The Lasker-Noether Theorem in the Category $U(H^*)$

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SUMMARY: We prove the Lasker-Noether Theorem in the category $U(H^*)$ of unstable $H^* \otimes F^*$-modules. Along the way, we generalize Lam's $j$-functor to the context of modules.

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Let $F$ be a Galois field of characteristic $p$ with $q$ elements. Consider a faithful representation of degree $n$

$$\rho : G \rightarrow GL(n, F)$$

of a finite group $G$. Then $G$ acts via $\rho$ on the vector space $V = F^n$, and hence on the ring of polynomial functions

$$F[V] = F[x_1, \ldots, x_n]$$
in $n$ variables via

$$gf(v) = f(\rho(g)^{-1}v) \quad \forall f \in F[V], \; v \in V, \; g \in G.$$ 

The ring of polynomials invariant under this action is denoted by $F[V]^G$. By a classical theorem of Emmy Noether, any ring of invariants $F[V]^G$ is Noetherian, [11]. Since the groundfield $F$ is finite, the full general linear group $GL(n, F)$ is finite, and is moreover present in every ring of invariants $F[V]^G$. In 1911 Leonard E. Dickson, [3], proved that the ring of invariants of $GL(n, F)$

$$F[V]^{GL(n, F)} = F[d_{n,0}, \ldots, d_{n,n-1}] = D^*(n),$$
is a polynomial ring in $d_{n,0}, \ldots, d_{n,n-1}$, which are now called the Dickson classes. The algebra $D^*(n)$ is called the Dickson algebra.

As described in Chapters 10 and 11 in [12], or in the introduction of [8], $F[V]^G$ inherits from $F[V]$ an unstable action of the Steenrod algebra $P^*$, i.e., $F[V]^G$ is an objects in $\mathcal{K}_{fg}$, the category of finitely generated unstable (graded connected commutative $F$-) algebras over $P^*$. By the Imbedding Theorem 8.1.5 in [8] every object $H^*$ in $\mathcal{K}_{fg}$ contains a fractal\footnote{Recall that a fractal of the Dickson algebra is}

$$D^*(n)^Q \hookrightarrow H^*$$
is an integral extension, where $t$ can be choosen to be zero, if $H^*$ is $P^*$-inseparably closed. Therefore every finitely generated unstable $F$-algebra over the Steenrod algebra can be considered as a module over $D^*(n)^Q$, i.e., as a finitely generated module over a Noetherian ring. In classical theory every such module has a primary decomposition. In this paper we prove the $P^*$-invariant version of this statement in its most general form: Let $H^*$ be an unstable Noetherian algebra over the
Steenrod algebra. Then every finitely generated unstable $H^*$-module $M$ has a primary decomposition

$$M = Q_1 \cap \ldots \cap Q_m,$$

consisting of unstable primary components $Q_1, \ldots, Q_m$. Moreover, the associated prime ideals

$$\text{Rad}(Q_i : M) \subseteq H^* \quad \forall \ i = 1, \ldots, m,$$

are $P^*$-invariant ideals, i.e., ideals that are closed under the action of the Steenrod algebra.

This solves a long open problem, see [10] and Section 6 in [13], that has some surprising immediate applications, see [9].

§1. Lam's $\mathscr{J}$ for Modules

Let $H^*$ be an unstable algebra over the Steenrod algebra. Note that, in this first section, we do not need to assume that the ground ring $H^*$ is Noetherian. We follow the notation introduced in [14] and denote by $\mathcal{U}(H^*)$ the category of unstable $H^* \odot P^*$-modules. The semitensor product\(^2\) was introduced by W.S. Massey and F.P. Peterson in [5], Definition 2.5. It summarizes that we are looking at objects $M$ that are

1. left $H^*$-modules, as well as,
2. unstable left modules over the Steenrod algebra $P^*$, and

both structures are compatible in the sense that

$$P^l(hm) = \sum_{i+k=l} P^i(h) P^k(m)$$

for every element $P^l \in P^*$, $h \in H^*$ and $m \in M$. In other words, the map

$$H^* \otimes M \longrightarrow M,$$

defining the $H^*$-module structure on $M$, is a homomorphism of left $P^*$-modules.

