ON DAVIS-JANUSZKIEWICZ HOMOTOPY TYPES I; FORMALITY AND RATIONALISATION

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ABSTRACT. For an arbitrary simplicial complex K, Davis and Januszkiewicz have defined a family of homotopy equivalent CW-complexes whose integral cohomology rings are isomorphic to the Stanley-Reisner algebra of K. Subsequently, Buchstaber and Panov gave an alternative construction, which they showed to be homotopy equivalent to Davis and Januszkiewicz's examples. It is therefore natural to investigate the extent to which the homotopy type of a space X is determined by having such a cohomology ring. We begin this study here, in the context of model category theory. In particular, we extend work of Franz by showing that the singular cochain algebra of X is formal as a differential graded noncommutative algebra. We then specialise to the rationals, by proving the corresponding property for Sullivan's *commutative* cochain algebra; this confirms that the rationalisation of X is unique. In a sequel, we will consider the uniqueness of X at each prime separately, and apply Sullivan's arithmetic square to produce global results in special families of cases.

1. Introduction

Over the last decade, work of Davis and Januszkiewicz [7] has popularised homotopy theoretical aspects of toric geometry amongst algebraic topologists. The results of [7] have been surveyed by Buchstaber and Panov in [4], where several further applications were developed. Their constructions have led us to consider the uniqueness of certain associated homotopy types, and our aim is to begin that study here; we shall address issues of formality and rationalisation, where the answers are most general. In a sequel [19], we discuss the problem prime by prime, and deduce global results for families of special cases by appealing to Sullivan's arithmetic square. The conclusions there are less clear-cut, and suggest further problems that seem to be of considerable difficulty.

We work over an arbitrary commutative ring R with identity, and consider a universal set V of *vertices* $v_1, \ldots, v_m$, ordered by their subscripts. The vertices masquerade as algebraically independent variables, which generate a graded polynomial algebra $S_R(V)$ over R. The grading is defined by assigning each of the generators a common dimension, which we usually take to be 2. A function $M\colon V \to \mathbb{N}$ is known as a *multiset* on V, with cardinality $|M| := \sum_j M(v_j)$; it may be represented by the monomial $v_M := \prod_V v^{M(v)}$, or by the n-tuple of constituent vertices $(v_{j_1}, \ldots, v_{j_n})$, where $j_1 \leq \cdots \leq j_n$ and $n = |M|$. So $S_R(V)$ is generated additively by the v_M, and v_M is squarefree precisely when M is a genuine subset.

Key words and phrases. Formality, Davis-Januszkiewicz spaces, homotopy colimit, model category, rationalisation, Stanley-Reisner algebra.

A simplicial complex K on V consists of a finite set of faces $\sigma \subseteq V$, closed with respect to the formation of subsets $\tau \subseteq \sigma$. Alternatively, we may interpret K as the set of squarefree monomials $v_\sigma := \prod_\sigma v$, which is closed under factorisation. Every simplicial complex generates a simplicial set $K_\bullet$; for each $n \geq 0$, the n-simplices K_n contain all M of cardinality $n+1$ whose support is a face of K. The face and degeneracy operators delete and repeat the appropriate vertices respectively.

The Stanley-Reisner algebra $R[K]$ (otherwise known as the *face ring* of K) is an important combinatorial invariant. It is defined as the quotient

$$S_R(V)/(v_U : U \notin K), \tag{1.1}$$

and is therefore generated additively by the simplices of $K_\bullet$. The algebraic properties of $R[K]$ encode a host of combinatorial features of K, and are discussed in detail by Bruns and Herzog [3] and Stanley [23], for example. If K is the simplex on V, then $R[K]$ is the polynomial algebra $S_R(V)$.

