THE 2-COMPACT GROUPS IN THE A-FAMILY ARE N-DETERMINED

JESPER M. MOLLE

Abstract. The 2-compact groups associated to central quotients of $SU(n+1)$, $n \geq 1$, are shown to be determined up to isomorphism by their maximal torus normalizers.

1. Introduction

A 2-compact group is a 2-complete connected based space $BX$ such that $H^*(\Omega BX; \mathbb{F}_2)$ is finite where $\Omega BX$ is the loop space [6]. It is customary, though sometimes confusing, to refer to $BX$ by the symbol $X$.

Any 2-compact group $BX$ comes equipped with a maximal torus normalizer $BN(X) \to BX$ where $BN(X)$ is the Borel construction $BT(X) \to BN(X) \to BW(X)$ for the action of the Weyl group $W(X)$ on the maximal torus $T(X)$ [6, 9.8]. Does $BN(X)$ determine $BX$?

The answer to this question is "no" for the following reason. Let $G$ be a Lie group and $N(G) \to G$ its Lie group maximal torus normalizer. Assuming that the component group $\pi_0(G)$ is a finite 2-group, $\hat{BG}$ is a 2-compact group and $B\hat{N}(G) \to \hat{BG}$ its 2-compact group maximal torus normalizer. (For any Lie group $H$, $B\hat{H}$ stands for the partial 2-completion of the classifying space $BH$ for $H$.) Since there are distinct Lie groups, such as $O(2n)$ and $SO(2n+1)$, with isomorphic maximal torus normalizers, there are also distinct 2-compact groups, such as $\hat{O}(2n)$ and $\hat{SO}(2n+1)$, with isomorphic maximal torus normalizers. Thus we need to replace the maximal torus normalizer by a more delicate invariant which retains information about component groups. The maximal torus normalizer pair is a candidate for such a more delicate invariant.

For a 2-compact group $BX$ let $BX_0$, the identity component of $X$, denote the universal covering space of $BX$. Since $BX_0$ is again a 2-compact group, it has a maximal torus normalizer $BN(X_0) \to BX_0$. The maximal torus normalizers of $X$ and $X_0$ are related by a commutative diagram

$$
\begin{array}{ccc}
BN(X_0) & \longrightarrow & BX_0 \\
\downarrow & & \downarrow \\
BN(X) & \longrightarrow & BX \\
\downarrow & & \downarrow \\
B\pi_0(X) & \longrightarrow & B\pi_0(X)
\end{array}
$$

where the columns are fibration sequences. The fibration $BN(X_0) \to BN(X) \to B\pi_0(X)$, called the maximal torus normalizer pair associated to $BX$, has the built-in property that it fully informs about the component group of $X$. Does the maximal torus normalizer pair determine the 2-compact group up to isomorphism?

Focusing on the following properties for a 2-compact group $X$,

1. $X$ is determined by $(N(X), N(X_0))$
2. Automorphisms of $X$ are determined by their restrictions to $N(X)$
3. Automorphisms of $X$ are determined by their restrictions to $T(X)$

we shall say that

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\[ \text{X is totally N-determined if it satisfies (1) and (2)} \]
\[ \text{X is uniquely N-determined if it satisfies (1) and (3)} \]

In this terminology, one might formulate the conjecture that all 2-compact groups are totally N-determined, all connected 2-compact groups even uniquely N-determined. Here is an infinite family of simple 2-compact groups corroborating the conjecture.

1.1. \textbf{Theorem.} \textit{The simple 2-compact group PGL}(n + 1, C), \( n \geq 1 \), is uniquely N-determined and its automorphism group \( \text{Aut}(\text{PGL}(n + 1, C)) \) equals \( \mathbb{Z}_2^n \) for \( n > 1 \) and \( \mathbb{Z}^n \mathbb{Z}_2^n \) for \( n = 1 \).

It immediately follows (3.2, 4.3) that the Lie group PGL\((n + 1, C) = \text{PSL}(n + 1, C)\) occuring in Theorem 1.1 can be replaced by any central quotient of \( \text{SL}(n + 1, C) \). Indeed, the methods used here are not confined to simple, or semi-simple, 2-compact groups.

1.2. \textbf{Corollary.} [22, 1.9] \textit{The 2-compact group GL}(n, C) is uniquely N-determined and its automorphism group \( \text{Aut}(\text{GL}(n, C)) \) equals \( \text{Aut}_{\mathbb{Z}_2^\Sigma_n}(\mathbb{Z}_2^n) \) for \( n > 2 \) and \( \mathbb{Z}^n \mathbb{Z}_2^n \text{Aut}_{\mathbb{Z}_2^\Sigma}(\mathbb{Z}_2^n) \) for \( n = 2 \).

The methods are not even confined to the connected cases. For instance, it follows from Lemma 4.1 that the 2-compact group \( \text{GL}(n, C) \times C_2 \), where \( C_2 \) acts on \( \text{GL}(n, C) \) by complex conjugation, is totally N-determined.

See [31, 33, 34] for classification results for other 2-compact groups (with polynomial \( F_2 \)-cohomology). The results for the automorphism groups are not new [18] but reproved here.

2. \textbf{Generalities}

This sections contains the fundamental definitions and the first general results. Whereas \( p \)-compact groups are determined by their maximal torus normalizers [29, 1] when \( p > 2 \), a finer invariant is needed for 2-compact groups as there are examples (2.2) of distinct 2-compact groups with identical maximal torus normalizers.

2.1. \textbf{Maximal torus normalizer pairs.} Let \( N_0 \to N \) be a maximal rank normal monomorphism between two extended 2-compact tori, meaning simply that there exists a short exact sequence of loop spaces \( N_0 \to N \to \pi \) for some finite group \( \pi \). We say that \((N, N_0)\) is a maximal torus normalizer pair for the 2-compact group \( X \), and we write \( N(X, X_0) = (N, N_0) \), if there exists a morphism of loop space short exact sequences

\[ \begin{array}{c}
N_0 \longrightarrow N \longrightarrow \pi \\
\downarrow j_0 \quad \quad \quad \downarrow j \quad \quad \quad \quad \quad \quad \downarrow \cong \\
X_0 \longrightarrow X \longrightarrow \pi_0(X)
\end{array} \]

where \( j \) and \( j_0 \) are maximal torus normalizers for \( X \) and its identity component \( X_0 \). A maximal torus normalizer pair for \( X \) determines the maximal torus \( T(X) \), the Weyl groups, \( W(X) \) and \( W(X_0) \), of \( X \) and \( X_0 \), the component group \( \pi_0(X) = N(X)/N(X_0) = W(X)/W(X_0) \), and [7, 7.5] the center \( Z(X_0) \to X_0 \) of \( X_0 \).

2.2. \textbf{Example.} 1. Since \( N(\text{SO}(2n + 1)) \subseteq \text{O}(2n) \subseteq \text{SO}(2n + 1) \), \( \text{O}(2n) \) and \( \text{SO}(2n + 1) \) have the same maximal torus normalizer. Their maximal torus normalizer pairs are distinct, however, for \( \text{SO}(2n + 1) \) is connected and \( \text{O}(2n) \) disconnected.

2. More generally [14], let \( G \) be any compact connected Lie group and \( N(G) \) its maximal torus normalizer. If \( N(G) \) is not maximal, there exists a compact Lie group \( H \) such that \( N(G) \subseteq H \subseteq G \). The two compact Lie groups, \( G \) and \( H \), have isomorphic maximal torus normalizers but distinct maximal torus normalizer pairs as \( H \) is non-connected.

3. The Weyl groups for \( \text{SO}(2n + 1) \) and \( \text{Sp}(n) \), \( n \geq 3 \), are isomorphic as reflection groups but \( N(\text{SO}(2n + 1)) \) is a split and \( N(\text{Sp}(n)) \) a non-split extension [3]. Thus connected 2-compact groups can not be classified by their Weyl group alone.

2.3. \textbf{The Adams–Mahmud homomorphism.} For a 2-compact group (or extended 2-compact torus) \( X \), we let \( \text{Hom}(X) = [BX, *; BX] \) denote the monoid of homotopy classes of endomorphisms of \( X \). The \textit{automorphism group} \( \text{Aut}(X) \subseteq [BX, *; BX] \) of \( X \) is the group of invertible elements in \( \text{Hom}(X) \) and the \textit{outer automorphism group} \( \text{Out}(X) = \text{Aut}(X)/\pi_0(X) \subseteq [BX; BX] \) is the group of conjugacy classes of automorphisms of \( X \).
Let $X$ be a 2-compact group with maximal torus normalizer pair $(N, N_0)$. Turn the maximal torus normalizer $Bj : BN \to BX$ into a fibration. Any automorphism $f : X \to X$ of the 2-compact group $X$ restricts to an automorphism $AM(f) : N \to N$ of the maximal torus normalizer, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N)$ of the identity component $X_0$ of $X$, such that the diagram

\[
\begin{array}{ccc}
BN & \xrightarrow{B(AM(f))} & BN \\
Bj \downarrow & & \downarrow Bj \\
BX & \xrightarrow{Bf} & BX
\end{array}
\]

commutes up to based homotopy \cite[§3]{26}. The Adams–Mahmud homomorphism is the resulting homomorphism

\[
(2.4) \quad AM : \text{Aut}(X) \to W(X_0) \backslash \text{Aut}(N)
\]

of automorphism groups.

The automorphism group of $N$ sits \cite[5.2]{24} in a short exact sequence

\[
(2.5) \quad 0 \to H^1(W(X); \hat{T}(X)) \to \text{Aut}(N) \xrightarrow{\text{res}} \text{Aut}(W(X), \hat{T}(X), e(X)) \to 1
\]

where the normal subgroup to the left consists of all automorphisms of $N$ that induce the identity on homotopy groups and the group to the right consists of all pairs $(\alpha, \theta) \in \text{Aut}(W(X)) \times \text{Aut}(\hat{T}(X))$ such that $\theta$ is $\alpha$-linear and the induced automorphism $H^2(\alpha^{-1}, \theta)$\cite[6.7.6]{35} preserves the extension class $e(X) \in H^2(W(X); \hat{T}(X))$. The image of $W(X_0)$ in $\text{Aut}(N)$ does not intersect the subgroup $H^1(W(X); \hat{T}(X))$ (as $W(X_0)$ is represented faithfully in $\text{Aut}(\hat{T}(X))$\cite[6, 9.7]{6}) so there is an induced short exact sequence

\[
(2.6) \quad 0 \to H^1(W(X); \hat{T}(X)) \to W(X_0) \backslash \text{Aut}(N) \xrightarrow{\text{res}} W(X_0) \backslash \text{Aut}(W(X), \hat{T}(X), e(X)) \to 1
\]

whose middle term is the target of the Adams–Mahmud homomorphism. In particular, if $X$ is connected, this short exact sequence

\[
(2.7) \quad 0 \to H^1(W(X); \hat{T}(X)) \to \text{Out}(N) \to W(X) \backslash \text{Aut}(W(X), \hat{T}(X), e(X)) \to 1
\]

has the group $\text{Out}(N) = W(X) \backslash \text{Aut}(N)$ of outer automorphisms of $N$ as its middle term. The group $\text{Aut}(W(X), \hat{T}(X), 0)$, which is the normalizer $N_{GL(L(X))}(W(X))$ of $W(X)$ in $GL(L(X))$, $L(X) = \pi_2(BT(X))$, fits into an exact sequence

\[
Z(W(X)) \backslash \text{Aut}_{Z_2W(X)}(L(X)) \to W(X) \backslash N_{GL(L(X))}(W(X)) \to \text{Out}(W(X))
\]

where, by Schur’s lemma, $\text{Aut}_{Z_2W(X)}(L(X)) = Z_2^2$ if $X$ is simple.

2.8. **Totally $N$-determined 2-compact groups.** We are now ready to formulate the concept of $N$-determinism that will be used in this paper.

2.9. **Definition.** Let $X$ be a 2-compact group with maximal torus normalizer pair $(N, N_0)$.

1. $X$ has $N$-determined automorphisms if the Adams–Mahmud homomorphism \cite[(2.4)]{26} for $X$ is injective and $\pi_\ast(N)$-determined automorphisms if $AM^{-1}(H^1(W(X); \hat{T}(X)))$ is trivial.
2. $X$ is $N$-determined if for any other 2-compact group $X'$ with maximal torus normalizer pair $(N, N_0)$ there exist an isomorphism $f : X \to X'$ and an automorphism $\alpha \in \pi_0(N) \backslash \text{Aut}(N)$ with $\pi_\ast(B\alpha) = 1$ such that the diagram

\[
(2.10) \quad \begin{array}{ccc}
BN & \xrightarrow{B\alpha} & BN \\
Bj \downarrow & & \downarrow Bj' \\
BX & \xrightarrow{Bf} & BX'
\end{array}
\]

commutes up to based homotopy.

3. $X$ is totally $N$-determined if it has $N$-determined automorphisms and is $N$-determined.

A totally $N$-determined 2-compact group is

- uniquely $N$-determined if it has $\pi_\ast(N)$-determined automorphisms (i.e. $H^1(W(X); \hat{T}(X)) \cap AM(\text{Aut}(X)) = \{1\}$)
• strongly $N$-determined if $H^1(W(X); \hat{T}(X)) \subset \text{AM}(\text{Aut}(X))$

Thus a totally $N$-determined $p$-compact group is both uniquely and strongly $N$-determined if $H^1(W(X); \hat{T}(X)) = 0$.

For a compact connected Lie group $G$, the cohomology group $H^1(W(G); \hat{T}(G))$ is always an elementary abelian $2$-group [20, 1.1]. For instance, this first cohomology group has order $2$ for $G = \text{PSU}(4)$ [19, Appendix B] (7.2), generated by an involution $\alpha$, say, of $N(\text{PSU}(4))$. The unique solution to diagram (2.10) is

$$\begin{align*}
N(\text{PSU}(4)) & \xrightarrow{\alpha} N(\text{PSU}(4)) \\
\text{PSU}(4) & \xrightarrow{j} \text{PSU}(4)
\end{align*}$$

when we use the morphisms $j$, induced by an inclusion of Lie groups, and $j' = j\alpha$ for maximal torus normalizers. $\text{PSU}(4)$ is a uniquely but not strongly $N$-determined $2$-compact group.

2.11. Proposition. Suppose that the $2$-compact group $X$ is totally $N$-determined.

1. For fixed $\alpha \in \text{Aut}(N)$ with $\pi_*(B\alpha) = 1$ there is at most one isomorphism $f : X \to X'$ such that diagram in 2.9.(2) based homotopy commutes.

2. The pair $(f, \alpha)$ in 2.9.(2) is unique $\iff X$ is uniquely $N$-determined.

3. It is always possible to use $\alpha = 1$ in 2.9.(2) $\iff H^1(W(X); \hat{T}(X)) \subset \text{AM}(\text{Aut}(X))$.

4. $W(X_0) \setminus \text{Aut}(N) = H^1(W(X); \hat{T}(X)) \cdot \text{AM}(\text{Aut}(X))$

Proof. 1. If $(f_1, \alpha)$ and $(f_2, \alpha)$ are two solutions to (2.10), then $\text{AM}(f_2^{-1}f_1)$ is the identity and $f_1 = f_2$ as $\text{AM}$ is assumed injective.

2. Suppose that the condition is satisfied and let $(f_1, \alpha_1)$ and $(f_2, \alpha_2)$ be two solutions to 2.9.(2). Then $\text{AM}(f_2^{-1}f_1) = \alpha_2^{-1}\alpha_1 \in W(X_0) \setminus \text{Aut}(N)$ belongs to both $\text{AM}(\text{Aut}(X))$ and $H^1(W(X); \hat{T}(X))$ and is therefore trivial. Thus $\text{AM}(f_2^{-1}f_1) = 1$ and $f_2 = f_1$ as $\text{AM}$ is injective. Conversely, if $\text{AM}(f) \neq 0$ lies in $H^1(W(X); \hat{T}(X))$ for some $f \in \text{Aut}(X)$ then $(f, \text{AM}(f))$ and $(1, 0)$ are two solutions to 2.9.(2) with $X' = X$ and $j' = j$.

3. Let $\alpha \in H^1(W(X); \hat{T}(X))$. If we can always find an isomorphism under $N$, then there exists an isomorphism $f \in \text{Aut}(X)$ such that $fj = j\alpha$. This means that $\text{AM}(f) = \alpha$. Conversely, let $(f, \alpha)$ be a solution to 2.9.(2). If $H^1(W(X); \hat{T}(X)) \subset \text{AM}(\text{Aut}(X))$ then $\text{AM}(g) = \alpha$ for an automorphism $g \in \text{Aut}(X)$. According to the commutative diagram

$$\begin{align*}
\text{BN} & \xrightarrow{B\alpha} \text{BN} \\
\text{BX} & \xrightarrow{B\alpha} \text{BX}
\end{align*}$$

$f^{-1}g : X \to X'$ is an isomorphism under $N$.

