The zero divisors of the Cayley–Dickson algebras over the real numbers

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Abstract. In this paper we describe algebraically the zero divisors of the Cayley - Dickson algebras $A_n = \mathbb{R}^{2^n}$ for $n \geq 4$ over the real numbers.

Introduction. The Cayley–Dickson algebra $A_n$ over $\mathbb{R}$ is an algebra structure on $\mathbb{R}^{2^n}$ given inductively by the formulae:

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^{2^n} = \mathbb{R}^{2^{n-1}} \times \mathbb{R}^{2^{n-1}}$ then

$$xy = (x_1y_1 - y_2x_2, y_2x_1 + x_2y_1)$$

where

$$\overline{x} = (\overline{x_1}, -x_2).$$

Therefore $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$ complex numbers, $A_2 = \mathbb{H}$ the Hamilton quaternions, $A_3 = \mathbb{O}$ the Cayley octonians, etc. This four algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are known as the classical Cayley–Dickson algebras and their main distinctive feature of these is:

**Hurwitz Theorem:** Let “$\| \|” denote the euclidean norm in $\mathbb{R}^{2^n}$. Then

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\[ \| xy \| = \| x \| \| y \| \ \forall x \text{ and } y \text{ in } A_n \text{ if and only if } n = 0, 1, 2, 3. \] (See [5] and [7]).

That is, for \( n \geq 4 \) this norm–preserving formula is not true in general and this opens the possibility of the existence of zero divisors in \( A_n \) for \( n \geq 4 \) i.e. \( x \) and \( y \) non–zero elements in \( A_n \) such that \( xy = 0 \) e.g. let \( x = e_1 + e_{10} \) and \( y = e_{15} - e_4 \) in \( \mathbb{R}^{16} = A_4 \) where \( e_0, e_1, e_2, \ldots, e_{15} \) is the canonical basis in \( \mathbb{R}^{16} \).

In this paper we study the zero divisors in \( A_n \) for \( n \geq 4 \).

From the algebraic side these algebras are non–commutative for \( n \geq 2 \) and non–associative for \( n \geq 3 \). Moreover \( A_3 \) is alternative:

\[ x^2 y = x(xy) \text{ and } xy^2 = (xy)y \text{ for all } x \text{ and } y \text{ in } A_3 \]

and \( A_n \) for \( n \geq 4 \) is flexible:

\[ x(yx) = (xy)x \text{ for all } x \text{ and } y \text{ in } A_n. \]

Clearly

\[ \text{Associative } \Rightarrow \text{ Alternative } \Rightarrow \text{ Flexible} \]

and the backwards implications are not true. Introducing the “associator symbol”

\[ (a, b, c) = (ab)c - a(bc) \]

we have that alternativity property in \( A_3 \) is equivalent to the alternativity of the associator symbol in \( A_3 \) via polarization; flexibility means that \( (x, y, x) = 0 \) for all \( x \) and \( y \) in \( A_n \).

We will see that the absence of the norm–preserving property, the non–alternativity and the presence of zero divisors in \( A_n \) for \( n \geq 4 \) are very strongly related and in fact they determine one to each other.

Usually the zero divisors are present in different algebraic contexts
1) the trivial one: in the direct sum of algebras i.e. if $A$ and $B$ are algebras over $\mathbb{R}$, $A \oplus B$ has a lot of zero divisors e.g. the ones of the form $(a,0)$ and $(0,b)$.

2) In an associative (alternative) context as in $M_n(\mathbb{R})$ the $n \times n$ matrices over $\mathbb{R}$ where a god–given invariant called determinant says that $0 \neq A \in M_n(\mathbb{R})$ is a zero divisor if and only if $detA = 0$.

3) In a non– associative (non–alternative) context which becomes rather difficult, this is our case.

This paper has two chapters, in chapter one we give a general description of the zero divisors for $A_n \ (n \geq 4)$ studying the linear transformations $R_a, L_a : A_n \to A_n$ right and left multiplication respectively for $0 \neq a \in A_n$ fixed and we prove that each $a \neq 0$ (doubly pure) induce a direct sum decomposition of $A_n$ where one summand is a copy of $\mathbb{H}$ the quaternions, other summand is the elements of $A_n$ which “alternate” with $a$ and one third summand is the annihilator of $A \setminus KerL_a = KerR_a$ (see Theorem I.15).

This also implies that $dimKerL_a \equiv 0 \mod 4$ and $dimKerL_a \leq 2^n - 4$.

In Chapter II we construct some type of zero divisors (called special ones), in $A_4$ are all of them, which are pairs of alternatives elements, we describe all of them completely for $n \geq 4$ and relate these zero divisors with the multiplicative monomorphisms of $A_3$ to $A_n \ n \geq 3$.

In a second paper we will describe the topology of the subset of $\mathbb{R}^{2^{n+1}}$

$$ZD_1^n = \{(x,y) \in \mathbb{R}^{2^n} \times \mathbb{R}^{2^n} | xy = 0 \text{ and } |x| = |y| = 1\}$$
and relate this with suitable Stiefel manifolds. (Second paper is under review, the preprint is available under request).

We work also on a third paper on the subject: “Applications” where we use the algebraic and topological parts to construct bilinear normed maps, construct generalized Hopf maps, etc.

We thank to Fred Cohen, Sam Gitler and K.Y. Lam for many illuminating conversations and to Isidoro Gitler and José Martínez Bernal for the painstaking discussion on the linear algebra related with this subject.
I. Basic properties of the Cayley–Dickson Algebras and its zero divisors

\[ A_n \text{ denotes } \mathbb{R}^{2n} \text{ with the Cayley–Dickson multiplication (} n \geq 1): \]

\[ x = (x_1, x_2), \quad y = (y_1, y_2) \quad \text{in} \quad \mathbb{R}^{2n} = \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1} \]

\[ xy = (x_1 y_1 - y_2 x_2, y_2 x_1 + x_2 y_1) \quad \text{with} \]

\[ \overline{x} = (\overline{x}_1, -x_2) \]

\[ e_0 = (1, 0, \ldots, 0) \in A_n \text{ denotes the unit element.} \]

The Euclidian norm and inner product are given by

\[ \| x \|^2 = \overline{x} x = \| x_1 \|^2 + \| x_2 \|^2 \]

and

\[ \langle x, y \rangle = \frac{1}{2} (x \overline{y} + y \overline{x}), \]

respectively.

The trace is

For \( x \in A_n \)

\[ t(x) = x + \overline{x}, \]

i.e.

\[ t(x) = 2(\text{real part of } x) \quad \text{and} \quad t(x) = 2\langle x, e_0 \rangle. \]

Let \( x \) and \( y \) be in \( A_n \), \( x \) is orthogonal to \( y \) \( (x \perp y) \) if and only if

\[ x \overline{y} = -y \overline{x} \quad \text{or} \quad t(x \overline{y}) = 0 \quad \text{because} \quad \overline{\overline{y}} = y \quad \overline{x}. \]

Therefore elements of zero trace (purely imaginaries) are orthogonal if and only if they anti-commute i.e.

\[ x \perp y \iff xy = -yx \]
for $x$ and $y$ in $\mathbb{A}_n$ with $t(x) = 0$ i.e. $\overline{x} = -x$.

