The Noether Map

Mara D. Neusel  
DEPT. OF MATH. AND STATS.  
TEXAS TECH UNIVERSITY  
MS 1042  
LUBBOCK, TX 79409  
USA

MUFIT SEZER  
DEPARTMENT OF MATHEMATICS  
BOĞAZICI UNIVERSITESI  
BEBEK  
ISTANBUL  
TURKEY

October 23rd 2005

AMS CODE: 13A50 Invariant Theory, 20J06 Group Cohomology
KEYWORDS: Invariant Theory of Finite Groups, Integral Closure, Noether Map, Modular Invariant Theory, Orbit Chern Classes, Transfer, Projective $\mathbb{P}G$-Modules, Tate Cohomology, Degree Bounds, Cohen-Macaulay Defect
The first author is partially supported by NSA Grant No. H98230-05-1-0026

Typeset by $\LaTeX$
SUMMARY: Let $\rho: G \hookrightarrow GL(n, \mathbb{F})$ be a faithful representation of a finite group $G$. In this paper we study the image of the associated Noether map

$$\eta^G_G : \mathbb{F}[V(G)]^G \rightarrow \mathbb{F}[V]^G.$$ 

It turns out that the image of the Noether map characterizes the ring of invariants in the sense that its integral closure $\overline{\text{Im}(\eta^G_G)} = \mathbb{F}[V]^G$. This is true without any restrictions on the group, representation, or ground field. Furthermore, we show that the Noether map is surjective, i.e., its image integrally closed, if $V = \mathbb{F}^n$ is a projective $\mathbb{F}G$-module. Moreover, we show that the converse of this statement is true if $G$ is a $p$-group and $\mathbb{F}$ has characteristic $p$, or if $\rho$ is a permutation representation. We apply these results and obtain upper bounds on the Noether number and the Cohen-Macaulay defect of $\mathbb{F}[V]^G$. We illustrate our results with several examples.
Let $\rho: G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a faithful representation of a finite group $G$ over a field $\mathbb{F}$. The representation $\rho$ induces naturally an action of $G$ on the vector space $V = \mathbb{F}^n$ of dimension $n$ and hence on the ring of polynomial functions $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$. Our interest is focused on the subring of invariants

$$\mathbb{F}[V]^G = \{ f \in \mathbb{F}[V] \mid gf = f \ \forall \ g \in G \},$$

which is a graded connected Noetherian commutative algebra.

In the first section of this paper we introduce the Noether map and show that its image characterizes the ring of invariants. In Section 2 we consider projective $\mathbb{F}G$-modules $V$, and show that the Noether map is surjective in this case. The next section deals with the converse: In Section 3 we show that the Noether map is surjective if and only if $V$ is $\mathbb{F}G$-projective in the cases of $p$-groups and of permutation representations. In Section 4 we derive some results about degree bounds and the Cohen-Macaulay defect of $\mathbb{F}[V]^G$. Furthermore we present some examples.

§1. The Noether Map

Let $\rho: G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a representation of a group $G$ of order $d$. Let $\mathbb{F}[V]$ be the symmetric algebra on $V^*$. Denote by $\mathbb{F}G$ the group algebra. Let

$$V(G) = \text{Hom}_\mathbb{F}(\mathbb{F}G, V) \cong \mathbb{F}G \otimes V$$

be the coinduced module $\text{coind}_1^G(V)$. The group $G$ acts on $V(G)$ by left multiplication on the first component. We obtain a $G$-equivariant surjection

$$V(G) \twoheadrightarrow V, (g, v) \mapsto gv.$$  

Let us choose a basis $e_1, \ldots, e_n$ for $V$. Let $x_1, \ldots, x_n$ be the standard dual basis for $V^*$, and set $G = \{g_1, \ldots, g_d\}$. Then $V(G)$ can be written as

$$V(G) = \text{span}_\mathbb{F}\{e_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\},$$

and the map $(\star)$ translates into

$$V(G) \twoheadrightarrow V, \ e_j \mapsto g_j e_j.$$

Similarly, we have

$$V(G)^* = \text{span}_\mathbb{F}\{x_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, d\}$$

with

$$V(G)^* \twoheadrightarrow V^*, \ x_{ij} \mapsto g_j x_i.$$

We obtain a surjective $G$-equivariant map between the rings of polyn-
mial functions
\[ \eta_G : \mathbb{F}[V(G)] \to \mathbb{F}[V]. \]
The group $G$ acts on $\mathbb{F}[V(G)]$ by permuting the basis elements $x_{ij}$. By restriction to the induced ring of invariants, we obtain the classical Noether map, cf. Section 4.2 in [11],
\[ \eta_G^G : \mathbb{F}[V(G)]^G \to \mathbb{F}[V]^G. \]
We note that $V(G)$ is the $n$-fold regular representation of $G$. Thus $\mathbb{F}[V(G)]^G$ are the $n$-fold vector invariants of the regular representation of $G$.