4. For any automorphism $g$ of $N$ it is possible to find an automorphism $f$ of $X$ and an automorphism $\alpha$ of $N$ with $\pi_*(B\alpha) = 1$ such that the diagram

$$\begin{align*}
\text{BN} & \xrightarrow{B\alpha} \text{BN} \\
\text{BX} & \xrightarrow{B\alpha} \text{BX}
\end{align*}$$

commutes up to based homotopy. Thus $g = \text{AM}(f)\alpha$. \hfill \square

The subgroup $H^1(W(X); \hat{T}(X))$ is clearly normal so that

$W(X_0) \setminus \text{Aut}(N) \cong H^1(W(X); \hat{T}(X)) \rtimes \text{Aut}(X)$, \quad $\text{Aut}(X) \cong W(X_0) \setminus \text{Aut}(W(X), \hat{T}(X), c(X))$

for a uniquely $N$-determined $2$-compact group $X$. (The corresponding statement for compact connected Lie groups is true [14, 3.10]. It is already known that compact connected Lie groups perceived as $2$-compact groups have $\pi_*(N)$-determined automorphisms [18, 2.5].)
2.12. **Lemma.** Let $X$ be a $2$-compact group. Assume that the identity component $X_0$ is completely reducible [23, 3.4, 3.10] and that $\tilde{Z}(X_0) = T(X_0)^{W(X_0)}$.

1. $H^1(W(X); T(X)) \cap \text{AM}(\text{Aut}(X)) = H^1(W/W_0; \tilde{T}^{W_0})$.
2. If $H^1(W/W_0; \tilde{T}^{W_0}) \neq 0$ then $X$ does not have $\pi_*(N)$-determined automorphisms.
3. If $H^1(W/W_0; \tilde{T}^{W_0}) = 0$ and $X_0$ has $\pi_*(N)$-determined automorphisms, so does $X$.
4. If the monomorphism $\inf: H^1(W/W_0; \tilde{T}^{W_0}) \to H^1(W; \tilde{T})$ is an isomorphism, $H^1(W; \tilde{T}) \subseteq \text{AM}(\text{Aut}(X))$.

**Proof.** (1) and (2). This follows from the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1(W/W_0; \tilde{T}^{W_0}) & \longrightarrow & \text{Aut}(X) & \longrightarrow & \text{Aut}(\pi_0, X_0)_X & \longrightarrow & 1 \\
& & \downarrow & & \downarrow \text{AM} & & \downarrow & & \\
0 & \longrightarrow & H^1(W; \tilde{T}) & \longrightarrow & W_0 \setminus \text{Aut}(N) & \longrightarrow & W_0 \setminus \text{Aut}(W, \tilde{T}; e) & \longrightarrow & 1
\end{array}
$$

with exact rows. The upper row is [24, 5.2].

(3). We must show that $\text{Aut}(X) \xrightarrow{\text{AM}} W_0 \setminus \text{Aut}(W, \tilde{T}; e)$ is injective. The image of this homomorphism is contained in the subgroup $W_0 \setminus \text{Aut}(W, W_0, \tilde{T}; e)$ where $\text{Aut}(W, W_0, \tilde{T}; e)$ consists of those pairs $(\alpha, \theta) \in \text{Aut}(W; \tilde{T}; e)$ for which $\alpha(W_0) = W_0$. In the commutative diagram

$$
\begin{array}{cccccc}
\text{Aut}(X) & \longrightarrow & \text{Aut}(\pi_0, X_0)_X & \longrightarrow & \text{Aut}(\pi_0) \times \text{Aut}(X_0) & \\
& & \searrow & & \downarrow 1 \times \text{AM} & \\
W_0 \setminus \text{Aut}(W, W_0, \tilde{T}; e) & \longrightarrow & \text{Aut}(\pi_0) \times W_0 \setminus \text{Aut}(W_0, \tilde{T})
\end{array}
$$

the slanted arrow must be injective. We know from [24, 5.2] that $\text{Aut}(X) \cong \text{Aut}(\pi_0, X_0)_X$.

(4). This is clear from (2) and (3). \qed

2.13. **Example.** The $2$-compact group $\text{GL}(2n, \mathbf{R}) = \text{SL}(2n, \mathbf{R}) \ltimes \mathbf{Z}/2$, $n > 1$, does not have $\pi_*(N)$-determined automorphisms for $H^1(W/W_0; \tilde{T}^{W_0}) = H^1(\mathbf{Z}/2; \mathbf{Z}/2) = \mathbf{Z}/2$ is non-trivial. The maximal torus normalizer for $\text{GL}(2n, \mathbf{R})$ is the same as the one for $\text{SL}(2n + 1, \mathbf{R})$ so $H^1(W; \tilde{T})$ equals $\mathbf{Z}/2$ for $n = 2$ and $(\mathbf{Z}/2)^2$ for $n \geq 3$ (2.2, 7.3). The $2$-compact group $\text{GL}(2n + 1, \mathbf{R}) = \text{SL}(2n + 1, \mathbf{R}) \ltimes \mathbf{Z}/2$, $n > 0$, has $\pi_*(N)$-determined automorphisms for $\text{Aut}(\text{GL}(2n + 1, \mathbf{R})) = \text{Aut}(\text{SL}(2n + 1, \mathbf{R}))$ and $\text{SL}(2n + 1, \mathbf{R})$ has $\pi_*(N)$-determined automorphisms (as does any compact connected Lie group [18, 2.5]). The $2$-compact group $\text{PGL}(2n, \mathbf{R}) = \text{PSL}(2n, \mathbf{R}) \ltimes \mathbf{Z}/2$ has $\pi_*(N)$-determined automorphisms since the identity component has trivial center. In fact, $H^1(W(\text{PSL}(2n, \mathbf{R})) \rtimes \mathbf{Z}/2; \tilde{T}) \subseteq H^1(\mathbf{Z}/2; H^0(W; \tilde{T}))) = 0$ for $n \geq 5$ since $H^0(W; \tilde{T}) = 0 = H^1(W; \tilde{T})$ for $\text{PSL}(2n, \mathbf{R})$ by [13].

2.14. **Lemma.** Let $X$ be a connected $2$-compact group with maximal torus normalizer $j: N \to X$. Then $X$ is (uniquely) $N$-determined if and only if for any other connected $2$-compact group $X'$ with maximal torus normalizer $j': N \to X'$ there exists a (unique) morphism $f: X \to X'$ such that $\xymatrix{ j|T \
j'|T \ar@{=>}[u] \\
X \ar[r]^f & X'}$ commutes up to conjugacy.

**Proof.** The morphism $f: X \to X'$ in the above commutative diagram is in fact an isomorphism [8, 5.6] [27, 3.11]. The assumption of the lemma that $f$ be a morphism under $T$ means (use
$W \setminus [BT, BX] = [BT, BX] \ [25, 3.4] \ [8, 3.4]$) that $f$ admits a restriction $N(f)$ to $N$ which is the identity on $T$, i.e. such that

$$
\begin{array}{cccc}
BT & \rightarrow & BN & \rightarrow \ Bf \\
\downarrow & & \downarrow & \\
BN(f) & \rightarrow & Bf \\
BT & \rightarrow & BN & \rightarrow \ Bf'
\end{array}
$$

is homotopy commutative. But then also $\pi_0 N(f) : W \rightarrow W$ is the identity map for $W$ is faithfully represented as a group of operators on $T$ $[6, 9.7]$. Thus $\pi_*(BN(f))$ is the identity automorphism of $\pi_*(BN)$.

Assume that the isomorphism $f$ exists and is uniquely determined. In particular, the identity of $X$ is the only automorphism under $T$. That $f \in \text{Aut}(X)$ is a map under $T$ means precisely that $\text{AM}(f) \in H^1(W; \hat{T})$. Thus $X$ is uniquely $N$-determined by (2.11.2). Suppose, conversely, that $X$ has this property and let $f_0, f_1 : X \rightarrow X'$ be two isomorphisms under $T$. Then $f_1^{-1} f_0 \in \text{Aut}(X)$ is an isomorphism under $T$ so equals the identity. \hfill \Box

2.15. \textbf{Remark.} When the 2-compact group $X$ has $N$-determined automorphisms, also the un-based Adams–Mahmud homomorphism $\text{Out}(X) = W(X) \setminus \text{Aut}(X) \rightarrow \text{Out}(N) = \pi_0(N) \setminus \text{Aut}(N)$ is injective $[26, 3.7–3.9]$. 

2.16. \textbf{LHS 2-compact groups.} Let $N_0 \rightarrow N$ be maximal rank normal monomorphism between two extended 2-compact tori, i.e. a commutative diagram with rows and columns that are short exact sequences of loop spaces

$$
\begin{array}{ccc}
T & \rightarrow & T & \rightarrow \ \{1\} \\
\downarrow & & \downarrow & \\
N_0 & \rightarrow & N & \rightarrow \ W/W_0 \\
\downarrow & & \downarrow & \\
W_0 & \rightarrow & W & \rightarrow \ W/W_0
\end{array}
$$

where $T$ is a 2-compact torus and $W_0$ a normal subgroup of the finite group $W$. The 5-term exact sequence

$$0 \rightarrow H^1(W/W_0; \hat{T}W_0) \xrightarrow{\text{inf}} H^1(W; \hat{T}) \xrightarrow{\text{res}} H^1(W_0; \hat{T})^{W/W_0} \xrightarrow{d_2} H^2(W/W_0; \hat{T}W_0) \xrightarrow{\text{inf}} H^2(W; \hat{T})$$

is part of the Lyndon–Hochschild–Serre spectral sequence $[15]$ converging to $H^*(W; \hat{T})$.

2.17. \textbf{Definition.} The pair $(N, N_0)$ of extended 2-compact tori is LHS if the initial segment

$$0 \rightarrow H^1(W/W_0; \hat{T}W_0) \xrightarrow{\text{inf}} H^1(W; \hat{T}) \xrightarrow{\text{res}} H^1(W_0; \hat{T})^{W/W_0} \rightarrow 0$$

is a short exact sequence. A 2-compact group is LHS if its maximal torus normalizer pair is LHS.

Here are two ways to check if a given $p$-compact group $X$ is LHS (besides the evident situations where $\hat{T}W_0 = 0$ or $W = W_0 \times W/W_0$ is a direct product).

The inflation homomorphism is the composition

$$H^2(W/W_0; \hat{T}W_0) \rightarrow H^2(W/W_0; \hat{T}) \xrightarrow{H^2(W \rightarrow W/W_0)} H^2(W; \hat{T})$$

of a coefficient group homomorphism followed by the restriction homomorphism induced by the projection of $W$ onto the group of components $W/W_0$. If the Weyl group $W = W_0 \times W/W_0$ is a semi-direct product, $H^2(W \rightarrow W/W_0)$ is injective and therefore

(2.18) $H^1(W; \hat{T}) \rightarrow H^1(W_0; \hat{T})^{W/W_0}$ is surjective $\iff$

$$H^2(W/W_0; \hat{T}W_0) \rightarrow H^2(W/W_0; \hat{T})$$

is injective by exactness of the Lyndon–Hochschild–Serre spectral sequence.
Another possibility is to use the description of $H^1(W_0; \hat{T})$ from [13]. The short exact sequence

$$1 \rightarrow W_0 \rightarrow W \rightarrow W/W_0 \rightarrow 1$$

of abelian groups (where $H_2(W/W_0) = 0$ if $W/W_0$ has order two). The middle arrow in this exact sequence can be used to define a homomorphism

$$\text{Hom}(W, \hat{T}^W) = \text{Hom}(W_{ab}, (\hat{T}^{W_0})^{W/W_0}) \rightarrow \text{Hom}(((W_0)_{ab})_{W/W_0}, (\hat{T}^{W_0})^{W/W_0})$$

which fits into the commutative diagram

$$(2.19) \quad \begin{array}{ccc} H^1(W; \hat{T}) & \rightarrow & H^1(W_0; \hat{T})^{W/W_0} \\
\text{Hom}(W, \hat{T}^W) & \downarrow & \text{Hom}(W_0, \hat{T}^{W_0})^{W/W_0}
\end{array}$$

Here, the left vertical arrow, say, takes a homomorphism $W \rightarrow \hat{T}^W$ to the cohomology class represented by the crossed homomorphism $W \rightarrow \hat{T}^W \rightarrow \hat{T}$. Since the right vertical arrow is an epimorphism in many cases [13, 1.2, 1.3], this can sometimes be used to show that $H^1(W; \hat{T}) \rightarrow H^1(W_0; \hat{T})^{W/W_0}$ is surjective.

2.20. Example. 1. The 2-compact group $\frac{\text{GL}(m, C)}{\text{GL}(1, C)} \rtimes C_2$, $m \geq 1$, where the $C_2$-action switches the two $\text{GL}(m, C)$-factors, is LHS because (2.18) the map

$$H^2\left(C_2; \frac{\hat{S} \times \hat{S}}{S}\right) \rightarrow H^2\left(C_2; \frac{\hat{S}^m \times \hat{S}^m}{S}\right), \quad \hat{S} = Z/2^\infty,$$

can be identified to the identity on $H^3(C_2; \hat{S}) = Z/2$ since $H^0(C_2; \hat{S} \times \hat{S}) = 0 = H^0(C_2; \hat{S}^m \times \hat{S}^m)$ by Shapiro’s lemma. Moreover, $H^1(W/W_0; \hat{T}^{W_0}) = H^2\left(C_2; \frac{\hat{S} \times \hat{S}}{S}\right) = H^2(C_2; \hat{S}) = 0$.

2. The 2-compact group $\frac{\text{GL}(i_0, C)^2 \times \text{GL}(i_1, C)^2}{\text{GL}(1, C)} \rtimes C_2$, $i_0, i_1 \geq 1$, where $C_2$ acts diagonally by switching the two $\text{GL}(i_0, C)$-factors and the two $\text{GL}(i_1, C)$-factors, is LHS, again, because

$$H^2\left(C_2; \frac{\hat{S}^2 \times \hat{S}^2}{S}\right) \rightarrow H^2\left(C_2; \frac{(\hat{S}^{i_0})^2 \times (\hat{S}^{i_1})^2}{S}\right), \quad \hat{S} = Z/2^\infty,$$

can be identified to the identity on $H^3(C_2; \hat{S})$. Moreover, $H^1(W/W_0; \hat{T}^{W_0}) = H^2(C_2; \frac{\hat{S}^2 \times \hat{S}^2}{S}) = H^2(C_2; \hat{S}) = 0$.

3. The 2-compact group $\frac{\text{GL}(m, C)^4}{\text{GL}(1, C)} \rtimes (C_2 \times C_2)$, $m \geq 1$, where $C_2 \times C_2 = \langle (12)(34), (13)(24) \rangle$ permutes the four $\text{GL}(m, C)$-factors, is LHS. Again,

$$H^2\left(C_2 \times C_2; \frac{\hat{S}^2 \times \hat{S}^2}{S}\right) \rightarrow H^2\left(C_2 \times C_2; \frac{(\hat{S}^m)^2 \times (\hat{S}^m)^2}{S}\right), \quad \hat{S} = Z/2^\infty,$$

identifies to the identity on $H^3(C_2 \times C_2; \hat{S})$ by means of Shapiro’s lemma and the Künneth isomorphism. Moreover, $H^1(W/W_0; \hat{T}^{W_0}) = H^2\left(C_2 \times C_2; \frac{\hat{S}^2 \times \hat{S}^2}{S}\right) = H^2(C_2 \times C_2; \hat{S}) = H^2(C_2; \hat{S}) + H^1(C_2; Z/2) = H^1(C_2; \hat{S}) = Z/2$.

4. The 2-compact group $\text{GL}(2n, R) = \text{SL}(2n, R) \rtimes C_2$, $n \geq 2$, is LHS by (2.18). The homomorphism $Z/2 = H^2(C_2; Z/2) \rightarrow H^2(C_2, T) = Z/2$ is injective because the action of $C_2$ on $T = (Z/2^\infty)^n$ has $(-1, 1, \ldots, 1)$ as its matrix. The 2-compact group $\text{GL}(4, R) = \text{SL}(4, R) \rtimes C_2 = (\text{SL}(2, C) \rtimes \text{SL}(2, C)) \rtimes C_2$, in particular, is strongly, but not uniquely $N$-determined because $0 \neq H^1(W/W_0; \hat{T}^{W_0}) = H^1(W; \hat{T}) (2.12, 7.3)$. For $n > 2$, $\text{GL}(2n, R)$ can be neither uniquely nor strongly $N$-determined.
2.21. The center of the maximal torus normalizer. We need criteria to ensure that the center of the 2-compact group $X$ agrees with the center of its maximal torus normalizer.

2.22. Proposition. [29, 4.12] Let $X$ be a 2-compact group. If $Z(X_0) = Z(N(X_0))$ and $X_0$ has $N$-determined automorphisms, then $Z(X) = Z(N(X))$.

Assume from now on that $X$ is a connected 2-compact group. Let $N(X) \to X$ be the maximal torus normalizer and $Z \to N$ a central monomorphism such that also the composition $Z \to N(X) \to X$ is central. The action map $BZ \times BN(X) \to BN(X)$ induces an action $[BN(X), BZ] \times \text{Out}(N(X)) \to \text{Out}(N(X))$ of the group $[BN(X), BZ] \cong H^1(N(X); \hat{Z})$ on the set $\text{Out}(N(X))$. Let $[BN(X), BZ]_{(1)}$ denote the isotropy subgroup at $(1) \in \text{Out}(N(X))$.

