X))$. The image of $W(X_0)$ in $\operatorname{Aut}(N)$ does not intersect the subgroup $H^1(W(X); \check{T}(X))$ (as $W(X_0)$ is represented faithfully in $\operatorname{Aut}(\check{T}(X))$ [6, 9.7]) so there is an induced short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to W(X_0)\backslash \operatorname{Aut}(N) \xrightarrow{\pi_*} W(X_0)\backslash \operatorname{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.6}$$

whose middle term is the target of the Adams–Mahmud homomorphism. In particular, if X is *connected*, this short exact sequence

$$0 \to H^1(W(X); \check{T}(X)) \to \operatorname{Out}(N) \to W(X)\backslash \operatorname{Aut}(W(X), \check{T}(X), e(X)) \to 1 \tag{2.7}$$

has the group $\operatorname{Out}(N) = W(X)\backslash \operatorname{Aut}(N)$ of outer automorphisms of N as its middle term. The group $\operatorname{Aut}(W(X), \check{T}(X), 0)$, which is the normalizer $N_{\mathrm{GL}(L(X))}(W(X))$ of $W(X)$ in $\mathrm{GL}(L(X))$, $L(X) = \pi_2(BT(X))$, fits into an exact sequence

$$Z(W(X))\backslash \operatorname{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) \to W(X)\backslash N_{\mathrm{GL}(L(X))}(W(X)) \to \operatorname{Out}(W(X))$$

where, by Schur's lemma, $\operatorname{Aut}_{\mathbf{Z}_2 W(X)}(L(X)) = \mathbf{Z}_2^\times$ if X is simple.

2.8. **Totally N-determined 2-compact groups.** We are now ready to formulate the concept of N-determinism that will be used in this paper.

2.9. **Definition.** *Let X be a 2-compact group with maximal torus normalizer pair (N, N_0).*

1. X has N-determined automorphisms *if the Adams–Mahmud homomorphism* (2.4) *for X is injective and* $\pi_*(N)$-determined automorphisms *if* $\mathrm{AM}^{-1}(H^1(W(X); \check{T}(X))$ *is trivial.*
2. *X is* N-determined *if for any other 2-compact group X' with maximal torus normalizer pair (N, N_0) there exist an isomorphism $f\colon X \to X'$ and an automorphism $\alpha \in \pi_0(N)\backslash \operatorname{Aut}(N)$ with $\pi_*(B\alpha) = 1$ such that the diagram*

$$\begin{tikzcd} BN \arrow[r, "B\alpha", "\cong"'] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\ BX \arrow[r, "\cong", "Bf"'] & BX' \end{tikzcd} \tag{2.10}$$

commutes up to based homotopy.

3. *X is* totally N-determined *if it has N-determined automorphisms and is N-determined.*

A totally N-determined 2-compact group is

- *uniquely* N-determined if it has $\pi_*(N)$-determined automorphisms (i.e. $H^1(W(X); \check{T}(X)) \cap \mathrm{AM}(\operatorname{Aut}(X)) = \{1\}$)

- *strongly* N-determined if $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$

Thus a totally N-determined p-compact group is both uniquely and strongly N-determined if $H^1(W(X);\check{T}(X)) = 0$.

For a compact connected Lie group G, the cohomology group $H^1(W(G);\check{T}(G))$ is always an elementary abelian 2-group [20, 1.1]. For instance, this first cohomology group has order 2 for $G = \mathrm{PSU}(4)$ [19, Appendix B] (7.2), generated by an involution α, say, of $N(\mathrm{PSU}(4))$. The unique solution to diagram (2.10) is

$$\begin{tikzcd} N(\mathrm{PSU}(4)) \arrow[r, "\alpha"] \arrow[d, "j"'] & N(\mathrm{PSU}(4)) \arrow[d, "j'"] \\ \mathrm{PSU}(4) \arrow[r, equal] & \mathrm{PSU}(4) \end{tikzcd}$$

when we use the morphisms j, induced by an inclusion of Lie groups, and $j' = j\alpha$ for maximal torus normalizers. PSU(4) is a uniquely but not strongly N-determined 2-compact group.

2.11. **Proposition.** *Suppose that the* 2*-compact group* X *is totally* N*-determined.*

1. *For fixed* $\alpha \in \mathrm{Aut}(N)$ *with* $\pi_*(B\alpha) = 1$ *there is at most one isomorphism* $f\colon X \to X'$ *such that diagram in* 2.9.*(*2*) based homotopy commutes.*
2. *The pair* (f, α) *in in* 2.9.*(*2*) is unique* $\Leftrightarrow$ X *is uniquely* N*-determined.*
3. *It is always possible to use* $\alpha = 1$ *in* 2.9.*(*2*)* $\Leftrightarrow$ $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$.
4. $W(X_0)\backslash \mathrm{Aut}(N) = H^1(W(X);\check{T}(X)) \cdot \mathrm{AM}(\mathrm{Aut}(X))$

Proof. 1. If (f_1, α) and (f_2, α) are two solutions to (2.10), then $\mathrm{AM}(f_2^{-1}f_1)$ is the identity and $f_1 = f_2$ as AM is assumed injective.
2. Suppose that the condition is satisfied and let (f_1, α_1) and (f_2, α_2) be two solutions to 2.9.(2). Then $\mathrm{AM}(f_2^{-1}f_1) = \alpha_2^{-1}\alpha_1 \in W(X_0)\backslash \mathrm{Aut}(N)$ belongs to both $\mathrm{AM}(\mathrm{Aut}(X))$ and $H^1(W(X);\check{T}(X))$ and is therefore trivial. Thus $\mathrm{AM}(f_2^{-1}f_1) = 1$ and $f_2 = f_1$ as AM is injective. Conversely, if $\mathrm{AM}(f) \neq 0$ lies in $H^1(W(X);\check{T}(X))$ for some $f \in \mathrm{Aut}(X)$ then $(f, \mathrm{AM}(f))$ and $(1, 0)$ are two solutions to 2.9.(2) with $X' = X$ and $j' = j$.
3. Let $\alpha \in H^1(W(X);\check{T}(X))$. If we can always find an isomorphism under N, then there exists an isomorphism $f \in \mathrm{Aut}(X)$ such that $fj = j\alpha$. This means that $\mathrm{AM}(f) = \alpha$. Conversely, let (f, α) be a solution to 2.9.(2). If $H^1(W(X);\check{T}(X)) \subset \mathrm{AM}(\mathrm{Aut}(X))$ then $\mathrm{AM}(g) = \alpha$ for an automorphism $g \in \mathrm{Aut}(X)$. According to the commutative diagram

$$\begin{tikzcd} BN \arrow[d, "Bj"'] & BN \arrow[l, "B\alpha"'] \arrow[r, "B\alpha"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj'"] \\ BX & BX \arrow[l, "Bg"] \arrow[r, "Bf"'] & BX' \end{tikzcd}$$

$fg^{-1}\colon X \to X'$ is an isomorphism under N.
4. For any automorphism g of N it is possible to find an automorphism f of X and and automorphism α of N with $\pi_*(B\alpha) = 1$ such that the diagram

$$\begin{tikzcd} BN \arrow[r, "B\alpha"] \arrow[d, "Bj"'] & BN \arrow[r, "Bg"] & BN \arrow[d, "Bj"] \\ BX \arrow[rr, "Bf"'] & & BX \end{tikzcd}$$

commutes up to based homotopy. Thus $g = \mathrm{AM}(f)\alpha$. □

The subgroup $H^1(W(X);\check{T}(X))$ is clearly normal so that

$$W(X_0)\backslash \mathrm{Aut}(N) \cong H^1(W(X);\check{T}(X)) \rtimes \mathrm{Aut}(X), \quad \mathrm{Aut}(X) \cong W(X_0)\backslash \mathrm{Aut}(W(X), \check{T}(X), e(X))$$

for a uniquely N-determined 2-compact group X. (The corresponding statement for compact connected Lie groups is true [14, 3.10]. It is already known that compact connected Lie groups perceived as 2-compact groups have $\pi_*(N)$-determined automorphisms [18, 2.5].)

2.12. **Lemma.** *Let X be a 2-compact group. Assume that the identity component X_0 is completely reducible* [23, 3.4, 3.10] *and that $\check{Z}(X_0) = \check{T}(X_0)^{W(X_0)}$.*

1. $H^1(W(X); \check{T}(X)) \cap \mathrm{AM}(\mathrm{Aut}(X)) = H^1(W/W_0; \check{T}^{W_0})$.
2. *If $H^1(W/W_0; \check{T}^{W_0}) \neq 0$ then X does not have $\pi_*(N)$-determined automorphisms.*
3. *If $H^1(W/W_0; \check{T}^{W_0}) = 0$ and X_0 has $\pi_*(N)$-determined automorphisms, so does X.*
4. *If the monomorphism* $\inf \colon H^1(W/W_0; \check{T}^{W_0}) \to H^1(W; \check{T})$ *is an isomorphism,* $H^1(W; \check{T}) \subseteq \mathrm{AM}(\mathrm{Aut}(X))$.

Proof. (1) and (2). This follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(W/W_0; \check{T}^{W_0}) \arrow[r] \arrow[d, hook] & \mathrm{Aut}(X) \arrow[r] \arrow[d, "\mathrm{AM}"] & \mathrm{Aut}(\pi_0, X_0)_X \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & H^1(W; \check{T}) \arrow[r] & W_0 \backslash \mathrm{Aut}(N) \arrow[r] & W_0 \backslash \mathrm{Aut}(W, \check{T}; e) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. The upper row is [24, 5.2].
(3). We must show that

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[r, "\mathrm{AM}"] & W_0 \backslash \mathrm{Aut}(N) \arrow[r] & W_0 \backslash \mathrm{Aut}(W, \check{T}; e)
\end{tikzcd}$$

is injective. The image of this homomorphism is contained in the subgroup $W_0 \backslash \mathrm{Aut}(W, W_0, \check{T}; e)$ where $\mathrm{Aut}(W, W_0, \check{T}; e)$ consists of those pairs $(\alpha, \theta) \in \mathrm{Aut}(W; \check{T}; e)$ for which $\alpha(W_0) = W_0$. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Aut}(\pi_0, X_0)_X \arrow[r, hook] \arrow[d] & \mathrm{Aut}(\pi_0) \times \mathrm{Aut}(X_0) \arrow[d, hook, "1 \times \mathrm{AM}"] \\
 & W_0 \backslash \mathrm{Aut}(W, W_0, \check{T}; e) \arrow[r] & \mathrm{Aut}(\pi_0) \times W_0 \backslash \mathrm{Aut}(W_0, \check{T})
\end{tikzcd}$$

the slanted arrow must be injective. We know from [24, 5.2] that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\pi_0, X_0)_X$.
(4). This is clear from (2) and (3). □

2.13. **Example.** The 2-compact group $\mathrm{GL}(2n, \mathbf{R}) = \mathrm{SL}(2n, \mathbf{R}) \rtimes \mathbf{Z}/2$, $n > 1$, does not have $\pi_*(N)$-determined automorphisms for $H^1(W/W_0; \check{T}^{W_0}) = H^1(\mathbf{Z}/2; \mathbf{Z}/2) = \mathbf{Z}/2$ is non-trivial. The maximal torus normalizer for $\mathrm{GL}(2n, \mathbf{R})$ is the same as the one for $\mathrm{SL}(2n+1, \mathbf{R})$ so $H^1(W; \check{T})$ equals $\mathbf{Z}/2$ for $n = 2$ and $(\mathbf{Z}/2)^2$ for $n \geq 3$ (2.2, 7.3). The 2-compact group $\mathrm{GL}(2n+1, \mathbf{R}) = \mathrm{SL}(2n+1, \mathbf{R}) \times \mathbf{Z}/2$, $n > 0$, has $\pi_*(N)$-determined automorphisms for [24, 5.4.(1)] $\mathrm{Aut}(\mathrm{GL}(2n+1, \mathbf{R})) = \mathrm{Aut}(\mathrm{SL}(2n+1, \mathbf{R}))$ and $\mathrm{SL}(2n+1, \mathbf{R})$ has $\pi_*(N)$-determined automorphisms (as does any compact connected Lie group [18, 2.5]). The 2-compact group $\mathrm{PGL}(2n, \mathbf{R}) = \mathrm{PSL}(2n, \mathbf{R}) \rtimes \mathbf{Z}/2$ has $\pi_*(N)$-determined automorphisms since the identity component has trivial center. In fact, $H^1(W(\mathrm{PSL}(2n, \mathbf{R})) \rtimes \mathbf{Z}/2; \check{T}) \subseteq H^1(\mathbf{Z}/2; H^0(W; \check{T})) + H^0(\mathbf{Z}/2; H^1(W; \check{T})) = 0$ (for $n \geq 5$) since $H^0(W; \check{T}) = 0 = H^1(W; \check{T})$ for $\mathrm{PSL}(2n, \mathbf{R})$ by [13].

2.14. **Lemma.** *Let X be a connected 2-compact group with maximal torus normalizer $j \colon N \to X$. Then X is (uniquely) N-determined if and only if for any other connected 2-compact group X' with maximal torus normalizer $j' \colon N \to X'$ there exists a (unique) morphism $f \colon X \to X'$ such that*

$$\begin{tikzcd}
 & T \arrow[dl, "j|T"'] \arrow[dr, "j'|T"] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd}$$

commutes up to conjugacy.

Proof. The morphism $f \colon X \to X'$ in the above commutative diagram is in fact an isomorphism [8, 5.6] [27, 3.11]. The assumption of the lemma that f be a morphism under T means (use

$W\backslash[BT,BX]=[BT,BX]$ [25, 3.4] [8, 3.4]) that f admits a restriction $N(f)$ to N which is the identity on T, i.e. such that

$$\begin{tikzcd}
BT \arrow[r] \arrow[d, equal] & BN \arrow[r, "Bj"] \arrow[d, "BN(f)"'] & BX \arrow[d, "Bf"] \\
BT \arrow[r] & BN \arrow[r, "Bj'"'] & BX'
\end{tikzcd}$$

is homotopy commutative. But then also $\pi_0N(f)\colon W\to W$ is the identity map for W is faithfully represented as a group of operators on T [6, 9.7]. Thus $\pi_*(BN(f))$ is the identity automorphism of $\pi_*(BN)$.

Assume that the isomorphism f exists and is uniquely determined. In particular, the identity of X is the only automorphism under T. That $f\in\operatorname{Aut}(X)$ is a map under T means precisely that $\operatorname{AM}(f)\in H^1(W;\check{T})$. Thus X is uniquely N-determined by (2.11.2). Suppose, conversely, that X has this property and let $f_0,f_1\colon X\to X'$ be two isomorphisms under T. Then $f_1^{-1}f_0\in\operatorname{Aut}(X)$ is an isomorphism under T so equals the identity. □

2.15**. Remark.** When the 2-compact group X has N-determined automorphisms, also the unbased Adams–Mahmud homomorphism $\operatorname{Out}(X)=W(X)\backslash\operatorname{Aut}(X)\to\operatorname{Out}(N)=\pi_0(N)\backslash\operatorname{Aut}(N)$ is injective [26, 3.7–3.9].

2.16. **LHS 2-compact groups.** Let $N_0\to N$ be maximal rank normal monomorphism between two extended 2-compact tori, i.e. a commutative diagram with rows and columns that are short exact sequences of loop spaces

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
N_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & W/W_0 \arrow[d, equal] \\
W_0 \arrow[r] & W \arrow[r] & W/W_0
\end{tikzcd}$$

where T is a 2-compact torus and W_0 a normal subgroup of the finite group W. The 5-term exact sequence

$$0\to H^1(W/W_0;\check{T}^{W_0})\xrightarrow{\text{inf}}H^1(W;\check{T})\xrightarrow{\text{res}}H^1(W_0;\check{T})^{W/W_0}\xrightarrow{d_2}H^2(W/W_0;\check{T}^{W_0})\xrightarrow{\text{inf}}H^2(W;\check{T})$$

is part of the Lyndon–Hochschild–Serre spectral sequence [15] converging to $H^*(W;\check{T})$.

