The Invariants of the Symplectic Groups

Mara D. Neusel

AG INVARIANTENTHEORIE
GERMANY
EMAIL:MDN@SUNRISE.UNI-MATH.GWDG.DE
MARAMARA@STEENROD.MAST.QUEENSU.CA
MNEUSEL@CFGAUSS.UNI-MATH.GWDG.DE
NEUSEL@MATH.UMN.EDU

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SUMMARY : *In this notes we study the invariant rings of the symplectic groups in odd characteristic in their tautological representation, and try to make the original paper by Carlisle and Kropholler more readable and understandable, i.e., the only new thing is the expository, in particular that/how and where the Steenrod algebra is used is my contribution.*

§1. Notation a[redacted]

Let $\mathbb{F}$ be a Galo[redacted]characteristic p with q[redacted] space over $\mathbb{F}$ of even dimensio[redacted] basis $x_1, y_1, \dots, x_l, y_l$. De[redacted]

$$f : V \times V \longrightarrow \mathbb{F}$$

[redacted] defined [redacted] n, $\mathbb{F}$),

[redacted]

[redacted] could write [redacted]

[redacted]iliz[redacted]

[redacted]).

[redacted]PROOF: The proof [redacted] of sym[redacted]

[redacted]nks to Dickson we [redacted]e set of ge[redacted]

[redacted]ristic two there [redacted]problems in defin[redacted]

[redacted]es not depend[redacted] the choice of f sin[redacted] $\mathbb{F}$)-orbit. For a proof of this [redacted]r pp. 234 -246.

[redacted]e symplectic g[redacted]

[redacted] **THEOREM 1.2** (L.E. Dickson): *The symplectic group* $(n, \mathbb{F})$ *is generated by the following matrices:*

$$\mathbf{S(k)} = (s_{i,j}(k)), \quad \forall\ k = 1, \dots, l,$$

[redacted]*here*

$$s_{i,j}(k) = \begin{cases} 1 & \text{if } i = j \neq k \\ 1 & \text{if } i = 2k-1,\ j = 2k \\ -1 & \text{if } i = 2k,\ j = 2k-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{S(k,}\ \lambda) = (s_{i,j}(k,\ \lambda)), \quad \forall\ k = 1, \dots, l \quad \text{and } \lambda \in \mathbb{F}^{\times},$$

$$s_{i,j}(k,\ \lambda) = \begin{cases} 1 & \text{if } i = j \\ \lambda & \text{if } i = 2k-1,\ j = 2k \\ 0 & \text{otherwise,} \end{cases}$$

[redacted] $k,\ l,\ \lambda)), \quad \forall\ k,\ l = 1, \dots, l$ [redacted]

[redacted] $1 \quad if\ i =$ [redacted] $if\ i$ [redacted]

[redacted]nd a col[redacted]n will, [redacted]kson classes [redacted]lass, [redacted]rs of the ri[redacted]nts.

PROPOSITION 2.1: *Fo*[redacted]*tural number* $i \in \mathbb{N}$

$$\xi$$ [redacted] $$:= \sum_{j=1}^{l}(x_j y_j^{q^i} - y_j x_j^{q^i}) \in \mathbb{F}[x_1,\ y_1, \dots, x_l,\ y_l]$$ [redacted]

[4] In dimension two the symplec[redacted] IF) is nothing but the special linear group SL(2, IF). To [redacted] this you could use Dickson's theorem to find that [redacted]oup is contained in the special linear, because all ele[redacted] have determin[redacted] one, and then use again Dickso[redacted]ave the same order. On the other hand you c[redacted]e the first s[redacted] show by ordinary explicite matr[redacted] special linear matrices are symplectic.[redacted]

PR[redacted] Observe that

$$[redacted]\ x_1 \wedge y_1 + \cdots + x\ [redacted]\ y_l \in \left(\Lambda^2(V^*)\right)^{(n,\ \mathbb{F})} \subseteq \left(\mathbb{F}[V] \otimes_{\mathbb{F}} \Lambda^*(V\right.\ [redacted]$$

is a no[redacted]d hence, by definition of the symplec[redacted]o[redacted]nt

[redacted]

[redacted] know tha[redacted]

[redacted]$\mathbb{F}) \hookrightarrow \mathrm{GL}($[redacted]

On the o[redacted]

[redacted]$1)$

is injective.[redacted] $\xi_{n,1}$, also

[redacted]$_n = \xi_n$.

[redacted]turn[redacted]nce the symplectic gro[redacted]

[redacted]rself, you might look it up (Yes[redacted]

[redacted]calculated.

[redacted]e denote the homogeneous part of d[redacted]

REMARK[redacted]

$$[\text{redacted}]^{1}\beta \ : \Lambda^2(V[\text{redacted}]$$

are injective[redacted]hat the symplectic group [redacted]
$\xi_{n,i+1}$'s

$$[\text{redacted}](n,\ \mathbb{F}) = \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,i+1}} \quad \forall\ i,$$

what in turn[redacted]he preceding Proposition 2.1.

Next we show[redacted]maximal possible number of the invariants just constructed, namel[redacted] are algebraic[redacted]pendent.

LEMMA 2[redacted]The polynomials

$$\xi_{n,1}, \dots, \xi_{n,n} \in \mathbb{F}[V]^{(n,\ \mathbb{F})}$$

are algebraic[redacted]independent.

PROOF: [redacted] this by calculating the determinant of the Ja[redacted] associated to[redacted]nomials. To this end note that for all $j = 1$[redacted]

$$\partial\xi_{n,j}[\text{redacted}] = v^{q^i}[\text{redacted}]$$

[redacted]r clas[redacted]

$$[\text{redacted}]\xi_{n,n}] \hookrightarrow \mathbb{F}[V]^{(n,\ \mathbb{F})}$$

[redacted]al[redacted]

[redacted]mark at[redacted]nt[redacted]symplectic matrix has determinant one, because the sym[redacted]oup is genera[redacted]nsvections, see Theorem 114 in [6] or, for a more modern t[redacted] e.g., [8] Lemma[redacted] and therefore

$$[\text{redacted}], \dots, \mathbf{d}_{n,\ n-1}] = \mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})} \hookrightarrow \mathbb{F}[V]^{(n,\ \mathbb{F})},$$

where $\mathbf{E}_n$ denot[redacted]orbit $V^* \setminus 0$ of $\mathrm{SL}(n,\ \mathbb{F})$ on V^*.

In a later sectio[redacted]invariants, namely

$$[\text{redacted}]\mathbf{E}_n,\ \mathbf{d}_{n,\ 1}, \dots, \mathbf{d}_{n,\ n}[\text{redacted}]$$

[redacted]

form a complete set of algebr[redacted]generators[7] of the ring of invariants $\mathbb{F}[V]$ [redacted] $^{(n,\ \mathbb{F})}$, in other words, we will show that

A[redacted] $\xi_{n,1},\dots,\xi_{n,n},\ \mathbf{E}_n,\ \mathbf{d}_{n,\,1},\dots,\mathbf{d}_{n,\,n-1}$ >,

where $\mathbb{F} < - >$ denotes the $\mathbb{F}$-[redacted]nerated by the stuff in the $< - >$-brackets (which is possibly *not* a polynomial algebra), is prec[redacted] of invariants.

We do this by determine also[redacted]ong these generators, which leads to a dozen pages [redacted] calculations. Then we still h[redacted]at's it: that takes another 6 pages, i.e., it's prob[redacted] not really pleasant to read all[redacted]y not pleasant to give a talk about it).

§3. A Bunch of Relations

In this section we calculate so[redacted]ts we have so far[redacted]

We need some preliminaries.

