The Noether Map II

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SUMMARY : *Let* $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a faithful representation of a finite group* G*. In this paper we proceed with the study of the image of the associated Noether map*

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

In [8] *it has been shown that the Noether map is surjective if* V *is a projective* $\mathbb{F}G$*-module. This paper deals with the converse. The converse is in general not true: we illustrate this with an example. However, for* p*-groups (where* p *is the characteristic of the ground field* $\mathbb{F}$*) as well as for permutation representations of any group the surjectivity of the Noether map implies the projectivity of* V*.*

Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group G of order d over a field $\mathbb{F}$. The representation ϱ induces naturally an action of G on the vector space $V = \mathbb{F}^n$ of dimension n and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{f \in \mathbb{F}[V]^G \mid gf = f \ \forall\, g \in G\},$$

which is a graded connected Noetherian commutative algebra. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \mathbb{F}G \otimes V$$

be the induced module. The group G acts on $V(G)$ by left multiplication on the first component. We obtain a G-equivariant surjection

$$(\bigstar) \qquad V(G) \longrightarrow V, \ (g, v) \longmapsto gv.$$

Let us choose a basis $e_1, \ldots, e_n$ for V. Let $x_1, \ldots, x_n$ be the standard dual basis for V^*, and set $G = \{g_1, \ldots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \mathrm{span}_{\mathbb{F}}\{e_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\},$$

and the map $(\bigstar)$ translates into

$$V(G) \longrightarrow V, \ e_{ij} \longmapsto g_j e_i.$$

Similarly, we have

$$V(G)^* = \mathrm{span}_{\mathbb{F}}\{x_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\}$$

with

$$V(G)^* \longrightarrow V^*, \ x_{ij} \longmapsto g_j x_i.$$

We obtain a surjective G-equivariant map between the rings of polynomial functions

$$\eta_G : \mathbb{F}[V(G)] \longrightarrow \mathbb{F}[V].$$

The group G acts on $\mathbb{F}[V(G)]$ by permuting the basis elements x_{ij}. By restriction to the induced ring of invariants, we obtain the classical Noether map, cf. Section 4.2 in [9],

$$\eta_G^G : \mathbb{F}[V(G)]^G \longrightarrow \mathbb{F}[V]^G.$$

We note that $V(G)$ is the n-fold regular representation of G. Thus $\mathbb{F}[V(G)]^G$ are the n-fold vector invariants of the regular representation of G.

In the classical nonmodular case, where $p \nmid d$, the map η_G^G is surjective, see Proposition 4.2.2 in [9]. This has been generalized in the sense that the Noether map is surjective if V is a projective $\mathbb{F}G$-module, see Proposition 3.1 in [8]. The converse may fail as we illustrate with the next example.

EXAMPLE: Let $\mathrm{GL}(2, \mathbb{F}_3)$ be the general linear group of 2×2 matrices with entries from the field with three elements. By Corollary 9.14 in [4] the top Dickson class $\mathbf{d}_{2,0}$ is in the image of the transfer. Hence it is in the image of the Noether map. In order to see that also the other Dickson class $\mathbf{d}_{2,1}$ is in the image of the Noether map, we note that $\mathrm{GL}(2, \mathbb{F}_3)$ contains a supgroup H of order 6 generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda \in \mathbb{F}^{\times}$. Denote these six elements by $h_1, \dots, h_6$. Then the stabilizer subgroup of the monomial

$$(h_1 \otimes x_1) \cdots (h_6 \otimes x_1) \in \mathbb{F}[V(G)]$$

is H. Direct computation yields

$$\eta_G^G(o((h_1 \otimes x_1) \cdots (h_6 \otimes x_1))) = -\mathbf{d}_{2,1}.$$

In the next section we prove that whenever G is a p-group or ϱ is a permutation representation the Noether map is surjective if and only if V is a projective $\mathbb{F}G$-module.