Recall that the category $\mathcal{U}(H^*)$ is abelian. In this section we want to generalize Lam's $\mathscr{J}$-functor to the context of modules, see [4] or Section 11.2 in [12]. Since this functor played a significant role in the proof of

\(^2\)The multiplication in the semitensor product is defined as follows

$$(h \odot P^l)(h' \odot P^k) := \sum_{i+j=l} h^i(h') \odot P^j P^k,$$

see (2.3) or (2.4) in [5].
the Lasker-Noether Theorem for ideals, [10], it should not surprise that we need it here also.

Let $M$ be an object in $\mathcal{U}(H^*)$. Denote by $\text{Mod}_{H^*}$ the category of $H^*$-modules, and by $\text{Mod}_{H^*}(M)$ its full subcategory of $H^*$-submodules of $M$. Let $N$ be an object in $\text{Mod}_{H^*}(M)$. By restriction, we can define the images of the Steenrod powers on the elements of $N$. However, the module $N$ might not be closed under this action. This motivates the following definition.

**Definition and Lemma 1.1**: Let $M$ be an unstable $H^* \circ P^*$-module. Let $N$ be an object in $\text{Mod}_{H^*}(M)$. We define

$$J(N) := \{ n \in N \mid P^i(n) \in N \quad \forall \ i \geq 0 \},$$

and iteratively

$$J_j(N) = J(J_{j-1}(N)) \quad \text{for} \ j \geq 2.$$

This leads to a descending chain

$$J_0(N) := N \supseteq J_1(N) \supseteq J_2(N) \supseteq \ldots$$

of $H^*$-modules in $\text{Mod}_{H^*}(M)$. We denote the intersection of this chain by

$$J_\infty(N) = \bigcap_{j \geq 0} J_j(N).$$

Then $J_\infty(N)$ is an unstable $H^* \circ P^*$-module, and moreover, the maximal $H^*$-submodule in $N$ that is closed under the action of the Steenrod algebra.

**Proof**: For $n_1, n_2 \in J_j(N)$ and $h_1, h_2 \in H^*$, we have

$$P^l(h_1 n_1 + h_2 n_2) = P^l(h_1 n_1) + P^l(h_2 n_2)$$

$$= \sum_{i+k=l} \left( P^i(h_1) P^k(n_1) + P^i(h_2) P^k(n_2) \right),$$

where we made use of the Cartan formulae. Since

$$P^i(h_1), P^i(h_2) \in H^* \quad \forall \ i \geq 0,$$

and

$$P^k(n_1), P^k(n_2) \in J_{j-1}(N) \quad \forall \ k \geq 0,$$

by definition of $J_j(N)$, we see that

$$P^l(h_1 n_1 + h_2 n_2) \in J_{j-1}(N) \quad \forall \ l \geq 0.$$

This in turn means that

$$h_1 n_1 + h_2 n_2 \in J_j(N),$$

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making \( J_j(N) \) into a \( H^\ast \)-module. Therefore we have a chain of \( H^\ast \)-modules in \( \text{Mod}_{H^\ast}(M) \):

\[
N = J_0(N) \supseteq J_1(N) \supseteq J_2(N) \supseteq \ldots \supseteq J_j(N) \supseteq \ldots \supseteq J_{\infty}(N).
\]

Finally we need to show that \( J_{\infty}(N) \) is closed under the action of the Steenrod algebra. To this end, let \( n \in J_{\infty}(N) \), then

\[
n \in J_j(N) \quad \forall \ j \geq 0.
\]

Hence

\[
P^l(n) \in J_{j-1}(N) \quad \forall \ j \geq 1, \ \forall \ l \geq 0,
\]

i.e.,

\[
P^l(n) \in J_{\infty}(N) \quad \forall \ l \geq 0,
\]

as claimed. The maximality of \( J_{\infty}(N) \) is by construction clear.