Define an alternating bilinear [redacted]

for n-tuples $F = (f_1,\dots,f_{2l})$ and [redacted]$) \in \Sigma_{2l}$

and take its centralizer in the sym[redacted]

$$C(\tau) < \Sigma_{2l}.$$

Next define

$$||F^{(1)},\dots,F^{(2l)}|| := \sum_{\sigma}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right),$$

where the sum runs over a set of coset representatives of $C(\tau)$ in Σ_{2l}.

LEMMA 3.1: *With the preceding notation we have*
(1) $||\cdots||$ *is independent of the choice of the coset representatives.*
(2) $||\cdots||$ *is multilinear and alternating.*
(3) $||e^{(1)},\dots,e^{(2l)}|| = 1$ *where* $e^{(i)}$ *denotes the* i*-th standard basis vector* $(0,\dots,0,\ 1,\ 0,\dots,0)$.

PROOF:

AD (1) : The centralizer of the element $\tau = (12)(34)\cdots(2l-1\ 2l) \in \Sigma_{2l}$ can be expressed in the following way: Take an embedding

$$\Sigma_l \hookrightarrow \Sigma_{2l},\ \sigma \longmapsto \bar{\sigma},$$

where we define the image for any $k = 1,\dots,l$ by

$$\bar{\sigma}(2k-1) := 2\sigma(k)-1$$
$$\bar{\sigma}(2k) := 2\sigma(k).$$

---

[7] Note that these generators arise in a very natural way: the Dickson classes, an Euler class, the form ξ_n that defines the group and its Steenrod powers.

Then

[redacted]

[redacted] 12), . . . , (2 [redacted] ntries [redacted] this is inde [redacted] tion. An [redacted] at our sum [redacted] here π is [redacted] ermute the [redacted] of the pairs

[redacted] at [redacted] at the form is

[redacted]

Consider the element $\sigma_2(2i_0\ 2i_0') \in \Sigma_{2l}/C(\tau)$. It gives the same summand as σ_2, namely

$$\prod_j < F^{(\sigma_2(2i_0\ 2i_0'))(2j-1)},\ F^{(\sigma_2(2i_0\ 2i_0'))(2j)} >$$

$$= < F,\ F^{\sigma_2(2i_0')} >< F,\ F^{\sigma_2(2i_0)} > \prod_{j\neq i_0,\ i_0'} < F^{(\sigma_2(2i_0\ 2i_0'))(2j-1)},\ F^{(\sigma_2(2i_0\ 2i_0'))(2j)} >,$$

but with the opposite sign, because $\mathrm{sign}(\sigma_2) = -\mathrm{sign}(\sigma_2(2i_0\ 2i_0'))$. So the two summands cancel. Altogether we have

$$||F^{(1)}, \ldots, F^{(2l)}|| = \sum_\sigma \left(\mathrm{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right)$$

$$= \sum_{\sigma_1} \left(\mathrm{sign}(\sigma_1) < F,\ F > \prod_{j\neq i_0} < F^{\sigma_1(2j-1)},\ F^{\sigma_1(2j)} > \right)$$

$$+ \sum_{\sigma_2} \Big(\mathrm{sign}(\sigma_2) < F,\ F^{\sigma_2(2i_0)} >< F,\ F^{\sigma_2(2i_0')} >$$

$$\left(\prod_{j\neq i_0,\ i_0'} < F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j-1)},\ F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j)} > - \prod_{j\neq i_0,\ i_0'} < F^{\sigma_2(2j-1)},\ F^{\sigma_2(2j)} >\right) = 0.$$

AD (3) : We have

$$< e^{(i)},\ e^{(j)} >= \begin{cases} +1 & \text{if } i \text{ is odd and } j = i+1 \\ -1 & \text{if } i \text{ is even and } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence all summands va[redacted]here σ = id

$||e^{(1)},\dots,\ e^{(2l)}|| = \sum_{\sigma}$ [redacted]

$= \prod_{j=1}^{l} <$ [redacted]

$= 1.$ [redacted]

That's all we claimed •

So, the first statement of [redacted]ut our definition of $||\cdots||$ is wellde[redacted]
second and third imply th[redacted]

[redacted] $,\dots,\ F^{(2l)}|| = \det \begin{bmatrix} F^{(1)} \\ : \\ F^{(2l)} \end{bmatrix}.$

We are now prepared to pr[redacted] following lemma.

PROPOSITION 3.2: *Wi*[redacted]*e notation above we have:*

(1) *The Euler class* **E**[redacted] $\in \mathbb{F}[\xi_{n,1},\dots,\ \xi_{n,n-1}]$ *is irreducible in* $\mathbb{F}[V]^{(n,\ \mathbb{F})}$.
(2) $\mathbf{E}_n \mathbf{d}_{n,\ i} \in \mathbb{F}[\xi_{n,1},$ [redacted] $\xi_{n,n}]$ *for all* $i = 1,\dots,\ n-1$.
(3) *If for some polyno*[redacted] $\in \mathbb{F}[V]$ *we have that* $\mathbf{E}_n f \in \mathbb{F}[\xi_{n,1},\dots,\ \xi_{n,n-1}]$, *then* $f \in \mathbb{F}[\xi_{n,1},\dots,\ \xi_{n,n-1}]$.

PROOF: We take the t[redacted]rder.

AD 1 : Define $2l$-tupl[redacted] $y_1^{q^{j-1}},\dots,\ x_l^{q^{j-1}},\ y_l^{q^{j-1}}) \in \times_{2l} \mathbb{F}[V]$ of polynomials for $j \in \mathbb{N}$. Th[redacted]
together with the precedi[redacted]Euler class $\mathbf{E}_n$ is, according to Dickson [6], given b[redacted]

$\mathbf{E}_n = \det \begin{bmatrix} x_1 \\ : \\ : \\ x_1^{q^{2l-1}} \end{bmatrix}$ [redacted]

$= \det \begin{bmatrix} F^{(1} \\ : \\ F^{(2l} \end{bmatrix}$ [redacted]

$= ||F^{(1)},\dots$ [redacted]

Since [redacted]

[redacted]), $F^{(j)} >= \xi^{q^{\min(}}_{n,|j-i|}$ [redacted]

as one [redacted] oice $|j-i| \le 2l-1$ we get the d[redacted]

$$\mathbf{E}_n \in \mathbb{F}[\xi_{n,1}, \dots, \xi_{n,n-1}],$$

i.e., th[redacted] $E = E(X_1, \dots, x_{n-1}) \in \mathbb{F}[X_1, \dots, X_{n-1}]$ in $n-1$ indetermi[redacted]

$$\mathbf{E}_n = E(\xi_{n,1}, \dots, \xi_{n,n-1}).$$

The E[redacted] clas[redacted] construction the product of linear forms where we take for each one dimensiona[redacted] vecto[redacted] bspac[redacted] V^* exactly one form. Moreover the symplectic group [redacted] $(n, \mathbb{F})$ acts transitively [8] on the se[redacted] hype[redacted] planes of V. Hence $\mathbf{E}_n$ is irreducible in $\mathbb{F}[V]$ [redacted] $^{(n, \mathbb{F})}$ and a fortiori in $\mathbb{F}[\xi_{n,1}, \dots, \xi_{n,n-1}]$.