2.17**. Definition.** *The pair (N,N_0) of extended 2-compact tori is LHS if the initial segment*

$$0\to H^1(W/W_0;\check{T}^{W_0})\xrightarrow{\text{inf}}H^1(W;\check{T})\xrightarrow{\text{res}}H^1(W_0;\check{T})^{W/W_0}\to 0$$

is a short exact sequence. A 2-compact group is LHS if its maximal torus normalizer pair is LHS.

Here are two ways to check if a given p-compact group X is LHS (besides the evident situations where $\check{T}^{W_0}=0$ or $W=W_0\times W/W_0$ is a direct product).

The inflation homomorphism is the composition

$$H^2(W/W_0;\check{T}^{W_0})\to H^2(W/W_0;\check{T})\xrightarrow{H^2(W\to W/W_0)}H^2(W;\check{T})$$

of a coefficient group homomorphism followed by the restriction homomorphism induced by the projection of W onto the group of components W/W_0. If the Weyl group $W=W_0\rtimes W/W_0$ is a semi-direct product, $H^2(W\to W/W_0)$ is injective and therefore

$$(2.18)\quad H^1(W;\check{T})\to H^1(W_0;\check{T})^{W/W_0}\text{ is surjective }\Leftrightarrow\ H^2(W/W_0;\check{T}^{W_0})\to H^2(W/W_0;\check{T})\text{ is injective}$$

by exactness of the Lyndon–Hochschild–Serre spectral sequence.

Another possibility is to use the description of $H^1(W_0;\check{T})$ from [13]. The short exact sequence $1 \to W_0 \to W \to W/W_0 \to 1$ of groups yields an exact sequence

$$H_2(W) \to H_2(W/W_0) \to ((W_0)_{\mathrm{ab}})_{W/W_0} \to W_{\mathrm{ab}} \to (W/W_0)_{\mathrm{ab}} \to 0$$

of abelian groups (where $H_2(W/W_0) = 0$ if W/W_0 has order two). The middle arrow in this exact sequence can be used to define a homomorphism

$$\begin{aligned}\mathrm{Hom}(W,\check{T}^W) = \mathrm{Hom}(W_{\mathrm{ab}},(\check{T}^{W_0})^{W/W_0}) &\to \mathrm{Hom}(((W_0)_{\mathrm{ab}})_{W/W_0},(\check{T}^{W_0})^{W/W_0}) \\ &\to \mathrm{Hom}(W_0,\check{T}^{W_0})^{W/W_0}\end{aligned}$$

which fits into the commutative diagram

$$\begin{tikzcd} H^1(W;\check{T}) \arrow[r] & H^1(W_0;\check{T})^{W/W_0} \\ \mathrm{Hom}(W,\check{T}^W) \arrow[u] \arrow[r] & \mathrm{Hom}(W_0,\check{T}^{W_0})^{W/W_0} \arrow[u] \end{tikzcd} \tag{2.19}$$

Here, the left vertical arrow, say, takes a homomorphism $W \to \check{T}^W$ to the cohomology class represented by the crossed homomorphism $W \to \check{T}^W \hookrightarrow \check{T}$. Since the right vertical arrow is an epimorphism in many cases [13, 1.2, 1.3], this can sometimes be used to show that $H^1(W;\check{T}) \to H^1(W_0;\check{T})^{W/W_0}$ is surjective.

2.20**. Example.** 1. The 2-compact group $\frac{\mathrm{GL}(m,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $m \geq 1$, where the C_2-action switches the two $\mathrm{GL}(m,\mathbf{C})$-factors, is LHS because (2.18) the map

$$H^2\Big(C_2;\frac{\check{S}\times\check{S}}{\check{S}}\Big) \to H^2\Big(C_2;\frac{\check{S}^m\times\check{S}^m}{\check{S}}\Big), \quad \check{S} = \mathbf{Z}/2^\infty,$$

can be identified to the identity on $H^3(C_2;\check{S}) = \mathbf{Z}/2$ since $H^{>0}(C_2;\check{S}\times\check{S}) = 0 = H^{>0}(C_2;\check{S}^m\times\check{S}^m)$ by Shapiro's lemma. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2;\frac{\check{S}\times\check{S}}{\check{S}}\Big) = H^2(C_2;\check{S}) = 0$.

2. The 2-compact group $\frac{\mathrm{GL}(i_0,\mathbf{C})^2\times\mathrm{GL}(i_1,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes C_2$, $i_0, i_1 \geq 1$, where C_2 acts diagonally by switching the two $\mathrm{GL}(i_0,\mathbf{C})$-factors and the two $\mathrm{GL}(i_1,\mathbf{C})$-factors, is LHS, again, because

$$H^2\Big(C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) \to H^2\Big(C_2;\frac{(\check{S}^{i_0})^2\times(\check{S}^{i_1})^2}{\check{S}}\Big), \quad \check{S} = \mathbf{Z}/2^\infty,$$

can be identified to the identity on $H^3(C_2;\check{S})$. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) = H^2(C_2;\check{S}) = 0$.

3. The 2-compact group $\frac{\mathrm{GL}(m,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes (C_2\times C_2)$, $m \geq 1$, where $C_2\times C_2 = \langle(12)(34),(13)(24)\rangle$ permutes the four $\mathrm{GL}(m,\mathbf{C})$-factors, is LHS. Again,

$$H^2\Big(C_2\times C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) \to H^2\Big(C_2\times C_2;\frac{(\check{S}^m)^2\times(\check{S}^m)^2}{\check{S}}\Big), \quad \check{S} = \mathbf{Z}/2^\infty,$$

identifies to the identity on $H^3(C_2\times C_2;\check{S})$ by means of Shapiro's lemma and the Künneth isomorphism. Moreover, $H^1(W/W_0;\check{T}^{W_0}) = H^2\Big(C_2\times C_2;\frac{\check{S}^2\times\check{S}^2}{\check{S}}\Big) = H^2(C_2\times C_2;\check{S}) = H^2(C_2;\check{S}) + H^1(C_2;\mathbf{Z}/2) = H^1(C_2;\check{S}) = \mathbf{Z}/2$.

4. The 2-compact group $\mathrm{GL}(2n,\mathbf{R}) = \mathrm{SL}(2n,\mathbf{R}) \rtimes C_2$, $n \geq 2$, is LHS by (2.18). The homomorphism $\mathbf{Z}/2 = H^2(C_2;\mathbf{Z}/2) \to H^2(C_2,\check{T}) = \mathbf{Z}/2$ is injective because the action of C_2 on $\check{T} = (\mathbf{Z}/2^\infty)^n$ has $(-1,1,\ldots,1)$ as its matrix. The 2-compact group $\mathrm{GL}(4,\mathbf{R}) = \mathrm{SL}(4,\mathbf{R}) \rtimes C_2 = (\mathrm{SL}(2,\mathbf{C}) \circ \mathrm{SL}(2,\mathbf{C})) \rtimes C_2$, in particular, is strongly, but not uniquely N-determined because $0 \neq H^1(W/W_0;\check{T}^{W_0}) = H^1(W;\check{T})$ (2.12, 7.3). For $n > 2$, $\mathrm{GL}(2n,\mathbf{R})$ can be neither uniquely nor strongly N-determined.

2.21. **The center of the maximal torus normalizer.** We need criteria to ensure that the center of the 2-compact group X agrees with the center of its maximal torus normalizer.

2.22. **Proposition.** [29, 4.12] *Let X be a 2-compact group. If $Z(X_0) = Z(N(X_0))$ and X_0 has N-determined automorphisms, then $Z(X) = Z(N(X))$.*

Assume from now on that X is a *connected* 2-compact group. Let $N(X) \to X$ be the maximal torus normalizer and $Z \to N$ a central monomorphism such that also the composition $Z \to N(X) \to X$ is central. The action map $BZ \times BN(X) \to BN(X)$ induces an action $[BN(X), BZ] \times \mathrm{Out}(N(X)) \to \mathrm{Out}(N(X))$ of the group $[BN(X), BZ] \cong H^1(\check{N}(X); \check{Z})$ on the set $\mathrm{Out}(N(X))$. Let $[BN(X), BZ]_{(1)}$ denote the isotropy subgroup at $(1) \in \mathrm{Out}(N(X))$.

2.23. **Lemma.** *If $Z(X) = Z(N(X))$ and $[BN(X), BZ]_{(1)} = 0$, then $Z(X/Z) = ZN(X/Z)$.*

Proof. Using [21, 4.6.4], the assumption of the lemma, and [29, 5.11], we get $Z(X/Z) = Z(X)/Z = Z(N(X))/Z = Z(N(X)/Z) = ZN(X/Z)$. □

2.24. **Remark.** Inspection shows that $Z(G) = ZN(G)$ for any *simply connected* compact Lie group G; see [5, 1.4] for a conceptual proof of this fact. In fact, $Z(G) = ZN(G)$ for any compact connected Lie group G containing no direct factors isomorphic to $\mathrm{SO}(2n+1)$ [20, 1.6].

2.25. **Example.** Let $X = \prod \mathrm{GL}(n_i, \mathbf{C})$ be a product of general linear groups and $Z = \mathbf{C}^\times \cdot (1, \dots, 1)$. Then $Z(X/Z) = ZN(X/Z)$ (2.24), unless $X = \mathrm{GL}(2, \mathbf{C})$, and, assuming that X/Z has N-determined automorphisms, $Z(X_{h\pi}) = ZN(X_{h\pi})$ for any 2-compact group $X_{h\pi}$ with X as its identity component (2.22). Indeed, the discrete approximation to $N(X)$ has the form $\check{N}(X) = \prod(\check{T}_i \rtimes \Sigma_{n_i}) = \check{T} \rtimes W$. Suppose that $(t, w) \in \check{N}(X)$ is such that $[(t, w), (s, 1)] \in \check{Z} = \mathbf{Z}/2^\infty$ for all $s \in \check{T}$. Then $(w-1)\check{T} \subseteq \check{Z}$, which means that w acts trivially on $\check{T}/\check{Z}$. But W is faithfully represented as a group of automorphisms of this maximal torus, so $w = 1$. Suppose therefore that $t \in \check{T}$ is such that $[(t, 1), (s, v)] \in \check{Z}$ for all $(s, v) \in \check{N}(X)$. Then $(v-1)t \in \check{Z}$ for all $v \in W$ and $v \to (v-1)t$ is an element of $H^1(W; \check{Z})$ which becomes trivial in $H^1(W; \check{T})$ where it is a principal crossed homomorphism. Actually, $H^1(W; \check{Z}) = \bigoplus H^1(\Sigma_{n_i}; \check{Z})$ is isomorphic to the subgroup $\bigoplus H^1(\Sigma_{n_i}; \check{T})$ of $H^1(W; \check{T})$.

3. 2-COMPACT GROUPS WITH N-DETERMINED AUTOMORPHISMS

Let X be a 2-compact group with maximal torus normalizer pair $N(X, X_0) = (N, N_0)$.

3.1. **Lemma.** [26, 4.2] *Suppose that*

1. *X_0 has N-determined automorphisms*
2. *$H^1\big(W/W_0; \check{Z}(X_0)\big) \to H^1\big(W/W_0; Z(\check{N}_0)\big)$ is injective*

Then X has N-determined automorphisms.

3.2. **Lemma.** [26, 4.8] *Suppose that X is connected. If the adjoint form $PX = X/Z(X)$ has $\pi_*(N)$-determined automorphisms, so does X.*

Proof. If $f \in \mathrm{Aut}(X)$ is an automorphism under $T(X)$, the induced automorphism $Pf \in \mathrm{Aut}(PX)$ is an automorphism under $T(PX)$, hence equals the identity, and the induced automorphism $Z(f) \in \mathrm{Aut}(ZX)$ is also the identity since the center $ZX \to X$ factors through the maximal torus $T(X) \to X$ [7, 7.5] [21, 4.3]. But then f itself is the identity for $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(PX) \times \mathrm{Aut}(ZX)$ [25, 4.3]. □

The functor $BC_X \colon \mathbf{A}(X) \to \mathbf{Top}$ takes an object (V, ν) of the Quillen category $\mathbf{A}(X)$ to its centralizer $BC_X(V, \nu) = \mathrm{map}(BV, BX)_{B\nu}$. The functor $\pi_j(BZC_X) \colon \mathbf{A}(X) \to \mathbf{Ab}$ takes (V, ν) into the abelian group $\pi_j(\mathrm{map}(BC_X(V, \nu), BX)_{e(\nu)}$ where $e(\nu) \colon BC_X(V, \nu) \to BX$ is the evaluation map.

3.3. **Lemma.** [26, 4.9] *Suppose that X is connected and centerless. If*

1. *$C_X(L, \lambda)$ has N-determined ($\pi_*(N)$-determined) automorphisms for each rank 1 object (L, λ) of $\mathbf{A}(X)$*
2. $\lim^1(\mathbf{A}(X); \pi_1(BZC_X)) = 0 = \lim^2(\mathbf{A}(X); \pi_2(BZC_X))$

Then X has N-determined ($\pi_*(N)$-determined) automorphisms.

Proof. Suppose first that each line centralizer has $\pi_*(N)$-determined automorphisms. Let $f\colon X \to X$ be an automorphism under the maximal torus $T \to X$. Since any monomorphism $\lambda\colon L \to X$, $L = \mathbf{Z}/2$, factors through the maximal torus, the commutative diagram

$$\begin{tikzcd}
 & & N \arrow[r] \arrow[dd, "\mathrm{AM}(f)"] & X \arrow[dd, "f"] \\
L \arrow[r, "\lambda^T"] & T \arrow[ur] \arrow[dr] & & \\
 & & N \arrow[r] & X
\end{tikzcd}$$

shows that $f\lambda = \lambda$ and gives a commutative diagram

$$\begin{tikzcd}
 & C_N(L) \arrow[r] \arrow[dd, "C_{\mathrm{AM}(f)}(L)"] & C_X(L) \arrow[dd, "C_f(L)"] \\
T \arrow[ur] \arrow[dr] & & \\
 & C_N(L) \arrow[r] & C_X(L)
\end{tikzcd}$$

of automorphisms under T. Thus $\mathrm{AM}(C_f(L)) = C_{\mathrm{AM}(f)}(L)\colon C_N(L) \to C_N(L)$. Now, $\pi_*(C_N(L))$ is a subgroup of $\pi_*(N)$ (for $\pi_1(C_N(L)) = \pi_1(N)$ and $\pi_0(C_N(L)) = W(X)(L)$ is [7, 7.6] [25, 3.2.(1)] the stabilizer subgroup at $L < \check{T}$ for the action of $W(X)$ on $\check{T}$) so $\pi_*(C_{\mathrm{AM}(f)}(L)) = 1$ and $C_f(L) \simeq 1_{C_X(L)}$ since $C_X(L)$ has $\pi_*(N)$-determined automorphisms. For any other object (V, ν) of $\mathbf{A}(X)$ of rank > 1, choose a line L in V. Since the monomorphism $\nu\colon V \to X$ canonically factors through $C_X(L)$ [6, 8.2] [29, 3.18], the commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dd, "f"] \\
V \arrow[urr, bend left, "\nu"] \arrow[r] \arrow[drr, bend right, "\nu"'] & C_X(L) \arrow[ur] \arrow[dr] & \\
 & & X
\end{tikzcd}$$

shows that $f\nu = \nu$ and the induced diagram

$$\begin{tikzcd}
 & C_X(V) \arrow[dd, "C_f(V)"] \\
C_{C_X(L)}(V) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & \\
 & C_X(V)
\end{tikzcd}$$

that $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity. The second assumption of the lemma assures that there are no obstructions to conjugating f to the the identity now that we know that the restriction of f to each of the centralizers is conjugate to the identity, see [26, 4.9].