Thus for all $x \in \mathbb{A}_n$ we have the characteristic equation:

$$x^2 - t(x)x + \|x\|^2 = 0$$

because

$$x^2 - (x + \overline{x})x + \overline{x}x = 0.$$ 

**Definition:** The associator of $a, b$ and $c$ in $\mathbb{A}_n$ is

$$(a, b, c) = (ab)c - a(bc)$$

which is linear in each variable.

For $n = 0, 1, 2$ the associator is identically zero.

For $\mathbb{A}_3 = \mathbb{O}$ the octonian numbers the associator is alternative i.e. for all $x$ and $y$ in $\mathbb{A}_3$

$$x^2y = x(xy) \quad \text{and} \quad xy^2 = (xy)y$$

then

$$0 = (x + y, x + y, z) = (x, x, z) + (y, y, z) + (x, y, z) + (y, x, z) \text{ then}$$

$$0 = (x, y, z) + (y, x, z) \quad \therefore \quad (x, y, z) = -(y, x, z).$$

Similarly $(x, y, z) = -(x, z, y) = (z, x, y) = -(z, y, x)$.

$\mathbb{A}_n$ for $n \geq 4$ is flexible i.e $x(yx) = (xy)x$ for all $x$ and $y$ in $\mathbb{A}_n$. This means that

$$(x, y, x) = 0 \quad \text{or equivalently} \quad (x, y, z) = -(z, y, x)$$

for all $x, y$ and $z$ in $\mathbb{A}_n$. 

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Notice that \((x, y, z) = 0\) if one of the entries \(x, y\) or \(z\) is real, that is, its imaginary part is zero.

**Lemma 1.1.** For all \(x, y\) and \(z\) in \(A_n\)

i) \(−(x, y, z) = (x, y, z) = (x, y, z) = (x, y, z)\)

ii) \(t((x, y, z)) = 0\)

iii) \(t(xy − yx) = 0\)

**Proof.** For \(n \leq 2\) the assertions are trivial. Let \(n \geq 3\) and \(x = x_r + x_I\) where \(x_r = \text{real part of } x\) and \(x_I = \text{imaginary part of } x\) so \(\overline{x} = x_r − x_I\).

\[
(x, y, z) = (x_r - x_I, y, z) = (x_r, y, z) - (x_I, y, z)
\]

\[
= 0 - (x_I, y, z) = -(x_r, y, z) - (x_I, y, z)
\]

\[
= -(x_r + x_I, y, z) = -(x, y, z).
\]

Similarly \(-(x, y, z) = (x, y, z) = (x, y, z)\). So we prove i).

To prove ii) we observe that

\[
\overline{(x, y, z)} = \overline{-x(yz + (xy)z)} = -(\overline{z} \overline{y} \overline{x} + \overline{y} \overline{x} \overline{z}) = -(\overline{z} \overline{y} \overline{x}) = (z, y, x) = -(x, y, z)
\]

by i) and flexibility.

Therefore \((x, y, z) + (x, y, z) = 0\) so ii) is proved.

If \(x\) or \(y\) are real \(xy = yx\) and \(t(xy - yx) = 0\). Suppose \(x\) and \(y\) are pure imaginary then

\[
\overline{xy - yx} = \overline{xy} - \overline{yx} = \overline{y} \overline{x} - \overline{xy} = yx - xy.
\]

\[
(xy - yx) + (xy - yx) = xy - yx + yx - xy = 0.
\]
On the other hand the symbol \([x, y] = xy - yx\) is bilinear, therefore decomposing \(x\) and \(y\) in the real and imaginary parts respectively, we are done with (iii).

Q.E.D.

**Lemma 1.2.** For all \(x, y, z\) and \(w\) in \(A_n\).

\[
x(y, z, w) + (x, y, z)w = (xy, z, w) - (x, yz, w) + (x, y, zw)
\]

**Proof.** (Adem’s paper [1]).

**Lemma 1.3.** For all \(x, y\) and \(z\) in \(A_n\)

\[
\langle x, yz \rangle = \langle x\overline{z}, y \rangle = \langle \overline{y}x, z \rangle
\]

**Proof.**

\[
2\langle x, yz \rangle = t(x(y\overline{z})) = t(x(\overline{z} \overline{y})) = t((x\overline{z})\overline{y})
\]

\[
= 2\langle x\overline{z}, y \rangle
\]

by Lemma 1.1 ii).

Similarly

\[
2\langle \overline{y}x, z \rangle = t((\overline{y}x)\overline{z}) = t(\overline{y}(x\overline{z})) = t((x\overline{z})\overline{y}) =
\]

\[
= 2\langle x\overline{z}, y \rangle
\]

using Lemma 1.1. iii) and ii).

Q.E.D.
Lemma 1.4. For all $x$ and $y$ in $A_n$.

$$\|xy\| = \|\overline{x}y\|$$

Proof. $\|xy\|^2 = \langle x, xy \rangle$ and $\|\overline{x}y\|^2 = \langle \overline{x}y, \overline{x}y \rangle$ using Lemma 1.3.

\[
\begin{align*}
\langle x, xy \rangle &= \langle \overline{x}(xy), y \rangle = \langle (\overline{x}(xy))\overline{y}, e_0 \rangle = \frac{1}{2} t((\overline{x}(xy))\overline{y}) \\
\langle \overline{x}y, \overline{x}y \rangle &= \langle x(\overline{x}y), y \rangle = \langle (x(\overline{x}y))\overline{y}, e_0 \rangle = \frac{1}{2} t((x(\overline{x}y))\overline{y})
\end{align*}
\]

thus

\[
\|xy\|^2 - \|\overline{x}y\|^2 = \frac{1}{2} t[(\overline{x}(xy) - x(\overline{x}y))\overline{y}]
\]

\[
\begin{align*}
&= \frac{1}{2} t[(- (\overline{x}, x, y) + (\overline{x}x)y + (x, \overline{x}, y) - (x\overline{x})y)\overline{y}] \\
&= \frac{1}{2} t[((x, x, y) + (x, \overline{x}, y)) + \|x\|^2 y - \|x\|^2 y)\overline{y}]
\end{align*}
\]

\[
\frac{1}{2} t(0\overline{y}) = 0.
\]

Thus $\|xy\|^2 = \|\overline{x}y\|^2$ and $\|xy\| = \|\overline{x}y\|$

Q.E.D.