2.23. Lemma. If $Z(X) = Z(N(X))$ and $[BN(X), BZ]_{(1)} = 0$, then $Z(X/Z) = Z(N(X/Z))$.

Proof. Using [21, 4.6.4], the assumption of the lemma, and [29, 5.11], we get $Z(X/Z) = Z(X)/Z = Z(N(X))/Z = Z(N(X)/Z) = Z(N(X/Z))$.

2.24. Remark. Inspection shows that $Z(G) = ZN(G)$ for any simply connected compact Lie group $G$; see [5, 1.4] for a conceptual proof of this fact. In fact, $Z(G) = ZN(G)$ for any compact Lie group $G$ containing no direct factors isomorphic to $SO(2n+1)$ [20, 1.6].

2.25. Example. Let $X = \prod \text{GL}(n_i, \mathbb{C})$ be a product of general linear groups and $Z = \mathbb{C}^\times$. Then $Z(X/Z) = ZN(X/Z)$ (2.22), unless $X = \text{GL}(2, \mathbb{C})$, and, assuming that $X/Z$ has $N$-determined automorphisms, $Z(X_{h\pi}) = ZN(X_{h\pi})$ for any 2-compact group $X_{h\pi}$ with $X$ as its identity component (2.22). Indeed, the discrete approximation to $N(X)$ has the form $\tilde{N}(X) = \prod (\hat{T}_i \times \Sigma_m_i) = \hat{T} \times W$. Suppose that $(t, w) \in \tilde{N}(X)$ is such that $[(t, w), (s, 1)] \in \hat{Z} = \mathbb{Z}/2\mathbb{Z}$ for all $s \in \hat{T}$. Then $(w - 1)\hat{T} \subseteq \hat{Z}$, which means that $w$ acts trivially on $\hat{T}/\hat{Z}$. But $W$ is faithfully represented as a group of automorphisms of this maximal torus, so $w = 1$. Suppose therefore that $t \in \hat{T}$ is such that $[(t, 1), (s, v)] \in \hat{Z}$ for all $s, v \in \tilde{N}(X)$. Then $(v - 1)t \in \hat{Z}$ for all $v \in W$ and $v \to (v - 1)t$ is an element of $H^1(W; \hat{Z})$ which becomes trivial in $H^1(W; \hat{T})$ where it is a principal crossed homomorphism. Actually, $H^1(W; \hat{Z}) = \bigoplus H^1(\Sigma_n; \hat{Z})$ is isomorphic to the subgroup $\bigoplus H^1(\Sigma_n; \hat{T})$ of $H^1(W; \hat{T})$.

3. 2-COMPACT GROUPS WITH $N$-DETERMINED AUTOMORPHISMS

Let $X$ be a 2-compact group with maximal torus normalizer pair $N(X, X_0) = (N, N_0)$.

3.1. Lemma. [26, 4.2] Suppose that

1. $X_0$ has $N$-determined automorphisms
2. $H^1(W/W_0; \hat{Z}(X_0)) \to H^1(W/W_0; \hat{Z}(N_0))$ is injective

Then $X$ has $N$-determined automorphisms.

3.2. Lemma. [26, 4.8] Suppose that $X$ is connected. If the adjoint form $PX = X/Z(X)$ has $\pi_*(N)$-determined automorphisms, so does $X$.

Proof. If $f \in \text{Aut}(X)$ is an automorphism under $T(X)$, the induced automorphism $Pf \in \text{Aut}(PX)$ is an automorphism under $T(PX)$, hence equals the identity, and the induced automorphism $Z(f) \in \text{Aut}(Z(X))$ is also the identity since the center $ZX \to X$ factors through the maximal torus $T(X) \to X$ [7, 7.5] [21, 4.3]. But then $f$ itself is the identity for $\text{Aut}(X)$ embeds into $\text{Aut}(PX) \times \text{Aut}(ZX)$ [25, 4.3].

The functor $BC_X : \mathcal{A}(X) \to \text{Top}$ takes an object $(V, \nu)$ of the Quillen category $\mathcal{A}(X)$ to its centralizer $BC_X(V, \nu) = \text{map}(BV, BX)_{\nu_0}$. The functor $\pi_j(BCX)_\nu : \mathcal{A}(X) \to \mathbb{A}$ takes $(V, \nu)$ into the abelian group $\pi_j(\text{map}(BCX(V, \nu), BX)_{\nu_0})$ where $e(\nu) : BC_X(V, \nu) \to BX$ is the evaluation map.

3.3. Lemma. [26, 4.9] Suppose that $X$ is connected and centerless. If

1. $C_X(L, \lambda)$ has $N$-determined $\pi_*(N)$-determined automorphisms for each rank 1 object $(L, \lambda)$ of $\mathcal{A}(X)$
2. $\lim^1(\mathcal{A}(X); \pi_1(BCX)) = 0 = \lim^2(\mathcal{A}(X); \pi_2(BCX))$

Then $X$ has $N$-determined $\pi_*(N)$-determined automorphisms.
Proof. Suppose first that each line centralizer has \( \pi_+(N) \)-determined automorphisms. Let \( f: X \to X \) be an automorphism under the maximal torus \( T \to X \). Since any monomorphism \( \lambda: L \to X \), \( L = \mathbb{Z}/2 \), factors through the maximal torus, the commutative diagram

\[
\begin{array}{ccc}
L & \xrightarrow{X} & T \\
\downarrow & & \downarrow \text{AM}(f) \\
N & \xrightarrow{f} & X
\end{array}
\]

shows that \( f \lambda = \lambda \) and gives a commutative diagram

\[
\begin{array}{ccc}
C_N(L) & \to & C_X(L) \\
\downarrow & & \downarrow \text{AM}(f)(L) \\
C_N(L) & \xrightarrow{f(L)} & C_X(L)
\end{array}
\]

of automorphisms under \( T \). Thus \( \text{AM}(C_f(L)) = C_{\text{AM}(f)}(L): C_N(L) \to C_N(L) \). Now, \( \pi_+(C_N(L)) \) is a subgroup of \( \pi_+(N) \) (for \( \pi_1(C_N(L)) = \pi_1(N) \) and \( \pi_0(C_N(L)) = W(X)(L) \) is \([7, 7.6] [25, 3.2.1]\) the stabilizer subgroup at \( L < \hat{T} \) for the action of \( W(X) \) on \( \hat{T} \) so \( \pi_+(C_{\text{AM}(f)}(L)) = 1 \) and \( C_f(L) \sim 1_{C_X(L)} \) since \( C_X(L) \) has \( \pi_+(N) \)-determined automorphisms. For any other object \((V, \nu)\) of \( A(X) \) of rank \( \geq 1 \), choose a line \( L \) in \( V \). Since the monomorphism \( \nu: V \to X \) canonically factors through \( C_X(L) \) \([6, 8.2] [29, 3.18]\), the commutative diagram

\[
\begin{array}{ccc}
V & \to & X \\
\downarrow \nu & & \downarrow f \\
C_X(L) & \to & X
\end{array}
\]

shows that \( f \nu = \nu \) and the induced diagram

\[
\begin{array}{ccc}
C_X(V) & \to & C_X(V) \\
\cong \downarrow & & \cong \downarrow \\
C_{C_X(L)}(V) & \to & C_f(V)
\end{array}
\]

that \( C_f(V): C_X(V) \to C_X(V) \) is conjugate to the identity. The second assumption of the lemma assures that there are no obstructions to conjugating \( f \) to the the identity now that we know that the restriction of \( f \) to each of the centralizers is conjugate to the identity, see \([26, 4.9]\). Suppose next that each line centralizer has \( N \)-determined automorphisms. Let \( f: X \to X \) be an automorphism such that the diagram

\[
\begin{array}{ccc}
N & \to & X \\
\downarrow f & & \downarrow \\
X & \to & X
\end{array}
\]

commutes up to conjugacy. For each line \( L \) in \( T \), the induced diagram

\[
\begin{array}{ccc}
C_X(L) & \to & C_X(L) \\
\downarrow & & \downarrow \\
C_N(L) & \to & C_f(L)
\end{array}
\]

also commutes up to conjugacy. By assumption, this means \((2.15)\) that the induced automorphisms \( C_f(L) \) of line centralizers are conjugate to the identity. As above, this implies that the induced
map \( C_f(V): C_X(V) \to C_X(V) \) is conjugate to the identity for any object \((V, \nu)\) of the Quillen category for \(X\) and that \(f\) is conjugate to the identity.

3.4. Lemma. [29, 9.4] If the two connected 2-compact groups \(X_1\) and \(X_2\) have \(N\)-determined \((\pi_\ast(N)\text{-determined})\) automorphisms, so does the product \(X_1 \times X_2\).

**Proof.** Since the statement concerning \(N\)-determined automorphisms is proved in [29, 9.4] we deal here only with the case of \(\pi_\ast(N)\)-determined automorphisms. Let \(f\) be an automorphism under \(T_1 \times T_2\) of the product 2-compact group \(X_1 \times X_2\). Then

\[
\begin{align*}
  f_1: X_1 &\to X_1 \times X_2 \overset{f}{\longrightarrow} X_1 \times X_2 \to X_1 \\
  f_2: X_2 &\to X_1 \times X_2 \overset{f}{\longrightarrow} X_1 \times X_2 \to X_2
\end{align*}
\]

are endomorphisms under the maximal tori and therefore conjugate to the respective identity maps. But \(f\) is [29, 9.3] in fact conjugate to the product morphism \((f_1, f_2)\) which is the identity.

\[ \square \]

4. \textit{N}-determined 2-Compact Groups

Let \(X\) be a 2-compact group with maximal torus normalizer pair \(N(X, X_0) = (N, N_0)\).

4.1. Lemma. Suppose that

1. \(X_0\) is uniquely \(N\)-determined.
2. \(X\) is LHS.
3. \(H^2(W/W_0, Z(X_0)) \to H^2(W/W_0, Z(N_0))\) is injective.

Then \(X\) is \(N\)-determined.

**Proof.** Let \(X'\) be another 2-compact group with maximal torus normalizer pair \((N, N_0)\). The assumption on the identity component \(X_0\) means (2.14) that there exists an isomorphism \(f_0: X_0 \to X_0'\) under \(T\). For any \(\xi \in W/W_0 = N/N_0 = X/X_0 = X'/X_0'\), the isomorphism \(\xi f_0 \xi^{-1}\) is also an isomorphism under \(T\) and thus \(\xi f_0 = f_0 \xi\) as \(X_0\) is uniquely \(N\)-determined. By the second assumption, the automorphism \(\alpha_0 = AM(f_0): N_0 \to N_0\) with \(\pi_\ast(B\alpha_0) = 1\) extends to an isomorphism \(\alpha: N \to N\) with \(\pi_\ast(B\alpha) = 1\).

Our aim is to find an isomorphism \(f: X \to X'\) to fill in the based homotopy commutative diagram

\[
\begin{array}{ccc}
BX_0 & \xrightarrow{Bf_0} & BX'_0 \\
\downarrow & & \downarrow \\
BX & \xrightarrow{Bf_0} & BX'
\end{array}
\]

\[
\begin{array}{ccc}
B\pi_0(X) & \xrightarrow{\cong} & B\pi_0(X') \\
\downarrow & & \downarrow \\
BN & \xrightarrow{BJ} & BX'
\end{array}
\]

where the isomorphism between the base 2-compact groups is given by the isomorphisms \(\pi_0(X) \leftarrow N/N_0 \to \pi_0(X')\). Since \(f_0\) is \(W/W_0\)-equivariant up to homotopy, \(map(BX_0, BX'_0)_{Bf_0}\) is a \(W/W_0\)-space. Composition with \(BX \xrightarrow{BJ} BN \xrightarrow{BJ'} BX'\) gives maps

\[
map(BX_0, BX'_0; Bf_0)^{hW/W_0} \xrightarrow{BJ'} \map(BN_0, BX'_0; B(j_0\alpha))^{hW/W_0}
\]

\[
\map(BN_0, BN_0; B\alpha_0)^{hW/W_0}
\]

of homotopy fixed point spaces. The space to the right is non-empty for it contains the isomorphism \(B\alpha: BN \to BN\). Using obstruction theory and the second assumption of the lemma, we see that also the homotopy fixed point space to the left is non-empty; it contains a morphism \(Bf: BX \to BX'\) under \(Bf_0: BX_0 \to BX'_0\) and over \(B\pi_0(X) \xrightarrow{\cong} B\pi_0(X')\) such that \(Bf \circ Bj\) and \( Bj \circ B\alpha\) are homotopic over \(B(N/N_0) \to B\pi_0(X')\). But since the fibre \(BX'_0\) of \(BX' \to B\pi_0(X')\) is simply connected this means that \(Bf \circ Bj\) and \( Bj \circ B\alpha\) are based homotopic maps \(BN \to BX'\). \(\square\)
4.2. **Example.** 1. Any 2-compact torus $T$ is strongly $N$-determined for if $j : T \to X$ is the maximal torus normalizer for the connected 2-compact group $X$, then $j$ is an isomorphism. Indeed, $H^*(BT; \mathbb{Q}_2) \cong H^*(BX; \mathbb{Q}_2) [6, 9.7(3)]$ and the connected space $X/T$ has cohomological dimension $\dim_{\mathbb{F}_2}(X/T) = 0 [7, 4.5, 5.6]$ so is a point.

2. Any 2-compact toral group $G$ is strongly $N$-determined: $G$ clearly has $N$-determined automorphisms as $G$ is its own maximal torus normalizer. If the 2-compact group $X$ has the same maximal torus normalizer pair $(G,T)$ as $G$, then $X$ is a 2-compact toral group and $j' : G \to X$ is an isomorphism. $G$ is uniquely $N$-determined if and only if $H^1(\pi_0(G); \hat{T}) = 0$. In particular, $\text{GL}(2, \mathbb{R})$ is uniquely and strongly $N$-determined.

4.3. **Lemma.** Let $X$ be a connected 2-compact group and $Z \to X$ its center. If $X/Z$ is $N$-determined, so is $X$.

**Proof.** Let $j : N \to X$ be the maximal torus normalizer for $X$ and $j' : N \to X'$ the maximal torus normalizer for some other connected 2-compact group $X'$. It suffices (2.14) to find a morphism $f : X \to X'$ under the maximal tori $X \to T \to X'$. The 2-discrete center $\hat{Z}$ of $X$ and $X'$ is contained in the the 2-discrete maximal torus $\hat{T}$ [7, 7.5]. Factoring out [6, 8.3] these central monomorphisms we obtain the commutative diagram

$$
\begin{array}{cccc}
\text{B}X & \cong & \text{B}X' & \cong \\
\text{B}(X/Z) & \text{B}(T/Z) & \text{B}(X'/Z) & \\
\end{array}
$$

where the vertical maps are fibrations with fibre $B\hat{Z}$, the total spaces, such as $B\hat{X}$, are the fibre-wise discrete approximations, and $f/Z : X/Z \to X'/Z$ is the isomorphism under $T/Z$ that exists because $X/Z$ is $N$-determined. Construct the fibration

$$
\text{map}(\text{B}Z, \text{B}Z; B1) \to \text{B}Z_{h(X/Z)} \to \text{B}(X/Z)
$$

whose sections are maps $B\hat{X} \to B\hat{X}'$ over $B(f/Z)$ and under $B\hat{Z}$. There are two other such fibrations related to this one shown in the commutative diagram

$$
\begin{array}{cccc}
\text{B}Z_{h(X/Z)} & \cong & \text{B}Z_{h(T/Z)} & \\
\text{B}(X/Z) & \text{B}(T/Z) & \\
\end{array}
$$

where the middle fibration is the pull-back along $B(i/Z)$ of the left fibration and the fibre over $b \in B(T/Z)$ of the right fibration consists of one component (remark about equivariance?) of the space of maps of the fibre $B\hat{h}_b$ over $b$ into the fibre $B\hat{X}'_{B(i'/Z)(b)}$ over $B(i'/Z)(b)$. The fibre equivalence $B\hat{h}^*$ is induced by $B1 : B\hat{T} \to B\hat{X}$. The middle fibration has a section $u'$ such that $B\hat{h}^* \circ u'$ is the section $B\hat{h}' : B\hat{T} \to B\hat{X}$ of the right fibration. We now have fibre maps

$$
\begin{array}{cc}
X/T & \text{u}|X/T \\
\text{B}(i/Z) & \\
\end{array}
$$

$$
\begin{array}{cc}
\text{B}(i/Z) & \\
\text{B}(X/Z) & \\
\end{array}
$$

$$
\begin{array}{cc}
\text{B}T & \text{B}Z_{h(X/Z)} \\
\text{B}(i/Z) & \\
\end{array}
$$
where $u$ is the composition of $u'$ and $BZ_{h(T/Z)} \rightarrow BZ_{h(X/Z)}$. The canonical map, given by constants, $BZ \rightarrow \text{map}(X/T, BZ)$ is a homotopy equivalence since $X/T$ is simply connected [21, 5.6] and hence a version [26, 6.6] of the Zabrodsky lemma implies that $u = v \circ B(i/Z)$ for some section $v: B(X/Z) \rightarrow BZ_{h(X/Z)}$ of the left fibration. The section $v$ is, after fibre-wise completion, a fibre map $BX \rightarrow BX'$ under $BT$. 