Suppose next that each line centralizer has N-determined automorphisms. Let $f\colon X \to X$ be an automorphism such that the diagram

$$\begin{tikzcd}
 & X \arrow[dd, "f"] \\
N \arrow[ur] \arrow[dr] & \\
 & X
\end{tikzcd}$$

commutes up to conjugacy. For each line L in T, the induced diagram

$$\begin{tikzcd}
 & C_X(L) \arrow[dd, "C_f(L)"] \\
C_N(L) \arrow[ur] \arrow[dr] & \\
 & C_X(L)
\end{tikzcd}$$

also commutes up to conjugacy. By assumption, this means (2.15) that the induced automorphisms $C_f(L)$ of line centralizers are conjugate to the identity. As above, this implies that the induced map $C_f(V)\colon C_X(V) \to C_X(V)$ is conjugate to the identity for any object (V,ν) of the Quillen category for X and that f is conjugate to the identity. □

3.4. **Lemma.** [29, 9.4] *If the two connected* 2*-compact groups* X_1 *and* X_2 *have* N*-determined* ($\pi_*(N)$*-determined) automorphisms, so does the product* $X_1 \times X_2$*.*

Proof. Since the statement conecerning N-determined automorphisms is proved in [29, 9.4] we deal here only with the case of $\pi_*(N)$-determined automorphisms. Let f be an automorphism under $T_1 \times T_2$ of the product 2-compact group $X_1 \times X_2$. Then

$$f_1\colon X_1 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_1$$

$$f_2\colon X_2 \to X_1 \times X_2 \xrightarrow{f} X_1 \times X_2 \to X_2$$

are endomorphisms under the maximal tori and therefore conjugate to the respective identity maps. But f is [29, 9.3] in fact conjugate to the product morphism (f_1, f_2) which is the identity. □

4. N-DETERMINED 2-COMPACT GROUPS

Let X be a 2-compact group with maximal torus normalizer pair $N(X, X_0) = (N, N_0)$.

4.1. **Lemma.** *Suppose that*

(1) X_0 *is uniquely* N*-determined.*
(2) X *is LHS.*
(3) $H^2\big(W/W_0, \check{Z}(X_0)\big) \to H^2\big(W/W_0, Z(\check{N}_0)\big)$ *is injective.*

Then X *is* N*-determined.*

Proof. Let X' be another 2-compact group with maximal torus normalizer pair (N, N_0). The assumption on the identity component X_0 means (2.14) that there exists an isomorphism $f_0\colon X_0 \to X_0'$ under T. For any $\xi \in W/W_0 = N/N_0 = X/X_0 = X'/X_0'$, the isomorphism $\xi f_0 \xi^{-1}$ is also an isomorphism under T and thus $\xi f_0 = f_0 \xi$ as X_0 is uniquely N-determined. By the second assumption, the automorphism $\alpha_0 = \mathrm{AM}(f_0)\colon N_0 \to N_0$ with $\pi_*(B\alpha_0) = 1$ extends to an isomorphism $\alpha\colon N \to N$ with $\pi_*(B\alpha) = 1$.

Our aim is to find an isomorphism $f\colon X \to X'$ to fill in the based homotopy commutative diagram

$$\begin{tikzcd}
BX_0 \arrow[r, "Bf_0", "\cong"'] \arrow[d] & BX_0' \arrow[d] \\
BX \arrow[r, dotted] \arrow[d] & BX' \arrow[d] \\
B\pi_0(X) \arrow[r, "\cong"'] & B\pi_0(X')
\end{tikzcd}$$

where the isomorphism between the base 2-compact groups is given by the isomorphisms $\pi_0(X) \leftarrow N/N_0 \to \pi_0(X')$. Since f_0 is W/W_0-equivariant up to homotopy, $\mathrm{map}(BX_0, BX_0')_{Bf_0}$ is a W/W_0-space. Composition with $BX \xleftarrow{Bj} BN \xrightarrow{Bj'} BX'$ gives maps

$$\mathrm{map}(BX_0, BX_0'; Bf_0)^{hW/W_0} \xrightarrow{Bj^*} \mathrm{map}(BN_0, BX_0'; B(j_0'\alpha))^{hW/W_0}$$

$$\xleftarrow[\simeq]{Bj'_*} \mathrm{map}(BN_0, BN_0; B\alpha_0)^{hW/W_0}$$

of homotopy fixed point spaces. The space to the right is non-empty for it contains the isomorphism $B\alpha\colon BN \to BN$. Using obstruction theory and the second assumption of the lemma, we see that also the homotopy fixed point space to the left is non-empty; it contains a morphism $Bf\colon BX \to BX'$ under $Bf_0\colon BX_0 \to BX_0'$ and over $B\pi_0(X) \xrightarrow{\cong} B\pi_0(X')$ such that $Bf \circ Bj$ and $Bj \circ B\alpha$ are homotopic over $B(N/N_0) \to B\pi_0(X')$. But since the fibre BX_0' of $BX' \to B\pi_0(X')$ is simply connected this means that $Bf \circ Bj$ and $Bj \circ B\alpha$ are based homotopic maps $BN \to BX'$. □

4.2. **Example.** 1. Any 2-compact torus T is strongly N-determined for if $j\colon T \to X$ is the maximal torus normalizer for the connected 2-compact group X, then j is an isomorphism. Indeed, $H^*(BT;\mathbf{Q}_2) \cong H^*(BX;\mathbf{Q}_2)$ [6, 9.7.(3)] and the connected space X/T has cohomological dimension $\mathrm{cd}_{\mathbf{F}_2}(X/T) = 0$ [7, 4.5, 5.6] so is a point.
2. Any 2-compact toral group G is strongly N-determined: G clearly has N-determined automorphisms as G is its own maximal torus normalizer. If the 2-compact group X has the same maximal torus normalizer pair (G,T) as G, then X is a 2-compact toral group and $j'\colon G \to X$ is an isomorphism. G is uniquely N-determined if and only if $H^1(\pi_0(G);\check{T}) = 0$. In particular, $\mathrm{GL}(2,\mathbf{R})$ is uniquely and strongly N-determined.

4.3. **Lemma.** *Let X be a connected 2-compact group and $Z \to X$ its center. If X/Z is N-determined, so is X.*

Proof. Let $j\colon N \to X$ be the maximal torus normalizer for X and $j'\colon N \to X'$ the maximal torus normalizer for some other connected 2-compact group X'. It suffices (2.14) to find a morphism $f\colon X \to X'$ under the maximal tori $X \xleftarrow{i} T \xrightarrow{i'} X'$. The 2-discrete center $\check{Z}$ of X and X' is contained in the the 2-discrete maximal torus $\check{T}$ [7, 7.5]. Factoring out [6, 8.3] these central monomorphisms we obtain the commutative diagram

$$\begin{tikzcd}
B\check{X} \arrow[d] & B\check{T} \arrow[l, "Bi"'] \arrow[r, "Bi'"] \arrow[d] & B\check{X}' \arrow[d] \\
B(X/Z) \arrow[rr, bend right, "B(f/Z)"'] & B(T/Z) \arrow[l, "B(i/Z)"'] \arrow[r, "B(i'/Z)"] & B(X'/Z)
\end{tikzcd}$$

where the vertical maps are fibrations with fibre $B\check{Z}$, the total spaces, such as $B\check{X}$, are the fibre-wise discrete approximations, and $f/Z\colon X/Z \to X'/Z$ is the isomorphism under T/Z that exists because $X?Z$ is N-determined. Construct the fibration

$$\mathrm{map}(B\check{Z}, B\check{Z}; B1) \to B\check{Z}_{h(X/Z)} \to B(X/Z)$$

whose sections are maps $BX \to BX'$ over $B(f/Z)$ and under $B\check{Z}$. There are two other such fibrations related to this one as shown in the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[r, equal] \arrow[d] & \mathrm{map}(B\check{Z}, B\check{Z}; B1) \arrow[d] \\
B\check{Z}_{h(X/Z)} \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[l] \arrow[r, "Bi^*"] \arrow[d] & B\check{Z}_{h(T/Z)} \arrow[d] \\
B(X/Z) & B(T/Z) \arrow[l, "B(i/Z)"] \arrow[r, equal] & B(T/Z)
\end{tikzcd}$$

where the middle fibration is the pull-back along $B(i/Z)$ of the left fibration and the fibre over $b \in B(T/Z)$ of the right fibration consists of one component (remark about equivariance?) of the space of maps of the fibre $B\check{T}_b$ over b into the fibre $B\check{X}'_{B(i'/Z)(b)}$ over $B(i'/Z)(b)$. The fibre equivalence Bi^* is induced by $Bi\colon B\check{T} \to B\check{X}$. The middle fibration has a section u' such that $Bi^* \circ u'$ is the section $Bi'\colon B\check{T} \to B\check{X}'$ of the right fibration. We now have fibre maps

$$\begin{tikzcd}
X/T \arrow[rr, "u|X/T"] \arrow[d] & & B\check{Z} \arrow[d] \\
B\check{T} \arrow[rr] \arrow[dr, "B(i/Z)"'] & & B\check{Z}_{h(X/Z)} \arrow[dl] \\
& B(X/Z) &
\end{tikzcd}$$

where u is the composition of u' and $B\check{Z}_{h(T/Z)} \to B\check{Z}_{h(X/Z)}$. The canonical map, given by constants, $B\check{Z} \to \mathrm{map}(X/T, B\check{Z})$ is a homotopy equivalence since X/T is simply connected [21, 5.6] and hence a version [26, 6.6] of the Zabrodsky lemma implies that $u = v \circ B(i/Z)$ for some section $v\colon B(X/Z) \to B\check{Z}_{h(X/Z)}$ of the left fibration. The section v is, after fibre-wise completion, a fibre map $BX \to BX'$ under BT. $\square$

Let X_1 and X_2 be two connected 2-compact groups with trivial centers and $j_1\colon N_1 \to X_1$, $j_2\colon N_2 \to X_2$ their maximal torus normalizers. The Splitting Theorem [8, 1.4] says that if the monomorphism $j\colon N_1 \times N_2 \to X$ is the maximal torus normalizer for some connected 2-compact group X then there exist an isomorphism $s\colon X \to X_1 \times X_2$ and an automorphism α of $N_1 \times N_2$ such that the diagram

$$\begin{tikzcd}
N_1 \times N_2 \arrow[r, "\alpha", "\cong"'] \arrow[d, "j"'] & N_1 \times N_2 \arrow[d, "j_1 \times j_2"] \\
X \arrow[r, "\cong", "s"'] & X_1 \times X_2
\end{tikzcd}$$

commutes up to conjugacy. We record this in

4.4. **Lemma.** *The product of two N-determined connected 2-compact groups is N-determined.*

The problem is now reduced to the connected and center-less case. Consider therefore an extended 2-compact torus N and two connected, center-less 2-compact groups X and X' both having N as their maximal torus normalizer

$$\begin{tikzcd}
X & N \arrow[l, "j"'] \arrow[r, "j'"] & X'
\end{tikzcd} \tag{4.5}$$

For each toral object (V, ν) of $\mathbf{A}(X)$, let $\nu^N\colon V \to N$ be the unique preferred lift [27, 4.10] of ν (which factors through the identity component of N) and let (V, ν') be the object defined by $\nu' = j \circ \nu^N\colon V \to X'$ as in the commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\nu"'] \arrow[d, "\nu^N"] \arrow[dr, "\nu'"] & \\
X & N \arrow[l, "j"] \arrow[r, "j'"'] & X'
\end{tikzcd}$$

The functor $\mathbf{A}(X)^{\leq t} \to \mathbf{A}(X')^{\leq t}$ that takes the object (V, ν) to the object (V, ν') and is the identity on morphisms is an equivalence of toral Quillen categories [29, 2.8].

4.6. **Theorem.** *In the situation of (4.5), assume the following:*

1. *Centralizers of all toral rank ≤ 2 objects of $\mathbf{A}(X)$ have N-determined automorphisms.*
2. *There exists a self-homotopy equivalence $\alpha \in H^1(W; \check{T}) \subseteq \mathrm{Out}(N)$ such that for every object $(L, \lambda) \in \mathrm{Ob}(\mathbf{A}(X))$ of rank 1 the diagram*

$$\begin{tikzcd}
C_N(\lambda^N) \arrow[r, "\alpha|C_N(\lambda^N)"] \arrow[d, "j|C_N(\lambda^N)"'] & C_N(\lambda^N) \arrow[d, "j'|C_N(\lambda^N)"] \\
C_X(\lambda) \arrow[r, "f_\lambda"'] & C_{X'}(\lambda')
\end{tikzcd}$$

   commutes for some isomorphism f_λ.
3. *For any non-toral rank 2 object (V, ν) of $\mathbf{A}(X)$ the composite monomorphism*

$$\nu'_L\colon\ V \xrightarrow{\overline{\nu}(L)} C_X(L, \nu|L) \xrightarrow[\cong]{f(L,\nu|L)} C_{X'}(L, (\nu|L)') \xrightarrow{\mathrm{res}} X'$$

and the induced isomorphism $f_{\nu,L}\colon C_X(V,\nu) \to C_{X'}(V,\nu'_L)$ *defined by the commutative diagram*

$$\begin{tikzcd}
C_{C_X(L,\nu|L)}(V,\overline{\nu}(L)) \arrow[r, "C_{f(L,\nu|L)}"] \arrow[d, "\cong"'] & C_{C_{X'}(L,(\nu|L)')}(V, f(L,\nu|L)\circ\overline{\nu}(L)) \arrow[d, "\cong"] \\
C_X(V,\nu) \arrow[r, "f_{\nu,L}"'] & C_{X'}(V,\nu'(L))
\end{tikzcd}$$

do not depend on the choice of line $L < V$*.*

(4) $\lim^2(\mathbf{A}(X);\pi_1(BZC_X)) = 0 = \lim^3(\mathbf{A}(X);\pi_2(BZC_X))$.

Then there exists an isomorphism $f\colon X \to X'$ *under* T *(*2.14*).*

Proof. The idea is that the isomorphisms $f_\lambda\colon C_X(\lambda) \to C_{X'}(\lambda')$ on the line centralizers restrict to isomorphisms $f_\nu\colon C_X(\nu) \to C_{X'}(\nu')$ for all centralizers in the $\mathbf{F}_2$-homology decomposition

$$\operatorname{hocolim}_{\mathbf{A}(X)} BC_X(\nu) \to BX$$

of BX. These locally defined isomorphisms combine to a globally defined isomorphism $BX \to BX'$.

First observe that the isomorphisms f_λ on the line centralizers are uniquely determined by the cohomology class $\alpha \in H^1(W;\check{T})$ (2.11.(1)).

Let now (V,ν) be a rank 2 object of $\mathbf{A}(X)$ and L a line in the plane V. If (V,ν) is *toral*, define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be the isomorphism induced by $f_{\nu|L}\colon C_X(L,\nu|L) \to C_{X'}(L,(\nu|L)'$. Since f_ν is an isomorphism under $\alpha|C_N(V,\nu^N)$ it does not depend on the choice of L in V (2.11.(1)). If (V,ν) is *non-toral*, define ν' to be ν'_L and define $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ to be $f_{\nu,L}$. By assumption 4.6.(3), the monomorphism ν' and the isomorphism $f_{\nu,L}$ are independent of the choice of L.