In the classical nonmodular case, where $p \nmid d$, the map $\eta_G$ is surjective, see Proposition 4.2.2 in [11]. This does not remain true in the modular case as we illustrate in the next example.

**Example 1:** Let $\rho : \mathbb{Z}/2 \to \text{GL}(3, \mathbb{F}_2)$ be the 3-dimensional representation of $\mathbb{Z}/2$ over the field with two elements afforded by the matrix
\[
\rho(g) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Then
\[ \mathbb{F}[x_1, x_2, x_3]_{\mathbb{Z}/2} = \mathbb{F}[x_1 + x_2, x_1 x_2, x_3] \]
and
\[ \mathbb{F}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}]_{\mathbb{Z}/2} = \mathbb{F}[x_{11} + x_{12}, x_{11} x_{12} + x_{12} x_{11}, x_{11} x_{21} x_{31} + x_{12} x_{22} x_{32}], \]
where $i \in \mathbb{Z}/3$, cf. Example 2 in Section 2.3, [11] or Example 1 in Section 3.2, loc.cit. We obtain
\[ \text{Im}(\eta_{\mathbb{Z}/2}) = \mathbb{F}[x_1 + x_2, x_1 x_2, x_3^2, (x_1 + x_2)x_3]. \]
Thus the Noether map is no longer surjective, because the invariant $x_3$ is not in its image. However, note that the integral closure of the image of the Noether map is the ring of invariants $\mathbb{F}[V]^G$. This is always true as we see in this section.

Recall the **transfer map**
\[ \text{Tr}^G : \mathbb{F}[V] \to \mathbb{F}[V]^G; \ f \mapsto \sum_{g \in G} gf, \]
see, e.g., Section 2.2. in [11]. By construction the transfer is an $\mathbb{F}[V]^G$-module homomorphism. We denote by
\[ \mathbb{F}[\text{Im}(\text{Tr}^G)] \subseteq \mathbb{F}[V]^G \]
THE NOETHER MAP

the subalgebra generated by the image of the transfer.

We observe that any element

$$\frac{f_1}{f_2} \in \mathbb{F}(V)$$

can be written as the quotient of some polynomial by an invariant polynomial in the following way

$$\frac{f_1}{f_2} = \frac{f_1 N(f_2)}{N(f_2)}$$

where $N(f) = \prod_{g \in G} gf$ denotes the \textbf{Norm} of $f$. This allows us to extend

the transfer to a map of $\mathbb{F}(V)^G$-modules between the respective fields of fractions

$$\text{Tr}^G : \mathbb{F}(V) \to \mathbb{F}(V)^G; \frac{f_1}{f_2} \mapsto \sum_{g \in G} gf_1$$

where we assume that $f_2 \in \mathbb{F}[V]^G$.

\textbf{PROPOSITION 1.1:} We have that

$$\mathbb{F}(\text{Tr}^G(\mathbb{F}(V))) = \mathbb{F}[\text{Im(Tr}^G)] = \mathbb{F}(V)^G,$$

where $\mathbb{F}[\cdot]$ denotes the field of fractions functor.

\textbf{PROOF:} Let $\frac{\text{Tr}^G(f_1)}{\text{Tr}^G(f_2)} \in \mathbb{F}[\text{Im(Tr}^G)]$. Then

$$\frac{\text{Tr}^G(f_1)}{\text{Tr}^G(f_2)} = \text{Tr}^G \left( \frac{f_1}{\text{Tr}^G(f_2)} \right) \in \text{Tr}^G(\mathbb{F}(V)).$$

To prove the reverse inclusion take an element

$$\frac{\text{Tr}^G(f_1)}{\text{Tr}^G(f_2)} \in \text{Tr}^G(\mathbb{F}(V)),$$

where $f_2 \in \mathbb{F}[V]^G$. Choose a polynomial $f \in \mathbb{F}[V]$ such that $\text{Tr}^G(f) \neq 0$.
(Recall that the transfer map is never zero by Proposition 2.2.4 in [11].)

Then we have

$$\text{Tr}^G \left( \frac{f_1}{\text{Tr}^G(f_2)} \right) \in \mathbb{F}[\text{Im(Tr}^G)].$$