For each K, Davis and Januszkiewicz defined the notion of a toric space over the cone on the barycentric subdivision of K, and showed that the cohomology of such a space is related to $R[K]$. The relationship follows from their application of the Borel construction, which creates a family of spaces whose cohomology ring (with coefficients in R) is isomorphic to the Stanley-Reisner algebra. All members of the family are homotopy equivalent to a certain universal example, and we refer to any space which shares their common homotopy type as a Davis-Januszkiewicz space. The isomorphisms equip $R[K]$ with a natural grading, which agrees with that induced from $S_R(V)$. Subsequently, Buchstaber and Panov [4] defined a CW-complex whose cohomology ring is also isomorphic to $R[K]$. They confirmed that their complex is a Davis-Januszkiewicz space by giving an explicit homotopy equivalence with the universal example. In [21], their space is described as the pointed colimit $\operatorname{colim}^+(B^K)$ of a certain $\mathrm{CAT}(K)$-diagram B^K, which assigns the cartesian product B^σ to each face σ of K. Here B denotes the classifying space of the circle, otherwise known as complex projective space CP^∞.

We say that a space X *realises* the Stanley-Reisner algebra of K whenever there is an algebra isomorphism $H^*(X;R) \cong R[K]$. We denote the rationalisation of X by X_0, and write $\mathrm{Aut}_{ho}(X) < \mathrm{End}_{ho}(X)$ for the homotopy classes of self-equivalences of X, considered as a subgroup of the homotopy classes of self-maps with respect to composition. We refer to the group of unimodular complex numbers as T, in order to distinguish it from the underlying circle S^1.

The contents of this article are as follows.

In Section 2 we describe our notation and prerequisites, including those aspects of model category theory which provide a useful context for exponential diagrams and their cohomology. We also explain why it is more convenient to work with the unpointed colimit $c(K) := \operatorname{colim}(B^K)$. We introduce the Stanley-Reisner algebra in Section 3, and show that the Bousfield-Kan spectral sequence for $H^*(c(K);R)$ collapses by analysing higher limits of certain $\mathrm{CAT}(K)$-diagrams. In Section 4 we apply similar techniques to prove the formality of the singular cochain algebra $C^*(c(K);R)$. Finally, we specialise to the case $R = \mathbb{Q}$ in Section 5, where we confirm that Sullivan's commutative cochain algebra $A_{PL}(c(K))$

is formal in the commutative sense. We deduce that $\mathbb{Q}[K]$ determines the rationalisation $c(K)_0$ uniquely, and discuss $\mathrm{Aut}_{ho}(c(K)_0)$ in a few simple cases.

We would like to develop the results of Section 4 in the model category of E_∞ algebras, which are emerging as the integral analogue of differential graded commutative algebras over $\mathbb{Q}$. Recent work of Mandell [17] shows that $c(K)$ is classified by the E_∞ algebra $C^*(c(K);\mathbb{Z})$, at least up to weak equivalence of nilpotent spaces. As we shall explain in the sequel, we do *not* believe that $C^*(c(K);R)$ is always formal in this stricter sense; nevertheless, it may be true for our families of special cases.

Both authors are especially grateful to the organisers of the International Conference on Algebraic Topology, which was held on the Island of Skye in June 2001. The Conference provided the opportunity for valuable discussion with several colleagues, amongst whom Octavian Cornea, Kathryn Hess, and Taras Panov deserve special mention. Without that remarkable and stimulating environment, our work could not have begun. We should also thank the London Mathematical Society for supporting the Transpennine Topology Triangle, whose meetings have speeded up the completion of this article, and are helping us to prepare its sequel.

2. Background

We begin by establishing our notation and prerequisites, recalling various aspects of Davis-Januszkiewicz spaces. We refine results of [21] in the context of model category theory, referring readers to [9] and [14] for background details. Following [24], we adopt the model category TOP of k-spaces and continuous functions as our topological workplace. Weak equivalences induce isomorphisms in homotopy, fibrations are Serre fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations. Every function space Y^X is endowed with the corresponding k-topology. Many of the spaces we consider have a distinguished basepoint $*$, and we write TOP_+ for the model category of pairs $(X,*)$ and basepoint preserving maps. We often require the inclusion of $*$ to be a closed cofibration, in which case X is said to be *well-pointed*; this is automatic when X is a CW-complex and $*$ its 0–skeleton.