This construction respects morphisms in $\mathbf{A}(X)$. Consider first, for instance, a morphism $\beta\colon (L_1,\lambda_1) \to (L_2,\lambda_2)$ between two lines in X. Then $\lambda_1 = \lambda_2\beta$ and $\lambda_1^N = \lambda_2^N\beta$. The commutative diagram of isomorphisms

$$\begin{tikzcd}
C_X(\lambda_1) \arrow[ddd, "f_{\lambda_1}"'] & & & C_X(\lambda_2) \arrow[lll, "\beta^*"'] \arrow[ddd, "f_{\lambda_2}"] \\
& C_N(\lambda_1^N) \arrow[ul] \arrow[d, "\alpha|C_N(\lambda_1^N)"'] & C_N(\lambda_2^N) \arrow[l, "\beta^*"'] \arrow[ur] \arrow[d, "\alpha|C_N(\lambda_2^N)"] & \\
& C_N(\lambda_1^N) \arrow[dl] & C_N(\lambda_2^N) \arrow[l, "\beta^*"] \arrow[dr] & \\
C_{X'}(\lambda'_1) & & & C_{X'}(\lambda'_2) \arrow[lll, "\beta^*"]
\end{tikzcd}$$

shows that $(\beta^*)^{-1}\circ f_{\lambda_1}\circ\beta^* = f_{\lambda_2}$ for they are both isomorphism under $(\beta^*)^{-1}\circ\alpha|C_N(\lambda_1^N)\circ\beta^* = \alpha|C_N(\lambda_2^N)$. Second, by the very definition of f_ν, the diagram

$$\begin{tikzcd}
C_X(V,\nu) \arrow[r, "f_\nu"] \arrow[d] & C_{X'}(V,\nu') \arrow[d] \\
C_X(L,\nu|L) \arrow[r, "f_{\nu|L}"'] & C_{X'}(L,(\nu|L)')
\end{tikzcd}$$

commutes whenever $L < V$ and (V,ν) is (toral or non-toral) rank 2 object of $\mathbf{A}(X)$.

We have now defined natural isomorphisms $f_\nu\colon C_X(V,\nu) \to C_{X'}(V,\nu')$ for all objects $(V,\nu) \in \mathrm{Ob}(\mathbf{A}(X))$ of rank ≤ 2. For any other object (E,ε) of $\mathbf{A}(X)$, choose a line $L < E$ and proceed as for toral rank 2 objects. That is, define $\varepsilon'\colon E \to X'$ to be the monomorphism

$$\begin{tikzcd}
E \arrow[r, "\overline{\varepsilon}(L)"] & C_X(E,\varepsilon|L) \arrow[r, "f_{\varepsilon|L}"] & C_{X'}(E,(\varepsilon|L)') \arrow[r, "\mathrm{res}"] & X'
\end{tikzcd}$$

and define $f_\varepsilon\colon C_X(E,\varepsilon)\to C_{X'}(E,\varepsilon')$ to be the isomorphism

$$\begin{tikzcd}
C_{C_X(E,\varepsilon|L)}(\overline{\varepsilon}(L)) \arrow[r, "(f_{\varepsilon|L})_*"] \arrow[d, "\cong"'] & C_{C_{X'}(E,(\varepsilon|L)')}(f_{\varepsilon|L}\circ\overline{\varepsilon}(L)) \arrow[d, "\cong"] \\
C_X(E,\varepsilon) \arrow[r, "f_\varepsilon"'] & C_{X'}(E,\varepsilon')
\end{tikzcd}$$

induced by $f_{\varepsilon|L}$. If L_1 and L_2 are two distinct lines in E, let $P=\langle L_1,L_2\rangle$ be the plane generated by them. Then the commutative diagram

$$\begin{tikzcd}
 & C_X(L_1,\varepsilon|L_1) \arrow[r, "f_{\varepsilon|L_1}", "\cong"'] & C_{X'}(L_1,(\varepsilon|L_1)') \arrow[dr, "\mathrm{res}"] & \\
P \arrow[ur, "\overline{\varepsilon}(L_1)"] \arrow[r, "\overline{\varepsilon}(P)"] \arrow[dr, "\overline{\varepsilon}(L_2)"'] & C_X(P,\varepsilon|P) \arrow[u] \arrow[d] \arrow[r, "\cong", "f_{\varepsilon|P}"'] & C_{X'}(P,(\varepsilon|P)') \arrow[u] \arrow[d] \arrow[r, "\mathrm{res}"] & X' \\
 & C_X(L_2,\varepsilon|L_2) \arrow[r, "\cong", "f_{\varepsilon|L_2}"'] & C_{X'}(L_2,(\varepsilon|L_2)') \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

shows that neither $(E,\varepsilon')\in\mathrm{Ob}(\mathbf{A}(X'))$ nor the isomorphism f_ε depend on the choice of line in E. Thus we have constructed a collection of centric [4] maps

$$\mathrm{B}C_X(V,\nu)\to\mathrm{B}X',\quad (V,\nu)\in\mathrm{Ob}(\mathbf{A}(X)),\tag{4.7}$$

that are homotopy invariant under $\mathbf{A}(X)$-morphisms. The vanishing (4.6.(4)) of the obstruction groups means [36] that these homotopy $\mathbf{A}(X)$-invariant maps can be realized by a map

$$\mathrm{B}f\colon \mathrm{B}X\xleftarrow{\cong}\operatorname{hocolim}\mathrm{B}C_X\to\mathrm{B}X'$$

such that $f\circ\mathrm{res}=\mathrm{res}\circ f_\nu$ for all $(V,\nu)\in\mathrm{Ob}(\mathbf{A}(X))$. In particular, f is a map under T and an isomorphism (2.14). $\square$

4.8. **Verification of condition 4.6.(2).** Define $\mathbf{A}_{\mathrm{LHS}}(X)^{\le t}$ to be the full subcategory of the toral Quillen category $\mathbf{A}(X)^{\le t}=\mathbf{A}(W,t)$ [29, 2.2] generated by all objects ν whose centralizers $C_X(\nu)$ are LHS and totally N-determined. For such an object, the solutions to the isomorphism problem

$$\begin{tikzcd}
C_N(\nu^N) \arrow[r, "\alpha_\nu"] \arrow[d] & C_N(\nu^N) \arrow[d] \\
C_X(\nu) \arrow[r, "f_\nu"'] & C_{X'}(\nu')
\end{tikzcd}\tag{4.9}$$

define a subset $\{\alpha_\nu\}$ of $H^1(W;\check{T})(\nu)$ and (2.12.(1)) an element $\overline{\alpha}_\nu$ of $H^1(W_0;\check{T})^{W/W_0}(\nu)$. These elements respect the morphisms in $\mathbf{A}_{\mathrm{LHS}}(X)^{\le t}$ (because the restriction of a solution is a solution) so they represent an element $(\overline{\alpha}_\nu)$ of the limit group. If the two homomorphisms

$$H^1(W(X);\check{T}(X))\to\lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\le t};H^1(W;\check{T}))\to\lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\le t};H^1(W_0;\check{T})^{W/W_0})$$

are surjective, this elements is the image og an element $\alpha\in H^1(W(X);\check{T}(X))$. This means that th isomorphism problems (4.9) have a coherent solution where $\alpha_\nu=\alpha|C_N(\nu^N)$ is the restriction of α for all objects ν of $\mathbf{A}_{\mathrm{LHS}}(X)^{\le t}$.

We can therefore replace 4.6.(1) and 4.6.(2) by

- $C_X(\nu)$ is LHS and totally N-determined for each toral elementary abelian 2-subgroup (V,ν) of X of rank ≤ 2
- $\lim^1(\mathbf{A}_{\mathrm{LHS}}(X)^{\le t};H^1(W_0;\check{T})^{W/W_0})=0$

The first property ensures that

$$\begin{aligned}H^1(W(X);\check{T}(X))\to\lim{}^0(\mathbf{A}_{\mathrm{LHS}}(X)^{\le t};H^1(W;\check{T}))&\cong\lim{}^0(\mathbf{A}(X)^{\le t}_{\le 2};H^1(W;\check{T}))\\&\cong\lim{}^0(\mathbf{A}(X)^{\le t};H^1(W;\check{T}))\end{aligned}$$

is an isomorphism [10, 8.1] [32] and the second property that there is an exact sequence

$$0 \to \lim^0 H^1(W/W_0; \check{T}^{W_0}) \to \lim^0 H^1(W; \check{T}) \to \lim^0 H^1(W_0; \check{T})^{W/W_0} \to 0,$$

where the limits are taken over over $\mathbf{A}_{\mathrm{LHS}}(X)^{\leq t}$ or $\mathbf{A}(X)^{\leq t}_{\leq 2}$. It is sometimes possible to compute the above $\lim^1$-term by means of Oliver's cochain complex [32].

4.10. **Verification of condition 4.6.(3).** The following observation can sometimes be useful in the verification of condition 4.6.(3).

4.11**. Lemma.** *Let (V, ν) be a non-toral rank 2 object of $\mathbf{A}(X)$ and $L < V$ a line in V. Write C_3 for the Sylow 3-subgroup of $\mathrm{GL}(V)$. Suppose that*

1. *$C_3 \subseteq \mathbf{A}(X)(V, \nu) \cap \mathbf{A}(X')(V, \nu')$*
2. *$f_{\nu,L} \colon C_X(V, \nu) \to C_{X'}(V, \nu')$ is C_3-equivariant*

Then condition 4.6.(3) is satisfied.

Proof. Let L_1 and $L_2 = L$ be lines in V. Choose an automorphism α of (V, ν) that takes L_1 to L_2. Then $\nu'_{L_2}\alpha = \nu' - L_1$ and $f_{\nu,L_1} = C_X(\alpha) \circ f_{\nu,L_2} \circ C_X(\alpha)^{-1}$ (4.15). □

The following lemma assures that condition 4.11.(1) holds.

4.12**. Lemma.** *Suppose that*

1. *There is (up to conjugacy) a unique monomorphism $\lambda \colon \mathbf{Z}/2 \to X$ with non-connected centralizer*
2. *There is (up to conjugacy) a unique non-toral monomorphism $\nu \colon (\mathbf{Z}/2)^2 \to X$*

Then the same holds for X' and $\mathbf{A}(X)(V, \nu) = \mathrm{GL}(V) = \mathbf{A}(X')(V, \nu')$ for the non-toral object (V, ν) of $\mathbf{A}(X)$.

Proof. Let $\nu' \colon (\mathbf{Z}/2)^2 \to X'$ be a non-toral monomorphism from a rank elementary abelian 2-group into X' and let $i_1 \colon \mathbf{Z}/2 \to (\mathbf{Z}/2)^2$ be the inclusion into the first summand. Then $\nu' i_1$ corresponds to λ under the bijection between rank 1 objects of $\mathbf{A}(X)$ and $\mathbf{A}(X')$, i.e. $\nu' i_1 = \lambda'$. Moreover, the diagram

(4.13)

$$\begin{tikzcd}
X & C_X(\mathbf{Z}/2, \lambda) \arrow[l, "\mathrm{res}"'] \arrow[rr, "f_\lambda", "\cong"'] & & C_{X'}(\mathbf{Z}/2, \nu' i_1) \arrow[r, "\mathrm{res}"] & X' \\
 & & (\mathbf{Z}/2)^2 \arrow[ull, bend left, "\nu"] \arrow[ul, "\overline{\nu}(\mathbf{Z}/2)"'] \arrow[ur, "\overline{\nu}'(\mathbf{Z}/2)"] \arrow[urr, bend right, "\nu'"'] & &
\end{tikzcd}$$

is commutative. To see this, observe that $\nu = \mathrm{res} \circ f_\lambda^{-1} \circ \overline{\nu}'(\mathbf{Z}/2)$ by uniqueness of ν, and $\overline{\nu}(\mathbf{Z}/2) = f_\lambda^{-1} \circ \overline{\nu}'(\mathbf{Z}/2)$ by uniqueness of canonical factorizations under $\mathbf{Z}/2$ [28, 3.9]. We conclude that $\nu' = \mathrm{res} \circ \overline{\nu}'(\mathbf{Z}/2) = \mathrm{res} \circ f_\lambda \circ \overline{\nu}(\mathbf{Z}/2)$ is uniquely determined up to conjugacy. □

Note in connection with condition 4.11.(2) that by mapping $(\mathbf{Z}/2)^2$ into middle part of diagram (4.13) we see that $f((\mathbf{Z}/2)^2, \nu)$ is a map under the canonical factorization in the sense that

(4.14)

$$\begin{tikzcd}
 & (\mathbf{Z}/2)^2 \arrow[dl, hook', "\overline{\nu}((\mathbf{Z}/2)^2)"'] \arrow[dr, "\overline{\nu}'((\mathbf{Z}/2)^2)"] & \\
C_X((\mathbf{Z}/2)^2, \nu) \arrow[rr, "\cong", "f((\mathbf{Z}/2)^2,\nu)"'] & & C_{X'}((\mathbf{Z}/2)^2, \nu')
\end{tikzcd}$$

commutes where the canonical monomorphisms, $\overline{\nu}((\mathbf{Z}/2)^2)$ and $\overline{\nu}'((\mathbf{Z}/2)^2)$, are $\mathrm{GL}(E)$-equivariant. Thus the restriction of $f(V, \nu)$ to V is C_3-equivariant.

4.15. **Canonical factorizations.** Let $\nu\colon V \to X$ be a monomorphism from an elementary abelian p-group to the p-compact group X. The canonical factorization of ν through its centralizer is the central monomorphism $\overline{\nu}(V)\colon V \to C_X(V,\nu)$ whose adjoint is $V \times V \xrightarrow{+} V \xrightarrow{\nu} X$ [6, 8.2]. If $\alpha\colon (V_1,\nu_1) \to (V_2,\nu_2)$ is a morphism in $\mathbf{A}(X)$ then the canonical factorizations are related by a commutative diagram

(4.16)
$$\begin{tikzcd}
V_1 \arrow[r, "\overline{\nu}_1(V_1)"] \arrow[d, "\alpha"'] & C_X(V_1,\nu_1) \arrow[r, "\mathrm{res}"] & X \arrow[d, equal] \\
V_2 \arrow[r, "\overline{\nu}_2(V_2)"'] & C_X(V_2,\nu_2) \arrow[r, "\mathrm{res}"'] \arrow[u, "C_X(\alpha)"] & X
\end{tikzcd}$$

and we shall write $\overline{\nu}_2(V_1)\colon V_2 \to C_X(V_1,\nu_1)$ for $C_X(\alpha)\circ\overline{\nu}_2(V_2)$ and call it the canonical factorization of ν_2 through the centralizer of ν_1. The induced diagram

(4.17)
$$\begin{tikzcd}
C_{C_X(V_2,\nu_2)}(V_2,\overline{\nu}_2(V_2)) \arrow[r, "C_{C_X(\alpha)}", "\cong"'] \arrow[d, "\cong"'] & C_{C_X(V_1,\nu_1)}(V_2,\overline{\nu}_2(V_1)) \arrow[r, "C_{C_X(V_1,\nu_1)}(\alpha)"] & C_{C_X(V_1,\nu_1)}(V_1,\overline{\nu}_1(V_1)) \arrow[d, "\cong"] \\
C_X(V_2,\nu_2) \arrow[rr, "C_X(\alpha)"'] & & C_X(V_1,\nu_1)
\end{tikzcd}$$

is a factorization of $C_X(\alpha)$.

5. THE QUILLEN CATEGORY OF $\mathrm{PGL}(n+1,\mathbf{C})$

For W is a finite group acting on a finite $\mathbf{F}_2$-vector space V, define $\mathbf{A}(W,V)$ to be the category whose objects are non-trivial subspaces of V and whose morphisms are group homomorphisms induced by the W-action; the morphism set $\mathbf{A}(W,V)(V_1,V_2)$ is the set of orbits $\overline{W}(V_1,V_2)/W(V_1,V_1)$ for the action of the point-wise stabilizer group $W(V_1,V_1)=\{w\in W \mid wv=v \text{ for all } v\in V\}$ on the set $\overline{W}(V_1,V_2)=\{w\in W \mid wV_1\subseteq V_2\}$.

5.1. **The toral subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$.** An object (V,ν) of the Quillen category of the 2-compact group X is *toral* if the monomorphism $\nu\colon V\to X$ is conjugate to a monomorphism that factors through the maximal torus $T(X)$ of X. Let $\mathbf{A}(X)^{\leq t}$ denote the full subcategory of $\mathbf{A}(X)$ generated by all toral objects. We shall determine this toral subcategory in case $X=\mathrm{PGL}(n+1,\mathbf{C})$.