We come to the second equality. Since $\mathbb{F}([\text{Im(Tr}^G)] \subset \mathbb{F}[V]^G$ we have that

$$\mathbb{F}[\text{Im(Tr}^G)] \subset \mathbb{F}(V)^G.$$

To prove the reverse inclusion, let $\frac{f_1}{f_2} \in \mathbb{F}(V)^G$ where without loss of gen-
equality $f_1, f_2 \in \mathbb{F}[V]$. Let $\text{Tr}^G(f) \neq 0$ for some suitable $f \in \mathbb{F}[V]$. Thus

$$\frac{f_1}{f_2} = \frac{\text{Tr}^G(f) f_1}{\text{Tr}^G(f) f_2} = \frac{\text{Tr}^G(ff_1)}{\text{Tr}^G(ff_2)} \in \mathbb{F}[\mathbb{F}[\text{Im}(\text{Tr}^G)]]$$

as desired. □

**Proposition 1.2:** The integral closure of the image of the Noether map is the ring of invariants

$$\overline{\text{Im}(\eta^G)} = \mathbb{F}[V]^G.$$

**Proof:** By Proposition 1.1 and Lemma 4.2.1 in [11] we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{F}[\text{Im}(\text{Tr}^G)] & \subseteq & \mathbb{F}[\text{Im}(\eta^G)] \subseteq \mathbb{F}[V]^G \subset \mathbb{F}[V] \\
\downarrow & & \downarrow \\
\mathbb{I}\mathbb{F}(\mathbb{F}[\text{Im}(\text{Tr}^G)]) & = & \mathbb{I}\mathbb{F}(\mathbb{F}[\text{Im}(\eta^G)]) = \mathbb{F}(V)^G \subset \mathbb{F}(V).
\end{array}$$

Let $x_1, \ldots, x_n \in V^*$ be a basis. Then the coefficients of the polynomials

$$F_i(X) = \prod_{g \in G} (X - gx_i),$$

are the orbit chern classes of $x_i$ counted with multiplicities

$$\sigma_1(x_i) = \text{Tr}^G(x_i), \ldots, \sigma_d(x_i) = \text{N}(x_i).$$

Thus they are in the image of $\eta^G$. Denote by $A$ the $\mathbb{F}$-algebra generated by these coefficients. By construction $A$ is finitely generated, thus noetherian. Furthermore $\mathbb{F}[V]$ is finitely generated as an $A$-module, thus as an $\text{Im}(\eta^G)$-module since $A \subseteq \text{Im}(\eta^G)$. Therefore the extension

$\text{Im}(\eta^G) \subset \mathbb{F}[V]$

is finite, and

$$\overline{\text{Im}(\eta^G)} = \mathbb{F}[V]^G$$

as desired. □

We close this section with an immediate corollary of the preceding result:

**Corollary 1.3:** The Krull dimension of the image of the Noether map coincides with the Krull dimension of the ring of invariants, which in turn is equal to $n = \dim_{\mathbb{F}} V$. □

**Addendum:** Define a map $E : \mathbb{F}[V] \to \mathbb{F}[V(G)]^G$, $x_i \mapsto \sum_{j=1}^d x_j$. Then we obtain a
THE NOETHER MAP

commutative triangle as follows:

\[
\begin{array}{ccc}
\mathbb{F}[V(G)]^G & \xrightarrow{\eta^G} & \mathbb{F}[V]^G \\
\downarrow{\mathrm{Tr}^G} & & \\
\mathbb{F}[V] & &
\end{array}
\]

If \(p \nmid d\), then the preceding diagram proves that the Noether map is surjective, since the transfer is surjective, see Lemma 4.2.1 in [11]. We want to add the following observation:

**Proposition 1.4:** The algebra generated by the image of the transfer map is equal to the image of the Noether map if and only if \(V\) is a nonmodular \(\mathbb{F}G\)-module.

**Proof:** By Lemma 4.2.1 in [11] the image of the transfer is always contained in the image of the Noether map. Thus if \(p \nmid |G|\), then the transfer is surjective, and hence the Noether map. If \(p \mid |G|\), then the transfer is no longer surjective. Indeed, the height of the image of the transfer is at most \(n - 1\), see Theorem 6.4.7 in [11]. Thus the Krull dimension of \(\mathbb{F}[\text{Im}(\mathrm{Tr}^G)]\) is strictly less than \(n\). On the other hand the Krull dimension of the image of the Noether map is \(n\) by Proposition 1.2. Thus they cannot be equal.

\(\Box\)

§2. Projective Modules

In this section we want to study the question when the Noether map is surjective.

We note that the \(\mathbb{F}G\) module \(V\) is projective if and only if its dual vector space \(V^*\) is injective which in turn is equivalent to projective because \(G\) is a finite group. We will make frequently use of this fact in what follows.

**Proposition 2.1:** If \(V\) is a projective \(\mathbb{F}G\)-module, then the Noether map is surjective.

**Proof:** By construction we have a short exact sequence of \(\mathbb{F}G\)-modules as follows

\[
0 \rightarrow W^* \rightarrow V(G)^* \rightarrow V^* \rightarrow 0.
\]

Since \(V^*\) is projective, this sequence splits and

\[
V(G)^* \cong V^* \oplus W^* \xrightarrow{\text{pr}} V^*
\]

as \(\mathbb{F}G\)-modules. Taking invariants we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathbb{F}[V(G)]^G & \xrightarrow{\varphi^*} & \mathbb{F}[V \oplus W]^G \\
\downarrow{\eta^G} & & \downarrow{\text{pr}^*} \\
\mathbb{F}[V]^G & &
\end{array}
\]

Thus \(\eta^G\) is surjective because \(\varphi^*\) as well as \(\text{pr}^*\) are. \(\Box\)
Remark: Since nonmodular $\mathbb{F}G$-modules are always projective we recover the classical result that $\eta_G^G$ is surjective for every nonmodular representation of $G$.