Given a small category A, we refer to a covariant functor $D\colon \mathrm{A}\to\mathrm{R}$ as an A-*diagram* in R. Such diagrams are themselves the objects of a category $[\mathrm{A},\mathrm{R}]$, whose morphisms are natural transformations of functors. We may interpret any object X of R as a constant diagram, which maps every object of A to X and every morphism to the identity. We describe A as *finite* whenever the total collection of morphisms is finite.

Examples 2.1. (1) For each integer $n\geq 0$, the category $\mathrm{ORD}(n)$ has objects 0, 1, …, n, equipped with a single morphism $k\to m$ when $k\leq m$. An $\mathrm{ORD}(n)$-diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \tag{2.2}$$

consists of n composable morphisms in R.

(2) The category Δ has objects $(n) := \{0,1,\ldots,n\}$ for $n\geq 0$, and morphisms the nondecreasing functions; then Δ^{op}- and Δ-diagrams are simplicial and cosimplicial objects

of R respectively. In particular, $\Delta\colon \Delta \to \mathrm{TOP}$ is the cosimplicial space which assigns the standard n-simplex $\Delta(n)$ to each object (n).

Given objects X_0 and X_1 of R, we write the set of morphisms $X_0 \to X_1$ as $\mathrm{R}(X_0, X_1)$; when R is small, the diagrams (2.2) also form a set $\mathrm{R}_n(X_0, X_n)$, which reduces to $\mathrm{R}(X_0, X_1)$ when $n = 1$. For any R it is often convenient to abbreviate $[\Delta^{op}, \mathrm{R}]$ to SR, and write a generic simplicial object as $D_\bullet$. In particular, SSET denotes the category of simplicial sets $Y_\bullet$.

From this point on we work with an abstract simplicial complex K, whose faces σ are subsets of the vertices V. We assume that the empty face belongs to K, and write $K^\times$ when it is expressly omitted. The integer $|\sigma| - 1$ is known as the *dimension* of σ, and written $\dim \sigma$; its maximum value is the dimension $\dim K$ of K. For any integer $d \geq 1$, the *d–skeleton* $K^{(d)}$ of K is the subcomplex of faces whose dimension satisfies $\dim \sigma \leq d$; it has dimension d. When K contains every subset of V, we may call it the *simplex* $\Delta(V)$ on V. Each face of K therefore determines a subsimplex $\Delta(\sigma)$, whose *boundary* (or $(\dim \sigma - 1)$–skeleton) $\partial(\sigma)$ is given by deleting the subset σ. We also require the *link* $\ell_K(\sigma)$, whose faces consist of those $\tau \setminus \sigma$ for which $\sigma \subseteq \tau$ in K.

Definition 2.3. For any simplicial complex K, the small category $\mathrm{CAT}(K)$ has objects the faces of K and morphisms the inclusions $i_{\sigma,\tau}\colon \sigma \subseteq \tau$. The empty face $\varnothing$ is an initial object, and the *maximal faces* μ admit only identity morphisms. The opposite category $\mathrm{CAT}^{op}(K)$ has morphisms $p_{\tau,\sigma} := i^{op}_{\sigma,\tau}\colon \tau \supseteq \sigma$, and $\varnothing$ is final.

The nondegerate simplices of the nerve $N_\bullet\mathrm{CAT}(K)$ form the cone on the barycentric subdivision K', and those of $N_\bullet\mathrm{CAT}(K^\times)$ correspond to the subcomplex K'. So the classifying space $B\mathrm{CAT}(K)$ (formed by realising the nerve) is a contractible CW-complex, and $B\mathrm{CAT}(K^\times)$ is a subcomplex homeomorphic to K. We shall study $\mathrm{CAT}(K)$- and $\mathrm{CAT}^{op}(K)$-diagrams D in various algebraic and topological categories R. Usually, R is *pointed* by an object $*$, which is both initial and final; unless stated otherwise, we then assume that $D(\varnothing) = *$.