5.2. **Lemma.** *The monomorphism $\nu\colon V\to\mathrm{PGL}(n+1,\mathbf{C})$ is toral if and only if it lifts to a morphism $V\to\mathrm{GL}(n+1,\mathbf{C})$. If n is even, all objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ are toral.*

Proof. All objects of $\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))$ are toral by complex representation theory. Any monomorphism $V\to(\mathbf{C}^\times)^{n+1}/\mathbf{C}^\times$ lifts to $(\mathbf{C}^\times)^{n+1}$ since $\mathbf{C}^\times$ is a divisible abelian group. If n is even, $\mathrm{PGL}(n+1,\mathbf{C})=\mathrm{SL}(n+1,\mathbf{C})$ as 2-compact groups and all monomorphisms $V\to\mathrm{SL}(n+1,\mathbf{C})$ are toral by complex representation theory. □

5.3. **Proposition.** [29, 2.8] *The inclusion $t(\mathrm{PGL}(n+1,\mathbf{C}))\to T(\mathrm{PGL}(n+1,\mathbf{C}))$ induces an equivalence of categories $\mathbf{A}(\Sigma_{n+1},t(\mathrm{PGL}(n+1,\mathbf{C})))\to\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}$.*

Proof. The functor is the identity on morphisms. Any morphism between two toral objects $V_1\to V_2$ of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ is induced from the action by a Weyl group element. □

5.4. **Corollary.** *When $n>1$, the limits $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t};\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})}))=0$ and $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}));\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})}))$ is isomorphic to*

$$\lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))_{\not\leq t};\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t})\cong \lim^i(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}));\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})})_{\not\leq t})$$

for all $i\geq 0$ and $j=1,2$.

Proof. For any non-trivial toral subgroup $V \subseteq t(\mathrm{PGL}(n+1,\mathbf{C}))$ we have by (2.25) that

$$\pi_j(\mathrm{B}ZC_{\mathrm{PGL}(n+1,\mathbf{C})}) = H^{2-j}\big(\Sigma_{n+1}(V); L(\mathrm{PGL}(n+1,\mathbf{C}))\big), \quad j=1,2,$$

because the 2-discrete toral group

$$\begin{aligned}\check{Z}C_{\mathrm{PGL}(n+1,\mathbf{C})}(V) = ZC_{\check{N}(\mathrm{PGL}(n+1,\mathbf{C}))}(V) &= Z\big(\check{T}(\mathrm{PGL}(n+1,\mathbf{C})) \rtimes \Sigma_{n+1}(V)\big) \\ &= H^0(\Sigma_{n+1}(V); \check{T}(\mathrm{PGL}(n+1,\mathbf{C})))\end{aligned}$$

and consequently the higher limits of these functors $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t} \to \mathbf{Ab}$ are trivial while for $i=0$ we get $H^{2-j}(\Sigma_{n+1}; L(\mathrm{PGL}(n+1,\mathbf{C})))$ which is trivial for $n>1$. Apply [29, 2.11] to get the isomorphisms. □

Let E be a non-trivial elementary abelian 2-group and $\mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C}))$ the set of functions $i\colon E^\vee \to \mathbf{N}$ taking the dual $E^\vee = \mathrm{Hom}(E,\mathbf{C}^\times)$ of E into the natural numbers such that $\sum_{f\in E^\vee} i(f) = n+1$. This set supports group actions

$$E^\vee \times \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \longrightarrow \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \longleftarrow \mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \times \mathrm{GL}(E)$$

given by $g\cdot i = i\circ\tau_g$, $g\in E^\vee$, and $i\cdot A = i\circ A^\vee$, $A\in\mathrm{GL}(E)$, where $\tau_g(f) = g+f$ and $A^\vee(f) = f\circ A^{-1}$ for all linear forms $f\in E^\vee$. The identity $\tau_{A^\vee(g)}A^\vee = A^\vee\tau_g$ gives $(g\cdot i)\cdot A = ((A^{-1})^\vee g)\cdot(i\cdot A)$.

We say that a subset S of linear forms on E has trivial equalizer, and write $\mathrm{Eq}(S)=0$, if S contains at least two elements and all the elements of S agree only on the trivial element of E.

5.5. **Proposition.** *Let E be a non-trivial elementary abelian 2-group.*

(1) *The set of conjugacy classes of toral monomorphisms $\nu\colon E\to\mathrm{PGL}(n+1,\mathbf{C})$ corresponds bijectively to the set*

$$E^\vee\backslash\{i\in\mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \mid \mathrm{Eq}(S(i))=0\}$$

of $E^\vee$-orbits.

(2) $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i) = \{A\in\mathrm{GL}(E) \mid (E^\vee i)\cdot A^\vee = E^\vee i\}$.

(3) $\pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i)) = \{\zeta\in E^\vee \mid \zeta\cdot i = i\}$.

(4) *The set of isomorphism classes of $\dim_{\mathbf{F}_p} E$-dimensional toral objects of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ corresponds bijectively to the set*

$$E^\vee\backslash\{i\in\mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C})) \mid \mathrm{Eq}(S(i))=0\}/\mathrm{GL}(E)$$

of $E^\vee\times\mathrm{GL}(E)$-orbits.

Proof. 1. Let $\nu\colon E\to\mathrm{PGL}(n+1,\mathbf{C})$ be a toral monomorphism and $\mu\colon E\to\mathrm{GL}(n+1,\mathbf{C})$ any lift of ν to $\mathrm{GL}(n+1,\mathbf{C})$. The representation μ is a sum of linear characters

$$\mu = \sum_{f\in E^\vee} i_\mu(f)f$$

for some function $i_\mu\in\mathrm{Rep}(E,\mathrm{GL}(n+1,\mathbf{C}))$. The condition that $\mu(E)$ intersects the center $\mathbf{C}^\times$ trivially, translates to $\mathrm{Eq}(S(i_\mu))=0$ (or, equivalently, $S(i_\mu)$ spans $V^\vee$ and $V=\bigcup_{f\in S(i_\mu)}\ker f$).

Any other lift of ν has the form $\zeta\mu$ for some $\zeta\in E^\vee$. We have $i_{\zeta\mu} = \zeta\cdot i_\mu$ for

$$(\zeta\mu)(v) = \sum i_\mu(f)\zeta(v)f(v) = \sum(i_\mu\circ\tau_\zeta)(f)f(v) = \sum(\zeta\cdot i_\mu)(f)f(v)$$

for all $v\in V$. (Also, $S(\tau_\zeta i_\mu) = \tau_\zeta S(i_\mu)$ so the equalizer subspace does not change.)

2. An automorphism $A\in\mathrm{GL}(E)$ preserves the conjugacy class of $\nu\colon E\to\mathrm{PGL}(n+1,\mathbf{C})$ if and only if $\mu A(v) = \zeta(v)\mu(v)$ for some $\zeta\in E^\vee$ (depending on A). Since

$$(\mu A)(v) = \sum i_\mu(f)(fA)(v) = \sum(i_\mu\circ A^\vee)(f)f(v) = \sum(i_\mu\cdot A)(f)f(v)$$

for all $v\in V$, this means that $i_\mu\cdot A = \zeta\cdot i_\mu$. Then $(g\cdot i_{mu})\cdot A = ((A^{-1})^\vee g)\cdot(i_{mu}\cdot A) = ((A^{-1})^\vee g)\cdot(\zeta\cdot i_{mu}) = ((A^{-1})^\vee g+\zeta)\cdot i_{mu}\in E^\vee i_{mu}$ for all $g\in E^\vee$.

3. The component group of $C_{PGL(n+1,\mathbf{C})}(E^\vee i_\mu)$ is [29, 5.11.(2)] isomorphic to the group of $\zeta\in E^\vee$ for which μ and $\zeta\mu$ are conjugate in $\mathrm{GL}(n+1,\mathbf{C})$. For the traces, this means that $i_\mu = \zeta\cdot i_\mu$. □

5.6. **Remark.** 1. Since $i\tau_f\tau_\zeta = i\tau_f \Leftrightarrow i\tau_\zeta = i$, the right hand side for the equation in 5.5.(3) remains the same for all elements of the orbit $E^\vee i$.
2. Let $A \in \mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i)$ so that $iA^\vee = i\tau_\zeta$ for some $\zeta \in E^\vee$. Then

$$i\tau_{A^\vee(g)} = i \Leftrightarrow i\tau_{A^\vee(g)}A^\vee = iA^\vee \Leftrightarrow iA^\vee\tau_g = iA^\vee \Leftrightarrow i\tau_\zeta\tau_g = i\tau_\zeta \Leftrightarrow i\tau_g = i$$

for any $g \in E^\vee$, meaning that $A^\vee(g) \in \pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i)) \Leftrightarrow g \in \pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i))$. Thus $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(E^\vee i)^{\mathrm{op}}$ acts on $\pi_0(C_{\mathrm{PGL}(n+1,\mathbf{C})}(E^\vee i))$.

5.7. **Example.** (Toral lines and planes in $\mathrm{PGL}(m,\mathbf{C})$) Let $P(m,k)$ be the number of ways to write $m = i_0 + i_1 + \cdots + i_k$ as a sum of k integers $i_0, i_1, \ldots, i_k$ such that $1 \le i_0 \le i_1 \le \cdots \le i_k$. There are $P(m,2) = [m/2]$ toral lines and $P(m,3) + P(m,4)$ toral planes in $\mathrm{PGL}(m,\mathbf{C})$. The $P(m,2)$ toral lines of type $i = (i_0,i_1)$ with $i_0, i_1 > 0$ and $i_0 + i_1 = m$ have these Quillen automorphism groups and centralizer component groups:

(i_0,i_1)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(L) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(L)) = 1$
(i_0,i_0)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(L) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(L)) = L^\vee$

The non-connected rank 1 centralizer is

$$C_{\mathrm{PGL}(m,\mathbf{C})}(L) = \frac{\mathrm{GL}(i_0,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes L^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(L) \overset{7.7}{=} L$$

The $P(m,3) + P(m,4)$ toral planes of type $i = (i_0,i_1,i_2,i_3)$ with $i_0,i_1,i_2 > 0$, $i_3 \ge 0$, and $i_0 + i_1 + i_2 + i_3 = m$ have these Quillen automorphism groups and centralizer component groups:

(i_0,i_1,i_2,i_3)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = 1$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
(i_0,i_0,i_2,i_3)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = C_2$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
(i_0,i_0,i_0,i_3)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = \mathrm{GL}(V)$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$
(i_0,i_0,i_2,i_2)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = C_2$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = L^\vee$ for some line $L < V$.
(i_0,i_0,i_0,i_0)**:** $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V) = \mathrm{GL}(V)$, $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = V^\vee$

If $V = \mathbf{F}_2^2$ is a plane, then $V^\vee$ and $\mathrm{GL}(V) \cong \Sigma_3$ together generate all permutations of the four letters (i_0,i_1,i_2,i_3). Thus there are $P(m,3)$ isomorphism classes of the form $(i_0,i_1,i_2,0)$, $i_0,i_1,i_2 > 0$, $i_0 + i_1 + i_2 = m$ and $P(m,4)$ isomorphism classes of the form (i_0,i_1,i_2,i_3), $i_0,i_1,i_2,i_3 > 0$, $i_0+i_1+i_2+i_3 = m$. If, for instance $i = (i_0,i_0,i_0,i_3)$ with $i_0 \ne i_3$, then $V^\vee i = i\mathrm{GL}(V)$ contains four elements, so $\pi_0 \cong V^\vee$ and $\mathrm{Aut} = \mathrm{GL}(V)$. In all cases, $\pi_0 C_{\mathrm{PGL}(m,\mathbf{C})}(V,\nu) = \pi_0 ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\nu)$; this is clear in case $\pi_0(C_{\mathrm{PGL}(m,\mathbf{C})}(V)) = 1$ is trivial and in the remaining two cases it is a direct check.

The character table for $V = C_2 \times C_2 = \{e_0, e_2, e_2, e_3 = e_1 + e_2\}$

	e_0	e_1	e_2	e_3
ρ_0	1	1	1	1
ρ_1	1	1	-1	-1
ρ_2	1	-1	1	-1
ρ_3	1	-1	-1	1

contains four linear characters $V^\vee = \{\rho_0,\rho_1,\rho_2,\rho_3\}$. In the list above, (i_0,i_1,i_2,i_3) means $i_0\rho_0 + i_1\rho_1 + i_2\rho_2 + i_3\rho_3$. Non-connected $\mathrm{PGL}(m,\mathbf{C})$-centralizers only occur for induced $\mathrm{GL}(m,\mathbf{C})$-representations:

$$(i_0,i_0,i_2,i_2) = \begin{cases} \mathrm{ind}_{\langle e_1\rangle}^V(i_0\rho_0 + i_2\rho_1) & i_0 \ne i_2 \\ \mathrm{ind}_{\{0\}}^V(i_0\rho_0) = i_0\mathrm{reg}_V & i_0 = i_2 \end{cases}$$

In the first case, the centralizer

$$C_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) = \frac{\mathrm{GL}(i_0,\mathbf{C})^2 \times \mathrm{GL}(i_1,\mathbf{C})^2}{\mathrm{GL}(1,\mathbf{C})} \rtimes L^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) \overset{7.7}{=} \frac{\mathrm{GL}(1,\mathbf{C}) \times \mathrm{GL}(1,\mathbf{C})}{\mathrm{GL}(1,\mathbf{C})} \times L,$$

is LHS and has $\pi_*(N)$-determined automorphisms (2.20). In the second case, we have a pure rank 2 object, the only rank 1 sub-object is $2i_0$ times the regular representation of C_2. Its centralizer

$$C_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) = \frac{\mathrm{GL}(i_0,\mathbf{C})^4}{\mathrm{GL}(1,\mathbf{C})} \rtimes V^\vee, \quad ZC_{\mathrm{PGL}(m,\mathbf{C})}(V,\rho) \overset{7.5}{=} V,$$

is LHS but does not does not have $\pi_*(N)$-determined automorphisms (2.20).

5.8. **The non-toral subcategory of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$.** For 2-compact group X, let $\mathbf{A}(X)_{\not\leq t}$ denote the full subcategory of $\mathbf{A}(X)$ on all non-toral objects and their sub-objects. We determine this non-toral subcategory $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))_{\not\leq t}$ in case $X = \mathrm{PGL}(n+1,\mathbf{C})$.

For any non-trivial elementary abelian 2-group V in $\mathrm{PGL}(n+1,\mathbf{C})$, let $[\ ,\ ]\colon V\times V\to\mathbf{F}_2$ be the symplectic bilinear form [16, II.9.1] given by $[u\mathbf{C}^\times, v\mathbf{C}^\times] = [u,v]$ for all $u\mathbf{C}^\times$, $v\mathbf{C}^\times \in V$. (The elements $[u,v]$ and u^2 lie in the center $\mathbf{C}^\times$ of $\mathrm{GL}(n+1,\mathbf{C})$ so that $E = [u^2,v] = [u,v]^u[u,v] = [u,v]^2$ and thus $[u,v]\in\mathbf{C}^\times$ has order 2. Therefore $[u,v] = [u,v]^{-1} = [v,u]$.)

5.9. **Lemma.** *V in $\mathrm{PGL}(n+1,\mathbf{C})$ is toral* $\Leftrightarrow [V,V] = 0$

Proof. Let $e_i\mathbf{C}^\times$, $1\leq i\leq d$, be a basis for V. Since $\mathbf{C}^\times$ is divisible, we can assume that each $e_i\in\mathrm{GL}(n+1,\mathbf{C})$ has order 2. If $[V,V]=0$, these e_is commute and span a lift to $\mathrm{GL}(n+1,\mathbf{C})$ of $V\subseteq\mathrm{PGL}(n+1,\mathbf{C})$. □

An extra special 2-group is of *positive type* if it is isomorphic to a central product of dihedral groups D_8 of order 8.