Corollary 2.2: Let $\rho: G \hookrightarrow GL(p, \mathbb{F})$ be a permutation representation of the finite group $G$ over a field $\mathbb{F}$ of characteristic $p$. Then $\eta_G^G$ is surjective.

Proof: Let $\psi: \Sigma_p \hookrightarrow GL(p, \mathbb{F})$ be the defining representation of the symmetric group in $p$ letters. Since $\rho$ is a permutation representation we have that $\rho(G) \leq \psi(\Sigma_p) \leq GL(p, \mathbb{F})$.

Since $V = \mathbb{F}^p$ is a projective $\Sigma_p$-module it is projective as a $\mathbb{F}G$-module. Thus by Proposition 2.1 the Noether map $\eta_G^G$ is surjective. □

Example 1: If $\psi: \Sigma_p \hookrightarrow GL(n, \mathbb{F})$ is the defining representation of the symmetric group in $n$ letter over a field of characteristic $p$, where $p < n$, then neither $V$ is projective as a module over $\Sigma_n$ nor is $\eta_{\Sigma_n}^{\Sigma_n}$ surjective. The latter is true because in degree one we have

$$\mathbb{F}[V(\Sigma_n)]_{(1)}^{\Sigma_n} = \text{span}_\mathbb{F}\{\sum_{j=1}^{n!} x_{ij} | i = 1, \ldots, n\}$$

and thus

$$\eta_{\Sigma_n}^{\Sigma_n}(\sum_{j=1}^{n!} x_{ij}) = (n-1)! \sum_{i=1}^{n} x_i \equiv 0 \text{ mod } p.$$ 

Therefore the first elementary symmetric function $e_1 = x_1 + \cdots + x_n \in \mathbb{F}[V]^{\Sigma_n}$ is not hit. Therefore, $V$ is not $\mathbb{F}\Sigma_n$-projective. This is not a new result: For the defining representation $\psi: \Sigma_n \hookrightarrow GL(n, \mathbb{F})$, $V = \mathbb{F}^n$ is a projective $\mathbb{F}\Sigma_n$-module if and only if $p \geq n$. This follows from Corollary 7 on Page 33 of [1]. See Theorem 3.5 in Section 3 for a generalization of this.

Example 2: Let $\psi: A_n \hookrightarrow GL(n, \mathbb{F})$ be the defining representation of the alternating group in $n$ letters over a field of characteristic $p$. By Corollary 2.2 the Noether map $\eta_{A_n}^{A_n}$ is surjective if $n \leq p$. We want to check what happens if $n > p$.

We start by considering the Noether map

$$\eta_{A_n}^{A_n}: \mathbb{F}[V(A_n)]^{A_n} \rightarrow \mathbb{F}[V]^{A_n}$$

1 For a graded object $A$ we denote the homogeneous degree $i$-part by $A(i)$.
THE NOETHER MAP

in degree one. We have

$$\mathbb{F}[V(A_n)]^{A_n}_{(1)} = \text{span}_\mathbb{F}\{\sum_{j=1}^{A_n} x_{ij} \mid i = 1, \ldots, n\}$$

and

$$\mathbb{F}[V]^{A_n}_{(1)} = \text{span}_\mathbb{F}\{e_1 = x_1 + \cdots + x_n\}.$$ 

Thus we have

$$\eta_{A_n}^{A_n}(\sum_{j=1}^{A_n} x_{ij}) = |\text{Stab}_{A_n}(x_i)| e_1 = |A_{n-1}| e_1 = \frac{(n-1)!}{2} e_1.$$

Thus the elementary symmetric function $e_1$ is in the image of the Noether map if and only if

$$\frac{(n-1)!}{2} \in \mathbb{F}^\times.$$ 

This in turn happens exactly when

1. $p$ is odd and $p \geq n$,
2. $p = 2$ and $n \leq 4$.

We know already that the Noether map is surjective in the first case. If $p$ is even and $n \leq 3$ we are in the nonmodular case, so the Noether map is again surjective. Thus the only case that we have to check by hand is the defining representation of $A_4$ over a field of characteristic 2.