Let $X_1$ and $X_2$ be two connected 2-compact groups with trivial centers and $j_1: N_1 \rightarrow X_1$, $j_2: N_2 \rightarrow X_2$ their maximal torus normalizers. The Splitting Theorem [8, 1.4] says that if the monomorphism $j: N_1 \times N_2 \rightarrow X$ is the maximal torus normalizer for some connected 2-compact group $X$ then there exist an isomorphism $s: X \rightarrow X_1 \times X_2$ and an automorphism $\alpha$ of $N_1 \times N_2$ such that the diagram

$$
\begin{array}{ccc}
N_1 \times N_2 & \xrightarrow{\alpha} & N_1 \times N_2 \\
\downarrow j & & \downarrow j_1 \times j_2 \\
X & \xrightarrow{s} & X_1 \times X_2
\end{array}
$$

commutes up to conjugacy. We record this in

4.4. Lemma. The product of two $N$-determined connected 2-compact groups is $N$-determined.

The problem is now reduced to the connected and center-less case. Consider therefore an extended 2-compact torus $N$ and two connected, center-less 2-compact groups $X$ and $X'$ both having $N$ as their maximal torus normalizer

$$(4.5)$$

$$X \xleftarrow{j} N \xrightarrow{j'} X'$$

For each toral object $(V, \nu)$ of $\mathbf{A}(X)$, let $\nu^N: V \rightarrow N$ be the unique preferred lift [27, 4.10] of $\nu$ (which factors through the identity component of $N$) and let $(V, \nu')$ be the object defined by $\nu' = j \circ \nu^N: V \rightarrow X'$ as in the commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\nu'} & V \\
\downarrow \nu^N & & \downarrow \nu' \\
X & \xrightarrow{j} & N \xrightarrow{j'} X'
\end{array}
$$

The functor $\mathbf{A}(X)^{\leq t} \rightarrow \mathbf{A}(X')^{\leq t}$ that takes the object $(V, \nu)$ to the object $(V, \nu')$ and is the identity on morphisms is an equivalence of toral Quillen categories [29, 2.8].

4.6. Theorem. In the situation of (4.5), assume the following:

1. Centralizers of all toral rank $\leq 2$ objects of $\mathbf{A}(X)$ have $N$-determined automorphisms.
2. There exists a self-homotopy equivalence $\alpha \in H^1(W; T) \subseteq \text{Out}(N)$ such that for every object $(L, \lambda) \in \text{Ob}(\mathbf{A}(X))$ of rank 1 the diagram

$$
\begin{array}{ccc}
C_N(\lambda^N) & \xrightarrow{\alpha} & C_N(\lambda^N) \\
\downarrow j|C_N(\lambda^N) & & \downarrow j'|C_N(\lambda^N) \\
C_X(\lambda) & \xrightarrow{f_\lambda} & C_{X'}(\lambda')
\end{array}
$$

commutes for some isomorphism $f_\lambda$.
3. For any non-toral rank 2 object $(V, \nu)$ of $\mathbf{A}(X)$ the composite monomorphism

$$
\nu'_L: V \xrightarrow{\nu_L} C_X(L, \nu|L) \xrightarrow{f(L, \nu|L)} C_{X'}(L, (\nu|L)', \nu'_L) \xrightarrow{\text{res}} X'
$$
and the induced isomorphism \( f_{\nu, L} : C_X(V, \nu) \to C_{X'}(V, \nu') \) defined by the commutative diagram

\[
\begin{array}{ccc}
C_{C_X(L, \nu|L)}(V, \nu|L) & \xrightarrow{C_{f(L, \nu|L)}} & C_{C_{X'}(L, \nu'|L)}(V, f(L, \nu|L) \circ \nu|L)) \\
\cong & & \cong \\
C_X(V, \nu) & \xrightarrow{f_{\nu, L}} & C_{X'}(V, \nu'(L))
\end{array}
\]

do not depend on the choice of line \( L < V \).

(4) \( \lim^2(\mathbf{A}(X); \pi_1(BZC_X)) = 0 = \lim^3(\mathbf{A}(X); \pi_2(BZC_X)) \).

Then there exists an isomorphism \( f : X \to X' \) under \( T \) (2.14).

**Proof.** The idea is that the isomorphisms \( f_\lambda : C_X(\lambda) \to C_{X'}(\lambda') \) on the line centralizers restrict to isomorphisms \( f_\nu : C_X(\nu) \to C_{X'}(\nu') \) for all centralizers in the \( F_2 \)-homology decomposition

\[
\mathrm{hocolim}_{\mathbf{A}(X)} BC_X(\nu) \to BX
\]

of \( BX \). These locally defined isomorphisms combine to a globally defined isomorphism \( BX \to BX' \).

First observe that the isomorphisms \( f_\lambda \) on the line centralizers are uniquely determined by the cohomology class \( \alpha \in H^1(W; \overline{T}) \) (2.11.(1)).

Let now \( (V, \nu) \) be a rank 2 object of \( \mathbf{A}(X) \) and \( L \) a line in the plane \( V \). If \( (V, \nu) \) is toral, define \( f_\nu : C_X(V, \nu) \to C_{X'}(V, \nu') \) to be the isomorphism induced by \( f_{\nu, L} : C_X(L, \nu|L) \to C_{X'}(L, \nu'|L) \).

Since \( f_\nu \) is an isomorphism under \( \alpha|C_N(V, \nu|V') \) it does not depend on the choice of \( L \) in \( V \) (2.11.(1)).

If \( (V, \nu) \) is non-toral, define \( \nu' \) to be \( \nu'_L \) and define \( f_\nu : C_X(V, \nu) \to C_{X'}(V, \nu') \) to be \( f_{\nu, L} \). By assumption 4.6.(3), the monomorphism \( \nu' \) and the isomorphism \( f_{\nu, L} \) are independent of the choice of \( L \).

This construction respects morphisms in \( \mathbf{A}(X) \). Consider first, for instance, a morphism \( \beta : (L_1, \lambda_1) \to (L_2, \lambda_2) \) between two lines in \( X \). Then \( \lambda_1 = \lambda_2 \beta \) and \( \lambda_1^N = \lambda_2^N \beta \). The commutative diagram of isomorphisms

\[
\begin{array}{ccc}
C_X(\lambda_1) & \xrightarrow{\beta^*} & C_X(\lambda_2) \\
\downarrow f_{\lambda_1} & & \downarrow f_{\lambda_2} \\
C_N(\lambda_1^N) & \xrightarrow{\beta^*} & C_N(\lambda_2^N) \\
\downarrow \alpha|C_N(\lambda_1^N) & & \downarrow \alpha|C_N(\lambda_2^N) \\
C_{X'}(\lambda_1') & \xrightarrow{\beta^*} & C_{X'}(\lambda_2') \\
\downarrow \beta^* & & \downarrow \beta^* \\
C_N(\lambda_1^N) & \xrightarrow{\beta^*} & C_N(\lambda_2^N)
\end{array}
\]

shows that \( (\beta^*)^{-1} \circ f_{\lambda_1} \circ \beta^* = f_{\lambda_2} \) for they are both isomorphism under \( (\beta^*)^{-1} \circ \alpha|C_N(\lambda_1^N) \circ \beta^* = \alpha|C_N(\lambda_2^N) \). Second, by the very definition of \( f_\nu \), the diagram

\[
\begin{array}{ccc}
C_X(V, \nu) & \xrightarrow{f_\nu} & C_{X'}(V, \nu') \\
\downarrow & & \downarrow \\
C_X(L, \nu|L) & \xrightarrow{f_{\nu, L}} & C_{X'}(L, \nu'|L')
\end{array}
\]

commutes whenever \( L < V \) and \( (V, \nu) \) is (toral or non-toral) rank 2 object of \( \mathbf{A}(X) \).

We have now defined natural isomorphisms \( f_\nu : C_X(V, \nu) \to C_{X'}(V, \nu') \) for all objects \( (V, \nu) \in \mathrm{Ob}(\mathbf{A}(X)) \) of rank \( \leq 2 \). For any other object \( (E, \varepsilon) \) of \( \mathbf{A}(X) \), choose a line \( L < E \) and proceed as for toral rank 2 objects. That is, define \( \varepsilon' : E \to X' \) to be the monomorphism

\[
E \xrightarrow{\pi(L)} C_X(E, \varepsilon|L) \xrightarrow{f_{\varepsilon|L}} C_{X'}(E, (\varepsilon|L)'), \xrightarrow{\text{res}} X'
\]
and define \( f_x : C_X(E, \varepsilon) \to C_{X'}(E, \varepsilon') \) to be the isomorphism
\[
C_{C_X(E, \varepsilon)}(\overline{L}) \xrightarrow{(f_{\varepsilon|L})_*} C_{C_{X'}(E, \varepsilon')}((f_{\varepsilon|L}) \circ \overline{L})
\]
induced by \( f_{\varepsilon|L} \). If \( L_1 \) and \( L_2 \) are two distinct lines in \( E \), let \( P = \langle L_1, L_2 \rangle \) be the plane generated by them. Then the commutative diagram
\[
\begin{array}{ccc}
C_X(L_1, \varepsilon|L_1) & \xrightarrow{f_{\varepsilon|L_1}} & C_{X'}(L_1, (\varepsilon|L_1)') \\
\uparrow \quad & & \quad \uparrow \res \\
C_X(P, \varepsilon|P) & \xrightarrow{f_{\varepsilon|P}} & C_{X'}(P, (\varepsilon|P)') \\
\downarrow \quad & & \quad \downarrow \res \\
C_X(L_2, \varepsilon|L_2) & \xrightarrow{f_{\varepsilon|L_2}} & C_{X'}(L_2, (\varepsilon|L_2)')
\end{array}
\]
shows that neither \((E, \varepsilon') \in \text{Ob}(A(X'))\) nor the isomorphism \( f_x \) depend on the choice of line in \( E \).

Thus we have constructed a collection of centric [4] maps
\[
(4.7) \quad B C_X(V, \nu) \to B X', \quad (V, \nu) \in \text{Ob}(A(X)),
\]
that are homotopy invariant under \( A(X) \)-morphisms. The vanishing \((4.6.4)\) of the obstruction groups means [36] that these homotopy \( A(X) \)-invariant maps can be realized by a map
\[
B f' : B X \xrightarrow{\simeq} \text{hocolim} B C_X \to B X'
\]
such that \( f \circ \res = \res \circ f_{\nu} \) for all \((V, \nu) \in \text{Ob}(A(X))\). In particular, \( f \) is a map under \( T \) and an isomorphism \((2.14)\). \( \Box \)

4.8. **Verification of condition 4.6.(2).** Define \( A_{\text{LHS}}(X)^{\leq t} \) to be the full subcategory of the toral Quillen category \( A(X)^{\leq t} = A(W, t) \) [29, 2.2] generated by all objects \( \nu \) whose centralizers \( C_X(\nu) \) are LHS and totally \( N \)-determined. For such an object, the solutions to the isomorphism problem
\[
(4.9) \quad C_N(\nu^N) \xrightarrow{\alpha_\nu} C_N(\nu^N) \xrightarrow{f_{\nu}} C_X(\nu''),
\]
define a subset \( \{\alpha_\nu\} \) of \( H^1(W; \check{T})(\nu) \) and \((2.12.1)\) an element \( \tau_\nu \) of \( H^1(W_0; \check{T})^{W/W_0}(\nu) \). These elements respect the morphisms in \( A_{\text{LHS}}(X)^{\leq t} \) (because the restriction of a solution is a solution) so they represent an element \( (\tau_\nu) \) of the limit group. If the two homomorphisms
\[
H^1(W(X); \check{T}(X)) \to \lim^0(A_{\text{LHS}}(X)^{\leq t}; H^1(W; \check{T})) \to \lim^0(A_{\text{LHS}}(X)^{\leq t}; H^1(W_0; \check{T})^{W/W_0})
\]
are surjective, this elements is the image of an element \( \alpha \in H^1(W(X); \check{T}(X)) \). This means that the isomorphism problems \((4.9)\) have a coherent solution where \( \alpha_\nu = \alpha|C_N(\nu^N) \) is the restriction of \( \alpha \) for all objects \( \nu \) of \( A_{\text{LHS}}(X)^{\leq t} \).

We can therefore replace 4.6.(1) and 4.6.(2) by
- \( C_X(\nu) \) is LHS and totally \( N \)-determined for each toral elementary abelian 2-subgroup \((V, \nu) \) of \( X \) of rank \( \leq 2 \)
- \( \lim^1(A_{\text{LHS}}(X)^{\leq t}; H^1(W_0; \check{T})^{W/W_0}) = 0 \)

The first property ensures that
\[
H^1(W(X); \check{T}(X)) \to \lim^0(A_{\text{LHS}}(X)^{\leq t}; H^1(W; \check{T})) \cong \lim^0(A(X)^{\leq t}; H^1(W; \check{T})) \cong \lim^0(A(X)^{\leq t}; H^1(W; \check{T}))
\]
is an isomorphism \([10, 8.1] [32]\) and the second property that there is an exact sequence
\[
0 \to \lim^0 H^1(W/W_0; T^{W_0}) \to \lim^0 H^1(W; T) \to \lim^0 H^1(W_0; T)^{W/W_0} \to 0,
\]
where the limits are taken over over \(A_{\text{LHS}}(X)^{\leq 1}\) or \(A(X)^{\leq 1}\). It is sometimes possible to compute the above \(\lim^1\)-term by means of Oliver’s cochain complex \([32]\).

4.10. **Verification of condition 4.6.(3).** The following observation can sometimes be useful in the verification of condition 4.6.(3).

4.11. **Lemma.** Let \((V, \nu)\) be a non-toral rank 2 object of \(A(X)\) and \(L < V\) a line in \(V\). Write \(C_3\) for the Sylow 3-subgroup of \(GL(V)\). Suppose that

1. \(C_3 \subseteq A(X)(V, \nu) \cap A(X')(V, \nu')\)
2. \(f_{\nu, L}: C_X(V, \nu) \to C_X(V, \nu')\) is \(C_3\)-equivariant

Then condition 4.6.(3) is satisfied.

**Proof.** Let \(L_1\) and \(L_2 = L\) be lines in \(V\). Choose an automorphism \(\alpha\) of \((V, \nu)\) that takes \(L_1\) to \(L_2\). Then \(\nu'_{L_2} = \nu' - L_1\) and \(f_{\nu, L_1} = C_X(\alpha) \circ f_{\nu, L_2} \circ C_X(\alpha)^{-1}\) \((4.15)\).

The following lemma assures that condition 4.11.(1) holds.

4.12. **Lemma.** Suppose that

1. There is (up to conjugacy) a unique monomorphism \(\lambda: \mathbb{Z}/2 \to X\) with non-connected centralizer
2. There is (up to conjugacy) a unique non-toral monomorphism \(\nu: (\mathbb{Z}/2)^2 \to X\)

Then the same holds for \(X'\) and \(A(X)(V, \nu) = GL(V) = A(X')(V, \nu')\) for the non-toral object \((V, \nu)\) of \(A(X)\).

**Proof.** Let \(\nu': (\mathbb{Z}/2)^2 \to X'\) be a non-toral monomorphism from a rank elementary abelian 2-group into \(X'\) and let \(i_1: \mathbb{Z}/2 \to (\mathbb{Z}/2)^2\) be the inclusion into the first summand. Then \(\nu'i_1\) corresponds to \(\lambda\) under the bijection between rank 1 objects of \(A(X)\) and \(A(X')\), i.e. \(\nu'i_1 = \lambda\). Moreover, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{res}} & C_X(\mathbb{Z}/2, \lambda) \\
\text{res} & & \text{res} \\
\downarrow & \text{f}|_{\mathbb{Z}/2} & \downarrow \text{f}|_{\mathbb{Z}/2} \\
(\mathbb{Z}/2)^2 & \cong & C_X((\mathbb{Z}/2)^2, \nu') \\
\end{array}
\]

is commutative. To see this, observe that \(\nu = \text{res} \circ f^{-1}_\lambda \circ \overline{\nu}(\mathbb{Z}/2)\) by uniqueness of \(\nu\), and \(\overline{\nu}(\mathbb{Z}/2) = f^{-1}_\lambda \circ \overline{\nu}(\mathbb{Z}/2)\) by uniqueness of canonical factorizations under \(\mathbb{Z}/2\) \([28, 3.9]\). We conclude that \(\nu' = \text{res} \circ \overline{\nu}(\mathbb{Z}/2) = \text{res} \circ f_{\lambda} \circ \overline{\nu}(\mathbb{Z}/2)\) is uniquely determined up to conjugacy.