5.10. **Lemma.** [12, 3.1] [29, 5.4] *Let $\nu\colon V\to\mathrm{PGL}(n,\mathbf{C})$ be a non-toral monomorphism of a non-trivial elementary abelian 2-group V into $\mathrm{PGL}(n+1,\mathbf{C})$. Then there exists a morphism of short exact sequences of groups*

$$\begin{tikzcd}
1 \arrow[r] & Z(P) \arrow[r] \arrow[d, hook] & PE \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, hook, "\nu"] & 1 \\
1 \arrow[r] & \mathbf{C}^\times \arrow[r] & \mathrm{GL}(n+1,\mathbf{C}) \arrow[r] & \mathrm{PGL}(n+1,\mathbf{C}) \arrow[r] & 1
\end{tikzcd}$$

where PE is the direct product of an extra special 2-group $P\subseteq\mathrm{GL}(n+1,\mathbf{C})$ of positive type and an elementary abelian 2-group $E\subseteq\mathrm{GL}(n+1,\mathbf{C})$ with $P\cap E = \{1\} = [P,E]$.

Write $\mathbf{C}^{n+1} = \mathbf{C}^{2^d}\otimes\mathbf{C}^m$ for some $d>0$ and some $m\geq 0$. Let the extra-special 2-group 2_+^{1+2d} act faithfully on the first factor of the tensor product and let the (possibly trivial) elementary abelian 2-group E act faithfully on the second factor such that no non-trivial element of E acts as scalar multiplication. This makes $\mathbf{C}^{n+1}$ a $\mathbf{C}[2_+^{1+2d}\times E]$-module. The image of the group $2_+^{1+2d}\times E\subseteq\mathrm{GL}(n+1,\mathbf{C})$ in $\mathrm{PGL}(n+1,\mathbf{C})$ is a non-toral elementary abelian 2-group (5.9) and any non-toral elementary abelian 2-group in $\mathrm{PGL}(n+1,\mathbf{C})$ has this form (5.10).

Let $G = \langle P,E,i\rangle = P\circ C_4\times E$ be the group generated by E and the central product $P\circ C_4$ of P and the cyclic group $C_4 = \langle i\rangle\subseteq\mathbf{C}^\times$ with $\mathbf{Z}/2$ amalgamated. The image of G in $\mathrm{PGL}(n+1,\mathbf{C})$ is V and $q(v\mathbf{C}^\times) = v^2$, $v\in G$, is a quadratic form on V such that $q(u\mathbf{C}^\times + v\mathbf{C}^\times) = q(u\mathbf{C}^\times) + q(v\mathbf{C}^\times) + [u\mathbf{C}^\times, v\mathbf{C}^\times]$ for all $u\mathbf{C}^\times, v\mathbf{C}^\times\in V$.

5.11. **Lemma.** *$\mathbf{A}(\mathrm{GL}(n+1,\mathbf{C}))(G,G)\to\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(V,V)$ is surjective.*

Proof. Suppose that $B\in\mathrm{GL}(n+1,\mathbf{C})$ is such that $V^{B\mathbf{C}^\times} = V$. Then $G^B\subseteq G\cdot\mathbf{C}^\times$: For any $g\in G$ there exist $h\in G$ and $z\in\mathbf{C}^\times$ such that $g^B = hz$. But since G has exponent 4, $z^4 = 1$ so $z\in C_4$ and $g^B\in G$. □

A monomorphic conjugacy class $\nu\colon V\to\mathrm{PGL}(n+1,\mathbf{C})$ is said to be a $(2d+r,r)$ object of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ if the underlying symplectic vector space of (V,ν) is isomorphic to $V = H^d\times V^\perp$ where H denotes the symplectic plane over $\mathbf{F}_2$ and $\dim_{\mathbf{F}_p} V^\perp = r$ [16, II.9.6] (so that $\dim_{\mathbf{F}_p} V = r+2d$). An (r,r) object is the same thing as an r-dimensional toral object. We write $\mathrm{Sp}(V)$ or $\mathrm{Sp}(2d+r,r)$ (abbreviated to $\mathrm{Sp}(2d)$ if $r=0$) for the group of linear automorphisms of V that preserve the symplectic form.

5.12. **Corollary.** *Suppose that $n+1 = 2^d m$ for some natural numbers $d\geq 1$ and $m\geq 1$.*

1. *There is up to isomorphism a unique $(2d,0)$ object H^d of $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$, and*

$$\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))(H^d) = \mathrm{Sp}(2d), \quad C_{\mathrm{PGL}(n+1,\mathbf{C})}(H^d) = H^d\times\mathrm{PGL}(m,\mathbf{C})$$

for this object.

(2) *Isomorphism classes of* $(2d+r,r)$, $r>0$, *objects* V *of* $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$ *correspond bijectively to isomorphism classes of* (r,r) *objects* $V^\perp$ *of* $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$, *and*

$$\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))(V) = \begin{pmatrix} \mathrm{Sp}(2d) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(V^\perp) \end{pmatrix}$$

$$C_{\mathrm{PGL}(2^dm,\mathbf{C})}(V) = V/V^\perp \times C_{\mathrm{PGL}(m,\mathbf{C})}(V^\perp)$$

for these objects.

Proof. 1. The group $2^{1+2d}_+ \circ 4$ has [17, 7.5] 2^{1+2d} characters of degree 1 and 2 irreducible characters of degree 2^d (interchanged by the action of $\mathrm{Out}(2^{1+2d}_+ \circ 4) \cong \mathrm{Sp}(2d) \times \mathrm{Aut}(C_4)$ [11, pp. 403–404]) given by

$$\chi_\lambda(g) = \begin{cases} 2^d\lambda(g) & g \in C_4 \\ 0 & g \notin C_4 \end{cases}$$

where $\lambda\colon C_4 \to \mathbf{C}^\times$ is an injective group homomorphism ($\lambda(i) = \pm i$). The linear characters vanish on the derived group $2 = [2^{1+2d}_+ \circ 4, 2^{1+2d}_+ \circ 4]$ but the irreducible characters of degree 2^d do not. Thus the only faithful representations of $2^{1+2d}_+ \circ 4$ with central centers are multiples $m\chi_\lambda$ of χ_λ for a fixed λ. Phrased slightly differently, $\mathrm{GL}(m2^d,\mathbf{C})$ contains up to conjugacy a unique subgroup with central center isomorphic to $2^{1+2d}_+ \circ 4$. For this group and its image H^d in $\mathrm{PGL}(2^dm,\mathbf{C})$ we have

$$\mathbf{A}(\mathrm{GL}(m2^d,\mathbf{C}))(2^{1+2d}_+ \circ 4, 2^{1+2d}_+ \circ 4) \cong \mathrm{Sp}(2d) \cong \mathbf{A}(\mathrm{PGL}(m2^d,\mathbf{C}))(H^d,H^d)$$

$$C_{\mathrm{GL}(m2^d,\mathbf{C})}(2^{1+2d}_+ \circ 4) \cong \mathrm{GL}(m,\mathbf{C}), \quad C_{\mathrm{PGL}(m2^d,\mathbf{C})}(H^d) \cong H^d \times \mathrm{PGL}(m,\mathbf{C})$$

where the last isomorphism is a consequence of [29, 5.9].

2. The $(2d+r,r)$ object (V,ν) of $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$ and the $(r,0)$ object $(V^\perp,\nu^\perp)$ of $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))$ correspond to each other iff there is an m-dimensional representation $\mu\colon V^\perp \to \mathrm{GL}(m,\mathbf{C})$ such that $\mathbf{C}^{2^d} \otimes \mu$ is a lift of $\nu|V^\perp$ and μ a lift of $\nu^\perp$. According to 5.10 any lift of $\nu|V^\perp$ has this form for some μ uniquely determined up to the action of $(V^\perp)^\vee$.

We use 5.11 to calculate the Quillen automorphism group of a $(2d+r,r)$ object $H^d \times V^\perp$ of $\mathbf{A}(\mathrm{PGL}(2^dm,\mathbf{C}))$. Let $H^d \times V^\perp$ be covered by the group $P \circ C_4 \times V^\perp$ as in 5.10. Let α be an automorphism of $P \circ C_4$, let β be any homomorphism of the form $P \circ C_4 \to H^d \to V^\perp$, and let γ be any Quillen automorphism of $(V^\perp,\nu^\perp)$. Choose a homomorphism $\zeta_1\colon P \circ C_4 \to H^d \times C_4/C_2 \to C_4$ such that $\lambda(\zeta_1(x)\alpha(x)) = \lambda(x)$ for all $x \in C_4$ and a homomorphism $\zeta_2\colon V^\perp \to C_4$ such that $\lambda(\zeta_2(v))\mu(\gamma(v)) = \mu(v)$ for all $v \in V^\perp$. Then the automorphism of $P \circ C_4$ that takes (x,v) to $(\zeta_1(x)\zeta_2(v)\alpha(x), \beta(x)+\gamma(v))$ preserves the trace of $\chi_\lambda \# \mu$ and therefore the automorphism induced on the quotient is a Quillen automorphism of $H^d \times V^\perp$. Conversely, any automorphism of $P \circ C_4 \times V^\perp$ takes the center $C_4 \times V^\perp$ isomorphically to itself and hence it is of the form $(x,v) \to (\zeta(x,v)\alpha(x), \beta(x)+\gamma(v))$ for some automorphism α of $P \circ C_4$, some homomorphism $\beta\colon P \circ C_4 \to V^\perp$ vanishing on C_4, and some homomorphism $\zeta\colon P \circ C_4 \times V^\perp \to C_4$. Such an automorphism preserves the trace of $\chi_\lambda \# \mu$ iff $\lambda(\zeta(x,v)\alpha(x)) = \mu(\gamma(v))$ for all $(x,v) \in Z(P \circ C_4 \times V^\perp) = C_4 \times V^\perp$. But this means that the induced automorphism of $H^d \times V^\perp$ is of the stated form. □

5.13. **Example.** (Oliver's cochain complex [32]) The non-toral objects of $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))$ of rank ≤ 4 are

- One $(2,0)$ object H, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(H) = \mathrm{Sp}(2) = \mathrm{GL}(2,\mathbf{F}_2)$, $\pi_0 = H$.
- $P(m,2)$ $(3,1)$ objects V, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(V) = \mathrm{Sp}(3,1)$, $\pi_0 = V/V^\perp$ or V.
- $P(m,3)+P(m,4)$ $(4,2)$ objects E, $\mathbf{A}(\mathrm{PGL}(2m,\mathbf{C}))(E) = \begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp) \end{pmatrix}$, $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp) = 1$, C_2, $\mathrm{GL}(E)$, $\pi_0 = E/E^\perp$, $E/E^\perp$ or E/L, $E/E^\perp$ or E.
- One $(4,0)$ object if m is even.

The $(2,0)$ object H contributes

$$\mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H),H) \cong \mathbf{F}_2$$

The $(3,1)$ objects V contribute

$$\mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V) \cong \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V/V^\perp) \cong \mathbf{F}_2$$

The $(4,2)$ objects E with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=1$ contribute

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\cong\mathbf{F}_2^2$$

and the $(4,2)$ objects E with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=C_2$ contribute

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/L)\cong\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\cong\mathbf{F}_2$$

The $(4,0)$ object (if it exists) and the $(4,2)$ objects with $\mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp)=\mathrm{GL}(E)$ do not contribute to the cochain complex for the corresponding Hom-groups are trivial. Thus the cochain complex for computing higher limits of the functor $\pi_1(BZC_{\mathrm{PGL}(2m,\mathbf{C})})$ will have the form

$$(5.14)\quad 0\to\mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H),H)\xrightarrow{\delta^1}\prod_{[m/2]}\mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V),V/V^\perp)\xrightarrow{\delta^2}$$
$$\prod\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & 1\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\times\prod\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0\\ * & C_2\end{pmatrix}}(\mathrm{St}(E),E/E^\perp)\to\cdots$$

To show vanishing of the relevant higher limits it suffices to show that δ^1 is injective and that the rank of δ^2 is $P(m,2)-1$.

6. N-DETERMINISM OF THE A-FAMILY

By inductively applying 3.3 and 4.6 we show that the 2-compact groups $\mathrm{PGL}(n+1,\mathbf{C})$, $n\geq 1$, are uniquely N-determined.

6.1. **Lemma.** *Suppose that $n+1=2m\geq 2$ is even.*

(1) *There is a unique monomorphism conjugacy class $\lambda\colon\mathbf{Z}/2\to\mathrm{PGL}(n+1,\mathbf{C})$ with disconnected centralizer. The centralizer of this monomorphism is $\mathrm{GL}(m,\mathbf{C})^2/\mathbf{C}^\times\rtimes\mathbf{Z}/2$*

(2) *There is a unique monomorphism conjugacy class $\nu\colon H\to\mathrm{PGL}(n+1,\mathbf{C})$, $H=(\mathbf{Z}/2)^2$, such that ν is non-toral. The centralizer of this monomorphism is $H\times\mathrm{PGL}(m,\mathbf{C})$ and the Quillen automorphism group is $\mathrm{GL}(H)$.*

Proof. Use that any monomorphism of $\mathbf{Z}/2$ into $\mathrm{PGL}(n+1,\mathbf{C})$ lifts to $\mu\colon\mathbf{Z}/2\to\mathrm{GL}(n+1,\mathbf{C})$. The only possibility is that $\mu=m\cdot\mathrm{reg}$ is a direct sum of regular representations. The result for non-toral rank 2 objects in $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))$ is a special case of 5.10. □

6.2. **Lemma.** *Suppose that $\mathrm{PGL}(r+1,\mathbf{C})$ is uniquely N-determined for all $0\leq r<n$. Then $\mathrm{PGL}(n+1,\mathbf{C})$, $n\geq 1$, satisfies conditions 4.6.(1), 4.6.(2), and 4.6.(3).*

Proof. We shall verify 4.6.(1) and 4.6.(2) by establishing the alternative two conditions from 4.8.

Let (V,ν) be a toral elementary abelian 2-subgroup of $\mathrm{PGL}(n+1,\mathbf{C})$ of rank ≤ 2 and $C(\nu)=C_{\mathrm{PGL}(n+1,\mathbf{C})}(\nu)$ its centralizer. We have seen that $C(\nu)$ is LHS (2.20) and that $\check{Z}(C(\nu)_0)=\check{Z}(N_0(C(\nu)))$ as $C(\nu)_0$ does not contain a direct factor isomorphic to $\mathrm{GL}(2,\mathbf{C})/\mathrm{GL}(1,\mathbf{C})=\mathrm{SO}(3)$ (2.24, 5.7). The identity component $C(\nu)_0$ has $\pi_*(N)$-determined automorphisms according to 3.2 and 3.4, and $C(\nu)$ has N-determined automorphisms by 3.1. The identity component $C(\nu)_0$ is N-determined according to 4.3 and 4.4, and $C(\nu)$ is N-determined by 4.1. Thus $C(\nu)$ is LHS and totally N-determined.

The functor $H^1(W/W_0;\check{T}_0^W)$ is zero on $\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2}$ except on the object $(V,\nu)=(i_0,i_0,i_0,i_0)$, when $n+1=4i_0$, where it has value $\mathbf{Z}/2$. However, this object has Quillen automorphism group $\mathrm{GL}(V)$ and since the only $\mathrm{GL}(V)$-equivariant homomorphism $\mathrm{St}(V)=V\to\mathbf{Z}/2$ is the trivial homomorphism, $\lim^1(\mathbf{A}(\mathrm{PGL}(n+1,\mathbf{C}))^{\leq t}_{\leq 2};H^1(W/W_0;\check{T}_0^W))=0$ follows from Oliver's cochain complex [32].