We note that the 2-Sylow subgroup of $A_4$ is the Klein-Four-Group $\mathbb{Z}/2 \times \mathbb{Z}/2$. When we restrict $\psi |_{\mathbb{Z}/2 \times \mathbb{Z}/2}$ we obtain the regular representation of $\mathbb{Z}/2 \times \mathbb{Z}/2$. Thus $V$ is $\mathbb{F}(\mathbb{Z}/2 \times \mathbb{Z}/2)$-projective. Therefore, $V$ is $\mathbb{F}A_4$-projective. Hence the Noether map is surjective. Indeed, a short calculation shows that

$$\eta_{A_4}^{A_4}(o(x_{11})) = 3e_1 = e_4,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{12})) = e_2,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{21}x_{31})) = 3e_3 = e_5,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{12}x_{13}x_{14})) = 3e_4 = e_6,$$
$$\eta_{A_4}^{A_4}(o(x_{11}x_{21}x_{31})) = o(x_{1}^3x_2^2x_3),$$

where $o(-)$ denotes the orbit sum of $\cdot$, and $g_1 = (1)$, $g_2 = (12)(34)$, $g_3 = (13)(24)$, and $g_4 = (14)(23)$. 

7
§3. $p$-Groups and Permutation Representations

For nonmodular representations the Noether map is always surjective and $V$ is always projective. Therefore, we restrict ourselves to modular representations in what follows.

In this section we want to show that the converse Proposition 2.1 is true in the case of $p$-groups $P$ and in the case of permutation representations. The next two results settle the case of $P \cong \mathbb{Z}/p$.

**Lemma 3.1:** Let $P$ be a cyclic $p$-group, and let $F$ have characteristic $p$. Then

$$\text{Im}(\text{Tr}_P(1)) \subsetneq F[V]^P$$

unless $V$ is the $k$-fold regular representation of $P$ for some $k \in \mathbb{N}$.

**Proof:** Since the transfer is additive it suffices to consider indecomposable modules only.

Let the order of the group be $p^s$. Then up to isomorphism there are exactly $p^s$ indecomposable $FP$-modules $V_1, \ldots, V_{p^s}$ with $\dim_F V_i = i$. The action of $P$ on $V_i$ is afforded by the matrix consisting of one Jordan block with 1's on the diagonal and superdiagonal. Note that $V_i^P = V_1$ for all $i$.

Set $\Delta = g - 1$ where $g \in P$ is a generator. Then

$$\Delta(V_i^*) = \begin{cases} V_{i-1}^* & \text{for } i = 2, \ldots, p^s \\ 0 & \text{for } i = 1. \end{cases}$$

Since, $\text{Tr}_P = \Delta^{p^s-1}$, we obtain

$$\text{Tr}_P(V_i^*) = \Delta^{p^s-1}(V_1^*) = \begin{cases} 0 & \text{for } i = 1, \ldots, p^s - 1 \\ V_1^* & \text{for } i = p^s \end{cases}$$

as desired. $\square$

In Theorem 3.2 [8] (and the following remark) a more precise version of the preceding result is shown: the transfer is surjective in degrees prime to the characteristic in the case of $k$-fold regular representations. We obtain the following corollary that we note here for later reference.

**Corollary 3.2:** Let $\rho : G \to \text{GL}(n, F)$ be a faithful representation of a finite group. Let $i \in F^\times$. Then

$$\text{Im}(\eta^G_{\rho(i)}) = \text{Im}(\text{Tr}^G(i)).$$
**The Noether Map**

**Proof**: By construction we obtain a commutative diagram as follows:

\[
\begin{array}{ccc}
\mathbb{F}[V(G)]^G |_{(i)} & \xrightarrow{\eta_G|_{(i)}} & \mathbb{F}[V]^G |_{(i)} \\
\text{Tr}_G|_{(i)} & & \text{Tr}_G|_{(i)} \\
\mathbb{F}[V(G)] |_{(i)} & \xrightarrow{\eta_G|_{(i)}} & \mathbb{F}[V] |_{(i)}.
\end{array}
\]

By Theorem 3.2 [8] and the remark following it the transfer map on the left

\[\text{Tr}_G|_{(i)} : \mathbb{F}[V(G)] |_{(i)} \longrightarrow \mathbb{F}[V(G)]^G |_{(i)}\]

is surjective. By construction the lower map \(\eta_G|_{(i)}\) is surjective. Thus the result follows. □

Even though Proposition 3.4 contains the following result as a special case, we want to leave the proof in, because it is so simple and uses just some linear algebra, cf. Lemma 3.2 in [6].

**Proposition 3.3**: Let \(G = P\) a cyclic \(p\)-group. Then the following are equivalent

1. The Noether map is surjective.
2. The Noether map is surjective in degree one.
3. \(V\) is a projective \(FP\)-module.

**Proof**: The implication (1) \(\Rightarrow\) (2) is trivial. The implication (3) \(\Rightarrow\) (1) was proven in Proposition 2.1. Thus we need to show that \(V\) is projective if \(\eta_P|_{(1)}\) is surjective.