Corollary 1.5. For all $x$ and $y$ in $A_n$.

$$\|xy\| = \|\overline{x}y\| = \|x\overline{y}\| = \|y\overline{x}\|.$$

Proof. Follows from the lemma recalling that

$$\|z\| = \|\overline{z}\| \quad \forall z \in A_n \quad \text{so} \quad \|xy\| = \|\overline{x}y\| = \|\overline{y}x\| = \|yx\|$$

Q.E.D.
Corollary 1.6. (Elementary facts on zero divisors).

For \( x \) and \( y \) in \( A_n \) \( n \geq 4 \).

1. \( xy = 0 \leftrightarrow yx = 0 \leftrightarrow \overline{xy} = 0 \leftrightarrow \overline{x\overline{y}} = 0 \).

2. If \( x \neq 0 \) and \( xy = 0 \) then \( t(y) = 0 \).

3. If \( y \neq 0 \) and \( xy = 0 \) then \( t(x) = 0 \).

4. \( x^2 = 0 \) if and only if \( x = 0 \).

Proof. 1. By Corollary 1.5. \( \| xy \| = \| yx \| = \| \overline{xy} \| = \| \overline{x\overline{y}} \| \) so \( xy = 0 \leftrightarrow yx = 0 \leftrightarrow \overline{xy} = 0 \leftrightarrow \overline{x\overline{y}} = 0 \).

2. \( xt(y) = x(y + \overline{y}) = xy + x\overline{y} = 0 \) if \( xy = 0 \) but \( t(y) \) is a real number so \( t(y) = 0 \).

3. \( t(x)y = (x + \overline{x})y = xy + \overline{xy} = 0 \) if \( xy = 0 \) but \( t(x) \) is a real number so \( t(x) = 0 \).

4. If \( x^2 = x \cdot x = 0 \) then \( t(x) = 0 \) and \( x = -\overline{x} \) and

\[ \| x \|^2 = \overline{x}x = -x^2 \quad \therefore \quad x^2 = 0 \leftrightarrow x = 0. \]

Q.E.D.

Notation: \( \triangleleft A_n = \{ x \in A_n | t(x) = 0 \} \).

Definition. \( 0 \neq x \in \triangleleft A_n \) is a zero divisor if there exists \( 0 \neq y \in \triangleleft A_n \) with \( xy = 0 \).
Notice that we don’t have to distinguish between left and right zero divisors because $xy = 0$ if and only if $yx = 0$.

For $x \in \mathbb{A}_n$, we define the following linear transformations.

$$L_x, R_x : \mathbb{A}_n \rightarrow \mathbb{A}_n \quad \text{by}$$

$$L_x(y) = xy \quad \text{and} \quad R_x(y) = yx \quad \forall \quad y \in \mathbb{A}_n$$

Clearly $L_x$ and $R_x$ are linear.

**Proposition 1.7.** For $x \in \mathcal{O}\mathbb{A}_n$, $L_x$ and $R_x$ are skew-symmetric.

**Proof.** Let $y$ and $z$ in $\mathbb{A}_n$. Then

$$\langle L_x(y), z \rangle = \langle xy, z \rangle = \langle y, R(z) \rangle = \langle y, -zx \rangle$$

$$= \langle y, -L_x(z) \rangle.$$

$$\langle R_x(y), z \rangle = \langle yx, z \rangle = \langle y, L(z) \rangle = \langle y, -zx \rangle$$

$$= \langle y, -R_x(z) \rangle.$$

Q.E.D.

**Remark:** $x \in \mathcal{O}\mathbb{A}_n$ is a zero divisor if and only if $\{0\} \neq \text{Ker} L_x$ and $\{0\} \neq \text{Ker} R_x$ also notice that $\text{Ker} L_x = \text{Ker} R_x$.

**Lemma 1.8.** If $x \in \mathbb{A}_n$ is zero divisor and $x = (x_1, x_2)$ in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ then $\bar{x} = (x_2, x_1)$ is zero divisor in $\mathbb{A}_n$.

**Proof.** Consider $y \neq 0$ in $\mathbb{A}_n$ such that $xy = 0$ and $y = (y_1, y_2)$ so

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1 - \overline{y_2}x_2, y_2x_1 + x_2\overline{y_1}) = (0, 0).$$
Consider the following product in $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1} = \mathbb{A}_n$

$(-\overline{y}_2, y_1)(x_2, x_1) = (-\overline{y}_2 x_2 - \overline{x}_1 y_1, -x_1 \overline{y}_2 + y_1 \overline{x}_2)$

$= (x_1 y_1 - \overline{y}_2 x_2, x_2 \overline{y}_1 + y_2 \overline{x}_1)$

Recall that $t(x) = t(x_1) = 0$. Therefore $(-\overline{y}_2, y_1)(x_2, x_1) = (0, 0)$ and $\bar{x} = (x_2, x_1)$ is a zero divisor.

\[ \text{Q.E.D.} \]

**Corollary 1.9.** If $x \in \mathbb{A}_n$ is a zero divisor and $x = (x_1, x_2)$ then $t(x) = t(x_1) = t(x_2) = 0$.

**Proof.** We already know that $t(x) = t(x_1) = 0$.

Now $\bar{x}$ is a zero divisor then $t(\bar{x}) = t(x_2) = 0$.

\[ \text{Q.E.D.} \]

**Definition.** $a \in \mathbb{A}_n$ with $a = (a_1, a_2) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ is *doubly pure* if $t(a_1) = 0$ and $t(a_2) = 0$.

So any zero divisor is doubly pure.

Let $e_0$ be the neutral element in $\mathbb{A}_{n-1}$ so the neutral element in $\mathbb{A}_n$ is $e_0 = (e_0, 0) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$.

Denote by $\overline{e}_0$ the element $\overline{e}_0 \doteq (0, e_0) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ i.e. $\overline{e}_0 = e_{2n-1}$ in the canonical basis of $\mathbb{R}^{2^n}$.

**Lemma 1.10** Let $a \in \circ \mathbb{A}_n$ i.e. $t(a) = 0$. Then $a$ is doubly pure if and only
if \( a \perp \tilde{e}_0 \).

**Proof.** Calculating \( a\tilde{e}_0 = (a_1, a_2)(0, e_0) = (-a_2, a_1) \) and 
\( \tilde{e}_0 a = (0, e_0)(a_1, a_2) = (-\pi_2, \pi_1) = (-\pi_2, -a_1) \) because \( t(a) = 0 \) and 
\( \pi_1 = -a_1 \).

If \( a \) is doubly pure \( t(a_2) = 0 \) and \( \pi_2 = -a_2 \) so \( \tilde{e}_0 a = (a_2, -a_1) = -a\tilde{e}_0 \). Conversely if \( a \perp \tilde{e}_0 \) then \( a\tilde{e}_0 = -\tilde{e}_0 a \) and \( (-a_2, a_1) = (\pi_2, \pi_1) \) then \( \pi_2 = -a_2 \) and \( t(a_2) = 0 \).

Q.E.D.