A[redacted] i[redacted] again thanks to Dickson)

$$\mathbf{E}_n \mathbf{d}_{n,\,i} = \det \begin{bmatrix} x_1^{q^{i_1}} & y_1^{q^{i_1}} & \cdots & x_l^{q^{i_1}} & y_l^{q^{i_1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{q^{i_n}} & y_1^{q^{i_n}} & \cdots & x_l^{q^{i_n}} & y_l^{q^{i_n}} \end{bmatrix}$$

for $0 \le$ [redacted] here $i_j \neq i$ we can proceed as in (1) and get

$$\text{[redacted]} \prod^{l} < F^{\sigma(2j-1)}, F^{\sigma(2j)} > \Bigg) \in \mathbb{F}[\xi_n \text{[redacted]}$$

In oth[redacted] rminants

[redacted] $\in \mathbb{F}[$ [redacted]

such t[redacted]

for all i [redacted]

AD [redacted]

where t[redacted] of vector sp[redacted]

where $e_1, .$ [redacted] Lemma 2.3 the polynomials $\varphi(\xi_{n,1}) = \xi_n$ [redacted]. Therefore the image of φ generates a subal[redacted] dim($\mathbb{F}[x_1, \dots, y_{l-1}]$ the kernel of φ is a prime ideal [redacted] e kernel. By (1) the class $\mathbf{E}_n$ is irreducible, and therefo[redacted] nique factorization domain, i.e.,

$$\text{[redacted]}) \subset \mathbb{F}[\xi_{n,1}, \dots, \xi_{n,n-1}].$$

[8] This is just a[redacted], see, e.g., Section 20 in [1], or Lemma 3 in Section 6.9 of [8].

Denote by $<\mathbf{E}_n> \subset \mathbb{F}[V]$ the principle ideal generated by the Euler class in the full polynomial ring $\mathbb{F}[V]$. Certainly we have

$$(\mathbf{E}_n) \subseteq <\mathbf{E}_n> \cap\, \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}].$$

Since for every element in the big ideal $f\mathbf{E}_n \in <\mathbf{E}_n> \cap\, \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$ *(i.e., for all $f \in \mathbb{F}[V]$)* we have that $\varphi(f\mathbf{E}_n) = 0$, the two ideals in question must be equal, i.e., the polynomial $f \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$, as we wanted to show •

We need to be more precise about the polynomials $\mathcal{D}_{n,i}$ occuring in part (2).

LEMMA 3.3: *For $i = 0, \ldots, n-1$ the polynomials $\mathcal{D}_{n,i}(X_1, \ldots, X_n)$ are linear in X_n with leading coefficient*

$$\mathcal{D}_{n-2,i-1}(X_1, \ldots, X_n)^q.$$

PROOF: Recall that

$$\begin{aligned}
\mathbf{E}_n\mathbf{d}_{n,\,i} &= \det\begin{bmatrix} x_1^{q^{i_1}} & y_1^{q^{i_1}} & \cdots & x_l^{q^{i_1}} & y_l^{q^{i_1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{q^{i_n}} & y_1^{q^{i_n}} & \cdots & x_l^{q^{i_n}} & y_l^{q^{i_n}} \end{bmatrix} \\
&= \det\begin{bmatrix} F^{(1)} \\ \vdots \\ F^{(2l)} \end{bmatrix} \\
&= ||F^{(1)}, \ldots, F^{(2l)}|| \\
&= \sum_{\sigma}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
&= \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
&\quad + \sum_{\{\sigma,|\sigma(2j_0-1)-\sigma(2j_0)-1| = n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \xi_{n,n}\right) \\
&= \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
&\quad + \left(\sum_{\{\sigma,|\sigma(2j_0-1)-\sigma(2j_0)-1| = n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right)\right)\xi_{n,n}.
\end{aligned}$$

Therefore $\mathbf{E}_n\mathbf{d}_{n,\,i}$ is linear in $\xi_{n,\,n}$. To find the leading coefficient we have to work a bit harder and calculate the sum

$$\sum_{\{\sigma,|\sigma(2j_0-1)-\sigma(2j_0)-1| = n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right).$$

Note first of all that we can assume without loss of generality that $j_0 = l$ (if not we replace σ by $\sigma\gamma$ where $\gamma \in C(\tau)$ interchanges j_0 and l). So we have to calculate

$$\sum_{\{\sigma,|\sigma(2l-1)-\sigma(2l)-1| = n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l-1} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right),$$

where the exponents $\sigma(2j-1),\ \sigma(2j)\in\{2,\ldots,\ n-1\}$. That means that the involved vectors F *all* are q-th powers (because $F^{(1)}$ does not occur anymore) of, say, $\tilde{F}$, i.e., we have

$$\left(\sum_{\{\sigma,|\sigma(2l-1)-\sigma(2l)-1|=n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l-1}<\tilde{F}^{\sigma(2j-1)},\ \tilde{F}^{\sigma(2j)}>\right)\right)^q.$$

Note that we are still summing over coset representatives σ of the centralizer $C(\tau)$ in Σ_{2l}, with the only restriction that our σ's look[9] like $(n-1\ 1)(n)\tilde{\sigma}$, where $\tilde{\sigma}\in\Sigma_{2l-2}$. We have to convince ourselves that the elements $\tilde{\sigma}$ run over a complete set of coset representatives (exactly once) of $C(\tau_{2l-2})$ in Σ_{2l-2}, where we set $\tau_{2l-2}:=(12)\cdots(2l-3\ 2l-2)\in\Sigma_{2l-2}$. Define a map

$$\Sigma_{2l-2}/C(\tau_{2l-2})\longrightarrow\left\{(n-1\ 1)(n)\tilde{\sigma}\,C(\tau_{2l})\right\},\ \tilde{\sigma}\,C(\tau_{2l-2})\longmapsto(n-1\ 1)(n)\tilde{\sigma}\,C(\tau_{2l}).$$

This map is obviously injective. We define a splitting via

$$\left\{(n-1\ 1)(n)\tilde{\sigma}\,C(\tau_{2l})\right\}\longrightarrow\Sigma_{2l-2}/C(\tau_{2l-2}),\ (n-1\ 1)(n)\tilde{\sigma}\,C(\tau_{2l})\longmapsto\tilde{\sigma}\,C(\tau_{2l-2}).$$

This map is equally injective, because if we take two different elements

$$(n-1\ 1)(n)\tilde{\sigma_1}C(\tau_{2l})\neq(n-1\ 1)(n)\tilde{\sigma_2}C(\tau_{2l}),$$

then also $\tilde{\sigma_1}C(\tau_{2l-2})\neq\tilde{\sigma_2}C(\tau_{2l-2})$.

Coming back to our coefficient we summarize

$$\begin{aligned}
\mathbf{E}_n\mathbf{d}_{n,\ i} = & \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1|\neq n,\forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l}<F^{\sigma(2j-1)},\ F^{\sigma(2j)}>\right)\\
& +\left(\sum_{\{\sigma,|\sigma(2j_0-1)-\sigma(2j_0)-1|=n\}}\left(\operatorname{sign}(\sigma)\prod_{j=1,\neq j_0}^{l}<F^{\sigma(2j-1)},\ F^{\sigma(2j)}>\right)\right)\xi_{n,n}\\
= & \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1|\neq n,\forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l}<F^{\sigma(2j-1)},\ F^{\sigma(2j)}>\right)\\
& +\left(\sum_{\{\sigma,|\sigma(2l-1)-\sigma(2l)-1|=n\}}\operatorname{sign}(\sigma)\prod_{j=1}^{l-1}\left(<\tilde{F}^{\sigma(2j-1)},\ \tilde{F}^{\sigma(2j)}>\right)^q\right)\xi_{n,\ n}\\
= & \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1|\neq n,\forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l}<F^{\sigma(2j-1)},\ F^{\sigma(2j)}>\right)\\
& +\left(\sum_{\tilde{\sigma}\in C(\tau_{2l-2})}\left(\operatorname{sign}(\tilde{\sigma})\prod_{j=1}^{l-1}<\tilde{F}^{\tilde{\sigma}(2j-1)},\ \tilde{F}^{\tilde{\sigma}(2j)}>\right)\right)^q\xi_{n,\ n}\\
= & \sum_{\{\sigma,|\sigma(2j-1)-\sigma(2j)-1|\neq n,\forall j\}}\left(\operatorname{sign}(\sigma)\prod_{j=1}^{l}<F^{\sigma(2j-1)},\ F^{\sigma(2j)}>\right)\\
& +\left(\mathcal{D}_{n-2,\ i-1}(\xi_{n,\ 1},\ldots,\ \xi_{n,\ n-2})^q\right)\xi_{n,\ n},
\end{aligned}$$

[9] The (n) emphasizes that the σ's fix n.

where in the last step we used that

$$\tilde{F}^j = (x_1^{q^{i_j}-1}, \dots, y_l^{q^{i_j}-1}) \quad \forall\, j$$

by construction. Since the invariants $\xi_{n,1}, \dots, \xi_{n,n}$ are algebraically independent by Lemma 2.3 this proves, for all $i = 0, \dots, n-1$,

$$\mathcal{D}_{n,i}(X_1, \dots, X_n) = \left(\mathcal{D}_{n-1,i-1}(X_1, \dots, X_{n-2})\right)^q X_n + \text{junk},$$

where junk does not depend on X_n •

A similar construction leads to another relation. For that we need the following lemma.