When $n+1=2m$ is even, we verify condition 4.6.(3) by applying 4.11. Let X' be a connected 2-compact group with maximal torus normalizer $j'\colon N(\mathrm{PGL}(n+1,\mathbf{C})\to X$. Since the first item in 4.11 is satisfied by 4.12 and 6.1, it suffices to show that the isomorphism (from 4.6.(3))

$$f_{\nu,L}\colon C_{\mathrm{PGL}(2m,\mathbf{C})}(H)=H\times\mathrm{PGL}(m,\mathbf{C})\to C_{X'}(H,\nu')$$

defined by choosing one of the three lines L in H, is C_3-equivariant. Now [24]

$$\mathrm{Aut}(H \times \mathrm{PGL}(m, \mathbf{C})) = \mathrm{GL}(H) \times \mathrm{Aut}(\mathrm{PGL}(m, \mathbf{C}))$$

so that $f_{\nu,L}$ is C_3-equivariant if $\pi_0 f_{\nu,L}$ and the restriction of $f_{\nu,L}$ to the identity components are C_3-equivariant. Here, $\mathrm{Aut}(\mathrm{PGL}(m, \mathbf{C})) = \mathbf{Z}_2^\times$ (or $\mathbf{Z}_2^\times/\{\pm 1\}$ if $m = 2$) since $\mathrm{PGL}(m, \mathbf{C})$ has $\pi_*(N)$-determined automorphisms by induction hypothesis so C_3 must act trivially on the identity components for purely group theoretic reasons. The commutative triangle (4.14)

$$\begin{tikzcd}
 & \pi_0(H) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\pi_0(C_{\mathrm{PGL}(2m,\mathbf{C})}(H, \nu)) \arrow[rr, "\pi_0(f_{\nu,L})"'] & & \pi_0(C_{X'}(H, \nu'))
\end{tikzcd}$$

in which the slanted arrows, representing the canonical factorizations, are C_3-equivariant (even $\mathrm{GL}(H)$-equivariant) shows that $\pi_0(f_{\nu,L})$ is C_3-equivariant. □

We shall next compute the higher limits from 3.3.(2) and 4.6.(4) by means of 5.4 and the cochain complex 5.14 from [32]. As 5.4 is not valid for $\mathrm{PGL}(2, \mathbf{C})$ we first consider this case separately.

6.3. **Proposition.** *The 2-compact group* $\mathrm{PGL}(2, \mathbf{C})$ *is uniquely N-determined.*

Proof. The functor $C_{\mathrm{PGL}(2,\mathbf{C})}$ takes the Quillen category of $\mathrm{PGL}(2, \mathbf{C})$, consisting (5.7, 5.13, 6.1) of one toral line, L, and one non-toral plane, H,

(6.4)
$$\begin{tikzcd}
L \arrow[r] & H \arrow[loop right, "\mathrm{GL}(H)"]
\end{tikzcd}$$

to the diagram

(6.5)
$$\begin{tikzcd}
\mathrm{GL}(1, \mathbf{C})^2/\mathrm{GL}(1, \mathbf{C}) \rtimes C_2 & H \arrow[l] \arrow[loop right, "\mathrm{GL}(H)^{\mathrm{op}}"]
\end{tikzcd}$$

of uniquely N-determined 2-compact groups. The 2-compact toral group to the left is uniquely N-determined (4.2) because $H^1(C_2; \mathbf{Z}/2^\infty) = 0$ for the non-trivial action. The center functor takes this diagram back to the starting point (6.4) for which the higher limits vanish [29, 12.7.4]. $\mathrm{PGL}(2, \mathbf{C})$ is thus uniquely N-determined by 3.3 and 4.6. □

6.5. **Lemma.** *The low degree higher limits of the functors* $\pi_j(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})$, $j = 1, 2$, *are:*

(1) $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1, \mathbf{C})), \pi_1(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $i = 1, 2$,
(2) $\lim^i(\mathbf{A}(\mathrm{PGL}(n+1, \mathbf{C})), \pi_2(BZC_{\mathrm{PGL}(n+1,\mathbf{C})})) = 0$ *for* $i = 2, 3$,

for all $n \geq 1$.

Let $V = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3$ be a 3-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{F}_2[1]$ be the 21-dimensional $\mathbf{F}_2$-vector space on all length one flags $[P > L]$ and $\mathbf{F}_2[0]$ the 14-dimensional $\mathbf{F}_2$-vector space on all length zero flags, $[P]$ or $[L]$, of non-trivial and proper subspaces of V. The Steinberg module $\mathrm{St}(V)$ over $\mathbf{F}_2$ for V is the $2^3 = 8$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[1] \to \mathbf{F}_2[0]$ given by $d[P > L] = [P] + [L]$. Define $f_1 = \overline{f}_1 | \mathrm{St}(V)\colon \mathrm{St}(V) \to V$ as the restriction to $\mathrm{St}(V)$ of the linear map $\overline{f}_1\colon \mathbf{F}_2[1] \to V$ with values

$$\overline{f}_1[P > L] = \begin{cases} L & P \cap P^\perp = 0 \\ 0 & \text{otherwise} \end{cases}$$

on the basis vectors.

Let $E = \mathbf{F}_2 e_1 + \mathbf{F}_2 e_2 + \mathbf{F}_2 e_3 + \mathbf{F}_2 e_4$ be a 4-dimensional vector space over $\mathbf{F}_2$ with basis $\{e_1, e_2, e_3, e_4\}$ and (degenerate) symplectic inner product matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{F}_2[2]$ be the 315-dimensional $\mathbf{F}_2$-vector space on all length two flags $[V > P > L]$ and $\mathbf{F}_2[1]$ the also 315-dimensional $\mathbf{F}_2$-vector space on all length one flags, $[P > L]$ or $[V > L]$ or $[V > P]$, of non-trivial, proper subspaces of E. The Steinberg module $\mathrm{St}(E)$ over $\mathbf{F}_2$ for E is the $2^6 = 64$-dimensional kernel of the linear map $d\colon \mathbf{F}_2[2] \to \mathbf{F}_2[1]$ given by $d[V > P > L] = [P > L] + [V > L] + [V > P]$. Define $F_1 = \overline{F}_1 | \mathrm{St}(E)\colon \mathrm{St}(E) \to E$ as the restriction to $\mathrm{St}(E)$ of the linear map $\overline{F}_1\colon \mathbf{F}_2[2] \to E$ with values

$$\overline{F}_1[V > P > L] = \begin{cases} L & P \cap P^\perp = 0, V \cap V^\perp = \mathbf{F}_2 e_3 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

on the basis elements. Define $F_2 = \overline{F}_2 | \mathrm{St}(E)\colon \mathrm{St}(E) \to E$ similarly but replace the condition $V \cap V^\perp = \mathbf{F}_2 e_3$ by $V \cap V^\perp = \mathbf{F}_2 e_4$. The linear maps F_1 and F_2 are $\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}$-equivariant because this group preserves the symplectic inner product on E and preserves $V^\perp = \mathbf{F}_2 \langle e_3, e_4 \rangle$ pointwise.

6.7. **Lemma.** *Let f_1 and F_1, F_2 be the linear maps defined above.*

(1) *The vector f_1 is a basis vector for*

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(V), V) \cong \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(V), V/V^\perp) \cong \mathbf{F}_2$$

(2) *The set $\{F_1, F_2\}$ is a basis for*

$$\mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(E), E) \cong \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(E), E/E^\perp) \cong \mathbf{F}_2^2$$

The sum $F_1 + F_2$ is the linear map defined as in (6.6) but with condition $V \cap V^\perp = \mathbf{F}_2 e_3$ replaced by $V \cap V^\perp = \mathbf{F}_2(e_3 + e_4)$.

Proof. This can be directly verified by machine computation. □

6.8. **Proposition.** *The differentials in the cochain complex 5.14 are given as follows:*

(1) *Let H be the $(2,0)$ object and V a $(3,1)$ object of $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$. The V-component of the coboundary map*

$$\delta^1_V\colon \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \to \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(V), V)$$

is an isomorphism of 1-dimensional $\mathbf{F}_2$-vector spaces.

(2) *Let V be the $(4,2)$ object of $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$ corresponding (5.12, 5.7) to the two dimensional toral object $(1, i-1, m-i, 0)$ of $\mathbf{A}(\mathrm{PGL}(m, \mathbf{C}))$, $1 < i \leq m/2$, $m \geq 4$. Then $\delta^2_E(x_i) = (x_1 + x_i)F_1 + (x_1 + x_{i-1})F_2$ where*

$$\delta^2_E\colon \textstyle\prod_{1 \leq i \leq m/2} \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(V_i), V_i) \to \mathrm{Hom}_{\begin{pmatrix}\mathrm{Sp}(2) & 0 \\ * & 1\end{pmatrix}}(\mathrm{St}(E), E)$$

is the E-component of the coboundary map and $(x_i) \in \prod_{1 \leq i \leq m/2} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V)$.

Proof. 1. The non-zero vector in $\mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H)$ is the restriction to $\mathrm{St}(H) \subseteq \mathbf{F}_2^3$ of the linear map $\mathbf{F}_2[0] = \mathbf{F}_2^3 \to H$ that takes a basis vector $[L]$ in $\mathbf{F}_2^3$ to $L \in H$. In the composition

$$\mathrm{St}(V) \to \bigoplus_{V > P} \mathrm{St}(P) \to \bigoplus_{V > P} P \xrightarrow{+} V$$

the middle maps $\mathrm{St}(P) \to P$ equal the map just described if $P < V$ is non-toral, $P \cap P^\perp = 0$, and are trivial if $P < V$ is toral, $P \cap P^\perp = P$. This is precisely the map f_1.

2. For any non-toral three dimensional subspace V of E we have either

- $V \cap V^\perp = \mathbf{F}_2 e_3$, and then $V = V_i$, or,
- $V \cap V^\perp = \mathbf{F}_2 e_4$, and then $V = V_{i-1}$, or,
- $V \cap V^\perp = \mathbf{F}_2(e_3 + e_4)$, and then $V = V_1$,

and thus the composite linear map

$$\mathrm{St}(E) \to \bigoplus_{E>V} \mathrm{St}(V) \xrightarrow{\oplus x_i} \bigoplus V \xrightarrow{+} E$$

equals $x_iF_1 + x_{i-1}F_2 + x_1(F_1 + F_2) = (x_1 + x_i)F_1 + (x_{i-1} + x_1)F_2$. □

Proof of Lemma 6.5. Since we already know that these higher limits vanish when $n+1$ is odd (5.4) we can assume that $n + 1 = 2m$ is even.
1. In Oliver's cochain complex 5.14, the coboundary map δ^1 is injective and $\ker \delta^2$ is 1-dimensional by 6.8 when $m \geq 4$. See 6.3 for the case $m = 1$. For $m = 2$ and $m = 3$, the cochain complexes 5.14 reduce to

$$0 \to \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V/V^\perp) \to 0$$

$$0 \to \mathrm{Hom}_{\mathrm{Sp}(2)}(\mathrm{St}(H), H) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathrm{Sp}(3,1)}(\mathrm{St}(V), V/V^\perp) \to \mathrm{Hom}_{\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathrm{GL}(E^\perp) \end{pmatrix}}(\mathrm{St}(E), E/E^\perp) = 0$$

with two non-trivial groups, both 1-dimensional $\mathbf{F}_2$-vector spaces, and with just one differential δ^1 which is an isomorphism (6.8). Thus the higher limits vanish in these cases as well.
2. Oliver's cochain complex for computing these higher limits over $\mathbf{A}(\mathrm{PGL}(2m, \mathbf{C}))$ involve the $\mathbf{Z}_2$-modules

$$\mathrm{Hom}_{\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & \mathbf{A}(\mathrm{PGL}(m,\mathbf{C}))(E^\perp \end{pmatrix}}(\mathrm{St}(E), \pi_2(BZC_{\mathrm{PGL}(2m,\mathbf{C})}(E))), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

that are submodules of finite products of $\mathbf{Z}_2$-modules of the form

$$\mathrm{Hom}_{\begin{pmatrix} \mathrm{Sp}(2) & 0 \\ * & 1 \end{pmatrix}}(\mathrm{St}(E), \mathbf{Z}_2), \quad \dim_{\mathbf{F}_2} E = 3, 4,$$

where the action on $\mathbf{Z}_2$ is trivial. According to the computer program magma, these latter modules are trivial. □

Proof of Theorem 1.1. By induction over n using 3.3 and 4.6. The start of the induction is provided by 6.3. Use (2.7) to compute the automorphism group. □

Proof of Corollary 1.2. The connected 2-compact group $\mathrm{GL}(n, \mathbf{C})$ is uniquely N-determined because (3.2, 4.3) its adjoint form $\mathrm{PGL}(n, \mathbf{C})$ is (1.1). Since the maximal torus normalizer for $\mathrm{GL}(n, \mathbf{C})$ is a split extension, we get (2.7) that $\mathrm{Aut}(\mathrm{GL}(n, \mathbf{C}))$ is isomorphic to $Z(\Sigma_n)\backslash \mathrm{Aut}_{\mathbf{Z}_2\Sigma_n}(\mathbf{Z}_2^n)$. □

This finishes the discussion of the 2-compact groups in the A-family. The relevance of these are that they occur as centralizers of elementary abelian subgroups of many other 2-compact groups. Here is a result illustrating this.

6.9. **Theorem.** [34, 1.3] *The simple* 2*-compact group* G_2 *is uniquely* N*-determined and its automorphism group* $\mathrm{Aut}(\mathrm{G}_2)$ *equals* $\mathbf{Z}^\times\backslash\mathbf{Z}_2^\times \times C_2$.

Proof. The Quillen category $\mathbf{A}(\mathrm{G}_2)$ is equivalent to the category $\mathbf{A}(\mathrm{GL}(V), V)$ of all non-trivial subspaces of $V = \mathbf{F}_2^3$ [12, 6.1] [10, 1.6] [9, 5.3] and the value of centralizer functor BC_{G_2} on the three isomorphism classes of objects L, P, V is $\mathrm{SL}(4, \mathbf{R})$, $T \rtimes \mathbf{Z}/2$, V. The rank one centralizer, $\mathrm{SL}(4, \mathbf{R}) = \mathrm{SL}(2, \mathbf{C}) \circ \mathrm{SL}(2, \mathbf{C})$, is uniquely N-determined (6.3, 3.2, 3.4, 4.3, 4.4). Condition 4.6.(2) is satisfied because $H^1(W(X); \check{T}(X)) = 0$ for $X = \mathrm{G}_2$, $\mathrm{SL}(4, \mathbf{R})$ [13], 4.6.(1) and 4.6.(3) because the only rank two object in G_2 is toral and its centralizer is a 2-compact toral group. The functor $\pi_1(BZC_{\mathrm{G}_2})$ is the identity functor and $\pi_2(BZC_{\mathrm{G}_2})$ the zero functor so the obstruction groups vanish. Now 3.3 and 4.6 show that G_2 is uniquely N-determined. The short exact sequence (2.7) can be used to calculate the automorphism group. We have $\mathrm{Aut}(\mathrm{G}_2) = W(\mathrm{G}_2)\backslash N_{GL(2,Z_2)}(W(\mathrm{G}_2))$ as the extension class $e(\mathrm{G}_2) = 0$ [3]. Using the description of the root system from [2, VI.4.13] with short root $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and long root $\alpha_2 = 2\varepsilon - \varepsilon_2 - \varepsilon_3$ generating the integral lattice in $\mathbf{Z}_2^3$ one finds that

$$N_{GL(2,\mathbf{Z}_2)}(W(\mathrm{G}_2)) = \left\langle \mathbf{Z}_2^\times, A, W(\mathrm{G}_2) \right\rangle, \quad A = \sqrt{-3}\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

and therefore $\mathrm{Aut}(\mathrm{G}_2) = \mathbf{Z}_2^\times / \mathbf{Z}^\times \times C_2$ where the cyclic group of order two is generated by the exotic automorphism A interchanging the two roots. □

7. Miscellaneous

This section contains auxiliary results that are used at various places in the main argument of this paper.