Right multiplication by \( \tilde{e}_0 \) defines an orthogonal transformation of determinant one whose square is minus the identity linear transformation in \( A_n \).

For \( x \in A_n \) denotes by \( \tilde{x} = R_{\tilde{e}_0}(x) = (-x_2, x_1) \) if \( x = (x_1, x_2) \) so 
\[ \| \tilde{x} \| = \| x \| \]
and \( \tilde{x} = -x \).

And \( R_{\tilde{e}_0}^T = -R_{\tilde{e}_0} = R_{\tilde{e}_0}^{-1} \) and \( \text{det}(R_{\tilde{e}_0}) = (-1)^{2n} = 1 \).

**Proposition 1.11** For \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \) doubly pure elements in \( A_n \) we have:

1) \( a\tilde{e}_0 = \tilde{a}; \quad \tilde{e}_0 a = -\tilde{a} \).
2) \( a\tilde{a} = -\| a \|^2 \tilde{e}_0; \quad \tilde{a} a = \| a \|^2 \tilde{e}_0 \) so \( a \perp \tilde{a} \).
3) \( \tilde{a} b = -\tilde{a} b \).
4) If $a \perp b$ then $\tilde{a}b + \tilde{b}a = 0$.

5) If $\tilde{a} \perp b$ then $ab = \tilde{b}a$.

**Proof.** 1) Follows from definition.

2) $a\tilde{a} = (a_1, a_2)(-a_2, a_1) = (-a_1a_2 + a_1a_2, a_1^2 + a_2^2) = (0, -\|a\|^2)$

$\tilde{a}a = (-a_2, a_1)(a_1, a_2) = (-a_2a_1 + a_2a_1, -a_2^2 - a_1^2) = (0, \|a\|^2)$

$= \|a\|^2 \tilde{e}_0.$

Since $t(a) = t(\tilde{a}) = 0$ then $a\tilde{a} = -\tilde{a}a$ and $a \perp \tilde{a}$.

3) $\tilde{a}b = (-a_2, a_1)(b_1, b_2) = (-a_2b_1 + b_2a_1, -b_2a_2 - a_1b_1)$ so

$\tilde{a}b = (a_1b_1 + b_2a_2, b_2a_1 - a_2b_1) = ab \therefore \tilde{a}b = \tilde{b}a.$

4) If $ab = -ba$ then $\tilde{a}b = -\tilde{a}b = \tilde{b}a = -\tilde{b}a \therefore \tilde{a}b + \tilde{b}a = 0$.  

5) If $\tilde{a} \perp b$ then $0 = \tilde{a}b + \tilde{b}a = -\tilde{a}b + \tilde{b}a = ab + \tilde{b}a = ab - \tilde{b}a$ and $0 = \tilde{a}b + \tilde{b}a = -ab + \tilde{b}a$.

Q.E.D.

**Corollary 1.12** For $0 \neq a \in A_n$ (doubly pure)

$ax = 0$ if and only if $a\tilde{x} = 0 \quad \forall x$.

**Proof.** $ax = 0 \iff xa = 0 \iff \tilde{x}a = 0 \iff \tilde{x}a = 0 \iff a\tilde{x} = 0$.

Q.E.D.

**Theorem 1.13** For $a \in A_n$ doubly pure of norm one and $n \geq 2$. The vector space generated by $\{e_0, \tilde{a}, a, \tilde{e}_0\}$ is multiplicatively closed and isomorphic as
algebra to $A_2 = \mathbb{H}$ the quaternions.

**Proof.** Construct the following multiplication table

<table>
<thead>
<tr>
<th></th>
<th>$e_0$</th>
<th>$\tilde{a}$</th>
<th>$a$</th>
<th>$\tilde{e}_0$</th>
<th>$a\tilde{e}_0$</th>
<th>$=\tilde{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_0$</td>
<td>$e_0$</td>
<td>$\tilde{a}$</td>
<td>$a$</td>
<td>$\tilde{e}_0$</td>
<td>$\tilde{e}_0a$</td>
<td>$= -\tilde{a}$</td>
</tr>
<tr>
<td>$\tilde{a}$</td>
<td>$\tilde{a}$</td>
<td>$-e_0$</td>
<td>$\tilde{e}_0$</td>
<td>$-a$</td>
<td>since $\tilde{a}\tilde{e}_0$</td>
<td>$= \tilde{a} = -a$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$-\tilde{e}_0$</td>
<td>$-e_0$</td>
<td>$\tilde{a}$</td>
<td>$\tilde{e}_0\tilde{a}$</td>
<td>$= -\tilde{a} = a$</td>
</tr>
<tr>
<td>$\tilde{e}_0$</td>
<td>$\tilde{e}_0$</td>
<td>$a$</td>
<td>$-\tilde{a}$</td>
<td>$-e_0$</td>
<td>$a\tilde{a}$</td>
<td>$= -\tilde{e}_0$ and $\tilde{a}a = \tilde{e}_0$</td>
</tr>
</tbody>
</table>

Q.E.D.

Recall that $\text{Ker}L_{ra} = \text{Ker}L_a$ for all $r \neq 0$ in $\mathbb{R}$ and that $\text{Ker}L_a \neq \{0\}$ only if $a$ is doubly pure non-zero element for $n \geq 4$.

*From now on we assume that $n \geq 4, \| a \| = 1$ and $a$ is doubly pure element in $A_n$.*

$H_a$ denotes the copy of the quaternions inside of $A_n$ generated by $\{e_0, \tilde{a}, a, \tilde{e}_0\}$.

Let $T_a : A_n \to A_n$ be given by $T_a(y) = (a, a, y)$.

Because the associator is tri-linear, $T_a$ is a linear map and because $H_a$ is associative $T_a(H_a) = \{0\}$.

On the other hand $T_a(y) = a^2y - a(ay) = -y - L_a^2(y)$ so $T_a = -I - L_a^2$ and since $L_a$ is anti-symmetric then $T_a$ is symmetric.

$H_a^\perp$ denotes the orthogonal complement of $H_a$ in $A_n = \mathbb{R}^{2^n}$.

**Lemma 1.14** $T_a(H_a^\perp) \subset H_a^\perp$.

**Proof.** Let $0 \neq y \in H_a^\perp$ then $T_a(y) = (a, a, y)$.  

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By Lemma 1.2.

\[ a(a,a,y) + (a,a,a)y = (a^2,a,y) - (a,a^2,y) + (a,a,ay) \quad \therefore \]

then \( a(a,a,y) = (a,a,ay) \) because \( a^2 = -e_0 \) and

\[ \langle a,(a,a,y) \rangle = t(a(a,a,y)) = -t(a(a,a,y)) = -t((a,a,ay)) = 0 \]

because the trace of any asociator is zero (Lemma 1.1).

Similarly \( \tilde{a}(a,a,y) + (\tilde{a},a,a)y = \) (Sum of Associators) since \( (\tilde{a},a,a) = 0 \) because \( \mathbb{H}_a \) is associative then \( \tilde{a} \perp (a,a,y) \).