LEMMA 3.4: *Let $F^{(1)}, \dots, F^{(n+2)} \in \times_n \mathbb{F}[V]$ be $n+2$ n-tuples of polynomials. Then*

$$\wr\wr F^{(1)}, \dots, F^{(n+2)} \wr\wr := \sum_{k=2}^{n+2} (-1)^k < F^{(1)}, F^{(k)} > || F^{(2)}, \dots, \widehat{F^{(k)}}, \dots, F^{(n+2)} ||$$

defines an alternating multilinear form.

PROOF: Turn the F's into $(n+2)$-tuples of polynomials by adding two zero entries at the end and note that this does not change the value of $< F^j, F^k >$ for any j, k. We have seen in Lemma 3.1 that

$$|| F^{(1)}, \dots, F^{(n+2)} || := \sum_{\sigma} \left(\text{sign}(\sigma) \prod_{j=1}^{l+1} < F^{\sigma(2j-1)}, F^{\sigma(2j)} > \right)$$

is a multilinear alternating form. In every product the factor $< F^{(1)}, F^{(k)} >$ occurs for some k, i.e.,

$$\sigma_k(2j_0 - 1) = 1 \quad \text{and} \quad \sigma_k(2j_0) = k$$

(or vice versa) for some $j_0 = 1, \dots, l$. Without loss of generality we can assume that $\sigma_k(1) = 1$ and $\sigma_k(2) = k$ for otherwise we replace σ_k by $\sigma_k(2j_0 - 1\ 1)(2j_0\ 2)$, which represents the same coset. Hence we have

$$\wr\wr F^{(1)}, \dots, F^{(n+2)} \wr\wr = \sum_{\sigma} \text{sign}(\sigma) < F^{(1)}, F^{(k)} > \left(\prod_{j=2}^{l+1} < F^{\sigma(2j-1)}, F^{\sigma(2j)} > \right).$$

Like in the preceding lemma we replace σ_k by $\tilde{\sigma}$ where $\tilde{\sigma} \in \Sigma_{2l}$ permutes the set $\{3, \dots, 2l+2\}$ and we observe that the permutations $\tilde{\sigma}$ run through a complete set of coset representatives of $C(\tau_{2l})$ in Σ_{2l}. Moreover observe that $\text{sign}(\sigma) = (-1)^k \text{sign}(\tilde{\sigma})$, because the number of descents of σ is precisely k plus the number of descents of $\tilde{\sigma}$. Therefore we have

$$\begin{aligned}
&\wr\wr F^{(1)}, \dots, F^{(n+2)} \wr\wr \\
&\quad = \sum_{\sigma} \text{sign}(\sigma) < F^{(1)}, F^{(k)} > \prod_{j=2}^{l+1} < F^{\sigma(2j-1)}, F^{\sigma(2j)} > \\
&\quad = \sum_{k=2}^{n+2} (-1)^k < F^{(1)}, F^{(k)} > \left(\sum_{\tilde{\sigma}} \text{sign}(\tilde{\sigma}) \prod_{j=2}^{l+1} < F^{\tilde{\sigma}(2j-1)}, F^{\tilde{\sigma}(2j)} > \right) \\
&\quad = \sum_{k=2}^{n+2} (-1)^k < F^{(1)}, F^{(k)} > || F^{(2)}, \dots, \widehat{F^{(k)}}, \dots, F^{(n+2)} ||
\end{aligned}$$

as claimed •

Convention: Denote $\mathbf{d}_{n,}$[redacted]er $j \notin \{0,\dots,n\}$. Then setting $\mathcal{D}_{n,\,n} = \mathcal{E}_n$ makes the whole story consistent.

Proposition 3.5: *We have*

$$P_0 := \sum_{j=1}$$

[redacted]

$$\mathbf{d}_{n,j} = 0$$

in $\mathbb{F}[V]^{(n,\,\mathbb{F})}$.

Proof: We set

$$F = F^{(1)} =$$

[redacted]

$$y_1,\dots,x_l,\ y_l)$$

and for $j = 3,\dots,n+2$

$F^{(j}$ [redacted] $,\ x_l^{q^{j-2}},\ y_l^{q^{j-2}}).$

[redacted]

$.\,.\,,\ F^{(n+2)}||$

[redacted]

$$= \sum_{j=3}(-1)^j\xi_{n,j-2}\mathbf{E}_n$$

[redacted]

$$= \left(\sum_{j=3}^{n+2}(-1)^j\xi_{n,j-2}\mathbf{d}_{n,\,j-2}\right)\mathbf{E}_n.$$

Since $\mathbf{E}_n \neq 0 \in \mathbb{F}[V]^{(n,\,\mathbb{F})}$ we have

$$P_0 = \sum_{j=1}^{n}(-1)^j\xi_{n,j}\mathbf{d}$$

[redacted]

as claimed •

§4. Steenrod plays his game

We are going to calculate the Steenr[redacted]ploit Steenrod to find further rela[redacted]ns, and to show that the $\mathbb{F}$-alg[redacted]1

[redacted]der the acti[redacted]V], i.e., we will show that

[redacted]$_{n,\ n-1}$ >

[redacted]of the Dickson and Euler classes are known and as well polynomials in the [redacted]ses, see [11] Appendix A.2 and the references there, we are left to deal with the [redacted]$_{n,i}$. For simplicity of notation we make the following conventions:

CONVENTION: Let $\mathcal{P}^i \equiv 0$ whenever $i \notin \mathbb{N}_0$. Moreover, let $\xi_{n,i} = 0$ for $i \notin \mathbb{N}$.

LEMMA 4.1: *The Steenrod powers of the new classes $\xi_{n,i}$, $i \geq 1$, are given by the following formulae*

$$\mathcal{P}^j(\xi_{n,i}) = \begin{cases} \xi_{n,i}^q & \text{if } j = q^i + 1 \\ \xi_{n,i+1} & \text{if } j = q^i \\ \xi_{n,i-1}^q & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

PROOF: By straightforward calculation:

$$\begin{aligned}
\mathcal{P}^j(\xi_{n,i}) &= \sum_{k=1}^{l} \left(\mathcal{P}^j(x_k y_k^{q^i}) - \mathcal{P}^j(x_k^{q^i} y_k)\right) \\
&= \sum_{k=1}^{l} \left(x_k \mathcal{P}^j(y_k^{q^i}) - \mathcal{P}^j(x_k^{q^i}) y_k + x_k^q \mathcal{P}^{j-1}(y_k^{q^i}) - \mathcal{P}^{j-1}(x_k^{q^i}) y_k^q\right) \\
&= \sum_{k=1}^{l} \left(x_k (\mathcal{P}^{j/q^i}(y_k)^{q^i} - \mathcal{P}^{j/q^i}(x_k)^{q^i} y_k + x_k^q \mathcal{P}^{(j-1)/q^i}(y_k)^{q^i} - \mathcal{P}^{(j-1)/q^i}(x_k)^{q^i} y_k^q\right) \\
&= \begin{cases} \sum_{k=1}^{l} x_k y_k^{q^{i+1}} - x_k^{q^{i+1}} y_k & \text{if } j = q^i \\ \sum_{k=1}^{l} x_k^q y_k^{q^{i+1}} - x_k^{q^{i+1}} y_k^q & \text{if } j = q^i + 1 \\ \sum_{k=1}^{l} x_k^q y_k^{q^i} - x_k^{q^i} y_k^q & \text{if } j = 1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which was to be shown •

We evaluate the Steenrod derivations on our ξ's in the next lemma.