Before we proceed we present a general characterization:

PROPOSITION: *V is projective if and only if*

$$\eta_G^G : \mathbb{F}[\mathrm{End}(V)(G)]^G \longrightarrow \mathbb{F}[\mathrm{End}(V)]^G$$

is surjective.

PROOF: V is projective if and only if $\mathrm{End}(V)$ is projective by [2]. Thus the Noether map on that vector space is surjective by Proposition 3.1 in [8]. Conversely, if η_G^G is surjective, then it is surjective in degree one. Hence the transfer map is surjective in degree one by Corollary 1.2 below. In particular, the identity on V is in the image of the transfer. Thus V is projective by the Higman criterion, see, e.g., Proposition 3.6.4 in [3]. □

§1. p-Groups and Permutation Representations

In this section we want to show that the converse Proposition 3.1 in [8] is true in the case of p-groups P and in the case of permutation representations.

LEMMA 1.1: *Let P be a cyclic p-group, and let $\mathbb{F}$ have characteristic p. Then*

$$\mathrm{Im}(\mathrm{Tr}^P)_{(1)} = \mathbb{F}[V]^P_{(1)}$$

if and only if V is the k-fold regular representation of P for some $k \in \mathbb{N}$.

PROOF: Since the transfer is additive it suffices to consider indecomposable modules only.

Let the order of the group be p^s. Then up to isomorphism there are exactly p^s indecomposable $\mathbb{F}P$-modules $V_1, \ldots, V_{p^s}$ with $\dim_{\mathbb{F}} V_i = i$. The action of P on V_i is afforded by the matrix consisting of one Jordan block with 1's on the diagonal and superdiagonal. Note that $V_i^P = V_1$ for all i.

Set $\Delta = g - 1$ where $g \in P$ is a generator. Then

$$\Delta(V_i^*) = \begin{cases} V_{i-1}^* & \text{for } i = 2, \ldots, p^s \\ 0 & \text{for } i = 1. \end{cases}$$

Since, $\mathrm{Tr}^P = \Delta^{p^s-1}$, we obtain

$$\mathrm{Tr}^P(V_i^*) = \Delta^{p^s-1}(V_i^*) = \begin{cases} 0 & \text{for } i = 1, \ldots, p^s - 1 \\ V_1^* & \text{for } i = p^s \end{cases}$$

as desired. □

We obtain the following corollary that we note here for later reference.

COROLLARY 1.2: *Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Let $i \in \mathbb{F}^\times$. Then*

$$\mathrm{Im}(\eta_G^G|_{(i)}) = \mathrm{Im}(\mathrm{Tr}^G|_{(i)}).$$

PROOF: By construction we obtain a commutative diagram as follows

$$\begin{array}{ccc} \mathbb{F}[V(G)]^G|_{(i)} & \xrightarrow{\eta_G^G|_{(i)}} & \mathbb{F}[V]^G|_{(i)} \\ \uparrow{\scriptstyle \mathrm{Tr}^G|_{(i)}} & & \uparrow{\scriptstyle \mathrm{Tr}^G|_{(i)}} \\ \mathbb{F}[V(G)]|_{(i)} & \xrightarrow{\eta_G|_{(i)}} & \mathbb{F}[V]|_{(i)}. \end{array}$$

By Theorem 3.2 [7] and the remark following it the transfer map on the left

$$\mathrm{Tr}^G|_{(i)} : \mathbb{F}[V(G)]|_{(i)} \longrightarrow \mathbb{F}[V(G)]^G|_{(i)}$$

is surjective. By construction the lower map $\eta_G|_{(i)}$ is surjective. Thus the result follows. □

THEOREM 1.3: *Let $\rho : P \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a p-group over a field $\mathbb{F}$ of characteristic p. Then the following are equivalent:*

(1) *The Noether map is surjective.*
(2) *The Noether map is surjective in degree one.*
(3) *V is a projective $\mathbb{F}P$-module.*

PROOF: The implication (1) $\Rightarrow$ (2) is trivial. The implication (3) $\Rightarrow$ (1) was proven in Proposition 3.1 in [8]. Thus we need to show that V is projective if $\eta_P^P|_{(1)}$ is surjective.