Finally \( \tilde{e}_0(a,a,y) + (\tilde{e}_0,a,a)y = \) (Sum of Associators) since \( (\tilde{e}_0,a,a) = 0 \) because \( \mathbb{H}_a \) is associative then \( \tilde{e}_0 \perp (a,a,y) \).

Therefore \( (a,a,y) \in \mathbb{H}_a \perp \) if \( y \in \mathbb{H}_a \perp \)

Q.E.D.

\( \tilde{T}_a \) denotes the restriction of \( T_a \) to \( \mathbb{H}_a \perp \) so

\[ \tilde{T}_a : \mathbb{H}_a \perp \to \mathbb{H}_a \perp, \quad \tilde{T}_a = -I - L_a^2 \text{ is symmetric.} \]

**Theorem 1.15** For each \( a \in \mathbb{A}_n \) of norm one and doubly pure we have a direct sum decomposition.

\[ \mathbb{A}_n = \mathbb{R}^{2n} \cong \mathbb{H}_a \oplus \text{Ker} \tilde{T}_a \oplus \text{Ker} L_a \oplus \bigoplus V_\lambda \]

where

\[ V_\lambda = \{ x \in \mathbb{A}_n | a(ax) = -\lambda^2 x \}, \quad \lambda > 0, \lambda \neq 1 \]
Proof. Recall that $L_a$ is skew–symmetric and consequently $L_a^2$ is symmetric, negative semidefinite, that is for, all $x \in \mathbb{A}_n$

$$\langle L_a^2(x), x \rangle = -\langle L_a(x), L_a(x) \rangle = -|L_a(x)|^2 \leq 0$$

Therefore $a(ax) = 0 \Leftrightarrow ax = 0$ for $x \in \mathbb{A}_n$

On the other hand being $L_a^2$ symmetric no positive implies that all its eigenvalues are negative real numbers. So there exist $\{\lambda_0, \lambda_1, \ldots, \lambda_s\} \subset \mathbb{R}$ with $\lambda_0 = 0$ and $\lambda_1 = 1$ and eigenspaces

$$V_\lambda = \{x \in \mathbb{H}_a \mid a(ax) = -\lambda^2 x\}.$$ 

Recall that $L_a^2|\mathbb{H}_a = -I$.

And $V_0 = \text{Ker} L_a^2 = \text{Ker} L_a$ and $V_1 = \text{Ker} \tilde{T}_a$.

Q.E.D.

**Theorem 1.16.** For the $V_\lambda$’s as in the last theorem we have that for \( \lambda > 0 \)

$$\dim_{\mathbb{R}} V_\lambda \equiv 0 \mod 4.$$ 

**Proof.**

**CLAIM 1.** For $x \in V_\lambda$ with $\lambda > 0$ $y = \lambda^{-1}(ax)$ belongs $V_\lambda$, and $y \perp x$.

$ay = \lambda^{-1}a(ax) = \lambda^{-1}(-\lambda^2x) = -\lambda x$ and $a(ay) = -\lambda ax = -\lambda^2(\lambda^{-1}ax) = -\lambda^2y$. Now $\langle y, x \rangle = \langle \lambda^{-1}ax, x \rangle = \lambda^{-1}\langle ax, x \rangle = 0$.

**CLAIM 2.** If $x \in V_\lambda$ then $\bar{x} \in V_\lambda$. 

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For this we observe that: Since \( x \in \mathbb{H}_a^\perp, a \perp x, a \perp \tilde{x} \) and \( x \) is doubly pure then applying Proposition 1.11 (3) twice and noting that \( a(ax) = (xa)a \) by flexibility we have that \( a(a\tilde{x}) = (\tilde{x}a)a = -(ax)a = a(ax) = -\lambda^2\tilde{x} \).

Therefore if we take \( 0 \neq x \in V_\lambda \) for \( \lambda \neq 0 \) we may construct one orthogonal set

\[
\{x, y, \tilde{x}, \tilde{y}\}
\]

inside of \( V_\lambda \). Thus \( \dim_{\mathbb{R}} V_\lambda \equiv 0 \pmod{4} \quad \lambda > 0 \)

Q.E.D.

Corollary 1.17.

\[
2^n - 4 \geq \dim_{\mathbb{R}} \ker L_a \equiv 0 \pmod{4}
\]

**Proof.** Since \( \dim \mathbb{H}_a = 4, 2^n - 4 \geq \dim_{\mathbb{R}} \ker L_a \) and the last two theorems implies that

\[
\dim_{\mathbb{R}} \ker L_a \equiv 0 \pmod{4}.
\]

Q.E.D.
II. Zero divisors with alternative entries

Throughout this chapter $a$ and $b$ are nonzero elements in $\mathbb{A}_n$ for $n \geq 3$.

**Definition:** $a \in \mathbb{A}_n$ is alternative if $(a, a, x) = 0$ for all $x \in \mathbb{A}_n$.

The real elements ($\mathbb{R}e_0$) are alternative and since the associator symbol is linear in each variable we have that one element in $\mathbb{A}_n$ is alternative if and only if it’s imaginary part is alternative.

Therefore we restrict ourselves to the trace zero of norm one alternative elements in $\mathbb{A}_n$.

By Schafer [8] we know that the canonical basis $\{e_0, e_1, \ldots, e_{2^n-1}\}$ consists of alternative elements.

**Lemma 2.1.** If $a \in \mathbb{A}_n$ is alternative with $t(a) = 0$ and $|a| = 1$ then

i) $KerL_a = \{0\}$

ii) $L_a \in SO(2^n)$

iii) $ax = y$ has a unique solution for all $y \in \mathbb{A}_n$

iv) $L_a^2 = -I$

**Proof.** $0 = (a, a, x) = -x - a(ax)$ then $a(ax) = -x$ and if $ax = 0$ then $a(ax) = -x = 0$ and $KerL_a = \{0\}$ and $L_a$ is one to one. Now $L_a$ is skew-symmetric so

$$0 = \langle (a, a, x), x \rangle = \langle -x - a(ax), x \rangle = -\langle x, x \rangle - \langle a(ax), x \rangle$$
\[ = -|x|^2 + \langle ax, ax \rangle = -|x|^2 + |ax|^2 \]

then \(|L_a(x)| = |x|\) for all \(x \in \mathbb{A}_n\) and \(L_a \in O(n)\). Therefore \(L_a^{-1} = L_a^T = -L_a\) and \(L_a^2 = -I\) and \(L_a\) is similar to \(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\) for \(I\) and 0 in \(M_{2n-1}(\mathbb{R})\) which has determinant one.

Q.E.D.

Also we have similar results for \(R_a\), right multiplication.