LEMMA 4.2: *The Steenrod derivations act on the ξ's by*

$$\mathcal{P}^{\Delta_j}(\xi_{n,i}) = \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases} .$$

PROOF: By induction on j. For $j = 1$ we have

$$\begin{aligned}
\mathcal{P}^{\Delta_1}(\xi_{n,i}) &= \mathcal{P}^1(\xi_{n,i}) \\
&= \begin{cases} 0 & \text{if } i = 1 \\ \xi_{n,i-1}^q & \text{if } i > 1 \end{cases} ,
\end{aligned}$$

where we made use of the preceding lemma. Next take an $j > 1$. Then we get by using again the preceding lemma and the induction hypothesis

$$\begin{aligned}
\mathcal{P}^{\Delta_j}(\xi_{n,i}) &= \mathcal{P}^{\Delta_{j-1}} \mathcal{P}^{q^{j-1}}(\xi_{n,i}) - \mathcal{P}^{q^{j-1}} \mathcal{P}^{\Delta_{j-1}}(\xi_{n,i}) \\
&= \begin{cases} 0 - (-1)^{j-1} \mathcal{P}^{q^{j-1}} \left(\xi_{n,j-1-i}^{q^i}\right) & \text{if } j-1 > i \\ \mathcal{P}^{\Delta_{j-1}}(\xi_{n,i+1}) - 0 & \text{if } j-1 = i \\ 0 - (-1)^j \mathcal{P}^{q^{j-1}} \left(\xi_{n,i-j+1}^{q^{j-1}}\right) & \text{if } j-1 < i \end{cases}
\end{aligned}$$

$$= \begin{cases} (-1)^j \mathscr{P}^{q^{j-i-1}} \left(\xi_{n,j-1-i}\right)^{q^i} & \text{if } j-1 > i \\ (-1)^j \xi_{n,i+1-j+1}^{q^{j-1}} & \text{if } j-1 = i \\ (-1)^{j+1} \mathscr{P}^1 \left(\xi_{n,i-j+1}\right)^{q^{j-1}} & \text{if } j-1 < i \end{cases}$$

$$= \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i+1 \\ (-1)^j \xi_{n,1}^{q^{j-1}} & \text{if } j = i+1 \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases}$$

$$= \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases},$$

as we wanted •

REMARK: Since

$$\mathscr{P}^{\Delta_i} := \begin{cases} \mathscr{P}^1 & \text{if } i = 1 \\ [\mathscr{P}^{q^i}, \mathscr{P}^{\Delta_{i-1}}] & \text{if } i > 1 \end{cases}$$

the the action of $\mathscr{P}^{\Delta_i}$ and $\mathscr{P}^{q^i} \mathscr{P}^{q^{i-1}} \cdots \mathscr{P}^q \mathscr{P}^1$ agree on the classes in V^*. This can be used to give another proof of Lemma 4.2.

Now we can prove

PROPOSITION 4.3: *The $\mathbb{F}$-algebra A generated by $\xi_{n,1}, \ldots, \xi_{n,n}$, $\mathbf{E}_n$, $\mathbf{d}_{n,1}, \ldots, \mathbf{d}_{n,n-1}$ is an unstable algebra over the Steenrod algebra.*

[redacted] The only thing we need to show is that our algebra A is closed under the action of the [redacted] rest is inherited from $\mathbb{F}[V]$.

[redacted] $\xi_{n,n}$, i.e.,

[redacted] ppendix A.2 we have [redacted]

$$\mathscr{P}^j(\mathbf{d}_{n,i}) \in \mathbb{F}[\mathbf{d}_{n,0}, \ldots$$ [redacted] $\subseteq$ A.

[redacted] we consider $j \geq 0$ and $i = 1, \ldots, n-1$ and [redacted]

$$\mathscr{P}^j(\xi_{n,i}) \in \mathbb{F}[\xi_{n,1}, \ldots$$ [redacted]

by Proposition 2.1 •

This allows us to construct a second family of relations from P_0 given in [redacted] ition 3.5 in a very natural way.

COROLLARY 4.4: *In $\mathbb{F}[V]^{(n,\,\mathbb{F})}$ we have*

$$P_i := \sum_{j=i+1}^{n} \left((-1)^j \xi_{n,j-i}^{q^i} \mathbf{d}_{n,j}\right) - \sum_{j=0}^{i-1} \left((-1)^j \xi_{n,i}^{q^j}\right.$$ [redacted]

for $i = 0, \ldots, n-1$.

 [redacted]

PROOF: By Proposition 3.5 we know that

$$P_0 = 0.$$

Therefore all Steenrod powers of this polynomial are zero, and, by Proposition 4.3, are again polynomials in the algebra generators of A. Observe that

$$P_{i+1} = \mathcal{P}^{q^i}(P_i),$$

which is proved by a straighforward calculation, to wit:

$$\begin{aligned}
\mathcal{P}^{q^i}(P_i) &= \sum_{j=i+1}^{n} \left((-1)^j \mathcal{P}^{q^i}\left(\xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right)\right) - \sum_{j=0}^{i-1}\left((-1)^j \mathcal{P}^{q^i}\left(\xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right)\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\alpha}(\xi_{n,j-i}^{q^i})\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right) - \sum_{j=0}^{i-1}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\alpha}(\xi_{n,i-j}^{q^j})\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\frac{\alpha}{q^i}}(\xi_{n,j-i})^{q^i}\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right) - \sum_{j=0}^{i-1}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\frac{\alpha}{q^j}}(\xi_{n,i-j})^{q^j}\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\mathcal{P}^1(\xi_{n,j-i})^{q^i}\mathbf{d}_{n,j} + \xi_{n,j-i}^{q^i}\mathcal{P}^{q^i}(\mathbf{d}_{n,j})\right)\\
&\quad - \sum_{j=0}^{i-1}(-1)^j\left(\xi_{n,i-j}^{q^j}\mathcal{P}^{q^i}(\mathbf{d}_{n,j}) + \mathcal{P}^1(\xi_{n,i-j})^{q^j}\mathcal{P}^{q^i-q^j}(\mathbf{d}_{n,j}) + \mathcal{P}^{q^{i-j}}(\xi_{n,i-j})^{q^j}\mathbf{d}_{n,j}\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\xi_{n,j-i-1}^{q^{i+1}}\mathbf{d}_{n,j}\right) + (-1)^{i+1}\xi_{n,1}^{q^i}\mathbf{d}_{n,i} - \sum_{j=0}^{i-1}(-1)^j\left(\xi_{n,i-j+1}^{q^j}\mathbf{d}_{n,j}\right)\\
&= \sum_{j=i+2}^{n}(-1)^j\left(\xi_{n,j-i-1}^{q^{i+1}}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{i}(-1)^j\left(\xi_{n,i-j+1}^{q^j}\mathbf{d}_{n,j}\right)\\
&= P_{i+1},
\end{aligned}$$

where we made heavily use of the Cartan formulae, Lemma 4.1 and Appendix A.2 in [11] •

Note that Proposition 4.3 tells us *a priori* that all Steenrod powers of P_0 are again polynomials in the algebra generators of A. However, for their explicite description we had to calculate them anyway. In Section 6 it will turn out that we need only to consider $P_0, \ldots, P_{l-1}$. Moreover, note that

$$P_i \equiv \xi_{n,n-i}^{q^i} \quad \textbf{MOD}\ (\xi_{n,1}, \ldots, \xi_{n,n-i-1})$$

whenever $i \le l-1$.