Note in connection with condition 4.11.(2) that by mapping \((\mathbb{Z}/2)^2\) into middle part of diagram \((4.13)\) we see that \(f(\mathbb{Z}/2)^2, \nu)\) is a map under the canonical factorization in the sense that

\[
\begin{array}{ccc}
\overline{\nu}(\mathbb{Z}/2)^2 & \xrightarrow{f} & \overline{\nu}(\mathbb{Z}/2)^2 \\
\text{f}|_{\mathbb{Z}/2} & & \text{f}|_{\mathbb{Z}/2} \\
\downarrow & \downarrow & \downarrow \text{f}|_{\mathbb{Z}/2} \\
C_X((\mathbb{Z}/2)^2, \nu) & \cong & C_X((\mathbb{Z}/2)^2, \nu') \\
\end{array}
\]

commutes where the canonical monomorphisms, \(\overline{\nu}(\mathbb{Z}/2)^2\) and \(\overline{\nu}(\mathbb{Z}/2)^2\), are \(GL(E)\)-equivariant. Thus the restriction of \(f(V, \nu)\) to \(V\) is \(C_3\)-equivariant.
4.15. **Canonical factorizations.** Let \( \nu: V \to X \) be a monomorphism from an elementary abelian \( p \)-group to the \( p \)-compact group \( X \). The canonical factorization of \( \nu \) through its centralizer is the central monomorphism \( \varpi(V): V \to C_X(V, \nu) \) whose adjoint is \( V \times V \xrightarrow{\varpi} V \xrightarrow{\varpi} X \) \([6, 8.2]\). If \( \alpha: (V_1, \nu_1) \to (V_2, \nu_2) \) is a morphism in \( A(X) \) then the canonical factorizations are related by a commutative diagram

\[
\begin{array}{ccc}
V_1 & \xrightarrow{\varpi_1(V_1)} & C_X(V_1, \nu_1) & \xrightarrow{\text{res}} & X \\
\downarrow{\alpha} & & \downarrow{C_X(\alpha)} & & \downarrow{\text{res}} \\
V_2 & \xrightarrow{\varpi_2(V_2)} & C_X(V_2, \nu_2) & \xrightarrow{\text{res}} & X \\
\end{array}
\]

and we shall write \( \varpi_2(V_1): V_2 \to C_X(V_1, \nu_1) \) for \( C_X(\alpha) \circ \varpi_2(V_2) \) and call it the canonical factorization of \( \varpi_2 \) through the centralizer of \( \nu_1 \). The induced diagram

\[
\begin{array}{c}
C_X(V_2, \nu_2)(V_2, \varpi_2(V_2)) \xrightarrow{C_X(\alpha)} C_X(V_1, \nu_1)(V_2, \varpi_2(V_1)) \xrightarrow{C_X(\alpha)(\nu_1)} C_X(V_1, \nu_1)(V_1, \varpi_1(V_1)) \\
\downarrow{\cong} & & \downarrow{\cong} \\
C_X(V_2, \nu_2) & \xrightarrow{C_X(\alpha)} & C_X(V_1, \nu_1) \\
\end{array}
\]

is a factorization of \( C_X(\alpha) \).

5. **The Quillen category of \( \text{PGL}(n+1, \mathbb{C}) \)**

For \( W \) is a finite group acting on a finite \( \mathbb{F}_2 \)-vector space \( V \), define \( A(W, V) \) to be the category whose objects are non-trivial subspaces of \( V \) and whose morphisms are group homomorphisms induced by the \( W \)-action; the morphism set \( A(W, V)(V_1, V_2) \) is the set of orbits \( \overline{W}(V_1, V_2)/W(V_1, V_1) \) for the action of the point-wise stabilizer group \( W(V_1, V_1) = \{ w \in W | uv = v \text{ for all } v \in V \} \) on the set \( \overline{W}(V_1, V_2) = \{ w \in W | wV_1 \subseteq V_2 \} \).

5.1. **The toral subcategory of \( A(\text{PGL}(n+1, \mathbb{C})) \).** An object \( (V, \nu) \) of the Quillen category of the \( 2 \)-compact group \( X \) is **toral** if the monomorphism \( \nu: V \to X \) is conjugate to a monomorphism that factors through the maximal torus \( T(X) \) of \( X \). Let \( A(X)^{\leq t} \) denote the full subcategory of \( A(X) \) generated by all toral objects. We shall determine this toral subcategory in case \( X = \text{PGL}(n+1, \mathbb{C}) \).

5.2. **Lemma.** The monomorphism \( \nu: V \to \text{PGL}(n+1, \mathbb{C}) \) is toral if and only if it lifts to a morphism \( V \to \text{GL}(n+1, \mathbb{C}) \). If \( n \) is even, all objects of \( A(\text{PGL}(n+1, \mathbb{C})) \) are toral.

**Proof.** All objects of \( A(\text{GL}(n+1, \mathbb{C})) \) are toral by complex representation theory. Any monomorphism \( V \to (\mathbb{C}^\times)^{n+1}/\mathbb{C}^\times \) lifts to \( (\mathbb{C}^\times)^{n+1} \) since \( \mathbb{C}^\times \) is a divisible abelian group. If \( n \) is even, \( \text{PGL}(n+1, \mathbb{C}) = \text{SL}(n+1, \mathbb{C}) \) as \( 2 \)-compact groups and all monomorphisms \( V \to \text{SL}(n+1, \mathbb{C}) \) are toral by complex representation theory. \( \square \)

5.3. **Proposition.** \([29, 2.8]\) The inclusion \( t(\text{PGL}(n+1, \mathbb{C})) \to T(\text{PGL}(n+1, \mathbb{C})) \) induces an equivalence of categories \( A(\Sigma_{n+1}, t(\text{PGL}(n+1, \mathbb{C}))) \to A(\text{PGL}(n+1, \mathbb{C}))^{\leq t} \).

**Proof.** The functor is the identity on morphisms. Any morphism between two toral objects \( V_1 \to V_2 \) of \( A(\text{PGL}(n+1, \mathbb{C})) \) is induced from the action by a Weyl group element. \( \square \)

5.4. **Corollary.** When \( n > 1 \), the limits \( \lim^i(A(\text{PGL}(n+1, \mathbb{C}))^{\leq t}; \pi_j(BZ_{\text{PGL}(n+1, \mathbb{C})})) = 0 \) and \( \lim^i(A(\text{PGL}(n+1, \mathbb{C})); \pi_j(BZ_{\text{PGL}(n+1, \mathbb{C})})) \) is isomorphic to

\[
\lim^i(A(\text{PGL}(n+1, \mathbb{C})); \pi_j(BZ_{\text{PGL}(n+1, \mathbb{C})})^t) \cong \lim^i(A(\text{PGL}(n+1, \mathbb{C})); \pi_j(BZ_{\text{PGL}(n+1, \mathbb{C})}))
\]

for all \( i \geq 0 \) and \( j = 1, 2 \).
Proof. For any non-trivial toral subgroup $V \subseteq t(PGL(n+1, \mathbb{C}))$ we have by (2.25) that
\[
\pi_j(BZ_{PGL(n+1, \mathbb{C})}) = H^{2-j}(\Sigma_{n+1}(V); L(PGL(n+1, \mathbb{C}))), \quad j = 1, 2,
\]
because the 2-discrete toral group
\[
\check{Z}_{PGL(n+1, \mathbb{C})}(V) = ZC_{\check{N}(PGL(n+1, \mathbb{C}))}(V) = Z(\check{T}(PGL(n+1, \mathbb{C})) \times \Sigma_{n+1}(V))
\]
and consequently the higher limits of these functors $A(PGL(n+1, \mathbb{C})) \to \mathbb{A}b$ are trivial while for $i = 0$ we get $H^{2-j}(\Sigma_{n+1}; L(PGL(n+1, \mathbb{C})))$ which is trivial for $n > 1$. Apply [29, 2.11] to get the isomorphisms.

Let $E$ be a non-trivial elementary abelian 2-group and $\text{Rep}(E, GL(n+1, \mathbb{C}))$ the set of functions $i : E^\vee \to N$ taking the dual $E^\vee = \text{Hom}(E, \mathbb{C}^\times)$ of $E$ into the natural numbers such that $\sum_{f \in E^\vee} i(f) = n + 1$. This set supports group actions
\[
E^\vee \times \text{Rep}(E, GL(n+1, \mathbb{C})) \longrightarrow \text{Rep}(E, GL(n+1, \mathbb{C})) \leftarrow \text{Rep}(E, GL(n+1, \mathbb{C})) \times GL(E)
\]
given by $g \cdot i = i \circ \tau_g$, $g \in E^\vee$, and $i : A = i \circ A^\vee$, $A \in GL(E)$, where $
\tau_g(f) = g + f$ and $A^\vee(f) = f \circ A^{-1}$ for all linear forms $f \in E^\vee$. The identity $\tau_{(g \cdot i)} A^\vee = A^\vee \tau_i$ gives $(g \cdot i) \cdot A = ((A^{-1})^\vee g) \cdot (i \cdot A)$.

We say that a subset $S$ of linear forms on $E$ has trivial equalizer, and write $\text{Eq}(S) = 0$, if $S$ contains at least two elements and all the elements of $S$ agree only on the trivial element of $E$.

5.5. Proposition. Let $E$ be a non-trivial elementary abelian 2-group.

1. The set of conjugacy classes of toral monomorphisms $\nu : E \to PGL(n+1, \mathbb{C})$ corresponds bijectively to the set
\[
E^\vee \setminus \{i \in \text{Rep}(E, GL(n+1, \mathbb{C})) \mid \text{Eq}(S(i)) = 0\}
\]
of $E^\vee$-orbits.

2. $A(PGL(n+1, \mathbb{C}))(E^\vee i) = \{A \in GL(E) \mid (E^\vee i) \cdot A^\vee = E^\vee i\}$.

3. $\pi_0(C_{PGL(n+1, \mathbb{C})}(E^\vee i)) = \{\zeta \in E^\vee \mid \zeta \cdot i = i\}$.

4. The set of isomorphism classes of $dim_{E^\vee}$-dimensional toral objects of $A(PGL(n+1, \mathbb{C}))$ corresponds bijectively to the set
\[
E^\vee \setminus \{i \in \text{Rep}(E, GL(n+1, \mathbb{C})) \mid \text{Eq}(S(i)) = 0\} / GL(E)
\]
of $E^\vee \times GL(E)$-orbits.

Proof. 1. Let $\nu : E \to PGL(n+1, \mathbb{C})$ be a toral monomorphism and $\mu : E \to GL(n+1, \mathbb{C})$ any lift of $\nu$ to GL$(n+1, \mathbb{C})$. The representation $\mu$ is a sum of linear characters
\[
\mu = \sum_{f \in E^\vee} \mu_i(f) f
\]
for some function $i_{\mu} \in \text{Rep}(E, GL(n+1, \mathbb{C}))$. The condition that $\mu(E)$ intersects the center $\mathbb{C}^\times$ trivially, translates to $\text{Eq}(S(i_{\mu})) = 0$ (or, equivalently, $S(i_{\mu})$ spans $V$ and $V = \bigcup_{f \in S(i_{\mu})} \ker f$).

Any other lift of $\nu$ has the form $\mu \cdot i$ for some $\zeta \in E^\vee$. We have $i_{\mu \cdot i} = \zeta \cdot i_{\mu}$ for
\[
(\zeta \mu)(v) = \sum_{f \in E^\vee} i_{\mu}(f) \zeta(v) f(v) = \sum_{i \in S(i_{\mu})} (i \cdot \tau_i)(f) f(v) = \sum_{i \in S(i_{\mu})} (i \cdot \mu_i)(f) f(v)
\]
for all $v \in V$. (Also, $S(\tau_i i_{\mu}) = \tau_i S(i_{\mu})$ so the equalizer subspace does not change.)

2. An automorphism $A \in GL(E)$ preserves the conjugacy class of $\nu : E \to PGL(n+1, \mathbb{C})$ if and only if $\mu A(\nu) = \zeta(\nu) \mu(v)$ for some $\zeta \in E^\vee$ (depending on $A$). Since
\[
(\mu A)(v) = \sum_{f \in E^\vee} i_{\mu}(f) \mu(A)(v) = \sum_{i \in S(i_{\mu})} (i \cdot \mu_i)(A)(f) f(v) = \sum_{i \in S(i_{\mu})} (i \cdot \mu_i)(f) f(v)
\]
for all $v \in V$, this means that $i_{\mu} \cdot A = \zeta \cdot i_{\mu}$. Then $(g \cdot i_{\mu})(A) = ((A^{-1})^\vee g) \cdot (i_{\mu} \cdot A) = ((A^{-1})^\vee g) \cdot (\zeta \cdot i_{\mu}) = ((A^{-1})^\vee g + \zeta) \cdot i_{\mu} \in E^\vee i_{\mu}$ for all $g \in E^\vee$.

3. The component group of $C_{PGL(n+1, \mathbb{C})}(E^\vee i_{\mu})$ is [29, 5.11.(2)] isomorphic to the group of $\zeta \in E^\vee$ for which $\mu$ and $\zeta \mu$ are conjugate in GL$(n+1, \mathbb{C})$. For the traces, this means that $i_{\mu} = \zeta \cdot i_{\mu}$. □
5.6. Remark. 1. Since \(i \tau_f \tau_c = i \tau_f \leftrightarrow i \tau_c = i\), the right hand side for the equation in 5.5.(3) remains the same for all elements of the orbit \(E^\vee i\).

2. Let \(A \in A(PGL(n+1, C))(E^\vee i)\) so that \(iA^\vee = i\tau_c\) for some \(\zeta \in E^\vee\). Then
\[
i\tau_A^\vee(g) = i \leftrightarrow i\tau_{A^\vee(g)}A^\vee = iA^\vee \leftrightarrow i\tau_c\tau_g = i\tau_c \leftrightarrow i\tau_g = i
\]
for any \(g \in E^\vee\), meaning that \(A^\vee(g) \in \pi_0(C_{PGL(n+1, C)}(E^\vee i)) \leftrightarrow g \in \pi_0(C_{PGL(n+1, C)}(E^\vee i))\). Thus \(A(PGL(n+1, C))(E^\vee i)\) acts on \(\pi_0(C_{PGL(n+1, C)}(E^\vee i))\).

5.7. Example. (Toral lines and planes in \(PGL(m, C)\)) Let \(P(m, k)\) be the number of ways to write \(m = i_0 + i_1 + \cdots + i_k\) as a sum of \(k\) integers \(i_0, i_1, \ldots, i_k\) such that \(1 \leq i_0 \leq i_1 \leq \cdots \leq i_k\). There are \(P(m, 2) = \lfloor m/2 \rfloor\) toral lines and \(P(m, 3) + P(m, 4)\) toral planes in \(PGL(m, C)\). The \(P(m, 2)\) toral lines of type \(i = (i_0, i_1)\) with \(i_0, i_1 > 0\) and \(i_0 + i_1 = m\) have these Quillen automorphism groups and centralizer component groups:
\[
(i_0, i_1): A(PGL(m, C))(L) = 1, \pi_0(C_{PGL(m, C)}(L)) = 1
\]
\[
(i_0, i_0): A(PGL(m, C))(L) = 1, \pi_0(C_{PGL(m, C)}(L)) = L^\vee
\]
The non-connected rank 1 centralizer is
\[
C_{PGL(m, C)}(L) = \frac{GL(i_0, C)^2}{GL(1, C)} \times L^\vee, \quad ZC_{PGL(m, C)}(L) \cong L
\]
The \(P(m, 3) + P(m, 4)\) toral planes of type \(i = (i_0, i_1, i_2, i_3)\) with \(i_0, i_1, i_2 > 0, i_3 \geq 0\), and \(i_0 + i_1 + i_2 + i_3 = m\) have these Quillen automorphism groups and centralizer component groups:
\[
(i_0, i_1, i_2, i_3): A(PGL(m, C))(V) = 1, \pi_0(C_{PGL(m, C)}(V)) = 1
\]
\[
(i_0, i_0, i_0, i_0): A(PGL(m, C))(V) = GL(V), \pi_0(C_{PGL(m, C)}(V)) = 1
\]
\[
(i_0, i_0, i_0, i_0): A(PGL(m, C))(V) = GL(V), \pi_0(C_{PGL(m, C)}(V)) = L^\vee
\]
If \(V = \mathbb{F}_2^4\) is a plane, then \(V^\vee\) and \(GL(V) \cong \Sigma_3\) together generate all permutations of the four letters \((i_0, i_1, i_2, i_3)\). Thus there are \(P(m, 3)\) isomorphism classes of the form \((i_0, i_1, i_2, 0)\), \((i_0, i_1, i_2, 0)\), \((i_0, i_1, i_2, 0)\), \((i_0, i_1, i_2, 0)\), and \((i_0, i_1, i_2, 0)\) isomorphism classes of the form \((i_0, i_1, i_2, 0)\) with \(i_0 \neq i_2\), then \(V^\vee = iGL(V)\) contains four elements, so \(\pi_0 \cong V^\vee\) and \(Aut = GL(V)\). In all cases, \(\pi_0C_{PGL(m, C)}(V, \nu) = \pi_0ZC_{PGL(m, C)}(V, \nu)\); this is clear in case \(\pi_0(C_{PGL(m, C)}(V)) = 1\) is trivial and in the remaining two cases it is a direct check.