**Definition:** \(a \in \mathbb{A}_n\) is **special** if \(a\) is alternative, \(t(a) = 0\) and \(|a| = 1\).

If \(a \in \mathbb{A}_n\) is special then \(\{e_0, a\}\) generates a copy \(\mathbb{A}_1 = \mathbb{C}\) inside of \(\mathbb{A}_n\). So the special elements can be regarded as a closed subset of the manifold of the \(2^n \times 2\) matrices of rank two with real coefficients.

**Definition.** \(\{a, b\} \subset \mathbb{A}_n\) is a **special couple** if both \(a\) and \(b\) are special elements in \(\mathbb{A}_n\) and \(a\) is orthogonal to \(b\) \( (a \perp b) \) i.e. \(ab = -ba\).

**Notation:** \(V(a; b)\) denotes the vector space generated by the set

\(\{e_0, a, b, ab\}\),

obviously \(V(a; b)\) is fourth–dimensional: \(a \perp ab\) and \(b \perp ab\).

**Proposition 2.2.** \(V(a; b)\) is multiplicatively closed and isomorphic to \(\mathbb{A}_2 = \mathbb{H}\) the quaternions.

**Proof:** Since \(a \perp b, t(ab) = -t(ab) = 0\) and \(|ab|^2 = \langle ab, ab \rangle = \langle -a(ab), b \rangle = \langle b, b \rangle = |b|^2 = 1\). Therefore \(t(ab) = 0\) and \(|ab| = 1\). \(\{e_0, a, b, ab\}\) is an
orthonormal set. Clearly \( a(ab) = a^2b = -b \) and \( b(ab) = -b(ba) = -b^2a = a \).

So we have the following multiplication table

<table>
<thead>
<tr>
<th></th>
<th>( e_0 )</th>
<th>( a )</th>
<th>( b )</th>
<th>( ab )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_0 )</td>
<td>( e_0 )</td>
<td>( a )</td>
<td>( b )</td>
<td>( ab )</td>
</tr>
<tr>
<td>( a )</td>
<td>( e_0 )</td>
<td>( -e_0 )</td>
<td>( ab )</td>
<td>( -b )</td>
</tr>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( -ab )</td>
<td>( -e_0 )</td>
<td>( a )</td>
</tr>
<tr>
<td>( ab )</td>
<td>( ab )</td>
<td>( b )</td>
<td>( -a )</td>
<td>( -e_0 )</td>
</tr>
</tbody>
</table>

Q.E.D.

Therefore each special couple induce a multiplicative monomorphism from \( \mathbb{A}_2 = \mathbb{H} \) to \( \mathbb{A}_n \) and the set of special couples can be regarded as closed subset of the \( 2^n \times 4 \) matrices with real entries and rank four.

Depending on \( \{a, b\} \) special couple we define

\[
S : \mathbb{A}_n \to \mathbb{A}_n \quad \text{by} \quad S(y) = (a, y, b)
\]

First of all \( S \) is a linear map and \( V(a; b) \subset KerS \) because \( V(a; b) \cong \mathbb{H} \) is associative. Also \( S = R_b L_a - L_a R_b = [R_b, L_a] \) is skew–symmetric because \( R_b \) and \( L_a \) are skew–symmetric.

Therefore \( S^2(y) = S(S(y)) = 0 \) if and only if \( S(y) = 0 \) and \( S : V(a; b)^\perp \to V(a; b)^\perp \) because \( S(y) \in V(a; b) \) for \( y \in V(a; b) \). We are interested in calculate \( KerS \) and \( ImS \) for \( \{a, b\} \) special couple and \( S \) restricted to \( V(a, b)^\perp \).

**Theorem 2.3.** For \( \{a, b\} \) special couple and \( S \) and \( V(a, b) \) as above we have that

\[
Ker(S : V(a; b)^\perp \to V(a; b)^\perp) = KerL_{a+b} \oplus KerL_{a-b}.
\]
Proof. First of all we notice that \( \ker(L_a + b) \cap \ker(L_a - b) = \{0\} \). If \( 0 = (a + b)z = (a - b)z \) then \( az + bz = az - bz = 0 \) and \( 2bz = 0 \) and \( z = 0 \) because \( b \) is alternative. Now if \( (a + b)x = 0 \) then \( ax = -bx = xb \) for \( x \in V(a; b)^\perp \) then \( (a, x, b) = (ax)b - a(xb) = (xb)b - a(ax) = xb^2 - a^2x = 0 \). Similarly if \( (a - b)x = 0 \) then \( ax = bx = -xb \) for \( x \in V(a, b)^\perp \). Thus \( (a, x, b) = (ax)b - a(xb) = -(xb)b + a(ax) = -xb^2 + a^2x = 0 \). Therefore \( \ker(L_a + b) \oplus \ker(L_a - b) \subset \ker(S) \). Now if \( y \in V(a; b)^\perp \) with \( S(y) = 0 \) we have that

\[
(a + b)(y - a(yb)) = ay + by - a^2yb - b(a(yb))
= ay + by - by + ((ay)b)b
= ay + aby^2
= ay - ay
= 0
\]

because \( a(yb) = (ay)b \) and \( b \perp [(ay)b] \).

Similarly \( (a - b)(y + a(yb)) = 0 \) if \( S(y) = 0 \) and \( y \in V(a; b)^\perp \). Since \( y = \frac{1}{2}[(y - a(yb)) + (y + a(yb))] \) we have that: If \( S(y) = 0 \) and \( y \in V(a; b)^\perp \) then \( y \in \ker(L_{a+b}) \oplus \ker(L_{a-b}) \) and we are done.

Q.E.D.

Corollary 2.4 \( \dim_{\mathbb{R}} \ker(S) = 2 \dim \ker(L_{a+b}) \equiv 0 \mod 8 \).

Proof. By the proof of the Theorem 2.3. we know that the elements in \( \ker(L_{a+b}) \) are of the form \([y - (a(yb)]\) and the elements in \( \ker(L_{a-b}) \) are of the
form \( y + (a(yb)) \) for \( (a, y, b) = 0 \). Therefore the assignment

\[
y + a(yb) \mapsto y - a(yb)
\]

defines a linear isomorphism between \( \text{Ker}L_{a+b} \) and \( \text{Ker}L_{a-b} \). Therefore \( \dim \mathbb{R} \text{Ker}S = 2 \dim \text{Ker}L_{a+b} \) that by Theorem 1.15 is congruent with 0 module 8.

**Lemma 2.5** For \( \{a, b\} \) special couple in \( \mathbb{A}_n \) and \( y \in V(a; b)\perp \) we have that

\[
(a + b, a + b, y) = -(a, y, b) + 2(ay)b = (a, y, b) + 2a(yb)
\]
in short \( T_{a+b} = -S + 2R_bL_a = S + 2L_aR_b \) on \( V(a; b)\perp \)

**Proof.**

\[
(a + b, a + b, y) = (a, a, y) + (b, b, y) + (a, b, y) + (b, a, y) \\
= 0 + 0 + (ab + ba)y - a(by) - b(ay) \\
= a(yb) + (ay)b \\
= -(a, y, b) + 2(ay)b \\
= (a, y, b) + 2a(yb)
\]

Q.E.D.