By Corollary 3.2 we have that \(\text{Im}(\eta_G|_{(i)}) = \text{Im}(\text{Tr}_G|_{(i)})\). Since the transfer is surjective in degree one exactly when \(V\) is a \(k\)-fold regular representation by Lemma 3.1, we have that \(V\) is the \(k\)-fold regular representation and hence projective. □

**Theorem 3.4**: Let \(\rho : P \hookrightarrow \text{GL}(n, \mathbb{F})\) be a representation of a \(p\)-group over a field \(\mathbb{F}\) of characteristic \(p\). Then the following are equivalent:

1. The Noether map is surjective.
2. The Noether map is surjective in degree one.
3. \(V\) is a projective \(FP\)-module.

**Proof**: The implication (1) \(\Rightarrow\) (2) is trivial. The implication (3) \(\Rightarrow\) (1) was proven in Proposition 2.1. Thus we need to show that \(V\) is projective if \(\eta_P|_{(1)}\) is surjective.

Consider the short exact sequence of \(FP\)-modules

\[(*) \quad 0 \longrightarrow K^* \longrightarrow V(P)^* \xrightarrow{\eta_P|_{(1)}} V^* \longrightarrow 0.
\]
The module $V(P)$ is free and therefore cohomologically trivial. Thus the long exact cohomology sequence breaks up into

$$0 \rightarrow (K^*)^P \rightarrow (V(P)^*)^P \xrightarrow{\eta_P} (V^*)^P \rightarrow H^1(P, K^*) \rightarrow 0$$

and

$$H^i(P, V^*) \cong H^{i+1-i+1+i}(P, K^*) \quad \forall \ i \geq 1.$$ 

Since $\eta_P \mid_{\langle 1 \rangle}$ is surjective by assumption, we obtain

$$H^1(P, K^*) = 0.$$ 

Thus

$$\widetilde{H}^1(P, K^*) = H^1(P, K^*) = 0,$$

where $\widetilde{H}^*(-, -)$ denotes the Tate cohomology. Thus $K^*$ is a projective $\mathbb{F}_P$-module by Theorem 8.5, Chapter VI in [2]. Since $P$ is finite and $K^*$ finitely generated, this implies that $K^*$ is injective, see Corollary 2.7 in [3]. Thus the sequence $(\ast)$ splits and $V^*$ is projective as desired. □

We illustrate this result with an example.

**Example 1:** Let $\mathbb{F}$ be the field with $q$ elements of characteristic $p$. Let $P \leq \text{GL}(n, \mathbb{F})$ be a $p$-Sylow subgroup of the general linear group. With assume without loss of generality that $P$ consists of upper triangular matrices with 1’s on the diagonal. Then

$$\mathbb{F}[V(P)]^P = \text{span}_\mathbb{F}\{ o(x_{i1}) = \sum_{j=1}^{\frac{p}{p}} x_{ij} \mid i = 1, \ldots, n \}. $$

Thus

$$\eta_P(o(x_{i1})) = \sum_{j=1}^{\frac{p}{p}} g_j x_j$$

$$= \sum_{(a_{i+1}, \ldots, a_n) \in \mathbb{F}^{n-i}} (x_i + a_{i+1} x_{i+1} + \cdots + a_n x_n)$$

$$= q^{\frac{n(n-1)}{2}-(n-i)}(q^{n-i} x_i + q^{n-i-1} \left( \sum_{a_{i+1} \in \mathbb{F}} a_{i+1} x_{i+1} + \cdots + \sum_{a_n \in \mathbb{F}} a_n x_n \right)).$$

$$= q^{\frac{n(n-1)}{2}} x_i + q^{\frac{n(n-1)}{2}-1} \left( \sum_{a_{i+1} \in \mathbb{F}} a_{i+1} x_{i+1} + \cdots + \sum_{a_n \in \mathbb{F}} a_n x_n \right).$$

The factor $q^{\frac{n(n-1)}{2}}$ is nonzero if and only if $n = 0$ or $n = 1$. Since we are considering the modular case this cannot happen.
THE NOETHER MAP

The factor \( q^{\frac{n(n-1)}{2}} - 1 \) is nonzero if and only if \( n = 2 \).

Thus we proceed by having a closer look at the two-dimensional case:

We have by the above calculations

\[
\eta_p^G(o(x_{11})) = \sum_{j=1}^{p|} g_j x_1 = \sum_{a_2 \in \mathbb{F}} (x_1 + a_2 x_2) = (\sum_{a_2 \in \mathbb{F}} a_2) x_2,
\]

\[
\eta_p^G(o(x_{21})) = \sum_{j=1}^{p|} g_j x_2 = 0
\]

If \( p \) is odd then for every nonzero \( a_2 \in \mathbb{F} \) there exists a negative \(-a_2 \neq a_2\). Therefore

\[
\sum_{a_2 \in \mathbb{F}} a_2 = 0.
\]

If \( p = 2 \) then

\[
(\sum_{a_2 \in \mathbb{F}} a_2) x_2 = \begin{cases} x_2 & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}
\]

Thus we have that the Noether map is surjective if and only if \( n = 2 = p = q \). Explicitly we find

\[
\eta_p^G(o(x_{11})) = x_2 \quad \text{and} \quad \eta_p^G(o(x_{11} x_{12})) = x_1^2 + x_1 x_2.
\]

Note that in this case

\[
\text{Syl}_2(\text{GL}(2, \mathbb{F})) \cong \mathbb{Z}/2
\]

and our representation is projective.