Let \( I = (i_1, \ldots, i_k) \) be a multi index, and set

\[
P^I = p^{i_1} \ldots p^{i_k}.
\]

Then an element \( n \in N \) is in \( J_{\infty}(N) \) if, and only if,

\[
P^I(n) \in N \quad \forall \ \text{multi index} \ I.
\]

In the following series of technical lemmata we show that the category \( \mathcal{U}(H^\ast) \) of unstable \( H^\ast \odot P^\ast \)-modules is closed under certain standard module-theoretic operations. Moreover, we investigate the behaviour of such operations under the \( J_{\infty} \)-functor. Needless to say, we do this, because we will use these results later on.

**Lemma 1.2**: Let \( N, N' \) be objects in \( \text{Mod}_{H^\ast}(M) \), and \( M \) in \( \mathcal{U}(H^\ast) \). Then

\[
J_{\infty}(N \cap N') = J_{\infty}(N) \cap J_{\infty}(N').
\]

**Proof**: Take an element \( n \in J_{\infty}(N \cap N') \) then

\[
n \in N \cap N' \quad \text{and} \quad P^I(n) \in N \cap N' \quad \forall \ \text{multi index} \ I.
\]

This means

\[
n \in J_{\infty}(N) \cap J_{\infty}(N'),
\]

establishing the inclusion \( \subseteq \). The converse inclusion is proved by using this argument backward. 

**Lemma 1.3**: Let \( M \) and \( M' \) be objects in \( \mathcal{U}(H^\ast) \). Then so is their quotient

\[
\left( M' : M \right).
\]
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**Proof:** The quotient \((M' : M) \subseteq H^*\) is an ideal in \(H^*\). So we need to show that it is \(P^*\)-invariant. So, take an element \(h \in (M' : M)\), i.e., \(h \cdot M \subseteq M'\).

We claim that \(P^i(h) \in (M' : M)\), i.e., we claim that \(P^i(h) \cdot M \subseteq M'\) for every \(i \geq 0\). We proceed by induction on \(i\). Since \(P^0\) is the identity map, the case \(i = 0\) is trivial. However, we need to start our induction with \(i = 1\). Let \(m \in M\). Then we have by the Cartan formulae

\[
P^1(h)m = P^1(hm) - hP^1(m).
\]

Now, the first summand \(P^1(hm) \in M'\), because \(hm \in M'\) and \(M'\) is closed under the action of the Steenrod algebra. The second summand, \(hP^1(m)\), is equally in \(M'\), because \(m \in M\), therefore \(P^1(m) \in M\), and \(h \in (M' : M)\). Hence

\[
P^1(h)m \in M'
\]

for every \(m \in M\). This means that

\[
P^1(h) \in (M' : M).
\]

Let \(i > 1\). Then the Cartan formulae tell us that

\[
P^1(h)m = P^1(hm) - \sum_{k+l=i, k<l} P^k(h)P^l(m) \quad \forall m \in M.
\]

The sum on the right hand side is, by induction, in \(M'\). Since \(P^1(hm) \in M'\), because \(M'\) is in \(U(H^*)\), we conclude that also \(P^1(h)m \in M'\), in other words

\[
P^1(h) \in (M' : M) \quad \forall i \geq 0,
\]

as claimed. ☺

**Lemma 1.4:** Let \(M\) be an object in \(U(H^*)\), \(N\) in \(\text{Mod}_{H^*}(M)\). Then

\[
(J_\infty(N) : M) = J_\infty(N : M).
\]

**Proof:** Since \(J_\infty(N) \subseteq N\), we have

\[
(J_\infty(N) : M) := \{h \in H^* | hM \subseteq J_\infty(N)\} \\
\subseteq \{h \in H^* | hM \subseteq N\} = (N : M).
\]

Since \(J_\infty(N)\) and \(M\) are \(P^*\)-modules, so is their quotient, by the preceding Lemma 1.3. Therefore