Consider the short exact sequence of $\mathbb{F}P$-modules

$$(*) \qquad 0 \longrightarrow K^* \longrightarrow V(P)^* \stackrel{\eta_P|_{(1)}}{\longrightarrow} V^* \longrightarrow 0.$$

The module $V(P)$ is free and therefore cohomologically trivial. Thus the long exact cohomology sequence breaks up into

$$0 \longrightarrow (K^*)^P \longrightarrow (V(P)^*)^P \stackrel{\eta_P^P|_{(1)}}{\longrightarrow} (V^*)^P \longrightarrow \mathrm{H}^1(P, K^*) \longrightarrow 0$$

and

$$\mathrm{H}^i(P, V^*) \cong \mathrm{H}^{i+1}(P, K^*) \quad \forall\, i \geq 1.$$

Since $\eta_P^P|_{(1)}$ is surjective by assumption, we obtain

$$\mathrm{H}^1(P, K^*) = 0.$$

Thus K^* is a projective $\mathbb{F}P$-module (see, e.g., Proposition 4.4.11 in [10]. Since P is finite and K^* finitely generated, this implies that K^* is injective, see Corollary 2.7 in [5]. Thus the sequence $(*)$ splits and V^* is projective as desired. □

We illustrate this result with an example.

EXAMPLE 1: Let $\mathbb{F}$ be the field with q elements of characteristic p. Let $P \leq \mathrm{GL}(n,\ \mathbb{F})$ be a p-Sylow subgroup of the general linear group. With assume without loss of generality that P consists of all upper triangular matrices with 1's on the diagonal. Then

$$\mathbb{F}[V(P)]_{(1)}^P = \mathrm{span}_{\mathbb{F}}\{o(x_{i1}) = \sum_{j=1}^{|P|} x_{ij} \mid i = 1, \ldots, n\}.$$

Thus

$$\begin{aligned}
\eta_P^P(o(x_{i1})) &= \sum_{j=1}^{|P|} g_j x_i \\
&= \sum_{(a_{i+1}, \ldots, a_n) \in \mathbb{F}^{n-i}} (x_i + a_{i+1}x_{i+1} + \cdots + a_n x_n) \\
&= q^{\frac{n(n-1)}{2} - (n-i)}(q^{n-i}x_i + q^{n-i-1}\left(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n\right)).
\end{aligned}$$

$$= q^{\frac{n(n-1)}{2}} x_i + q^{\frac{n(n-1)}{2}-1} \left(\sum_{a_{i+1}\in\mathbb{F}} a_{i+1}x_{i+1} + \cdots + \sum_{a_n\in\mathbb{F}} a_n x_n \right)).$$

If $n \leq 1$ then P is the trivial group. Therefore V is $\mathbb{F}P$-projective and the Noether map is surjective.

If $n \geq 2$ then the factor $q^{\frac{n(n-1)}{2}}$ vanishes. The factor $q^{\frac{n(n-1)}{2}-1}$ is nonzero if and only if $n = 2$. Thus we proceed by having a closer look at the two-dimensional case: We have by the above calculations

$$\eta_P^P(o(x_{11})) = \sum_{j=1}^{|P|} g_j x_1 = \sum_{a_2\in\mathbb{F}} (x_1 + a_2x_2) = (\sum_{a_2\in\mathbb{F}} a_2)x_2,$$
$$\eta_P^P(o(x_{21})) = \sum_{j=1}^{|P|} g_j x_2 = 0$$

If p is odd then for every nonzero $a_2 \in \mathbb{F}$ there exists a negative $-a_2 \neq a_2$. Therefore

$$\sum_{a_2\in\mathbb{F}} a_2 = 0.$$

If $p = 2$ then

$$(\sum_{a_2\in\mathbb{F}} a_2)x_2 = \begin{cases} x_2 & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}$$

Thus we have that the Noether map is surjective if and only if $n = 2 = p = q$. Explicitely we find

$$\eta_P^P(o(x_{11})) = x_2 \quad \text{and} \quad \eta_P^P(o(x_{11}x_{12})) = x_1^2 + x_1x_2.$$

Note that in this case

$$\mathrm{Syl}_2(\mathrm{GL}(2, \mathbb{F}_2)) \cong \mathbb{Z}/2$$

and our representation is projective.