The character table for \(V = C_2 \times C_2 = \{e_0, e_2, e_2, e_3 = e_1 + e_2\}\) contains four linear characters \(V^\vee = \{\rho_0, \rho_1, \rho_2, \rho_3\}\). In the list above, \((i_0, i_1, i_2, i_3)\) means \(i_0 \rho_0 + i_1 \rho_1 + i_2 \rho_2 + i_3 \rho_3\). Non-connected \(PGL(m, C)\)-centralizers only occur for induced \(GL(m, C)\)-representations:
\[
(i_0, i_0, i_2, i_2) = \begin{cases}
\text{ind}_{V^\vee}^{V}(i_0 \rho_0 + i_1 \rho_1) & i_0 \neq i_2 \\
\text{ind}_{\{0\}}^{V}(i_0 \rho_0) & i_0 = i_2
\end{cases}
\]
In the first case, the centralizer
\[
C_{PGL(m, C)}(V, \rho) = \frac{GL(i_0, C)^2 \times GL(i_2, C)^2}{GL(1, C)} \times L^\vee, \quad ZC_{PGL(m, C)}(V, \rho) \cong GL(1, C) \times GL(1, C) \times L,
\]
is LHS and has \(\pi_s(N)\)-determined automorphisms \((2.20)\). In the second case, we have a pure rank 2 object, the only rank 1 sub-object is \(2i_0\) times the regular representation of \(C_2\). Its centralizer
\[
C_{PGL(m, C)}(V, \rho) = \frac{GL(i_0, C)^4}{GL(1, C)} \times V^\vee, \quad ZC_{PGL(m, C)}(V, \rho) \cong V,
\]
is LHS but does not be LHS does not have \(\pi_s(N)\)-determined automorphisms \((2.20)\).
5.8. The non-toral subcategory of $A(PGL(n + 1, \mathbb{C}))$. For 2-compact group $X$, let $A(X)_{\neq t}$ denote the full subcategory of $A(X)$ on all non-toral objects and their sub-objects. We determine this non-toral subcategory $A(PGL(n + 1, \mathbb{C}))_{\neq t}$ in case $X = PGL(n + 1, \mathbb{C})$.

For any non-trivial elementary abelian 2-group $V$ in $PGL(n + 1, \mathbb{C})$, let $[\cdot, \cdot] : V \times V \to F_2$ be the symplectic bilinear form [16, II.9.1] given by $[u, v] = [u, v]$ for all $u \in GL^\times, vz \in V$. (The elements $[u, v]$ and $uz$ lie in the center $C^\times$ of $GL(n + 1, \mathbb{C})$ so that $E = [u, v] = [u, v][u, v] = [u, v]^2$ and thus $[u, v] \in C^\times$ has order 2. Therefore $[u, v] = [u, v]^{-1} = [u, v]$.

5.9. Lemma. $V$ in $PGL(n + 1, \mathbb{C})$ is toral $\iff [V, V] = 0$

Proof. Let $e_i, C^x, 1 \leq i \leq d$, be a basis for $V$. Since $C^x$ is divisible, we can assume that each $e_i \in GL(n + 1, \mathbb{C})$ has order 2. If $[V, V] = 0$, these $e_i$s commute and span a lift to $GL(n + 1, \mathbb{C})$ of $V \subseteq PGL(n + 1, \mathbb{C})$.

An extra special 2-group is of positive type if it is isomorphic to a central product of dihedral groups $D_8$ of order 8.

5.10. Lemma. [12, 3.1] [29, 5.4] Let $\nu : V \to PGL(n, \mathbb{C})$ be a non-toral monomorphism of a non-trivial elementary abelian 2-group $V$ into $PGL(n + 1, \mathbb{C})$. Then there exists a morphism of short exact sequences of groups

$$
1 \longrightarrow Z(P) \longrightarrow PE \longrightarrow V \longrightarrow 1
$$

$$
1 \longrightarrow C^x \longrightarrow GL(n + 1, \mathbb{C}) \longrightarrow PGL(n + 1, \mathbb{C}) \longrightarrow 1
$$

where $PE$ is the direct product of an extra special 2-group $P \subseteq GL(n + 1, \mathbb{C})$ of positive type and an elementary abelian 2-group $E \subseteq GL(n + 1, \mathbb{C})$ with $P \cap E = \{1\} = [P, E]$.

Write $C^{n+1} = C^d \otimes C^m$ for some $d > 0$ and some $m \geq 0$. Let the extra-special 2-group $2^{1+2d}$ act faithfully on the first factor of the tensor product and let the (possibly trivial) elementary abelian 2-group $E$ act faithfully on the second factor such that no non-trivial element of $E$ acts as scalar multiplication. This makes $C^{n+1}$ a $C[2^{1+2d} \times E]$-module. The image of the group $2^{1+2d} \times E \subseteq GL(n + 1, \mathbb{C})$ in $PGL(n + 1, \mathbb{C})$ is a non-toral elementary abelian 2-group (5.9) and any non-toral elementary abelian 2-group in $PGL(n + 1, \mathbb{C})$ has this form (5.10).

Let $G = \langle P, E, a \rangle = P \circ C_4 \times E$ be the group generated by $E$ and the central product $P \circ C_4$ of $P$ and the cyclic group $C_4 = \langle a \rangle \subseteq C^x$ with $Z/2$ amalgamated. The image of $G$ in $PGL(n + 1, \mathbb{C})$ is $V$ and $q(vC^x) = v^2, v \in G$, is a quadratic form on $V$ such that $q(uC^x + vC^x) = q(uC^x) + q(vC^x) = [uC^x, vC^x]$ for all $uC^x, vC^x \in V$.

5.11. Lemma. $A(GL(n + 1, \mathbb{C}))(G, G) \to A(PGL(n + 1, \mathbb{C}))(V, V)$ is surjective.

Proof. Suppose that $B \in GL(n + 1, \mathbb{C})$ is such that $V^{BC^x} = V$. Then $G^B \subseteq G \cdot C^x$: For any $g \in G$ there exist $h \in G$ and $z \in C^x$ such that $g^B = hz$. But since $G$ has exponent 4, $z^4 = 1$ so $z \in C_4$ and $g^B \in G$. □

A monomorphic conjugacy class $\nu : V \to PGL(n + 1, \mathbb{C})$ is said to be a $(2d + r, r)$ object of $A(PGL(n + 1, \mathbb{C}))$ if the underlying symplectic vector space of $(V, \nu)$ is isomorphic to $V = H^d \times V^\perp$ where $H$ denotes the symplectic plane over $F_2$ and $\dim F_2 V^\perp = r$ [16, II.9.6] (so that $\dim F_2 V = r + 2d$). An $(r, r)$ object is the same thing as an $r$-dimensional toral object. We write $Sp(V)$ or $Sp(2d + r, r)$ (abbreviated to Sp(2d) if $r = 0$) for the group of linear automorphisms of $V$ that preserve the symplectic form.

5.12. Corollary. Suppose that $n + 1 = 2^d m$ for some natural numbers $d \geq 1$ and $m \geq 1$.

(1) There is up to isomorphism a unique $(2d, 0)$ object $H^d$ of $A(PGL(n + 1, \mathbb{C}))$, and

$$
A(PGL(n + 1, \mathbb{C}))(H^d) = Sp(2d), \quad C_{PGL(n+1,\mathbb{C})}(H^d) = H^d \times PGL(m, \mathbb{C})
$$

for this object.
(2) Isomorphism classes of \((2d + r, r), r > 0\), objects \(V\) of \(A(PGL(2^d m, C))\) correspond bijectively to isomorphism classes of \((r, r)\) objects \(V^\perp\) of \(A(PGL(m, C))\), and
\[
A(PGL(2^d m, C))(V) = \begin{pmatrix} \text{Sp}(2d) & 0 \\ \ast & A(PGL(m, C))(V^\perp) \end{pmatrix}
\]

\[
C_{PGL(2^d m, C)}(V) = V/V^\perp \times C_{PGL(m, C)}(V^\perp)
\]

for these objects.

Proof. 1. The group \(2_+^{1+2d} \circ 4\) has \([17, 7.5]\) \(2^{1+2d}\) characters of degree 1 and 2 irreducible characters of degree \(2^d\) (interchanged by the action of \(Out(2_+^{1+2d} \circ 4) \cong \text{Sp}(2d) \times \text{Aut}(C_4)\) [11, pp. 403–404]) given by
\[
\chi_\lambda(g) = \begin{cases} 2^d \lambda(g) & g \in C_4 \\ 0 & g \notin C_4 \end{cases}
\]
where \(\lambda : C_4 \to C^\times\) is an injective group homomorphism \((\lambda(i) = \pm i)\). The linear characters vanish on the derived group \(2 = [2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4]\) but the irreducible characters of degree \(2^d\) do not. Thus the only faithful representations of \(2_+^{1+2d} \circ 4\) with central centers are multiples \(m\lambda\) of \(\chi_\lambda\) for a fixed \(\lambda\). Phrased slightly differently, \(\text{GL}(m2^d, C)\) contains up to conjugacy a unique subgroup with central center isomorphic to \(2_+^{1+2d} \circ 4\). For this group and its image \(H^d\) in \(PGL(2^d m, C)\) we have
\[
A(\text{GL}(m2^d, C))(2_+^{1+2d} \circ 4, 2_+^{1+2d} \circ 4) \cong \text{Sp}(2d) \cong A(PGL(m2^d, C))(H^d, H^d)
\]
\[
C_{\text{GL}(m2^d, C)}(2_+^{1+2d} \circ 4) \cong \text{GL}(m, C), \quad C_{PGL(m2^d, C)}(H^d) \cong H^d \times PGL(m, C)
\]
where the last isomorphism is a consequence of \([29, 5.9]\).

2. The \((2d + r, r)\) object \((V, \nu)\) of \(A(PGL(2^d m, C))\) and the \((r, 0)\) object \((V^\perp, \nu^\perp)\) of \(A(PGL(m, C))\) correspond to each other iff there is an \(m\)-dimensional representation \(\mu : V^\perp \to \text{GL}(m, C)\) such that \(C^{2^d} \odot \mu\) is a lift of \(\nu|V^\perp\) and \(\mu\) a lift of \(\nu^\perp\). According to 5.10 any lift of \(\nu|V^\perp\) has this form for some \(\mu\) uniquely determined up to the action of \((V^\perp)^\vee\).

We use 5.11 to calculate the Quillen automorphism group of a \((2d + r, r)\) object \(H^d \times V^\perp\) of \(A(PGL(2^d m, C))\). Let \(H^d \times V^\perp\) be covered by the group \(P \circ C_4 \times V^\perp\) as in 5.10. Let \(\alpha\) be an automorphism of \(P \circ C_4\), let \(\beta\) be any homomorphism of the form \(P \circ C_4 \to H^d \to V^\perp\), and let \(\gamma\) be any Quillen automorphism of \((V^\perp, \nu^\perp)\). Choose a homomorphism \(\zeta : P \circ C_4 \to H^d \times C_4/C_2 \to C_4\) such that \(\lambda(\zeta(x)\alpha(x)) = \lambda(x)\) for all \(x \in C_4\) and a homomorphism \(\zeta : V^\perp \to C_4\) such that \(\lambda(\zeta(v))\mu(\gamma(v)) = \mu(v)\) for all \(v \in V^\perp\). Then the automorphism of \(P \circ C_4\) that takes \((x, v)\) to \((\zeta(x)\zeta(x)v)(\alpha(x), \beta(x) + \gamma(v))\) preserves the trace of \(\chi^\perp \mu\) and therefore the automorphism induced on the quotient is a Quillen automorphism of \(H^d \times V^\perp\). Conversely, any automorphism \(P \circ C_4 \times V^\perp\) takes the center \(C_4 \times V^\perp\) isomorphically to itself and hence it is of the form \((x, v) \to (\zeta(x, v)\alpha(x), \beta(x) + \gamma(v))\) for some automorphism \(\alpha\) of \(P \circ C_4\), some homomorphism \(\beta : P \circ C_4 \to V^\perp\) vanishing on \(C_4\), and some homomorphism \(\zeta : P \circ C_4 \to V^\perp\). Such an automorphism preserves the trace of \(\chi^\perp \mu\) iff \(\lambda(\zeta(x, v)\alpha(x)) = \mu(\gamma(v))\) for all \((x, v) \in Z(P \circ C_4 \times V^\perp) = C_4 \times V^\perp\). But this means that the induced automorphism of \(H^d \times V^\perp\) is of the stated form. □

5.13. Example. (Oliver’s cochain complex [32]) The non-toral objects of \(A(PGL(2m, C))\) of rank \(\leq 4\) are

- One \((2, 0)\) object \(H, A(PGL(2m, C))(H) = \text{Sp}(2) = \text{GL}(2, F_2), \pi_0 = H\).
- \(P(m, 2)\) \((3, 1)\) objects \(V, A(PGL(2m, C))(V) = \text{Sp}(3, 1), \pi_0 = V/V^\perp\) or \(V\).
- \(P(m, 3) + P(m, 4)\) \((4, 2)\) objects \(E, A(PGL(2m, C))(E) = \begin{pmatrix} \text{Sp}(2) & 0 \\ \ast & A(PGL(m, C))(E^\perp) \end{pmatrix}, \)
  \(A(PGL(m, C))(E^\perp) = 1, C_2, \text{GL}(E), \pi_0 = E/E^\perp, E/E^\perp\) or \(E/L, E/E^\perp\) or \(E\).
- One \((4, 0)\) object if \(m\) is even.

The \((2, 0)\) object \(H\) contributes
\[
\text{Hom}_{\text{Sp}(2)}(\text{St}(H), H) \cong F_2
\]
The \((3, 1)\) objects \(V\) contribute
\[
\text{Hom}_{\text{Sp}(3,1)}(\text{St}(V), V) \cong \text{Hom}_{\text{Sp}(3,1)}(\text{St}(V), V/V^\perp) \cong F_2
\]
The $(4, 2)$ objects $E$ with $A(PGL(m, C))(E^\perp) = 1$ contribute
\[ \text{Hom} \left( \left( \begin{array}{cc} \text{Sp}(2) & 0 \\ * & 1 \end{array} \right), (\text{St}(E), E/E^\perp) \right) \cong \mathbb{F}_2 \]
and the $(4, 2)$ objects $E$ with $A(PGL(m, C))(E^\perp) = C_2$ contribute
\[ \text{Hom} \left( \left( \begin{array}{cc} \text{Sp}(2) & 0 \\ * & C_2 \end{array} \right), (\text{St}(E), E/L) \right) \cong \text{Hom} \left( \left( \begin{array}{cc} \text{Sp}(2) & 0 \\ * & C_2 \end{array} \right), (\text{St}(E), E/E^\perp) \right) \cong \mathbb{F}_2 \]

The $(4, 0)$ object (if it exists) and the $(4, 2)$ objects with $A(PGL(m, C))(E^\perp) = \text{GL}(E)$ do not contribute to the cochain complex for the corresponding Hom-groups are trivial. Thus the cochain complex for computing higher limits of the functor $\pi_1(BZC_{PGL(2m, C)})$ will have the form
\[ \delta^1 \to \prod_{[m/2]} \text{Hom}_{Sp(2)}(\text{St}(H), H) \to \prod_{[3, 1]} \text{Hom}_{Sp(2)}(\text{St}(V), V/V^\perp) \delta^2 \]
\[ \prod \text{Hom} \left( \left( \begin{array}{cc} \text{Sp}(2) & 0 \\ * & 1 \end{array} \right), (\text{St}(E), E/E^\perp) \times \prod \text{Hom} \left( \left( \begin{array}{cc} \text{Sp}(2) & 0 \\ * & C_2 \end{array} \right), (\text{St}(E), E/E^\perp) \right) \to \cdots \right) \]

To show vanishing of the relevant higher limits it suffices to show that $\delta^1$ is injective and that the rank of $\delta^2$ is $P(m, 2) - 1$.


By inductively applying 3.3 and 4.6 we show that the 2-compact groups $PGL(n + 1, C)$, $n \geq 1$, are uniquely $N$-determined.

6.1. Lemma. Suppose that $n + 1 = 2m \geq 2$ is even.

(1) There is a unique monomorphism conjugacy class $\lambda: \mathbb{Z}/2 \to PGL(n + 1, C)$ with disconnected centralizer. The centralizer of this monomorphism is $\text{GL}(m, C)^2/C^\times \times \mathbb{Z}/2$.

(2) There is a unique monomorphism conjugacy class $\nu: H \to PGL(n + 1, C)$, $H = (\mathbb{Z}/2)^2$, such that $\nu$ is non-toral. The centralizer of this monomorphism is $H \times PGL(m, C)$ and the Quillen automorphism group is $\text{GL}(H)$.

Proof. Use that any monomorphism of $\mathbb{Z}/2$ into $PGL(n + 1, C)$ lifts to $\mu: \mathbb{Z}/2 \to \text{GL}(n + 1, C)$. The only possibility is that $\mu = m \cdot \text{reg}$ is a direct sum of regular representations. The result for non-toral rank 2 objects in $A(PGL(n + 1, C))$ is a special case of 5.10.