**Corollary 2.6** Let \( \{a, b\} \) be one special couple and \( y \in V(a, b)\perp \) in \( \mathbb{A}_n \).

\( (a + b, a + b, y) = 0 \) if and only if

\[
(a, b)(-\frac{1}{2}S(y), y) = (0, 0) \quad \text{in} \quad \mathbb{A}_{n+1},
\]

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Proof. By Lemma 2.5 \( T_{a+b}(y) = 0 \) \( \iff \) \( a(yb) = -(ay)b \) then

\[
(a,b)(a(yb), y) = (a(a(yb)) + yb, ya - b(a(yb)))
\]

\[
= (a^2yb + yb, ya - ((ay)b)b)
\]

\[
= (-yb + yb, ya - (ay)b^2)
\]

\[
= (0, 0)
\]

because \( y \in V(a, b)^\perp \).

Conversely if \((a, b)(x, y) = (0, 0)\) for \( y \in V(a; b)^\perp \) then \( ax + yb = 0 \) and \( ya - bx = 0 \) so

\[
ax = -yb \implies a(ax) = a^2x = -x = -a(yb) \therefore x = a(yb)
\]

\[
bx = ya \implies ay = xb \implies (ay)b = (xb)b = x^2b = -x.
\]

Therefore \( a(yb) + (ay)b = x - x = 0 \). And by the Lemma 2.5 \( T_{a+b}(y) = 0 \) and \( x = \frac{1}{2}S(y) \)

Q.E.D.

Theorem 2.7 Let \( \{a, b\} \) be one special couple in \( \mathbb{A}_n \) and

\( L_{(a,b)} : \mathbb{A}_{n+1} \rightarrow \mathbb{A}_{n+1} \) then

\[
\dim_{\mathbb{R}} \ker L_{(a,b)} \leq 2^n - 4 - 2 \dim_{\mathbb{R}} \ker L_{a+b}
\]

Proof. By the last corollary

\[
\ker L_{(a,b)} \cong \ker [T_{a+b} : V(a; b)^\perp \rightarrow V(a; b)^\perp]
\]

but \( \ker T_{a+b} \subset \text{Im}S \) restricted to \( V(a; b)^\perp \) and \( \text{Im}S \cong (\ker S)^\perp = (\ker L_{a+b} \oplus \ker L_{a-b})^\perp \) (by Theorem 2.3). Also we know that \( \ker L_{a+b} \cong \ker L_{a-b} \) and

\[
\dim_{\mathbb{R}} \ker L_{(a,b)} \leq \dim_{\mathbb{R}} \mathbb{A}_n - \dim_{\mathbb{R}} V(a, b) - 2 \dim_{\mathbb{R}} \ker L_{a+b}.
\]
Q.E.D.

Examples:

For \( n = 3 \). All element in \( \mathbb{A}_3 \) are alternative so \( \text{Ker}T_{a+b} = \mathbb{A}_3 \) for all \( a \) and \( b \). Then \( \dim \text{Ker}L_{(a,b)} = 4 \) for all special couple with \( S(y) \neq 0 \) for some \( y \in V(a;b)\perp \).

For \( n \geq 4 \). The top dimension \( 2^n - 4 \) is always realizable. For instance if \( a \) is one special element and doubly pure in \( \mathbb{A}_n \) then \( \bar{a} \) is also special and \( V(a;\bar{a}) = \mathbb{H}_a \) then \( \text{Ker}T_{a+\bar{a}} = \{0\} \) so \( \dim \text{Ker}L_{(a,\bar{a})} = 2^n - 4 \).

More generally if \( \{a, b\} \) is one special couple in \( \mathbb{A}_n \) with \( (a+b) \) alternative then \( \text{Ker}L_{(a,b)} \cong V(a;b)\perp \) i.e. \( \dim \text{Ker}L_{(a,b)} = 2^n - 4 \).

Now we analyze a more general case:

Given \( a \) and \( b \) alternative non-zero elements in \( \mathbb{A}_n \).

Under what conditions on \( a \) and \( b \), \( (a,b) \) is a zero divisor in \( \mathbb{A}_{n+1} \)?

Suppose that \( (a,b)(x,y) = (0,0) \) for \( x \neq 0 \) and \( y \neq 0 \). Then

1) \( t(a) = t(b) = 0 \) (Corollary 1.9)

2) \( |a| = |b| \)

\[ ax = -yb \Rightarrow |a||x| = |b||y| \quad \text{and} \]
\[ ya = bx \Rightarrow |a||y| = |x||b| \quad \text{so} \]
\[ |a|^2|y| = |a||x||b| = |b|^2|y| \quad \text{and} \quad |a| = |b|. \]

Notice that without losing generality we may assume that
\[ |a| = |b| = 1 \]

from now on.

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3) \( a \) and \( b \) are linearly independent.

Suppose that \( a = \lambda b \) for \( \lambda \neq 0 \) in \( \mathbb{R} \) so

\[
ax = -yb \Rightarrow \lambda bx = -yb
\]

and

\[
ya = bx \Rightarrow \lambda yb = bx \text{ so }
\]

\[
\lambda(bx) = \lambda(\lambda yb) = \lambda^2 yb = -yb \Rightarrow \lambda^2 = -1
\]

which is a contradiction.

4) \((a, y, b) = -2x\) and \((a, x, b) = 2y\). Recall that \((a, b)(x, y) = (0, 0) \Leftrightarrow \)

\((a, b)(-y, x) = (0, 0)\). Thus

\[
ax = -yb \quad \text{and} \quad xa = -by
\]

\[
ya = bx \quad ay = xb
\]

Then

\[
ax = -yb \implies a^2 x = -a(yb) \implies x = a(yb)
\]

\[
ay = xb \implies (ay)b = xb^2 \implies x = -(ay)b
\]

Also

\[
xa = -by \implies b(xa) = -b^2 y = y
\]

\[
ya = bx \implies ya^2 = (bx)a \implies -y = (bx)a
\]

Therefore \((a, y, b) = (ay)b-a(yb) = -x-x = -2x\) and \((a, x, b) = -(b, x, a) = -[(bx)a - b(xa)] = 2y\).

Notice that we also show that \( x \perp y \) because \( S \vdash (a, -, b) \) is skew-symmetric and that \( x \) and \( y \) belongs to \( \{e_0, a, b, ab\} \perp \) because \( S(e_0) = S(a) = S(b) = 0 \) and \( S(ab) = -aS(b) = 0 \).
5) $T_{a+b}(y) = (a + b, a + b, y) = -2(a, b)y$ and

\[
L_{a+b}^2(y) = (a + b)[(a + b)y] = -2y
\]

\[
(a + b)[(a + b)y] = (a + b)[ay + by] = a(ay) + b(by) + b(ay) + a(by)
\]

\[
= -y - y + x - x = -2y
\]

because $b(ay) = -(ay)b = x$ and $a(by) = -a(yb) = -x$.