\[
(J_\infty(N) : M) = J_\infty(J_\infty(N) : M) \subseteq J_\infty(N : M),
\]
by maximality of \( J_{\infty}(N : M) \) in \((N : M)\). This establishes the one inclusion. To show the reverse inclusion, we take an element \( h \in J_{\infty}(N : M) \). Then
\[
hM \subseteq N \quad \text{and} \quad P^I(h)M \subseteq N \quad \forall \text{ multi index } I.
\]
We need to show that \( h \in (J_{\infty}(N) : M) \), i.e.,
\[
hM \subseteq J_{\infty}(N).
\]
Since \( hM \subseteq N \), this means for every multi index \( I \) and every element \( m \in M \) we have to verify that
\[
P^I(hm) \in N.
\]
We employ the Cartan formulae, set \( I = (i_1, \ldots, i_k) \) and get
\[
P^I(hm) = P^I' \left( \sum_{j_k + l_k = i_k} P^{j_k}(h)P^{l_k}(m) \right),
\]
where \( I' = (i_1, \ldots, i_{k-1}) \). Hence, setting \( I'' = (i_1, \ldots, i_{k-2}) \) and iterating, we arrive at
\[
P^I(hm) = P^I'' \left( \sum_{j_k + l_k = i_k} \sum_{j_{k-1} + l_{k-1} = i_{k-1}} P^{j_{k-1}l_k}(h)P^{l_{k-1}}P^{l_k}(m) \right)
\]
\[
= \sum_{j_1 + l_1 = i_1} \cdots \sum_{j_k + l_k = i_k} P^{j_1\ldots j_k}(h)P^{j_1\ldots j_k}(m),
\]
for multi indices \( J = (j_1, \ldots, j_k) \) and \( L = (l_1, \ldots, l_k) \). Since \( M \) is an object in \( \mathcal{U}(H^*) \), we have \( P^L(m) \in M \) for all \( L \). Now, \( h \in J_{\infty}(N : M) \), i.e.,
\[
hm \in N \quad \forall \ m \in M \quad \text{and} \quad P^J(h)m \in N \quad \forall \ m \in M, \forall \ J.
\]
This means that the right hand side is an element of \( N \), hence so is the left, as desired. ☯

We collect these results, and extend them to

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**Proposition 1.5:** Let $M$, $M'$ be unstable $H^* \circ P^*$-modules, let $N$ be an object in $\text{Mod}_{H^*}(M)$. Then

1. $(M' : M) \subseteq H^*$ is $P^*$-invariant.
2. If $(N : M) \subseteq H^*$ is $P^*$-invariant, then there exists an unstable $H^* \circ P^*$-module $M_N \subseteq M$ such that

   $$(M_N : M) = (N : M).$$

**Proof:** Statement (1) is the contents of Lemma 1.3. To prove the second statement, recall from Lemma 1.4 that

$$J_\infty (N : M) = J_{\infty} \left( J_{\infty} (N : M) \right).$$

Choose $M_N = J_\infty (N)$ and finish the proof with Lemma 1.1.  

Denote by $\text{Rad}(-)$ the radical of $(-)$. We need the following result correlating the functor $J_\infty$ with $\text{Rad}$.

**Lemma 1.6:** Let $M$ be an object in $\mathcal{U}(H^*)$, and let $N$ an object in $\text{Mod}_{H^*}(M)$. Then

$$\text{Rad} \left( J_\infty (N) : M \right) = J_\infty \left( \text{Rad} (N : M) \right).$$

**Proof:** Let $h \in \text{Rad} \left( J_\infty (N) : M \right)$. Then

$$h^r M \subseteq J_\infty (N)$$

for some large $r \in \mathbb{N}$. Hence, a fortiori,

$$h^r M \subseteq N,$$

or,

$$h \in \text{Rad} (N : M).$$

This means

$$\text{Rad} \left( J_\infty (N) : M \right) \subseteq \text{Rad} (N : M).$$

By Lemma 1.3 $(J_\infty (N) : M)$ is $P^*$-invariant. Therefore so is its radical, by Lemma 1.4 in [10]. By maximality this implies that

$$\text{Rad} \left( J_\infty (N) : M \right) \subseteq J_\infty \left( \text{Rad} (N : M) \right).$$

Conversely, by Lemma 1.3 in [10] we have

$$J_\infty \left( \text{Rad} (N : M) \right) = \text{Rad} \left( J_\infty (N) : M \right).$$

We want to show that $\text{Rad} \left( J_\infty (N) : M \right) \subseteq \text{Rad} \left( J_\infty (N) : M \right)$. So, it is enough to show that

$$J_\infty (N : M) \subseteq \left( J_\infty (N) : M \right),$$
For that take an element $h \in J_{\infty}(N : M)$. Then, by definition, we have that

$$h \cdot M \subseteq N \quad \text{and} \quad p^I(h) \cdot M \subseteq N,$$

for every multi index $I = (i_1, \ldots, i_k)$. Let $m \in M$. We need to show that $h \cdot m \in J_{\infty}(N)$, i.e., we need to show that

$$p^I(h \cdot m) \in N \quad \forall \text{ multi index } I.$$

We induct on the length $k = |I|$, where $I = (i_1, \ldots, i_k)$.