6.2. Lemma. Suppose that $PGL(r + 1, C)$ is uniquely $N$-determined for all $0 \leq r < n$. Then $PGL(n + 1, C)$, $n \geq 1$, satisfies conditions 4.6.(1), 4.6.(2), and 4.6.(3).

Proof. We shall verify 4.6.(1) and 4.6.(2) by establishing the alternative two conditions from 4.8.

Let $(V, \nu)$ be a toral elementary abelian 2-subgroup of $PGL(n + 1, C)$ of rank $\leq 2$ and $C(\nu) = C_{PGL(n+1, C)}(\nu)$ its centralizer. We have seen that $C(\nu)$ is LHS (2.20) and that $\bar{Z}(C(\nu)_0) = \bar{Z}(N_0(C(\nu)))$ as $C(\nu)_0$ does not contain a direct factor isomorphic to $\text{GL}(2, C)/\text{GL}(1, C) = \text{SO}(3)$ (2.24, 5.7). The identity component $C(\nu)_0$ has $\pi_0(N)$-determined automorphisms according to 3.2 and 3.4, and $C(\nu)$ has $N$-determined automorphisms by 3.1. The identity component $C(\nu)_0$ is $N$-determined according to 4.3 and 4.4, and $C(\nu)$ is $N$-determined by 4.1. Thus $C(\nu)$ is LHS and totally $N$-determined.

The functor $H^1(W/W_0; T_0^W)$ is zero on $A(PGL(n + 1, C))^{\leq t}_{\leq 2}$ except on the object $(V, \nu) = (i_0, i_0, i_0, i_0)$, when $n + 1 = 4i_0$, where it has value $\mathbb{Z}/2$. However, this object has Quillen automorphism group $\text{GL}(V)$ and since the only $\text{GL}(V)$-equivariant homomorphism $\text{St}(V) = V \to \mathbb{Z}/2$ is the trivial homomorphism, $\lim^1(A(PGL(n + 1, C))^{\leq t}_{\leq 2}; H^1(W/W_0; T_0^W)) = 0$ follows from Oliver’s cochain complex [32].

When $n + 1 = 2m$ is even, we verify condition 4.6.(3) by applying 4.11. Let $X'$ be a connected 2-compact group with maximal torus normalizer $j': N(PGL(n + 1, C)) \to X$. Since the first item in 4.11 is satisfied by 4.12 and 6.1, it suffices to show that the isomorphism (from 4.6.(3))
\[ f_{\nu, L}: C_{PGL(2m, C)}(H) = H \times PGL(m, C) \to C_{X'}(H, \nu') \]
defined by choosing one of the three lines \( L \) in \( H \), is \( C_3 \)-equivariant. Now [24]

\[
\text{Aut}(H \times \text{PGL}(m, \mathbb{C})) = \text{GL}(H) \times \text{Aut}(\text{PGL}(m, \mathbb{C}))
\]

so that \( f_{\nu,L} \) is \( C_3 \)-equivariant if \( \pi_0(f_{\nu,L}) \) and the restriction of \( f_{\nu,L} \) to the identity components are \( C_3 \)-equivariant. Here, \( \text{Aut}(\text{PGL}(m, \mathbb{C})) = \mathbb{Z}_2^m \) (or \( \mathbb{Z}_2^m/\{\pm 1\} \) if \( m = 2 \)) since \( \text{PGL}(m, \mathbb{C}) \) has \( \pi_i(N) \)-determined automorphisms by induction hypothesis so \( C_7 \) must act trivially on the identity components for purely group theoretic reasons. The commutative triangle (4.14)

\[
\begin{array}{ccc}
\pi_0(H) & \cong & \pi_0(C_{\text{PGL}(2m, \mathbb{C})}(H, \nu)) \\
\pi_0(f_{\nu,L}) & \cong & \pi_0(C_\nu'(H, \nu'))
\end{array}
\]

in which the slanted arrows, representing the canonical factorizations, are \( C_3 \)-equivariant (even \( \text{GL}(H) \)-equivariant) shows that \( \pi_0(f_{\nu,L}) \) is \( C_3 \)-equivariant.

We shall next compute the higher limits from 3.3.(2) and 4.6.(4) by means of 5.4 and the cochain complex 5.14 from [32]. As 5.4 is not valid for \( \text{PGL}(2, \mathbb{C}) \) we first consider this case separately.

6.3. Proposition. The 2-compact group \( \text{PGL}(2, \mathbb{C}) \) is uniquely \( N \)-determined.

Proof. The functor \( C_{\text{PGL}(2, \mathbb{C})} \) takes the Quillen category of \( \text{PGL}(2, \mathbb{C}) \), consisting (5.7, 5.13, 6.1) of one toral line, \( L \), and one non-toral plane, \( H \),

\[
(6.4) \quad L \rightarrow H \bigcirc_{\text{GL}(H)}
\]

to the diagram

\[
(6.5) \quad \text{GL}(1, \mathbb{C})^2/\text{GL}(1, \mathbb{C}) \times C_2 \bigcirc_{\text{GL}(H)^{op}} H
\]

of uniquely \( N \)-determined 2-compact groups. The 2-compact toral group to the left is uniquely \( N \)-determined (4.2) because \( H^1(C_3; \mathbb{Z}/2^\infty) = 0 \) for the non-trivial action. The center functor takes this diagram back to the starting point (6.4) for which the higher limits vanish [29, 12.7.4]. \( \text{PGL}(2, \mathbb{C}) \) is thus uniquely \( N \)-determined by 3.3 and 4.6. \( \square \)

6.5. Lemma. The low degree higher limits of the functors \( \pi_j(B\text{ZC}_{\text{PGL}(n+1, \mathbb{C})}) \), \( j = 1, 2 \), are:

(1) \( \lim^i(A(\text{PGL}(n + 1, \mathbb{C})), \pi_1(B\text{ZC}_{\text{PGL}(n+1, \mathbb{C})})) = 0 \) for \( i = 1, 2 \),

(2) \( \lim^i(A(\text{PGL}(n + 1, \mathbb{C})), \pi_2(B\text{ZC}_{\text{PGL}(n+1, \mathbb{C})})) = 0 \) for \( i = 2, 3 \),

for all \( n \geq 1 \).

Let \( V = \mathbf{F}_2e_1 + \mathbf{F}_2e_2 + \mathbf{F}_2e_3 \) be a 3-dimensional vector space over \( \mathbf{F}_2 \) with basis \( \{e_1, e_2, e_3\} \) and (degenerate) symplectic inner product matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Let \( \mathbf{F}_2[1] \) be the 21-dimensional \( \mathbf{F}_2 \)-vector space on all length one flags \( [P > L] \) and \( \mathbf{F}_2[0] \) the 14-dimensional \( \mathbf{F}_2 \)-vector space on all length zero flags, \([P]\) or \([L]\), of non-trivial and proper subspaces of \( V \). The Steinberg module \( \text{St}(V) \) over \( \mathbf{F}_2 \) for \( V \) is the \( 2^{3} = 8 \)-dimensional kernel of the linear map \( d: \mathbf{F}_2[1] \to \mathbf{F}_2[0] \) given by \( d[P > L] = [P] + [L] \). Define

\[
f_1 = \overline{f_1}|\text{St}(V): \text{St}(V) \to V
\]

as the restriction to \( \text{St}(V) \) of the linear map \( \overline{f_1}: \mathbf{F}_2[1] \to V \) with values

\[
\overline{f_1}[P > L] = \begin{cases}
L & P \cap P^\perp = 0 \\
0 & \text{otherwise}
\end{cases}
\]

on the basis vectors.

Let \( E = \mathbf{F}_2e_1 + \mathbf{F}_2e_2 + \mathbf{F}_2e_3 + \mathbf{F}_2e_4 \) be a 4-dimensional vector space over \( \mathbf{F}_2 \) with basis \( \{e_1, e_2, e_3, e_4\} \) and (degenerate) symplectic inner product matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Let $F_2[2]$ be the 315-dimensional $F_2$-vector space on all length two flags $[V > P > L]$ and $F_2[1]$ the also 315-dimensional $F_2$-vector space on all length one flags, $[P > L]$ or $[V > L]$ or $[V > P]$, of non-trivial, proper subspaces of $E$. The Steinberg module $\text{St}(E)$ over $F_2$ for $E$ is the $2^6 = 64$-dimensional kernel of the linear map $d: F_2[2] \to F_2[1]$ given by $d[V > P > L] = [P > L] + [V > L] + [V > P]$. Define $F_1 = \text{St}(E): \text{St}(E) \to E$ as the restriction to $\text{St}(E)$ of the linear map $F_1: F_2[2] \to E$ with values

$$
(6.6) \quad F_1[V > P > L] = \begin{cases}
L & P \cap P^\perp = 0, V \cap V^\perp = F_2 e_3 \\
0 & \text{otherwise}
\end{cases}
$$

on the basis elements. Define $F_2 = \text{St}(E): \text{St}(E) \to E$ similarly but replace the condition $V \cap V^\perp = F_2 e_3$ by $V \cap V^\perp = F_2 e_4$. The linear maps $F_1$ and $F_2$ are $(\text{Sp}(2) \setminus 1)$-equivariant because this group preserves the symplectic inner product on $E$ and preserves $V^\perp = F_2 \langle e_3, e_4 \rangle$ pointwise.

6.7. Lemma. Let $f_1$ and $F_1, F_2$ be the linear maps defined above.

1. The vector $f_1$ is a basis vector for

$$
\text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(E) \right) \cong \text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(E), V/V^\perp \right) \cong F_2
$$

2. The set $\{F_1, F_2\}$ is a basis for

$$
\text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(E) \right) \cong \text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(E), E/E^\perp \right) \cong F_2^2
$$

The sum $F_1 + F_2$ is the linear map defined as in (6.6) but with condition $V \cap V^\perp = F_2 e_3$ replaced by $V \cap V^\perp = F_2 (e_3 + e_4)$.

Proof. This can be directly verified by machine computation.

6.8. Proposition. The differentials in the cochain complex 5.14 are given as follows:

1. Let $H$ be the $(2, 0)$ object and $V$ a $(3, 1)$ object of $\text{A}(\text{PGL}(2m, C))$. The $V$-component of the coboundary map

$$
\delta_1^V: \text{Hom}_{\text{Sp}(2)}(\text{St}(H), H) \to \text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(V), V \right)
$$

is an isomorphism of 1-dimensional $F_2$-vector spaces.

2. Let $V$ be the $(4, 2)$ object of $\text{A}(\text{PGL}(2m, C))$ corresponding (5.12, 5.7) to the two dimensional toral object $(1, i - 1, m - i, 0)$ of $\text{A}(\text{PGL}(m, C))$, $1 < i \leq m/2$, $m \geq 4$. Then

$$
\delta_2^V(\xi_i) = (x_i + x_i) F_1 + (x_i + x_{i-1}) F_2
$$

where $i = 1, \ldots, m/2$ and

$$
\delta_2^V: \text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(V), \text{St}(V) \right) \to \text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix}, \text{St}(E), E \right)
$$

is the $E$-component of the coboundary map and $(x_i) \in \prod_{1 \leq i \leq m/2} \text{Hom}_{\text{Sp}(3, 1)}(\text{St}(V), V)$.

Proof. 1. The non-zero vector in $\text{Hom}_{\text{Sp}(2)}(\text{St}(H), H)$ is the restriction to $\text{St}(H) \subseteq F_2^2$ of the linear map $F_2[0] = F_2^2 \to H$ that takes a basis vector $[L]$ in $F_2^2$ to $L \in H$. In the composition

$$
\text{St}(V) \to \bigoplus_{V > P} \text{St}(P) \to \bigoplus_{V > P} P^\perp \to V
$$

the middle maps $\text{St}(P) \to P$ equal the map just described if $P < V$ is non-toral, $P \cap P^\perp = 0$, and are trivial if $P < V$ is toral, $P \cap P^\perp = P$. This is precisely the map $f_1$.

2. For any non-toral three dimensional subspace $V$ of $E$ we have either

- $V \cap V^\perp = F_2 e_3$, and then $V = V_i$, or,
- $V \cap V^\perp = F_2 e_4$, and then $V = V_{i-1}$, or,
- $V \cap V^\perp = F_2 (e_3 + e_4)$, and then $V = V_1$, or.
and thus the composite linear map

$$\text{St}(E) \to \bigoplus_{E \in V} \text{St}(V) \oplus \bigoplus_{\oplus V \to E} E$$

equals x_1F_1 + x_{i-1}F_2 + x_1(F_1 + F_2) = (x_1 + x_i)F_1 + (x_{i-1} + x_1)F_2.

Proof of Lemma 6.5. Since we already know that these higher limits vanish when \(n+1\) is odd (5.4) we can assume that \(n+1 = 2m\) is even.

1. In Oliver’s cochain complex 5.14, the coboundary map \(\delta^1\) is injective and \(\ker \delta^2\) is 1-dimensional by 6.8 when \(m \geq 4\). See 6.3 for the case \(m = 1\). For \(m = 2\) and \(m = 3\), the cochain complexes 5.14 reduce to

$$0 \to \text{Hom}_{\text{Sp}(2)}(\text{St}(H), H) \xrightarrow{\delta^1} \text{Hom}_{\text{Sp}(3,1)}(\text{St}(V), V/V^\perp) \to 0$$

with two non-trivial groups, both 1-dimensional \(\text{F}_2\)-vector spaces, and with just one differential \(\delta^1\) which is an isomorphism (6.8). Thus the higher limits vanish in these cases as well.

2. Oliver’s cochain complex for computing these higher limits over \(A(\text{PGL}(2m, \text{C}))\) involves the \(\text{Z}_2\)-modules

$$\text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & A(\text{PGL}(m, \text{C}))(E^\perp) \end{pmatrix} \right)(\text{St}(E), \pi_2(\text{BZC}_{\text{PGL}(2m, \text{C})}(E)))$$

that are submodules of finite products of \(\text{Z}_2\)-modules of the form

$$\text{Hom} \left( \begin{pmatrix} \text{Sp}(2) & 0 \\ * & 1 \end{pmatrix} \right)(\text{St}(E), \text{Z}_2), \quad \dim_{\text{F}_2} E = 3, 4,$$

where the action on \(\text{Z}_2\) is trivial. According to the computer program magma, these latter modules are trivial. \(\square\)

Proof of Theorem 1.1. By induction over \(n\) using 3.3 and 4.6. The start of the induction is provided by 6.3. Use (2.7) to compute the automorphism group. \(\square\)

Proof of Corollary 1.2. The connected 2-compact group \(\text{GL}(n, \text{C})\) is uniquely \(N\)-determined because (3.2, 4.3) its adjoint form \(\text{PGL}(n, \text{C})\) is (1.1). Since the maximal torus normalizer for \(\text{GL}(n, \text{C})\) is a split extension, we get (2.7) that \(\text{Aut}(\text{GL}(n, \text{C}))\) is isomorphic to \(\text{Z}(\text{S}_n) \setminus \text{Aut}_{\text{Z}_2} \text{S}_n(\text{Z}_2^\alpha)\). \(\square\)

This finishes the discussion of the 2-compact groups in the \(A\)-family. The relevance of these are that they occur as centralizers of elementary abelian subgroups of many other 2-compact groups. Here is a result illustrating this.

6.9. Theorem. [34, 1.3] The simple 2-compact group \(G_2\) is uniquely \(N\)-determined and its automorphism group \(\text{Aut}(G_2)\) equals \(\text{Z}^\times \times \text{Z}_2^\times \times C_2\).

Proof. The Quillen category \(A(G_2)\) is equivalent to the category \(A(\text{GL}(V), V)\) of all non-trivial subspaces of \(V = \text{F}_2^8\) [12, 6.1] [10, 1.6] [9, 5.3] and the value of centralizer functor \(BC_{G_2}\) on the three isomorphism classes of objects \(L, P, V\) is \(\text{SL}(4, \text{R}), T \times \text{Z}/2, V\). The rank one centralizer, \(\text{SL}(4, \text{R}) = \text{SL}(2, \text{C}) \times \text{SL}(2, \text{C})\), is uniquely \(N\)-determined (6.3, 3.2, 3.4, 4.3, 4.4). Condition 4.6.2 is satisfied because \(H^1(W(X); \hat{T}(X)) = 0\) for \(X = G_2, \text{SL}(4, \text{R})\) [13], 4.6.1 and 4.6.3 because the only rank two object in \(G_2\) is toral and its centralizer is a 2-compact toral group. The functor \(\pi_1(BC_{G_2})\) is the identity functor and \(\pi_2(BC_{G_2})\) the zero functor so the obstruction groups vanish. Now 3.3 and 4.6 show that \(G_2\) is uniquely \(N\)-determined. The short exact sequence (2.7) can be used to calculate the automorphism group. We have \(\text{Aut}(G_2) = W(G_2) \setminus N_{\text{GL}(2, \text{Z}_2)}(W(G_2))\) as the extension class \(e(G_2) = 0\) [3]. Using the description of the root system from [2, VI.4.13]
with short root \( \alpha_1 = \varepsilon_1 - \varepsilon_2 \) and long root \( \alpha_2 = 2\varepsilon_2 - \varepsilon_3 \) generating the integral lattice in \( \mathbb{Z}_2^3 \) one finds that

\[
N_{GL(2, \mathbb{Z}_2)}(W(G_2)) = \langle \mathbb{Z}_2^x, A, W(G_2) \rangle, \quad A = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}
\]

and therefore \( Aut(G_2) = \mathbb{Z}_2^x / \mathbb{Z}_2^x \times C_2 \) where the cyclic group of order two is generated by the exotic automorphism \( A \) interchanging the two roots. \( \square \)

7. Miscellaneous

This section contains auxiliary results that are used at various places in the main argument of this paper.