§5. British $\mathcal{T}$

Recall from Proposition 3.2 (1) that the Euler class $\mathbf{E}_n$ is a polynomial in $\xi_{n,1}, \ldots, \xi_{n,n-1}$. So, there exists a polynomial $\mathcal{E}_n = \mathcal{E}_n(X_1, \ldots, X_{n-1})$ such that

$$\mathbf{E}_n = \mathcal{E}_n(\xi_{n,1}, \ldots, \xi_{n,n-1}).$$

The same proposition, part (2), shows that

$$\mathbf{E}_n\mathbf{d}_{n,\,i} \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}],$$

i.e., there exist polynomials $\mathcal{D}_{n,i} = \mathcal{D}_{n,i}(X_1, \dots, X_n)$ such that

$$\mathbf{E}_n \mathbf{d}_{n,\,i} = \mathcal{D}_{n,i}(\xi_{n,1}, \dots, \xi_{n,n}),$$

for all $i = 0, \dots, n-1$. Moreover, we have seen that these polynomials $\mathcal{D}_{n,i}$ are linear in their last indeterminant, which has coefficient

$$\left(\mathcal{D}_{n-2,i-1}(X_1, \dots, X_{n-2})\right)^q,$$

compare Lemma 3.3.

PROPOSITION 5.1: *There exist polynomials*

$$\mathcal{T}_{i,j} = \mathcal{T}_{i,j}(X_1, \dots, X_n) \in \mathbb{F}[X_1, \dots, X_n]$$

such that for $i,\ j = 1, \dots, l$ *we have*

$$\mathcal{D}_{n,i} = \sum_{k=0}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k}$$

and

$$\mathcal{T}_{i,j} = \mathcal{T}_{i,j}(X_1, \dots, X_{2(i+j)-3}) \in \mathbb{F}[X_1, \dots, X_{2(i+j)-3}],$$

i.e., $\mathcal{T}_{i,j}$ *depents only on the first* $2(i+j)-3$ *variables.*

PROOF: We construct the $\mathcal{T}_{i,j}$ by induction on j. Let $j = 1$. Then we define

$$\mathcal{T}_{i,1}(X_1, \dots, X_n) := \mathcal{E}_{2i}(X_1, \dots, X_n)^{q-1},$$

which is in $\mathbb{F}[X_1, \dots, X_{2i-1}]$, because the polynomial $\mathcal{E}_{2i}$ lives there. For $i = l$ this polynomial satisfies the desired relation

$$\mathcal{D}_{n,0} = \mathcal{E}_n^q = \mathcal{E}_n^{q-1}\mathcal{E}_n = \mathcal{T}_{l,1}\mathcal{E}_n.$$

For $i < l$ there is nothing more to prove.

Next take an $j > 1$ and assume $\mathcal{T}_{i,j}$ is defined for all $i = 1, \dots, l$ and all $j = 1, \dots, l-i$ such that the required relations hold. We then have by the induction hypothesis that

$$\mathcal{T}_{i,l-i+1}\mathcal{E}_n = \mathcal{D}_{n,l-i} - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k} \in \mathbb{F}[X_1, \dots, X_n].$$

We want to show that

$$\mathcal{T}_{i,l-i+1} \in \mathbb{F}[X_1, \dots, X_{n-1}].$$

From Proposition 3.2 (2) it follows that

$$\begin{aligned} \mathcal{T}_{i,l-i+1}\mathcal{E}_n &= \mathcal{D}_{n,l-i} - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k} \\ &= \mathcal{D}_{n,l-i}(\xi_{n,1}, \dots, \xi_{n,n}) - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k}(\xi_{n,1}, \dots, \xi_{n,n}) \end{aligned}$$

is an element in $\mathbb{F}[X_1, \dots, X_n]$. Since $k = 1, \dots, l-i$ we have by induction

$$\mathcal{T}_{i,l-k-i+1} \in \mathbb{F}[X_1, \dots, X_{2l-2k-1}].$$

Therefore, together with Lemma 3.3 our polynomial is linear in X_n with leading coefficient

$$\mathcal{D}_{n-2,l-i-1}(X_1,\ldots,X_{n-2})^q - \sum_{k=1}^{l-i} \mathcal{T}^{q^k}_{i,l-k-i+1}\mathcal{D}_{n-2,n-k-1}(X_1,\ldots,X_{n-2})^q =$$

$$\left(\mathcal{D}_{n-2,l-i-1}(X_1,\ldots,X_{n-2}) - \sum_{k=0}^{l-i-1} \mathcal{T}^{q^k}_{i,l-k-i}\mathcal{D}_{n-2,n-2-k}(X_1,\ldots,X_{n-2})\right)^q$$

$$= 0,$$

where the last equation follows from the induction hypothesis. Therefore

$$\mathcal{T}_{i,l-i+1}\mathcal{E}_n \in \mathbb{F}[X_1,\ldots,X_{n-1}]$$

which in turn implies that

$$\mathcal{T}_{i,l-i+1} \in \mathbb{F}[X_1,\ldots,X_{n-1}],$$

where the desired relation holds by construction •

§6. Some Algebra

In this section we do some algebra and prove that we have found all generators and relations of our ring of invariants. By what we have done so far we know that

$$\mathrm{A} = \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n},\ \mathbf{d}_{n,0},\ldots,\mathbf{d}_{n,n-1} >= \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n},\ \mathbf{d}_{n,1},\ldots,\mathbf{d}_{n,n-1} >,$$

compare Proposition 3.2 (1).[10]

Next we show that we can omit the Dickson classes of high degree.

LEMMA 6.1: *With th* [redacted] *eding notation we have*

[redacted] $\mathbf{d}_{n,l},\ldots,\mathbf{d}_{n,n-1} > .$

PROOF: Certainly A c [redacted]

$\mathbf{d}_n$ [redacted]

Proposition 5.1 hands us e [redacted]

[redacted] $\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k}$ [redacted]

for $i = 1,\ldots,l-1$. Since 2 [redacted] 0 we get,

$$\mathbf{d}_{n,l-i} = \sum_{k=0}^{l-i} \mathcal{T}^{q^k}_{i,l-i-k+}$$ [redacted] $$-k)-1)\mathbf{d}_{n,n-k}, \in \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n},\ \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n.n-1} >,$$

where we use that $i = 1,..$ [redacted]

Consider the remembering [redacted]

$$\rho : \mathrm{B}$$ [redacted] $$\mathbb{F}[\xi_{n,1},\ldots,\xi_{n,n},\ \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n,n-1}] \longrightarrow \mathbb{F}[V]^{(n,\ \mathbb{F})}.$$

[10] We forget for a moment that P_0 [redacted] in Proposition 3.5, cancels $\xi_{n,n}$.

Recall the remarks after Corollary 4.4

$$P_i \equiv \xi_{n,n-i}^{q^i} \ \mathbf{MOD}(\xi_{n,1}, \ldots, \xi_{n,n-i-1}),$$

whenever $i \le l-1$. Since the sequence

$$\xi_{n,1}, \ldots, \xi_{n,n}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

forms a regular sequence, so does the sequence

$$\xi_{n,1}, \ldots, \xi_{n,n-1}, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}.$$

Therefore also $\xi_{n,1}, \ldots, \xi_{n,n-1}^{q}, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$ forms a regular sequence and hence so does $\xi_{n,1}, \ldots, \xi_{n,n-2}, \ P_1, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$. Successively we get that

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ P_{l-1}, \ldots, P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

is a regular sequence, and in particular

$$P_0, \ldots, P_{l-1} \in \mathrm{B}$$

is a regular sequence. Hence we have shown

LEMMA 6.2: *The* $\mathbb{F}$*-algebra*

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1}]/\left(P_0, \ldots, P_{l-1}\right)$$

is a complete intersection of Krull dimension n*. In particular it is a Cohen-Macaulay algebra.*

PROOF: The Cohen-Macaulayness follows from the same calculation:

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ P_{l-1}, \ldots, P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

is a regular sequence, and hence in the quotient algebra

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right),$$

is a regular sequence of length n •

What we are about to do is to show that this algebra $\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$ is precisely the ring of invariants we are looking for, and, moreover the same as A.