**CASE $|I| = 1$**: Then $I = (i_1) = (i)$ and

$$p^I(hm) = \sum_{k+I = i} p^k(h)p^I(m).$$

Since $h \in J_{\infty}(N : M)$ we have that $p^k(h) \in J_{\infty}(N : M) \subseteq (N : M)$ for all $k$. Hence, because $p^I(m) \in M$ for all $I$

$$p^k(h) \cdot p^I(m) \in N \quad \forall k, \forall I.$$

Therefore

$$p^I(hm) = \sum_{k+I = i} p^k(h)p^I(m) \in N.$$

**CASE $|I| > 1$**: We rewrite $I = (I', i)$, where $I' = (i_1, \ldots, i_{k-1})$, $I = (i_1, \ldots, i_k)$ and $i = i_k$. We have

$$p^I(h \cdot m) = p^{I'}p^I(h \cdot m)$$

$$= p^{I'} \left( \sum_{k+I = i} p^k(h)p^I(m) \right)$$

$$= \sum_{k+I = i} p'^I \left( p^k(h)p^I(m) \right)$$

As in the preceding case we conclude that

$$p^I(m) \in M \quad \forall I \geq 0, \quad \text{and} \quad p^k(h) \in J_{\infty}(N : M) \quad \forall k \geq 0,$$

hence, by induction we have

$$p^{I'} \left( p^k(h)p^I(m) \right) \in N$$

and we are done. ☯
§2. Primary Unstable $H^* \otimes \mathcal{P}^*$-Modules

We want to show that an unstable Noetherian module over an unstable Noetherian algebra, $H^*$, has a primary decomposition consisting of unstable components. For that we follow the classical route as described in the Appendix to Chapter IV on page 252f of [15]. We start with recollecting some terminology. Let $H^*$ be an unstable Noetherian algebra over the Steenrod algebra. Let $M$ be an unstable $H^* \otimes \mathcal{P}^*$-module, and $Q \subseteq M$ a submodule. $Q$ is said to be primary, if whenever

$$ h \cdot m \in Q \quad \text{for } h \in H^*, \ m \in M $$

then

either $m \in Q$ or $h \in \text{Rad}(Q) := \text{Rad}(Q : M)$.

If a module $Q \subseteq M$ is primary, then the ideal

$$ (Q : M) := \{h \in H^* \mid hM \subseteq Q\} \subseteq H^* $$

is primary (but not conversely!)³. Let $M$ be Noetherian, and let $M' \subseteq M$ be unstable $H^* \otimes \mathcal{P}^*$-modules. As a $H^*$-submodule of $M$, the module $M'$ has a primary decomposition

$$ M' = Q_1 \cap \ldots \cap Q_m, $$

where $Q_1, \ldots, Q_m \subseteq M$ are primary $H^*$-submodules of $M$. The prime ideals

$$ p_i = \text{Rad}(Q_i : M) \subset H^* \quad \forall \ i = 1, \ldots, m $$

are called associated prime ideals of $M'$. Moreover, the decomposition is called irredundant if

$$ \bigcap_{i \neq j} Q_i \neq Q_j \quad \forall \ j = 1, \ldots, m. $$

It is called minimal if

$$ \text{Rad}(Q_i : M) \neq \text{Rad}(Q_j : M) \text{ whenever } i \neq j. $$

We want to show that the primary modules

$$ Q_1, \ldots, Q_m $$

as well as the associated prime ideals can be chosen to be $\mathcal{P}^*$-invariant. We start by showing that the associated prime ideals are $\mathcal{P}^*$-invariant.