7.1. The 2-compact toral groups \( O(2) \) and \( Pin(2) \). Let \( H = \{ a + bj | a, b \in \mathbb{C} \} \), where \( j^2 = -1 \) and \( ja = -aj \) for \( a \in \mathbb{C} \), be the quaternion algebra. The normalizer of \( \mathbb{C}^x \) in \( H^x \) is the Lie group \( N_{H^x}(\mathbb{C}^x) = \langle \mathbb{C}^x, j \rangle \) generated by the multiplicative Lie group \( \mathbb{C}^x \) and \( j \). The short exact sequence

\[
1 \rightarrow \mathbb{C}^x \rightarrow N_{H^x}(\mathbb{C}^x) \rightarrow \langle j \rangle / \langle -1 \rangle \rightarrow 1
\]
does not split for all elements of \( j\mathbb{C}^x \) have order 4. Its discrete approximation \( Pin(2) = \hat{N}_{H^x}(\mathbb{C}^x) = \langle \mathbb{Z}/2^\infty, j \rangle \subseteq \langle \mathbb{C}^x, j \rangle \subseteq H^x \), the non-split extension

\[
1 \rightarrow \mathbb{Z}/2^\infty \rightarrow \hat{N}_{H^x}(\mathbb{C}^x) \rightarrow \mathbb{Z}/2 \rightarrow 1
\]
of \( \mathbb{Z}/2 \) by \( \mathbb{Z}/2^\infty \), is the discrete approximation to 2-compact toral group \( Pin(2) \). The semi-direct product \( \hat{O}(2) = \mathbb{Z}/2^\infty \rtimes \mathbb{Z}/2 \) is the discrete approximation to the 2-compact toral group \( O(2) \) or to \( GL(2, \mathbb{R}) \).

7.2. Type \( A_n, n \geq 1 \). (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group \( PGL(n + 1, \mathbb{C}) = GL(n + 1, \mathbb{C}) / \mathbb{C}^x \) is the extended 2-discrete toral group

\[
\hat{N}(PGL(n + 1, \mathbb{C})) = \hat{U}(1)^{n+1} / \hat{U}(1) \times \Sigma_{n+1} = \hat{T} \times \Sigma_{n+1}
\]

where \( \hat{U}(1) = \mathbb{Z}/2^\infty \) is a discrete 2-torus of rank 1. In the coefficient sequence for \( \hat{U}(1) \rightarrow \hat{U}(1)^{n+1} \rightarrow \hat{T} \) we have \( H^\ast(\Sigma_{n+1}; \hat{U}(1)^{n+1}) \cong H^\ast(\Sigma_{n}; \hat{U}(1)) \) by Shapiro so that

\[
H^i(W; \hat{T}) \cong \ker(H^{i+1}(\Sigma_{n+1}; \mathbb{Z}/2^\infty) \rightarrow H^{i+1}(\Sigma_{n}; \mathbb{Z}/2^\infty))
\]

is trivial for \( n + 1 > 2(i + 1) \) by [30, 5.8, 6.7]. For small values of \( i \) we have

\[
H^0(W; \hat{T}) = \begin{cases} 0 & n \neq 1 \\ \mathbb{Z}/2 & n = 1 \end{cases} \quad \text{and} \quad H^1(W; \hat{T}) = \begin{cases} 0 & n \neq 3 \\ \mathbb{Z}/2 & n = 3 \end{cases}
\]
as can be seen by using that the Schur multiplier \( H_2(\Sigma_{n}; \mathbb{Z}) \) is of order 2 for \( n \geq 4 \) and trivial for \( 1 \leq n \leq 3 \) [16, V.25.12]. Thus the center \( ZN(PGL(n + 1, \mathbb{C})) \) of the maximal torus normalizer is trivial for \( n > 1 \) but cyclic of order 2 for \( n = 1 \). For \( n = 3 \), the crossed homomorphism \( \Sigma_4 \rightarrow \hat{U}(1)^4 / \hat{U}(1) \) whose values on the three generators \( (12), (23), (34) \in \Sigma_4 \) [16, I.19.7] are the columns of the matrix

\[
\begin{pmatrix} -1 & +1 & +1 \\ -1 & -1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}
\]
is not principal.

7.3. Type \( B_n, n \geq 2 \). (Cf. [20, 19, 13]) The discrete maximal torus normalizer for the center-less 2-compact group \( SL(2n + 1, \mathbb{R}) \) is the extended 2-discrete torus

\[
\hat{N}(SL(2n + 1, \mathbb{R})) = \hat{O}(2) \rtimes \Sigma_n = (\mathbb{Z}/2^\infty \times \mathbb{Z}/2) \rtimes \Sigma_n = (\mathbb{Z}/2^\infty)^n \times (\mathbb{Z}/2 \rtimes \Sigma_n)
\]

where \( \mathbb{Z}/2 \) acts on \( \mathbb{Z}/2^\infty \) by sign. There is an isomorphism

\[
H^1(\mathbb{Z}/2 \rtimes \Sigma_n; (\mathbb{Z}/2^\infty)^n) \cong \text{Hom}(\Sigma_{n-1}, \mathbb{Z}/2) \oplus \text{Hom}(\Sigma_n, \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2
\]

to the pair \((v, \chi) \in \mathbb{Z}/2 \oplus \text{Hom}(\Sigma_n, \mathbb{Z}/2)\) associates the derivation \( D(v, \chi) \) given by

\[
D(v, \chi)(\varepsilon_i, \sigma) = (v + \chi(\sigma), \ldots, v + \chi(\sigma), \chi(\sigma), v + \chi(\sigma), \ldots, v + \chi(\sigma))
\]
where \( \varepsilon_i \) is the \( i \)th canonical basis vector for \((\mathbb{Z}/2)^n\) and \( \chi(\sigma) \) is in the \( i \)th coordinate. To see this, use the exact sequence from the Lyndon–Hochschild–Serre spectral sequence

\[
0 \to H^1(\Sigma; (\mathbb{Z}/2)^n) \to H^1(\mathbb{Z}/2 \wr \Sigma; (\mathbb{Z}/2)^{2n}) \to H^1((\mathbb{Z}/2)^n; (\mathbb{Z}/2)^{2n})^\Sigma \to H^2(\Sigma; \mathbb{Z}/2) \to H^2((\mathbb{Z}/2)^n; \mathbb{Z}/2) \to \cdots
\]

where \( H^1(\Sigma; (\mathbb{Z}/2)^n) \cong \mathbb{Z}/2 \) \((n \geq 3)\) and also the third term is of order 2 as, in general,

\[
H^*(G^n; M^n) = H^*(G^n; M)^n = H^*(G^{n-1}; H^*(G; M)) = \cdots = H^*(G; \ldots; H^*(G; M) \cdots)^n
\]

for a group \( G \) and a \( G \)-module \( M \). This gives

\[
H^0(W; \bar{T}) = \mathbb{Z}/2, \quad H^1(W; \bar{T}) = \begin{cases} \mathbb{Z}/2 & n = 2 \\ (\mathbb{Z}/2)^2 & n \geq 3 \end{cases}
\]

in this case. The computation of \( H^0(W; \bar{T}) \) uses that the center of the maximal torus normalizer \( Z(\hat{N}(\text{SL}(2n+1, \mathbb{R}))) = Z(\hat{O}(2) \wr \Sigma_n) = Z\hat{O}(2) = \mathbb{Z}/2 \)

is cyclic of order two for all \( n \geq 2 \) (whereas \( \hat{Z}\text{SL}(2n+1, \mathbb{R}) = 0 \)).

7.4. The center of a semi-direct product. Let \( G \times \Sigma \) be the semi-direct product for the action \( \Sigma \to \text{Aut}(G) \) of the group \( \Sigma \) on the group \( G \). Let \( G^\Sigma = \{g \in G|\Sigma g = g\} \) and \( \Sigma_G = \{\sigma \in \Sigma|\sigma(g) = g \text{ for all } g \in G\} \).

7.5. Lemma. The center \( Z(G \times \Sigma) = G^\Sigma \times_{\text{Aut}(G)} Z(\Sigma) \) of \( G \times \Sigma \) is the pull-back

\[
\begin{array}{ccc}
Z(G \times \Sigma) & \longrightarrow & Z(\Sigma) \\
\downarrow & & \downarrow \\
G^\Sigma & \longrightarrow & \text{Aut}(G)
\end{array}
\]

of the action map restricted to the center of \( \Sigma \) along the map \( G^\Sigma \to \text{Aut}(G) \) given by inner automorphisms.

Proof. Suppose that \((g, \sigma) \in G \times \Sigma \) is in the center of \( G \times \Sigma \). Since

\[
(g, \sigma) \cdot (1, \tau) = (g, \sigma \tau) = (1, \tau) \cdot (g, \sigma) = (\tau(g), \tau \sigma)
\]

for all \( \tau \in \Sigma \), \( g \) is fixed by \( \Sigma \) and \( \sigma \) is central in \( \Sigma \). Moreover, from

\[
(g, \sigma) \cdot (h, 1) = (g \cdot \sigma(h), \sigma) = (h, 1) \cdot (g, \sigma) = (h, \sigma)
\]

we see that \( \sigma(h) = h^g \) for all \( h \in G \).

7.6. Corollary. If \( G \) is abelian, \( Z(G \times \Sigma) = G^\Sigma \times Z(\Sigma)_G \) is a direct product.

Proof. The bottom horizontal homomorphism \( G^\Sigma \to \text{Aut}(G) \) is trivial.

7.7. Corollary. Let \( G \) be a group and \( Z \neq G \) a central subgroup. Let the cyclic group \( C_p \) of prime order \( p \) act on \( G^p/Z \) by cyclic permutation. Then

\[
Z(G)/Z \times \{z \in Z|z^p = 1\} \cong Z(G^p/Z \times C_p)
\]

via the isomorphism that takes the element \( z \in Z \) of order \( p \) to \((1, z, \ldots, z^{p-1})Z \in G^p/Z \) and is the diagonal on \( Z(G)/Z \).

Proof. Observe that

\[
G/Z \times \{z \in Z|z^p = 1\} \cong (G^p/Z)^{C_p}
\]

via the isomorphism that takes \((gZ, z)\) to \(g(1, z, \ldots, z^{p-1})Z\). To see this, consider an element \((g_1, \ldots, g_p)Z\) which is fixed by \( C_p \). Then \((g_1, g_1, \ldots, g_p)Z = (g_p, g_1, \ldots, g_{p-1})Z\) so there exists an element \( z \in Z \) so that \( g_2 = g_1z, g_3 = g_2z = g_1z^2, \ldots, g_p = g_1z^{p-1}, g_1 = g_1z^p \). Therefore, \( z^p = 1\) and \((g_1, g_2, \ldots, g_p) = (g_1(1, z, \ldots, z^{p-1}))\).

Thus \( Z(G^p/Z \times C_p) \) is the pull back of the group homomorphisms

\[
G/Z \times \{z \in Z|z^p = 1\} \xrightarrow{\varphi} \text{Aut}(G^p/Z) \leftarrow C_p
\]

where \( \varphi(gZ, z)((g_1, \ldots, g_p)Z) = (g_1^g, \ldots, g_p^g)Z \). Let \( ((gZ, z), \sigma) \) be an element of the pull back. Assume that \( \sigma \) is non-trivial. Since \( p \) is a prime number, \( \sigma \) has no fixed points. The equation

\[
\forall g_1, \ldots, g_p \in G: (g_1^g, \ldots, g_p^g)Z = (g_{\sigma(1)}, \ldots, g_{\sigma(p)})Z
\]
shows that \( g_1^p Z = g_\sigma(1) Z \). This is impossible unless \( \sigma \) is the identity since otherwise we can find a \( g_1 \in Z \) and a \( g_\sigma(1) \notin Z \). Thus the permutation \( \sigma \) must be the identity. The requirement for \((gZ,z),1\) to be in the pull back is that
\[
\forall (g_1, \ldots, g_p) \in G^p \exists u \in Z : (g_1^p, g_2^p, \ldots, g_p^p) = (g_1 u, g_2 u, \ldots, g_p u)
\]
which implies that \([g_1,g] = u = [g_2,g] \) for all \( g_1, g_2 \in G \). If we take \( g_1 = 1 \) to be the identity, we see that \( g \) must be central. \( \square \)

7.8. **Action in Lie case.** Let \( \nu : V \to G \) be a monomorphism of a non-trivial elementary abelian \( p \)-group to a compact Lie group \( G \). There is a canonical map \( BC_G(\nu(V)) \to \text{map}(BV, BG)_{Br} \) from the classifying space of the Lie theoretic centralizer of \( \nu(V) \) to the mapping space component containing \( Bv \). Write \( c_g \) for conjugation with \( g \in G \).

7.9. **Lemma.** Suppose that \( \nu \alpha = c_g \nu \) for some element \( g \in G \) and some automorphism \( \alpha \in \text{GL}(V) \). Then conjugation by \( g \) takes \( C_G(\nu(V)) \) to \( C_G(c_g \nu(V)) = C_G(\nu \alpha(V)) = C_G(\nu(V)) \) and the diagram
\[
\begin{align*}
BC_G(\nu(V)) & \to \text{map}(BV, BG)_{Br} \\
Bc_g & \cong (Bo)^* \end{align*}
\]
is homotopy commutative.

**Proof.** The commutative diagram of Lie group morphisms
\[
\begin{array}{ccc}
V \times C_G(\nu(V)) & \xrightarrow{\nu \times 1} & \nu(V) \times C_G(\nu(V)) & \xrightarrow{\text{mult}} & G \\
\alpha \times c_g & & \downarrow & & \\
V \times C_G(\nu(V)) & \xrightarrow{\nu \times 1} & \nu(V) \times C_G(\nu(V)) & \xrightarrow{\text{mult}} & G
\end{array}
\]
induces a commutative diagram
\[
\begin{array}{ccc}
BV \times BC_G(\nu(V)) & \xrightarrow{B(\text{mult} \circ (\nu \times 1))} & BG \\
Bo \times Bc_g & & \\
BV \times BC_G(\nu(V)) & \xrightarrow{B(\text{mult} \circ (\nu \times 1))} & BG
\end{array}
\]
of classifying spaces. Taking adjoints, we obtain the homotopy commutative diagram
\[
\begin{array}{ccc}
BC_G(\nu(V)) & \to \text{map}(BV, BG)_{Br} \\
Bc_g & & \downarrow (Bo)^* \\
BC_G(\nu(V)) & \to \text{map}(BV, BG)_{Br}
\end{array}
\]
as claimed. \( \square \)

7.10. **Corollary.** Suppose that \( \mu : V \to N(G) \) is a monomorphism and that \( \mu \alpha = c_n \mu \) for some \( \alpha \in \text{GL}(V) \) and \( n \in N(G) \). Then
\[
w^{-1} = \pi_2((Bo)^*) : \pi_2(\text{BT}(G))^{\pi_0(\mu)(V)} \to \pi_2(\text{BT}(G))^{\pi_0(\mu)(V)}
\]
where \( w \in W(G) \) is the image of \( n \in N(G) \).

**Proof.** There is a commutative diagram
\[
\begin{array}{ccc}
\pi_2(\text{BT}) & \xrightarrow{\pi_2(\text{BN}(G))} & \pi_2(BC_{N(G)}(V, \mu)) & \xrightarrow{\cong} & \pi_2(\text{map}(BV, BN), B \mu) \\
\pi_2(\text{BT}) & \xrightarrow{\pi_2(\text{BN}(G))} & \pi_2(BC_{N(G)}(V, \mu)) & \xrightarrow{\cong} & \pi_2(\text{map}(BV, BN), B \mu)
\end{array}
\]

where $\pi_2(B\mathcal{C}_{N(G)}(V,\mu)) = \pi_2(B\mathcal{T}(G))^{\pi_0(\mu)(V)}$ denotes the fixed point group for the group action $\pi_0(\mu): V \to W(G) \subseteq \text{Aut}(\pi_2(B\mathcal{T}(G)))$. Since $B\mathcal{C}_n: BN \to BN$ is freely homotopic to the identity along the loop $w \in \pi_1(BN)$ its effect on the $\mathbb{Z}_p[\pi_1(BN)]$-module $\pi_2(BN)$ is multiplication by $w$. \hfill $\Box$

**References**


MATEMATISK INSTITUT, UNIVERSITETSPARKEN 5, DK–2100 KØBENHAVN

E-mail address: moller@math.ku.dk