The proof of the following lemma uses Nagata's theorem, see [9], i.e., one of the few existing standard methods to prove that a given ring is the desired ring of invariants, compare [10].

LEMMA 6.3: *The algebra* $\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$ *is a unique factorization domain.*

PROOF: We rewrite our system of relations $P_0, \ldots, P_{l-1}$ as a system of linear equations for the Dickson classes, i.e., the system

$$\begin{array}{rcl} P_0 & = & 0 \\ \ddots & & \ddots \\ P_i & = & 0 \\ \ddots & & \ddots \\ P_{l-1} & = & 0 \end{array}$$

is by Corollary 4.4 equivalent to

$$\begin{aligned}
\sum_{j=1}^{n}\left((-1)^j \xi_{n,j}\mathbf{d}_{n,j}\right) &= 0\\
&\ddots\\
\sum_{j=i+1}^{n}\left((-1)^j \xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{i-1}\left((-1)^j \xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right) &= 0\\
&\ddots\\
\sum_{j=l}^{n}\left((-1)^j \xi_{n,j-l+1}^{q^{l-1}}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{l-2}\left((-1)^j \xi_{n,l-1-j}^{q^j}\mathbf{d}_{n,j}\right) &= 0,
\end{aligned}$$

what in turn can be written as

$$\begin{aligned}
\sum_{j=l}^{n-1}\left((-1)^j \xi_{n,j}\mathbf{d}_{n,j}\right) + \sum_{j=1}^{l-1}\left((-1)^j \xi_{n,j}\mathbf{d}_{n,j}\right) + (-1)^n \xi_{n,n} &= 0\\
&\ddots\\
\sum_{j=l}^{n-1}\left((-1)^j \xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right) + \sum_{j=i+1}^{l-1}\left((-1)^j \xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{i-1}\left((-1)^j \xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right) + (-1)^n \xi_{n,n-i}^{q^i} &= 0\\
&\ddots\\
\sum_{j=l}^{n-1}\left((-1)^j \xi_{n,j-l+1}^{q^{l-1}}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{l-2}\left((-1)^j \xi_{n,l-1-j}^{q^j}\mathbf{d}_{n,j}\right) + (-1)^n \xi_{n,n-l+1}^{q^{l-1}} &= 0.
\end{aligned}$$

We set

$$\mathbf{M} = \begin{bmatrix} (-1)^l \xi_{n,l} & \cdots & (-1)^{n-1}\xi_{n,n-1}\\ \cdots & \cdots & \cdots\\ (-1)^l \xi_{n,l-i}^{q^i} & \cdots & (-1)^{n-1}\xi_{n,n-1-i}^{q^i}\\ \cdots & \cdots & \cdots\\ (-1)^l \xi_{n,1}^{q^{l-1}} & \cdots & (-1)^{n-1}\xi_{n,l}^{q^{l-1}} \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} 0 & \xi_{n,1} & \cdots & \cdots & (-1)^{i-1}\xi_{n,i-1} & \cdots & \cdots & (-1)^l \xi_{n,l-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \xi_{n,i} & -\xi_{n,i-1}^q & \cdots & (-1)^{i-1}\xi_{n,1}^{q^{i-1}} & 0 & (-1)^i \xi_{n,1}^{q^i} & \cdots & (-1)^l \xi_{n,l-1-i}^{q^i}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \xi_{n,l-1}^q & -\xi_{n\,l-2}^q & \cdots & (-1)^i \xi_{n,l-i}^{q^{i-1}} & \cdots & \cdots & (-1)^{l-2}\xi_{n,1}^{q^{l-2}} & 0 \end{bmatrix}$$

and get a system of linear equations as follows

$$\mathbf{M}\begin{bmatrix}\mathbf{d}_{n,l}\\ \ddots\\ \mathbf{d}_{n,n-1}\end{bmatrix} - \mathbf{N}\begin{bmatrix}\mathbf{d}_{n,0}\\ \ddots\\ \mathbf{d}_{n,l-1}\end{bmatrix} + (-1)^n\begin{bmatrix}\xi_{n,n}\\ \ddots\\ \xi_{n,n-l+1}^{q^{l-1}}\end{bmatrix} = \begin{bmatrix}0\\ \ddots\\ 0\end{bmatrix}.$$

Recall from Proposition 5.1 that the Dickson classes of high degree are given by

$$\begin{aligned}
\mathbf{d}_{n,j} &= \sum_{k=0}^{j} T_{l-j,j-k+1}^{q^k}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k}\\
&= T_{l-j,j+1}(\xi_{n,1},\ldots,\xi_{n,2l-1}) + \sum_{k=1}^{j} T_{l-j,j-k+1}^{q^k}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k}.
\end{aligned}$$

We put that in our system of linear equations, set $\mathcal{T}_{i,j}(\xi) := \mathcal{T}_{i,j}(\xi_{n,1}, \dots, \xi_{n,2(i+j)-1})$ for short and get

$$\begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \ddots \\ \mathbf{d}_{n,l-1} \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix}$$

$$= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} - \mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^q_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^q_{1,l-1}(\xi) \end{bmatrix} \mathbf{d}_{n,n-1} - \cdots$$

$$\cdots - \mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \mathbf{d}_{n,l+1} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix}.$$

Denote by

$$\mathbf{L} := \begin{bmatrix} \begin{matrix} 0 \\ \ddots \\ \ddots \\ 0 \end{matrix} & \left(\mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \right) & \cdots & \left(\mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^q_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^q_{1,l-1}(\xi) \end{bmatrix} \right) \end{bmatrix}$$

the $l \times l$ matrix with columns

$$\begin{pmatrix} 0 \\ \ddots \\ \ddots \\ 0 \end{pmatrix} \quad \left(\mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \right) \quad \cdots \quad \left(\mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^q_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^q_{1,l-1}(\xi) \end{bmatrix} \right).$$

Then

$$\begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \ddots \\ \mathbf{d}_{n,l-1} \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix}$$

$$= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{L} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix}$$

$$- \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix}$$

$$= \mathbf{T} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathcal{E}_n(\xi_{n,1}, \dots, \xi_{n,n-1})^{q-1} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix},$$

where we set

$$\mathbf{T} := \mathbf{M} - \mathbf{L}$$

and use that the top Dickson class $\mathbf{d}_{n,0}$ is nothing but the $(q-1)$-st power of the Euler class $\mathbf{E}_n$ what in turn is a polynomial in the first $n-1$ ξ's

$$\mathbf{d}_{n,0} = (\mathbf{E}_n)^{q-1} = E_n(\xi_{n,1}, \ldots, \xi_{n,n-1})^{q-1}$$

by Proposition 3.2. The matrix $\mathbf{T}$ is modulo $\xi_{n,1}, \ldots, \xi_{n,l-1}$ upper triangular with determinant

$$\det(\mathbf{T}) \equiv (-1)^{\frac{(3l-1)l}{2}} \xi_{n,l} \xi_{n,l}^{q} \cdots \xi_{n,l}^{q^{l-1}} =: \Delta \quad \mathbf{MOD}\ \xi_{n,1}, \ldots, \xi_{n,l-1}.$$

By Lemma 6.2 we know that

$$\xi_{n,l} \in \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$$

is not a zero divisor, hence so is Δ. Therefore we get by localizing at Δ an *inclusion*

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) \hookrightarrow \mathrm{B}[\Delta^{-1}]/\left(P_0, \ldots, P_{l-1}\right).$$

In the bigger algebra the system of equations given by the relations $P_0, \ldots, P_{l-1}$ can be *solved* for $\mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1}$ (well, we just inverted the determinant of the matrix of our system of linear equations) and hence

$$\mathrm{B}[\Delta^{-1}]/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}, \Delta^{-1}]$$

is a polynomial algebra, and in particular an integral domain. Hence

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$$

is also an integral domain.