**Theorem 3.5:** Let \( \rho : G \rightarrow \text{GL}(n, \mathbb{F}) \) be a permutation representation of a finite group of order \( d \). Then the Noether map \( \eta_G^G \) is surjective if and only if \( V = \mathbb{F}^n \) is projective.

**Proof:** By Proposition 2.1 we know that \( \eta_G^G \) is surjective if \( V \) is projective as \( \mathbb{F}G \)-module.

We show that the converse is also true as follows:

Let \( \eta_G^G \) be surjective, then its restriction to degree one, \( \eta_G^G |_{(1)} \), is also surjective:

\[
\eta_G^G |_{(1)} : (V(G)^*)^G \rightarrow (V^*)^G.
\]

We note that \( (V(G)^*)^G \) has an \( \mathbb{F} \)-basis consisting of

\[
o(x_{ij}) = \sum_{j=1}^{d} x_{ij} \quad \text{for } i = 1, \ldots, n.
\]
Therefore, the image under the Noether map is spanned by

\[ \eta^G_G \left( \sum_{j=1}^d x_{ij} \right) = k_i \circ(x_i) \text{ for } i = 1, \ldots, n, \]

where

\[ k_i = |\text{Stab}_G(x_i)| \]

is the order of the stabilizer of \( x_i \) in \( G \). Since \( \rho \) is a permutation representation, \((V^*)^G\) is spanned by the orbit sums of \( x_1, \ldots, x_n \). It follows that \( k_i \)'s are not zero, since the Noether map is surjective. Hence

\[ |\text{Stab}_G(x_i)| \neq 0 \mod p. \]

In other words, no element in a \( p \)-Sylow subgroup \( P \) of \( G \) fixes \( x_i, i = 1, \ldots, n \). Therefore

\[ o^P(x_i) = \text{Tr}^P(x_i) = \eta^P_P \mid_{(1)}(x_{i1}), \]

where \( o^P(\cdot) \) denotes the orbit sum under the action of \( P \), and \( g_1 \) is the identity element. Since \((V^*)^P\) is also spanned by the orbit sums of the \( x_i \)'s, we found in (\( \star \)) that \( \eta^P_P \mid_{(1)} \) is surjective. Therefore, \( \eta^P_P \) is surjective by Proposition 3.4. Hence \( V^* \) is a projective \( \mathbb{F}P \)-module, by the same Proposition 3.4. Since \( P \) is a \( p \)-Sylow subgroup of \( G \), the module \( V^* \) is projective as a \( \mathbb{F}G \)-module, see Corollary 3 on Page 66 of [1]. \( \square \)

§4. Applications and Examples

Let \( \rho : G \hookrightarrow \text{GL}(n, \mathbb{F}) \) be a faithful representation of a finite group of order \( d \). Set \( V = \mathbb{F}^n \). Recall that \( \beta(\mathbb{F}[V]^G) \) is the maximal degree of an \( \mathbb{F} \)-algebra generator of \( \mathbb{F}[V]^G \) in a minimal generating set, the so-called Noether number.

**Proposition 4.1:** If \( V \) is a projective \( \mathbb{F}G \)-module then

\[ \beta(\mathbb{F}[V]^G) \leq \max\{d, \binom{d}{2}\}. \]

**Proof:** If \( V \) is \( \mathbb{F}G \)-projective then the Noether map \( \eta^G_G \) is surjective by Proposition 2.1. Thus, since \( \eta^G_G \) is an \( \mathbb{F} \)-algebra map, a set of generators of \( \mathbb{F}[V(G)]^G \) is mapped onto a set of generators of \( \mathbb{F}[V]^G \). Since \( V(G) \) is a permutation module with \( n \) transitive components each of which has degree \( d \), it is generated by elements of degree at most \( \max\{d, \binom{d}{2}\} \), by Corollary 3.10.9 in [5] and the result follows. \( \square \)
**The Noether Map**

**Remark:** Let \( \rho : G \hookrightarrow \text{GL}(n, \mathbb{F}) \) be a representation of a finite group \( G \) of order \( d \). Assume that the characteristic of \( \mathbb{F} \) is zero or strictly larger than \( d \). (This is the strongly nonmodular case.) Then
\[
\beta(\mathbb{F}[V]^G) \leq \beta(\mathbb{F}[W]^G)
\]
where \( W \) is the regular \( \mathbb{F}G \)-module, see Theorem 4.1.4 in [11]. Thus our Proposition 4.1 is a characteristic-free generalization: for projective \( \mathbb{F}G \)-modules \( V \) of dimension \( n \), the upper bound for \( \beta(\mathbb{F}[V]^G) \) is given by \( \beta(\mathbb{F}[W]^G) \) where \( W \) is \( \oplus_n \mathbb{F}G \).