LEMMA 2.1: Let $M$ be Noetherian. Let $M' \subseteq M$ be objects in $\mathcal{U}(H^*)$, let

$$M' = Q_1 \cap \ldots \cap Q_m$$

be a primary decomposition of $M'$ as a $H^*$-module. Then the associated prime ideals of $M'$, i.e., the prime ideals

$$p_i := \text{Rad} (Q_i : M) \subseteq H^* \quad \forall \ i = 1, \ldots, m$$

are $P^*$-invariant.

**Proof:** Since $M' \subseteq M$ are objects in $\mathcal{U}(H^*)$, the ideal $(M' : M)$ in $H^*$ is $P^*$-invariant, by Lemma 1.3. As such, the ideal $(M' : M)$ has a $P^*$-invariant minimal irredundant primary decomposition by Theorem 3.5 in [10],

$$(M' : M) = \widehat{\mathfrak{p}}_1 \cap \ldots \cap \widehat{\mathfrak{p}}_k,$$

where the associated prime components are

$$\widehat{\mathfrak{p}}_j = \text{Rad} (\widehat{Q}_j) \subseteq H^*,$$

for $j = 1, \ldots, k$. Hence

$$\widehat{\mathfrak{p}}_1 \cap \ldots \cap \widehat{\mathfrak{p}}_k = \text{Rad}(M' : M) = \text{Rad}(Q_1 : M) \cap \ldots \cap \text{Rad}(Q_m : M) = p_1 \cap \ldots \cap p_m.$$

Hence, since both decompositions are minimal and irredundant, we have

$$k = m,$$

and, after possibly reordering,

$$\widehat{p}_i = \text{Rad}(Q_i : M) \quad \forall \ i = 1, \ldots, k,$$

cf Lemma 2.4.10 in [1]. ☯

The following result extends Lemma 1.5 in [10] to the context of modules.

PROPOSITION 2.2: Let $M$ be an unstable Noetherian $H^* \otimes P^*$-module. Let $Q$ in $\text{Mod}_{H^*}(M)$ be a primary module, such that its radical

$$p = \text{Rad}(Q : M) \subseteq H^*$$

is $P^*$-invariant. Then $J_\infty(Q)$ is primary in $\mathcal{U}(H^*)$ with radical $p$. 

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**Proof:** Let $Q$ be primary as a $H^*$-submodule of $M$. Then

$$q = (Q : M) \subseteq H^*$$

is a primary ideal with radical, $p \subseteq H^*$. By Theorem 3.3 in [7] we know that

$$I_\infty(q)$$

is a primary ideal with radical $I_\infty(p) = p$. Hence, by Proposition 1.5,

$$(I_\infty(Q) : M) = I_\infty(Q : M)$$

is a primary ideal with radical $p$. We need to show that

$$I_\infty(Q) \subseteq M$$

is a primary module. We have

$$p = I_\infty(p) = I_\infty(\text{Rad} (Q : M)) = \text{Rad} (I_\infty(Q) : M) \supseteq (I_\infty(Q) : M).$$

Let $h \in H^*$, $m \in M$ and $hm \in I_\infty(Q)$. We assume that $m \notin I_\infty(Q)$, and have to show that

$$h \in \text{Rad} (I_\infty(Q) : M) = p.$$ 

Note that we have a chain of modules

$$I_\infty(Q) \subseteq I_1(Q) \subseteq Q$$

with the same ($p^*$-invariant) prime radical

$$p = \text{Rad} (Q) = \text{Rad} (I_1(Q)) = I_\infty(\text{Rad} (Q)).$$

Therefore, by iteration, it is enough to show that $I_1(Q)$ is primary. To this end let $h \in H^*$ and $m \in M$ such that $hm \in I_1(Q)$, and assume that $m \notin I_1(Q)$. We need to show that

$$h \in \text{Rad} (I_1(Q) : M) = p.$$ 

**Case** $m \in Q \setminus I_1(Q)$: Then there exists an $i \in \mathbb{N}_0$ such that

$$P^1(m) \notin Q.$$ 

Let $i$ be minimal with this property. Then

$$P^1(hm) = \sum_{k+l=i, l < i} P^k(h)P^l(m) + hP^1(m).$$

Since $hm \in I_1(Q)$ we have that also $P^1(hm) \in Q$. By minimality of $i$ we know that $P^1(m) \in Q$ for all $l < i$. Therefore

$$hP^1(m) = P^1(hm) - \sum_{k+l=i, l < i} P^k(h)P^l(m) \in Q.$$
Since \( p^i(m) \notin Q \) by assumption we conclude, because \( Q \) is primary, that
\[
h \in \text{Rad} \left( Q : M \right)
= I_\infty \left( \text{Rad} \left( Q : M \right) \right)
= \text{Rad} \left( I_\infty(Q) : M \right)
= p
= \text{Rad} \left( J_1(Q) : M \right),
\]
where we used Lemma 2.1 and Lemma 1.6.