Next, we want to show that Δ is a prime element in $B/(P_0, \ldots, P_{l-1})$. Observe that the entries of the matrix $\mathbf{T}$ are polynomials in $\xi_{n,1}, \ldots, \xi_{n,n-1}$. However, $\xi_{n,n-1}$ occurs only in the top right corner, i.e., we can rewrite $\mathbf{T}$ as

$$\mathbf{T} := \begin{bmatrix} (-1)^l \xi_{n,l} & \cdots & (-1)^{n-2}\xi_{n,n-2} & (-1)^{n-1}\xi_{n,n-1} \\ & & & (-1)^{n-1}\xi_{n,n-2}^{q} + N_1 \\ & \mathbf{T^{cof}} & & \ddots \\ & & & (-1)^{n-1}\xi_{n,l}^{q^{l-1}} + N_{l-1} \end{bmatrix},$$

where we set

$$\mathbf{N}\begin{bmatrix} 0 \\ T_{l-1,1}^{q}(\xi) \\ \ddots \\ T_{1,l-1}^{q} \end{bmatrix}(\xi) = \begin{bmatrix} 0 \\ N_1 \\ \ddots \\ N_{l-1} \end{bmatrix}.$$

The cofactor matrix $\mathbf{T^{cof}}$ has determinant

$$\Delta^{\mathbf{cof}} \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-2}].$$

Therefore

$$\Delta^{\mathbf{cof}} \in B/(P_0, \ldots, P_{l-1}, \Delta)$$

is not a zero divisor (with a little help from Lemma 6.2). So, if we invert this determinant we get an *inclusion*

$$\mathrm{B}/(P_0, \ldots, P_{l-1}, \Delta) \hookrightarrow \mathrm{B}[(\Delta^{\mathbf{cof}})^{-1}]/(P_0, \ldots, P_{l-1}, \Delta).$$

In the localization the relations $P_1, \ldots, P_{l-1}$ can be solved for $\mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-2}$ and P_0 gives an equation for $\xi_{n,n}$. The determinant Δ is linear in $\xi_{n,n-1}$ and since we inverted the leading coefficient, namely Δ^{cof}, Δ can be solved for $\xi_{n,n-}$ [redacted] ence

$$\mathrm{B}[(\Delta^{\mathsf{cof}})^{-1}]/ \text{[redacted]} \xi_{n,n-2},\ \mathbf{d}_{n,n-1},\ (\Delta^{\mathsf{cof}})^{-1}]$$

is a polynomial ring and in pa [redacted]

is an integral domain, wha [redacted]

is a prime elem [redacted]

is a unique fact [redacted]

is, where we [redacted]

Finally, we a [redacted] cit de [redacted]

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THE [redacted]

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In Lemma [redacted]

is an inte [redacted]

Since the [redacted]

and in pa [redacted]

and at the [redacted]

[redacted] i.e.,

[redacted] $(n, \mathbb{F})$,

[redacted] is correct:

[redacted]

[redacted] all this [redacted]

[redacted] 7.1: [redacted] *given by*

[redacted]).

[redacted] ractions, [redacted]
[redacted] relation, i.e., w [redacted]
[redacted] ctions in $\xi_{n,1}, \ldots, \xi_{n,n}$ [redacted]

$$\mathbb{F}(V) \quad ^{(n,} \ldots$$ [redacted]

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[redacted] k at examples. If $n = 2$ [redacted] othing [redacted] ecial linea [redacted]

[redacted] 8.13 of [12] i [redacted] rk, namel [redacted]

[redacted] the next case: $n = 4$ [redacted]

[redacted] n is [redacted]

[redacted] 0 0 [redacted]
[redacted] 0 0 1 0 [redacted]
[redacted] 0 0 0 [redacted]

[redacted] the first Secti [redacted]
[redacted] is non Ab [redacted]

[redacted] $\xi_{4,3}$
[redacted] $- \xi_{4,3}^4$
[redacted] $\xi_{4,2} - \xi_{4,1}^3\xi_{4,4} - \xi_{4,3}^3\xi_{4,2}$.

[redacted] purely trancendental over $\mathbb{F}$.

[redacted] of this proposition gives you an explicite algorithm to find these expressions: note first that for $n = 4$ we have [redacted] (12)(34), the centralizer $C(\tau) = \{e, (12), (34), (12)(34), (13)(24), (1324), (1423), (14)(23)\}$ and a complete set of coset [redacted] presentatives in Σ_4 is given by e, (13) and (14). So, if you use the formula given in Proposition 3.2 you get the above expression.

Note that $\mathbf{E}_4$ is a polynomial in $\xi_{4,1}, \xi_{4,2}, \xi_{4,3}$ as Proposition 3.2 (1) predicts. The relation P_0 given in Proposition 3.5 reads as follows

$$P_0 = \xi_{4,1}\mathbf{d}_{4,1} + \xi_{4,2}\mathbf{d}_{4,2} + \xi_{4,3}\mathbf{d}_{4,3} + \xi_{4,4} = 0.$$

Corollary 4.4 hands us the remaining P_1, which is

$$P_1 = \xi_{4,1}^q\mathbf{d}_{4,2} - \xi_{4,2}^q\mathbf{d}_{4,3} + \xi_{4,3}^q - \xi_{4,1}\mathbf{d}_{4,0} = 0.$$

The british $\mathcal{T}$'s of Proposition 5.1, evaluated at $\xi_{4,1}, \ldots, \xi_{4,4}$ are given by

$$\mathcal{T}_{1,1} = \xi_{4,1}^2$$
$$\mathcal{T}_{1,2} = \mathbf{d}_{4,1} - \xi_{4,1}^6\mathbf{d}_{4,3} = \xi_{4,1}^{27}\xi_{4,2} - \xi_{4,2}^9\xi_{4,3} - \xi_{4,1}^7\xi_{4,2}^9 + \xi_{4,1}^6\xi_{4,2}\xi_{4,3}^3$$
$$\mathcal{T}_{2,1} = \mathbf{d}_{4,0} = \mathbf{E}_4^2 = \left(\xi_{4,1}^3\xi_{4,3} - \xi_{4,2}^4 + \xi_{4,1}^{10}\right)^2$$

The model algebra B of Section 6 is

$$\mathrm{B} = \mathbb{F}[\xi_{4,1}, \xi_{4,2}, \xi_{4,3}, \xi_{4,4}, \mathbf{d}_{4,2}, \mathbf{d}_{4,3}]/(P_0, P_1)$$

and the system of linear equations used in Lemma 6.3 looks like

$$\begin{bmatrix} \xi_{4,2} & \xi_{4,3} \\ \xi_{4,1}^3 & -\xi_{4,2}^3 - \xi_{4,1}^7 \end{bmatrix} \begin{bmatrix} \mathbf{d}_{4,2} \\ \mathbf{d}_{4,3} \end{bmatrix} + \begin{bmatrix} \xi_{4,1}\mathbf{E}_4^2 \\ \xi_{4,1}\,T_{1,2} \end{bmatrix} + \begin{bmatrix} \xi_{4,4} \\ \xi_{4,3}^3 \end{bmatrix} = 0.$$

So, we get

$$\Delta = -\xi_{4,2}\left(\xi_{4,2}^3 + \xi_{4,1}^7\right) - \xi_{4,1}^3\xi_{4,3}$$

and the cofactor determinant is just $\xi_{4,1}^3$.

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