For each face σ, the *overcategories* $\mathrm{CAT}(K){\downarrow}\sigma$ and $\mathrm{CAT}(K){\Downarrow}\sigma$ are given by restricting attention to those objects τ for which $\tau \subseteq \sigma$ and $\tau \subset \sigma$ respectively. The *undercategories* $\sigma{\downarrow}\mathrm{CAT}(K)$ and $\sigma{\Downarrow}\mathrm{CAT}(K)$ are defined likewise. It follows from the definitions that

$$\begin{aligned} \mathrm{CAT}(K){\downarrow}\sigma = \mathrm{CAT}(\Delta(\sigma)), \quad \mathrm{CAT}(K){\Downarrow}\sigma = \mathrm{CAT}(\partial(\sigma)),& \\ \sigma{\downarrow}\mathrm{CAT}(K) = \mathrm{CAT}(\ell_K(\sigma)) \quad \text{and} \quad \sigma{\Downarrow}\mathrm{CAT}(K) = \mathrm{CAT}(\ell_K(\sigma)^\times).& \end{aligned}$$

The dimension function may be interpreted as a functor $\dim\colon \mathrm{CAT}(K) \to \mathrm{ORD}(m)$, which is a *a linear extension* in the sense of [14]; thus $\mathrm{CAT}(K)$ is *direct* and $\mathrm{CAT}^{op}(K)$ is *inverse.*

For any model category R, we may therefore follow Hovey and impose an associated model structure on the category of diagrams $[\mathrm{CAT}(K), \mathrm{R}]$. Weak equivalences $e\colon C \to D$ are given *objectwise*, in the sense that $e(\sigma)\colon C(\sigma) \to D(\sigma)$ is a weak equivalence in R for every face σ of K. Fibrations are also given objectwise. To describe the cofibrations, we consider the restrictions of C and D to the overcategories $\mathrm{CAT}(\partial(\sigma))$, and write $L_\sigma C$ and $L_\sigma D$ for their respective colimits; L_σ is the *latching functor* of [14]. Then $g\colon C \to D$ is a cofibration whenever the induced maps

$$C(\sigma) \amalg_{L_\sigma C} L_\sigma D \longrightarrow D(\sigma) \tag{2.4}$$

are cofibrations in R for every face σ. Alternatively, the methods of Chacholski and Scherer [6] lead to the same model structure on $[\text{CAT}(K), \text{R}]$.

There is a dual model category structure on $[\text{CAT}^{op}(K), \text{R}]$, where weak equivalences and cofibrations are given objectwise. To describe fibrations $f\colon C \to D$, we consider the restrictions of C and D to the undercategories $\text{CAT}^{op}(\partial(\sigma))$, and write $M_\sigma C$ and $M_\sigma D$ for their respective limits; M_σ is the *matching functor* of [14]. Then f is a fibration whenever the induced maps

$$C(\sigma) \longrightarrow D(\sigma) \times_{M_\sigma D} M_\sigma C \tag{2.5}$$

are fibrations in R for every face σ.

Definition 2.6. For any CW-pair $(X, *)$, the *exponential pair* of diagrams (X^K, X_K) consists of functors

$$X^K\colon \text{CAT}(K) \longrightarrow \text{TOP}_+ \quad \text{and} \quad X_K\colon \text{CAT}^{op}(K) \longrightarrow \text{TOP}_+, \tag{2.7}$$

which assign the cartesian product X^σ to each face σ of K; the value of X^K on $i_{\sigma,\tau}$ is the cofibration $X^\sigma \to X^\tau$, where the superfluous coordinates are set to $*$, and the value of X_K on $p_{\tau,\sigma}$ is the fibration $X^\tau \to X^\sigma$, defined by projection. The pair are *twins*, in the sense that $X_K(i') \cdot X^K(i) = X^K(j') \cdot X_K(j)$ for every pullback square

$$\begin{array}{ccc} \sigma \cap \sigma' & \xrightarrow{j'} & \sigma' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i'} \\ \sigma & \xrightarrow[i]{} & \tau \end{array}$$

in $\text{CAT}(K)$.

The properties of twin diagrams are analogous to those of a Mackey functor. They include, for example, the fact that each $X^K(i)$ has left inverse $X_K(p)$, where $p = i^{op}$. Our applications in Theorem 3.12 are reminiscent of [15], where the acyclicity of certain Mackey functors is established.