**CASE** \( m \notin Q \) : By assumption, \( hm \in J_1(Q) \subseteq Q \), so, because \( Q \) is primary,
\[
h \in \text{Rad} \left( Q : M \right) = p.
\]
This shows that
\[
J_1(Q) \subseteq M
\]
is a primary \( H^* \)-module. Hence, iteratively, we get that \( I_\infty(Q) \subseteq M \) is primary in the category \( \mathcal{U}(H^*) \).

**THEOREM 2.3** (Lasker-Noether Theorem): Let \( H^* \) be an unstable Noetherian algebra over the Steenrod algebra. Let \( M \) be Noetherian, and let \( M' \subseteq M \) be unstable \( H^* \circ P^* \)-modules. Then \( M' \) admits a minimal irredundant primary decomposition in \( \mathcal{U}(H^*) \), i.e., all primary components, as well as the associated prime ideals are unstable \( H^* \circ P^* \)-modules.

**PROOF:** Choose a primary decomposition of \( M' \) as a \( H^* \)-module
\[
M' = Q_1 \cap \cdots \cap Q_m.
\]
By Lemma 2.1 we know that the associated prime ideals are \( P^* \)-invariant. Hence by Proposition 2.2 we have that \( I_\infty(Q_i) \) is a primary module with radical
\[
\text{Rad} \left( Q_i \right) := \text{Rad} \left( Q_i : M \right) = I_\infty \left( \text{Rad} \left( Q_i \right) \right).
\]
By Lemma 1.1 these are modules in \( \mathcal{U}(H^*) \). So
\[
M' = I_\infty(M') = I_\infty(Q_1 \cap \cdots \cap Q_m) = I_\infty(Q_1) \cap \cdots \cap I_\infty(Q_m),
\]
using Lemma 1.2, is a primary decomposition in the category \( \mathcal{U}(H^*) \). We make it irredundant by throwing away superfluous modules, and minimal by combining these modules which have the same radical.

**COROLLARY 2.4:** Let \( H^* \) be an unstable Noetherian algebra over the Steenrod algebra. Let \( M \) be an unstable Noetherian \( H^* \circ P^* \)-module. Let \( Q \) in \( \text{Mod}_{\mathcal{U}}(M) \) be a primary module, then \( I_\infty(Q) \) is primary in \( \mathcal{U}(H^*) \) with radical \( I_\infty \left( \text{Rad} \left( Q : M \right) \right) \).
Proof: Let $Q$ be a primary module in $\text{Mod}_H^+(M)$. Then, by definition, the ideal 

$$(Q : M) \subseteq H^*$$

is primary. From Theorem 3.3 in [7], we know that 

$$J_\infty(Q : M) \subseteq H^*$$

is $P^*$-invariant and primary with $P^*$-invariant radical 

$$J_\infty (\text{Rad} (Q : M)) = \text{Rad} (J_\infty(Q) : M),$$

where the last equality follows from Lemma 1.6. We need to show that $J_\infty(Q)$ is a primary module. To this end let 

$$J_\infty(Q) = Q_1 \cap \ldots \cap Q_m$$

be an irredundant and minimal primary decomposition of $J_\infty(Q)$. By Theorem 2.3 we can assume that the $Q_1, \ldots, Q_m$ are unstable modules. Hence the $P^*$-invariant primary ideal $(J_\infty(Q) : M)$ can be written as an intersection of $P^*$-invariant primary ideals 

$$\bigcap_{i=1}^m (Q_i : M).$$

By irredundancy and minimality we have that $m = 1$, as desired. ☐
References


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