Before proceeding to permutation representations, we want to mention two corollaries.

COROLLARY 1.4: *Let $\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a faithful representation of a finite group. Assume that the rings of invariants of G and its p-Sylow subgroup coincide in degree one. Then the Noether map is surjective if and only if V is $\mathbb{F}G$-projective.*

PROOF: If η_G^G is surjective, then it is surjective in degree one. Hence η_G^P is surjective in degree one by assumption. Therefore η_P^P is surjective in degree one by Proposition 2.1 in [8]. Thus V is projective by Theorem

1.3. The converse was shown in Proposition 3.1 in [8]. □

COROLLARY 1.5: *Let $G = H \times P$ be a direct product a p-group P and a p'-group H. Assume that P is a cyclic p-group. Consider a faithful representation ϱ of G over a field $\mathbb{F}$ of characteristic p such that V is indecomposable as an $\mathbb{F}P$-module. Then the Noether map is surjective if and only if V is $\mathbb{F}G$-projective.*

PROOF: If V is $\mathbb{F}G$-projective then the Noether map η_G^G is surjective by Proposition 3.1 in [8].

To prove the converse, let η_G^G be surjective. By Proposition 2.1 in [8] it is enough to show that the relative Noether map η_G^P is surjective. We proceed by contradiction and assume that η_G^P is not surjective. Then, by Proposition 2.1 in [8], the map η_P^P is not surjective. Hence V is not a projective $\mathbb{F}P$-module by Theorem 1.3.

Let σ be a generator for P. The isomorphism type of a P-module is determined by the Jordan canonical form of σ. Up to isomorphism there are $|P|$ indecomposable P modules $V_1, V_2, \ldots V_{|P|}$, where $\dim V_i = i$ and σ acts on V_i by a $i \times i$ matrix consisting of a single Jordan block with ones on the diagonal and superdiagonal. Moreover $V_{|P|}$ is the only indecomposable module which is projective. Thus by assumption we have that $V = V_n$ for $1 \le n < |P|$.

Let $x_1, x_2, \ldots, x_n$ be the basis of V such that

$$\sigma x_i = \begin{cases} x_1 & \text{if } i = 1 \\ x_{i-1} + x_i & \text{otherwise.} \end{cases}$$

Since the action of P commutes with the action of H and the action of H is nonmodular, it follows that $V = V_n$ is a direct sum of copies of isomorphic eigen spaces for H, and the variables $x_1, x_2, \ldots, x_n$ may be taken as eigen vectors. Let $\mathsf{N} = \prod_{g \in P} g(x_n)$ be the norm of x_n. Since p and $|H|$ are relatively prime, there exists positive integer m such that $m|P| \equiv -1 \text{MOD} |H|$. Consider the polynomial $x_1\mathsf{N}^m$. This polynomial is P-invariant since both x_1 and N are. Let $h \in H$. Then

$$h(x_1\mathsf{N}^m) = \lambda_h x_1 \lambda_h^{m|P|} \mathsf{N}^m = x_1\mathsf{N}^m.$$

It follows that $x_1\mathsf{N}^m$ is G-invariant.

Next we want to see that $x_1\mathsf{N}^m$ is not in the image of Tr^P. Since V is not projective, the fixed point x_1 is not in the image of Tr^P. The degree-one-component $\mathbb{F}[V]_{(1)}$ is a direct summand in $\mathbb{F}[V]_{m|P|+1}$ by multiplication by N, [6]. Thus the invariant $x_1\mathsf{N}^m$ is not in the image of Tr^P either. However, if a G-invariant polynomial is not in the image of Tr^P then it is not in the image of Tr^G.