The colimit $\operatorname{colim} X^K$ is a subcomplex of X^V, whose inclusion r_K is induced by interpreting the elements σ of K as faces of the $(m-1)$-simplex $\Delta(V)$. Composing r_K with any of the natural maps $X^\sigma \to \operatorname{colim} X^K$ yields the standard inclusion $X^\sigma \to X^V$. We note that $\operatorname{colim} X^K$ is pointed by $X^\varnothing$, otherwise known as the basepoint $*$, and is homeomorphic to the pointed colimit $\operatorname{colim}^+ X^K$ of [21].

We wish to study homotopy theoretic properties of $\operatorname{colim} X^K$ in favourable cases. Yet the colimit functor behaves particularly poorly in this context, because objectwise equivalent diagrams may well have homotopy inequivalent colimits. The standard procedure for dealing with this situation is to introduce the left derived functor, known as the *homotopy colimit*. Following [13], for example, $\operatorname{hocolim} X^K$ may be described by the two-sided bar construction $B(*, \text{CAT}(K), X^K)$ in TOP. We note that $\operatorname{hocolim} X^K$ is also pointed, and is related to the pointed homotopy colimit $\operatorname{hocolim}^+ X^K$ of [21] by the cofibration

$$B\text{CAT}(K) \longrightarrow \operatorname{hocolim} X^K \xrightarrow{f} \operatorname{hocolim}^+ X^K$$

of [2]. Since $B\mathrm{CAT}(K)$ is contractible, f is a weak equivalence. We may therefore concentrate on $\operatorname{hocolim} X^K$, and so avoid basepoint complications when working with function spaces in [19].

Lemma 2.8. *Every exponential diagram is cofibrant in* $[\mathrm{CAT}(K), \mathrm{TOP}]$.

Proof. The initial $\mathrm{CAT}(K)$-diagram in TOP is the constant diagram $*$, so X^K is cofibrant whenever the inclusion $* \to X^K$ is a cofibration. By (2.4), it suffices to show that the map $X^{\partial(\sigma)} \to X^\sigma$ is a cofibration for every face σ of K. But the map in question includes the fat wedge in the cartesian product, and the result follows. □

An immediate consequence of Lemma 2.8 is that the natural projection

$$\operatorname{hocolim} X^K \longrightarrow \operatorname{colim} X^K \tag{2.9}$$

is a homotopy equivalence. This illustrates one of the fundamental results of model category theory, and is sometimes called the Projection Lemma [25].

3. Integral Cohomology and Limits

In this section we work in the category MOD_R of R-modules, and study the cohomology of exponential diagrams B^K, where B is the classifying space of the circle T. For this case we abbreviate (2.9) to $hc(K) \to c(K)$. We focus on the relationship between the Stanley-Reisner algebra $R[K]$ and the Bousfield-Kan spectral sequence for $H^*(hc(K); R)$.

We begin by investigating the cohomology of $c(K)$. To simplify applications in later sections, we consider an arbitrary pair of twin diagrams (D^K, D_K),

$$D^K\colon \mathrm{CAT}^{op}(K) \longrightarrow \mathrm{MOD}_R \quad \text{and} \quad D_K\colon \mathrm{CAT}(K) \longrightarrow \mathrm{MOD}_R. \tag{3.1}$$

Thus $D^K(q') \cdot D_K(q) = D_K(p') \cdot D^K(p)$ for every pullback square $p \cdot q = p' \cdot q'$ in $\mathrm{CAT}^{op}(K)$; in particular, $D_K(i)$ is left inverse to $D^K(p)$ for every morphism $p = i^{op}$. Such pairs arise, for example, from any contravariant functor $D\colon \mathrm{TOP} \to \mathrm{MOD}_R$, by composing with the exponential twins (B^K, B_K). Then $(D^K, D_K) = (D \cdot B^K, D \cdot B_K)$, and functoriality ensures the diagrams are twins. In this case we may apply D to the natural maps $B^\sigma \to c(K) \xrightarrow{r_K} B^V$, and obtain homomorphisms

$$D(B^V) \xrightarrow{D(r_K)} D(c(K)) \xrightarrow{h_K} \lim D^K \tag{3.2}$$

in MOD_R.