7.1. **The 2-compact toral groups** $\mathrm{O}(2)$ **and** $\mathrm{Pin}(2)$**.** Let $\mathbf{H} = \{a + bj | a, b \in \mathbf{C}\}$, where $j^2 = -1$ and $ja = \overline{a}j$ for $a \in \mathbf{C}$, be the quaternion algebra. The normalizer of $\mathbf{C}^\times$ in $\mathbf{H}^\times$ is the Lie group $N_{\mathbf{H}^\times}(\mathbf{C}^\times) = \langle \mathbf{C}^\times, j \rangle$ generated by the multiplicative Lie group $\mathbf{C}^\times$ and j. The short exact sequence

$$1 \to \mathbf{C}^\times \to N_{\mathbf{H}^\times}(\mathbf{C}^\times) \to \langle j \rangle \,/\, \langle -1 \rangle \to 1$$

does not split for all elements of $j\mathbf{C}^\times$ have order 4. Its discrete approximation $\check{\mathrm{Pin}}(2) = \check{N}_{\mathbf{H}^\times}(\mathbf{C}^\times) = \langle \mathbf{Z}/2^\infty, j \rangle \subseteq \langle \mathbf{C}^\times, j \rangle \subseteq \mathbf{H}^\times$, the non-split extension

$$1 \to \mathbf{Z}/2^\infty \to \check{N}_{\mathbf{H}^\times}(\mathbf{C}^\times) \to \mathbf{Z}/2 \to 1$$

of $\mathbf{Z}/2$ by $\mathbf{Z}/2^\infty$, is the discrete approximation to 2-compact toral group $\mathrm{Pin}(2)$. The semi-direct product $\check{\mathrm{O}}(2) = \mathbf{Z}/2^\infty \rtimes \mathbf{Z}/2$ is the discrete approximation to the 2-compact toral group $\mathrm{O}(2)$ or to $\mathrm{GL}(2, \mathbf{R})$.

7.2. **Type** A_n**,** $n \geq 1$**.** (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group $\mathrm{PGL}(n+1, \mathbf{C}) = \mathrm{GL}(n+1, \mathbf{C})/\mathbf{C}^\times$ is the extended 2-discrete toral group

$$\check{N}(\mathrm{PGL}(n+1, \mathbf{C})) = \check{\mathrm{U}}(1)^{n+1}/\check{\mathrm{U}}(1) \rtimes \Sigma_{n+1} = \check{T} \rtimes \Sigma_{n+1}$$

where $\check{U}(1) = \mathbf{Z}/2^\infty$ is a discrete 2-torus of rank 1. In the coefficient sequence for $\check{U}(1) \to \check{U}(1)^{n+1} \to \check{T}$ we have $H^*(\Sigma_{n+1}; \check{U}(1)^{n+1}) \cong H^*(\Sigma_n; \check{U}(1))$ by Shapiro so that

$$H^i(W; \check{T}) \cong \ker\left(H^{i+1}(\Sigma_{n+1}; \mathbf{Z}/2^\infty) \xrightarrow{\mathrm{res}} H^{i+1}(\Sigma_n; \mathbf{Z}/2^\infty) \right)$$

is trivial for $n + 1 > 2(i + 1)$ by [30, 5.8, 6.7]. For small values of i we have

$$H^0(W; \check{T}) = \begin{cases} 0 & n \neq 1 \\ \mathbf{Z}/2 & n = 1 \end{cases} \quad \text{and} \quad H^1(W; \check{T}) = \begin{cases} 0 & n \neq 3 \\ \mathbf{Z}/2 & n = 3 \end{cases}$$

as can be seen by using that the Schur multiplier $H_2(\Sigma_n; \mathbf{Z})$ is of order 2 for $n \geq 4$ and trivial for $1 \leq n \leq 3$ [16, V.25.12]. Thus the center $ZN(\mathrm{PGL}(n+1, \mathbf{C}))$ of the maximal torus normalizer is trivial for $n > 1$ but cyclic of order 2 for $n = 1$. For $n = 3$, the crossed homomorphism $\Sigma_4 \to \check{\mathrm{U}}(1)^4/\check{\mathrm{U}}(1)$ whose values on the three generators $(12), (23), (34) \in \Sigma_4$ [16, I.19.7] are the columns of the matrix

$$\begin{pmatrix} -1 & +1 & +1 \\ -1 & -1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

is not principal.

7.3. **Type** B_n**,** $n \geq 2$**.** (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group $\mathrm{SL}(2n+1, \mathbf{R})$ is the extended 2-discrete torus

$$\check{N}(\mathrm{SL}(2n+1, \mathbf{R})) = \check{\mathrm{O}}(2) \wr \Sigma_n = (\mathbf{Z}/2^\infty \rtimes \mathbf{Z}/2) \wr \Sigma_n = (\mathbf{Z}/2^\infty)^n \rtimes (\mathbf{Z}/2 \wr \Sigma_n)$$

where $\mathbf{Z}/2$ acts on $\mathbf{Z}/2^\infty$ by sign. There is an isomorphism

$$H^1(\mathbf{Z}/2 \wr \Sigma_n; (\mathbf{Z}/2^\infty)^n) \cong \mathrm{Hom}(\Sigma_{n-1}, \mathbf{Z}/2) \oplus \mathrm{Hom}(\Sigma_n, \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

that to the pair $(v, \chi) \in \mathbf{Z}/2 \oplus \mathrm{Hom}(\Sigma_n, \mathbf{Z}/2)$ associates the derivation $D(v, \chi)$ given by

$$D(v, \chi)(\varepsilon_i, \sigma) = (v + \chi(\sigma), \ldots, v + \chi(\sigma), \chi(\sigma), v + \chi(\sigma), \ldots, v + \chi(\sigma))$$

where ε_i is the ith canonical basis vector for $(\mathbf{Z}/2)^n$ and $\chi(\sigma)$ is in the ith coordinate. To see this, use the exact sequence from the Lyndon–Hochschild–Serre spectral sequence

$$0 \to H^1(\Sigma_n; (\mathbf{Z}/2)^n) \to H^1(\mathbf{Z}/2 \wr \Sigma_n; (\mathbf{Z}/2^\infty)^n) \to H^1((\mathbf{Z}/2)^n; (\mathbf{Z}/2^\infty)^n)^{\Sigma_n}$$

where $H^1(\Sigma_n; (\mathbf{Z}/2)^n) \cong \mathbf{Z}/2$ $(n \geq 3)$ and also the third term is of order 2 as, in general,

$$H^*(G^n; M^n) = H^*(G^n; M)^n = H^*(G^{n-1}; H^*(G; M)) = \cdots = H^*(G; \dots; H^*(G; M) \cdots)^n$$

for a group G and a G-module M. This gives

$$H^0(W; \check{T}) = \mathbf{Z}/2, \qquad H^1(W; \check{T}) = \begin{cases} \mathbf{Z}/2 & n = 2 \\ (\mathbf{Z}/2)^2 & n \geq 3 \end{cases}$$

in this case. The computation of $H^0(W; \check{T})$ uses that the center of the maximal torus normalizer

$$Z(\check{N}(\mathrm{SL}(2n+1, \mathbf{R}))) = Z(\check{\mathrm{O}}(2) \wr \Sigma_n) = Z\check{\mathrm{O}}(2) = \mathbf{Z}/2$$

is cyclic of order two for all $n \geq 2$ (whereas $\check{Z}\mathrm{SL}(2n+1, \mathbf{R}) = 0$).

7.4. **The center of a semi-direct product.** Let $G \rtimes \Sigma$ be the semi-direct product for the action $\Sigma \to \mathrm{Aut}(G)$ of the group Σ on the group G. Let $G^\Sigma = \{g \in G | \Sigma g = g\}$ and $\Sigma_G = \{\sigma \in \Sigma | \sigma(g) = g \text{ for all } g \in G\}$.

7.5. **Lemma.** *The center* $Z(G \rtimes \Sigma) = G^\Sigma \times_{\mathrm{Aut}(G)} Z(\Sigma)$ *of* $G \rtimes \Sigma$ *is the pull-back*

$$\begin{tikzcd} Z(G \rtimes \Sigma) \arrow[r] \arrow[d] & Z(\Sigma) \arrow[d] \\ G^\Sigma \arrow[r] & \mathrm{Aut}(G) \end{tikzcd}$$

of the action map restricted to the center of Σ *along the map* $G^\Sigma \to \mathrm{Aut}(G)$ *given by inner automorphisms.*

Proof. Suppose that $(g, \sigma) \in G \times \Sigma$ is in the center of $G \rtimes \Sigma$. Since

$$(g, \sigma) \cdot (1, \tau) = (g, \sigma\tau) = (1, \tau) \cdot (g, \sigma) = (\tau(g), \tau\sigma)$$

for all $\tau \in \Sigma$, g is fixed by Σ and σ is central in Σ. Moreover, from

$$(g, \sigma) \cdot (h, 1) = (g \cdot \sigma(h), \sigma) = (h, 1) \cdot (g, \sigma) = (hg, \sigma)$$

we see that $\sigma(h) = h^g$ for all $h \in G$. □

7.6. **Corollary.** *If* G *is abelian,* $Z(G \rtimes \Sigma) = G^\Sigma \times Z(\Sigma)_G$ *is a direct product.*

Proof. The bottom horizontal homomorphism $G^\Sigma \to \mathrm{Aut}(G)$ is trivial. □

7.7. **Corollary.** *Let* G *be a group and* $Z \neq G$ *a central subgroup. Let the cyclic group* C_p *of prime order* p *act on* G^p/Z *by cyclic permutation. Then*

$$Z(G)/Z \times \{z \in Z | z^p = 1\} \cong Z(G^p/Z \rtimes C_p)$$

via the isomorphism that takes the element $z \in Z$ *of order* p *to* $(1, z, \dots, z^{p-1})Z \in G^p/Z$ *and is the diagonal on* $Z(G)/Z$.

Proof. Observe that

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\cong} \left(G^p/Z\right)^{C_p}$$

via the isomorphism that takes (gZ, z) to $g(1, z, \dots, z^{p-1})Z$. To see this, consider an element $(g_1, \dots, g_p)Z$ which is fixed by C_p. Then $(g_1, g_2, \dots, g_p)Z = (g_p, g_1, \dots, g_{p-1})Z$ so there exists an element $z \in Z$ so that $g_2 = g_1 z, g_3 = g_2 z = g_1 z^2, \dots, g_p = g_1 z^{p-1}, g_1 = g_1 z^p$. Therefore, $z^p = 1$ and $(g_1, g_2, \dots, g_p) = g_1(1, z, \dots, z^{p-1})$.

Thus $Z(G^p/Z \rtimes C_p)$ is the pull back of the group homomorphisms

$$G/Z \times \{z \in Z | z^p = 1\} \xrightarrow{\varphi} \mathrm{Aut}(G^p/Z) \leftarrow C_p$$

where $\varphi(gZ, z)((g_1, \dots, g_p)Z) = (g_1^g, \dots, g_p^g)Z$. Let $((gZ, z), \sigma)$ be an element of the pull back. Assume that σ is non-trivial. Since p is a prime number, σ has no fixed points. The equation

$$\forall g_1, \dots, g_p \in G \colon (g_1^g, \dots, g_p^g)Z = (g_{\sigma(1)}, \dots, g_{\sigma(p)})Z$$

shows that $g_1^g Z = g_{\sigma(1)}Z$. This is impossible unless σ is the identity since otherwise we can find a $g_1 \in Z$ and a $g_{\sigma(1)} \notin Z$. Thus the permutation σ must be the identity. The requirement for $((gZ,z),1)$ to be in the pull back is that

$$\forall(g_1,\dots,g_p)\in G^p \exists u \in Z\colon (g_1^g, g_2^g,\dots,g_p^g) = (g_1u, g_2u,\dots,g_pu)$$

which implies that $[g_1,g] = u = [g_2,g]$ for all $g_1,g_2 \in G$. If we take $g_1 = 1$ to be the identity, we see that g must be central. □

7.8. **Action in Lie case.** Let $\nu\colon V \to G$ be a monomorphism of a non-trivial elementary abelian p-group to a compact Lie group G. There is a canonical map $\mathrm{B}C_G(\nu(V)) \to \mathrm{map}(\mathrm{B}V,\mathrm{B}G)_{\mathrm{B}\nu}$ from the classifying space of the Lie theoretic centralizer of $\nu(V)$ to the mapping space component containing $\mathrm{B}\nu$. Write c_g for conjugation with $g \in G$.

7.9. **Lemma.** *Suppose that $\nu\alpha = c_g\nu$ for some element $g \in G$ and some automorphism $\alpha \in \mathrm{GL}(V)$. Then conjugation by g takes $C_G(\nu(V))$ to $C_G(c_g\nu(V)) = C_G(\nu\alpha(V)) = C_G(\nu(V))$ and the diagram*

$$\begin{tikzcd}
\mathrm{B}C_G(\nu(V)) \arrow[r] & \mathrm{map}(\mathrm{B}V,\mathrm{B}G)_{\mathrm{B}\nu} \arrow[d, "\cong"', "(\mathrm{B}\alpha)^*"] \\
\mathrm{B}C_G(\nu(V)) \arrow[r] \arrow[u, "\mathrm{B}c_g", "\cong"'] & \mathrm{map}(\mathrm{B}V,\mathrm{B}G)_{\mathrm{B}\nu}
\end{tikzcd}$$

is homotopy commutative.

Proof. The commutative diagram of Lie group morphisms

$$\begin{tikzcd}
V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"] \arrow[d, "\alpha\times c_g"'] & \nu(V)\times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"] & G \arrow[d, equal] \\
V \times C_G(\nu(V)) \arrow[r, "\nu\times 1"'] & \nu(V)\times C_G(\nu(V)) \arrow[r, "\mathrm{mult}"'] & G
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) \arrow[r, "\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))"] \arrow[d, "\mathrm{B}\alpha\times\mathrm{B}c_g"'] & \mathrm{B}G \arrow[d, equal] \\
\mathrm{B}V \times \mathrm{B}C_G(\nu(V)) \arrow[r, "\mathrm{B}(\mathrm{mult}\circ(\nu\times 1))"] & \mathrm{B}G
\end{tikzcd}$$

of classifying spaces. Taking adjoints, we obtain the homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{B}C_G(\nu(V)) \arrow[r] & \mathrm{map}(\mathrm{B}V,\mathrm{B}G)_{\mathrm{B}\nu} \arrow[d, "(\mathrm{B}\alpha)^*"] \\
\mathrm{B}C_G(\nu(V)) \arrow[r] \arrow[u, "\mathrm{B}c_g"] & \mathrm{map}(\mathrm{B}V,\mathrm{B}G)_{\mathrm{B}\nu}
\end{tikzcd}$$

as claimed. □

7.10. **Corollary.** *Suppose that $\mu\colon V \to N(G)$ is a monomorphism and that $\mu\alpha = c_n\mu$ for some $\alpha \in \mathrm{GL}(V)$ and $n \in N(G)$. Then*

$$w^{-1} = \pi_2((\mathrm{B}\alpha)^*)\colon \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)} \to \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$$

where $w \in W(G)$ is the image of $n \in N(G)$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{B}T) \arrow[r, equal] & \pi_2(\mathrm{B}N(G)) & \pi_2(\mathrm{B}C_{N(G)}(V,\mu)) \arrow[l, hook'] \arrow[r, "\cong"] & \pi_2(\mathrm{map}(\mathrm{B}V,\mathrm{B}N),\mathrm{B}\mu) \arrow[d, "\pi_2((\mathrm{B}\alpha)^*)"] \\
\pi_2(\mathrm{B}T) \arrow[r, equal] \arrow[u, "w"] & \pi_2(\mathrm{B}N(G)) \arrow[u, "\pi_2(\mathrm{B}c_n)"'] & \pi_2(\mathrm{B}C_{N(G)}(V,\mu)) \arrow[l, hook'] \arrow[r, "\cong"] \arrow[u, "\pi_2(\mathrm{B}c_n)"'] & \pi_2(\mathrm{map}(\mathrm{B}V,\mathrm{B}N),\mathrm{B}\mu)
\end{tikzcd}$$

where $\pi_2(\mathrm{B}C_{N(G)}(V,\mu)) = \pi_2(\mathrm{B}T(G))^{\pi_0(\mu)(V)}$ denotes the fixed point group for the group action $\pi_0(\mu)\colon V \to W(G) \subseteq \mathrm{Aut}(\pi_2(\mathrm{B}T(G)))$. Since $\mathrm{B}c_n\colon \mathrm{B}N \to \mathrm{B}N$ is freely homotopic to the identity along the loop $w \in \pi_1(\mathrm{B}N)$ its effect on the $\mathbf{Z}_p[\pi_1(\mathrm{B}N)]$-module $\pi_2(\mathrm{B}N)$ is multiplication by w. □

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