Since the degree of the polynomial $x_1\mathbf{N}^m$ is relatively prime to p, we have that it is not in the image of η_G^G by Corollary 1.2. This is a contradiction. □

COROLLARY 1.6: *Let $P \cong \mathbb{Z}/p$ and let V be an indecomposable P-module. Then the Noether map η_P^P is surjective in degrees divisible by p.*

PROOF: As above denote by $V = V_n$ the indecomposable $\mathbb{F}\mathbb{Z}/p$-modules and $x_1, x_2, \dots, x_n$ be the basis for V on which $\mathbb{Z}/p$ acts through a single Jordan block of dimension n. We note that

$$\mathbb{F}[V] = B \oplus \mathbf{N}\mathbb{F}[V]$$

as $\mathbb{F}P$-modules, where B consists of the polynomials of x_n-degree less than p, [6].

We proceed by induction on the degree. The decomposition

$$\mathbb{F}[V]^P_{(p)} = B^P_{(p)} \bigoplus \mathbf{N}\mathbb{F}[V]^P$$

yields that any invariant in degree p is a direct summand of a fixed point of a free module and the polynomial $\mathbf{N}$. Since fixed points of free modules and $\mathbf{N}$ are in the image of η_P^P, the result follows for degree p.

Using the decomposition for degree kp we have that

$$\mathbb{F}[V]^P_{(kp)} = B^P_{(kp)} \bigoplus \mathbf{N}\mathbb{F}[V]^P_{((k-1)p)}.$$

Since η_P^P is an algebra map, and $\mathbb{F}[V]^P_{((k-1)p)}$ is in the image of η_P^P by induction, the result follows. □

We turn to permutation representations.

THEOREM 1.7: *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a permutation representation of a finite group of order d. Then the Noether map η_G^G is surjective if and only if $V = \mathbb{F}^n$ is projective.*

PROOF: By Proposition 3.1 in [8] we know that η_G^G is surjective if V is projective as $\mathbb{F}G$-module.

We show that the converse is also true as follows:

Let η_G^G be surjective, then its restriction to degree one, $\eta_G^G|_{(1)}$, is also surjective:

$$\eta_G^G|_{(1)} : (V(G)^*)^G \longrightarrow (V^*)^G.$$

We note that $(V(G)^*)^G$ has an $\mathbb{F}$-basis consisting of

$$o(x_{ij}) = \sum_{j=1}^{d} x_{ij} \quad \text{for } i = 1, \ldots, n.$$

Therefore, the image under the Noether map is spanned by

$$\eta_G^G\left(\sum_{j=1}^{d} x_{ij}\right) = k_i o(x_i) = |\mathrm{Stab}_G(x_i)|\mathrm{Tr}^G(x_i) \quad \text{for } i = 1, \ldots, n,$$

where

$$k_i = |\mathrm{Stab}_G(x_i)|$$

is the order of the stabilizer of x_i in G. Since ϱ is a permutation representation, $(V^*)^G$ is spanned by the orbit sums of $x_1, \ldots, x_n$. It follows that k_i's are not zero, since the Noether map is surjective. Hence

$$|\mathrm{Stab}_G(x_i)| \not\equiv 0 \bmod p.$$

In other words, no element in a p-Sylow subgroup P of G fixes x_i, $i = 1, \ldots, n$. Therefore

$$(\maltese) \qquad o^P(x_i) = \mathrm{Tr}^P(x_i) = \eta_P^P|_{(1)}(x_{i1}),$$

where $o^P(-)$ denotes the orbit sum under the action of P, and g_1 is the identity element. Since $(V^*)^P$ is also spanned by the orbit sums of the x_i's, we found in ($\maltese$) that $\eta_P^P|_{(1)}$ is surjective. Therefore, η_P^P is surjective by Proposition 1.3. Hence V^* is a projective $\mathbb{F}P$-module, by the same Propositon 1.3. Since P is a p-Sylow subgroup of G, the module V^* is projective as a $\mathbb{F}G$-module, see Corollary 3 on Page 66 of [1]. □

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