The degree bound given above is sharp as we illustrate with the following example.

**Example 1:** Let \( A_3 \) be the alternating group in three letters. Let \( \mathbb{F} \) be a field containing a primitive 3rd root of unity \( \omega \in \mathbb{F} \). Then we obtain a faithful representation
\[
\rho : A_3 \hookrightarrow \text{GL}(1, \mathbb{F}), (123) \mapsto \omega.
\]
We have
\[
\mathbb{F}[x]^{A_3} = \mathbb{F}[x^3], \text{ and } \mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3} = \mathbb{F}[e_1, e_2, e_3, o(x_{11}x_{12})],
\]
where the \( e_i \)'s are the elementary symmetric functions in the \( x_{1j} \)'s. Thus
\[
\beta(\mathbb{F}[x]^{A_3}) = 3 = \beta(\mathbb{F}[x_{11}, x_{12}, x_{13}]^{A_3}) = \max\{3, \left(\begin{array}{c} 3 \\ 2 \end{array}\right)\}.
\]
Before we proceed we want to compare the degree bound given in Proposition 4.1 with the known general bounds, see [9] for an overview of this topic.

1. In the nonmodular case, we have that \( \beta(\mathbb{F}[V]^G) \leq |G| \) by Theorem 2.3.3 in [11]. This bound is better since
\[
|G| \leq \max\{n \cdot |G|, n\left(\frac{|G|}{2}\right)\}.
\]

2. The general degree bound given in Theorem 3.8.11 in [5] is
\[
\beta(\mathbb{F}[V]^G) \leq n(|G| - 1) + |G| n^{2n-1} n^{2n-1+1}.
\]
A short calculation shows that
\[
\max\{n \cdot |G|, n\left(\frac{|G|}{2}\right)\} \leq n(|G| - 1) + |G| n^{2n-1} n^{2n-1+1}.
\]
Thus the bound given in Proposition 4.1 is always better (where it applies).

3. If the ground field \( \mathbb{F} \) is finite of order \( q \), we have another general
degree bound given by:
\[
\beta(\mathbb{F}[V]^G) \leq \begin{cases} 
\frac{q^n-1}{q-1}(nq-n-1) & \text{if } n \geq 3, \\
2q^2 - q - 2 & \text{if } n = 2,
\end{cases}
\]
see Theorem 16.4 in [7]. This bound behaves worse than the one of Proposition 4.1 if \( q > |G| \).

(4) Finally in [4] a bound of a completely different flavor is proven. In particular it depends on a choice of a homogeneous system of parameters. In our Example 1 we found that the bound of Proposition 4.1 is sharp. If we apply Theorem 2.3 in [4] to this example we obtain
\[
\beta(\mathbb{F}[x]^{A_3}) \leq \text{degree}(f),
\]
where \( f \in \mathbb{F}[x]^{A_3} \) is a system of parameters. If we make the unlucky choice of \( f = x^9 \) the bound given in [4] is no longer sharp.

We denote by \( \text{CM defect}(\bullet) \) the Cohen-Macaulay defect. The following result tells us that the Cohen-Macaulay defect of the ring of invariants of \( n \) copies of the regular representation of a finite group \( G \) is an upper bound for the Cohen-Macaulay defect of the ring of invariants \( \mathbb{F}[V]^G \) in the case where \( V \) is projective.

**Proposition 4.2:** If \( V \) is \( \mathbb{F}G \)-projective then
\[
\text{CM defect}(\mathbb{F}[V]^G) \leq \text{CM defect}(\mathbb{F}[V(G)]^G).
\]

**Proof:** Since \( V \) is \( \mathbb{F}G \)-projective, we have the \( \mathbb{F}G \)-module decomposition
\[
V(G) = V \oplus K.
\]
Thus the result follows from [10]. □

**Remark:** The inequality in the preceding result is sharp since the Cohen-Macaulay defect of any nonmodular representation is zero.

**References**


THE NOETHER MAP


Mara D. Neusel
Department of Mathematics and Statistics
Mail Stop 1042
Texas Tech University
Lubbock TX 79409
USA
mara.d.neusel@ttu.edu

Mufti Sezer
Department of Mathematics
Boğaziçi Üniversitesi
Bebek
Istanbul
Turkey
mufit.sezer@boun.edu.tr