And

\[
T_{a+b}(y) = (a + b)^2 y - L_{a+b}^2(y)
\]

\[
= (a^2 + b^2 + ab + ba)y + 2y
\]

\[
= -2y - 2(a, b)y + 2y
\]

\[
= -2(a, b)y
\]

We collect enough necessary conditions to stablilsh.

**Theorem 2.9** Let $a$ and $b$ alternatives elements in $A_n$ of zero trace and norm one, then $(a, b) \in A_{n+1}$ is a zero divisor if and only if $\lambda = -2$ is an eigenvalue of $L_{a+b}^2$.

**Proof.**

$\Rightarrow$) Follows from the conditions enlisted above.

$\Leftarrow$) Let $y \neq 0$ in $A_n$ such that $L_{a+b}^2(y) = -2y$ then

\[
-2y = (a + b)[(a + b)y] = a^2y + by^2 + a(by) + b(ay)
\]

\[
= -2y + a(by) + b(ay)
\]
therefore

\[ a(by) = -b(ay). \]

On the other hand

\[
L_{a-b}^2(y) = (a-b)[(a-b)y] = (a^2 + b^2)y - (b(ay) + a(by))
\]

\[ = -2y. \]

then \( y \perp (a+b) \) and \( y \perp (a-b) \) and \( y \perp a \) and \( y \perp b \)

Consider the following product in \( \mathbb{A}_{n+1} \)

\[
(a, b)(-a(by), y) = (-a(a(by)) + yb, ya + b(a(by))
\]

\[ = (-a^2(by) + yb, ya - b(ay)) \]

\[ = (by + yb, ya - b^2(ay)) \]

\[ = (-yb, ya - ya) \]

\[ = (0, 0) \]

because \( a(by) = -b(ay), a \perp y \) and \( b \perp y \).

Q.E.D.

**Remark:** Condition 5) above shows that Theorem 2.9 implies Corollary 2.6
it is the case when \( \langle a, b \rangle = 0 \).

**Corollary 2.10** For \( a \) and \( b \) alternative elements in \( \mathbb{A}_n \) of zero trace and norm one

\[
\dim \ker L_{(a,b)} \leq 2^n - 4 - 2\dim \ker L_{a+b}
\]
Proof.

Notice that \( a(by) = -b(ay) \Leftrightarrow S(y) = -2(ay)b \) for \( y \in V(a, b)^\perp \) such that \((a, b)(x, y) = (0, 0)\) then

\[
\text{Ker} L_{(a, b)} \subset (\text{Ker} S)^\perp \cong (\text{Ker} L_{a+b} \oplus \text{Ker} L_{a-b})^\perp
\]

then \( \text{dim} \text{Ker} L_{(a, b)} \leq 2^n - 4 - 2\text{dim} \text{Ker} L_{a+b} \) because \( \text{Ker} L_{a+b} \cong \text{Ker} L_{a-b} \).

Q.E.D.

Remarks: Notice \( L^2_{a+b}(y) = -2y \) for \( a, b \) as in the Theorem 2.9 and \( y \neq 0 \) implies that \( a \) and \( b \) are linearly independent. If it would \( a = b \) or \( a = -b \) then \( L^2_{a+b} = -4I \) and \( L^2_{a-b} = 0 \).

There are examples of couples of alternatives linearly independent elements (of norm one and zero trace) in \( \mathbb{A}_n \) which no form a zero divisor in \( \mathbb{A}_{n+1} \). For instances put \( a = e_1 \) and \( b = \frac{e_1 + e_2}{\sqrt{2}} \) in \( \mathbb{A}_3 \) then \((a, b) \in \mathbb{A}_4 \) is neither a zero divisor nor an alternative element. (We thank to Paul Yiu for this piece of information).

Definition. A triple \( \{a, y, b\} \) in \( \mathbb{A}_n \) is special if it is an orthonormal set and

i) \((a, b)\) is an special couple

ii) \((ay)b = -a(yb)\)

Definition. A zero divisor \((a, b)\) in \( \mathbb{A}_{n+1} \) is special if \((a, b)(x, y) = (0, 0)\) implies that \( \{a, y, b\} \) is one special triple.

Proposition 2.11 Let \( a \) and \( b \) are alternatives of norm one in \( \mathbb{A}_n \) with
\((a, b)(x, y) = (0, 0)\) in \(\mathbb{A}_{n+1}\). If \((a + b)\) is alternative then \((a, b)\) is an special zero divisor.

**Proof.** By Theorem 2.9) \(T_{a+b}(y) = -2(a, b)y\). So \(0 = T_{a+b}(y) \Rightarrow a \perp b\) and \(\{a, b\}\) is one special couple on the other hand \((ay)b = -(ay)b\) by 4).

Q.E.D.

**Corollary 2.12** Any zero divisor (up to norm) in \(\mathbb{A}_4\) is special zero divisor.

Consider the vector subspace in \(\mathbb{A}_n\)

\[ V(a; y; b) = \langle \{e_0, a, b, ab, (ay)b, yb, ay, y\} \rangle \]

**Theorem 2.13** \(V(a; y, b)\) is a copy of \(\mathbb{A}_3 = \mathbb{O}\) the octonian numbers inside of \(\mathbb{A}_n\) if and only if \(\{a, y, b\}\) is an special triple.

**Proof.** Let’s make the following assignment.

\[
\begin{array}{cccccccc}
  e_0 & a & b & ab & (ay)b & yb & ay & y \\
  | & | & | & | & | & | & |
\end{array}
\]

we may easily see that this define an homomorphism of \(\mathbb{A}_3\) to \(\mathbb{A}_n\) if \(\{a, y, b\}\) is an special triple. Notice in fact that this a monomorphism from \(\mathbb{A}_3\) to \(\mathbb{A}_n\).

Conversely is easy to see that \(\{e_1, e_7, e_2\}\) form in special triple in \(\mathbb{A}_n\) for \(n \geq 3\).

Q.E.D.
Corollary 2.14 The set of zero divisor in $A_4$ (with entries of norm one) are homomorphic to $Aut(A_3) = G_2$ the exceptional Lie group of rank two.

Remarks: Notice that if $\{a, y, b\}$ is an special triple then $\{b, y, a\}$ is an special triple and $\{a, b, y\}$ is no necessarily an special triple. e.g.

$a = e_1$ $b = e_2$ $y = e_{15}$ in $A_4$, $(e_1, e_{15}, e_2) = 2e_{12}$ and

$(e_1e_{15})e_2 = -e_1(e_{15}e_2) = e_{12}$ but $(e_1, e_2, e_{15}) = 0$ and $(e_1e_2)e_{15} = e_1(e_2e_{15})$.

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