By way of example, we consider the case $D = H^{2j}(-, R)$, for any $j \geq 0$. For every face σ of K, the space B^σ is an Eilenberg-Mac Lane space $H(\mathbb{Z}^\sigma, 2)$, and may be expressed as the realisation of a simplicial abelian group $H_\bullet(\mathbb{Z}^\sigma, 2)$ whenever convenient [18]. As a CW-complex, the cells of B^σ are concentrated in even dimensions, and correspond to the simplices v_M of $\Delta(\sigma)_\bullet$. The cellular cohomology group $H^{2j}(B^\sigma; R)$ is therefore isomorphic to the free R-module generated by those v_M for which $|M| = j$ and the support of M is a subset of σ. The diagram D^K of (3.1) becomes

$$H^{2j}(B^K; R)\colon \mathrm{CAT}^{op}(K) \longrightarrow \mathrm{MOD}_R, \tag{3.3}$$

whose value on $p_{\tau,\sigma}$ is the homomorphism which fixes v_M whenever the support of M lies in σ, and annihilates it otherwise; the left inverse is the inclusion induced by D_K. When $D = H^{2j+1}(-,R)$, the diagram is zero.

In the case of cohomology, we may combine the diagrams (3.3) into a graded version

$$H^*(B^K;R)\colon \mathrm{CAT}^{op}(K) \longrightarrow \mathrm{GMOD}_R^+, \tag{3.4}$$

taking values in the category of augmented graded R-modules. The cup product on each of the constituent submodules $H^*(B^\sigma;R)$ is given by the product of monomials $x_Lx_M = x_{L+M}$, as follows from the case of a single vertex. In other words, the cohomology ring $H^*(B^\sigma;R)$ is isomorphic to the polynomial algebra $S_R(\sigma)$. So $H^*(B^K;R)$ actually takes values in the category GCA_R^+ of augmented graded commutative R-algebras, and maps the morphism $p_{\tau,\sigma}$ to the projection $S_R(\tau)\to S_R(\sigma)$; the left inverse is again inclusion.

The homomorphisms (3.2) may similarly be combined as

$$S_R(V) \xrightarrow{\;r_K^*\;} H^*(c(K);R) \xrightarrow{\;h_K\;} \lim H^*(B^K;R), \tag{3.5}$$

where the limit is taken in GMOD_R^+. Since (3.4) is a diagram of algebras, the limit inherits a multiplicative structure, and it is equally appropriate to interpret (3.5) in GCA_R^+. The composition $h_K\cdot r_K^*$ is induced by the projections $S_R(V)\to S_R(\sigma)$. In this case, we make one further observation.

Proposition 3.6. *The homomorphism r_K^* is epic, and its kernel is the ideal $(v_U : U\notin K)$.*

Proof. In each dimension $2j$, the cells x_M of B^V correspond to the multisets on V with $|M| = j$. The cells of $c(K)$ form a subset, given by those M for which x_σ divides x_M for some face σ of K. Hence r_K^* is epic, and its kernel is generated by the remaining cells. These are characterised by having no such factor, and therefore coincide with the $2j$–dimensional additive generators of the ideal $(v_U : U\notin K)$. □

So the Stanley-Reisner algebra (1.1) admits an isomorphism $R[K]\cong H^*(c(K);R)$, which plays a central rôle in [4].

Returning to our study of the twins (D^K, D_K), the following definition identifies an important property.

Definition 3.7. A diagram $F^K\colon \mathrm{CAT}^{op}(K)\to \mathrm{MOD}_R$ of R-modules (graded or otherwise) is *fat* if the natural map $F^K(\rho)\to \lim F^{\partial(\rho)}$ is an epimorphism for every face ρ of K.

The terminology acknowledges the relationship between $\partial(\rho)$ and the fat wedge described in Lemma 2.8.

Lemma 3.8. *The twin D^K is fat.*

Proof. We consider an arbitrary face ρ of K, whose vertices we label w_k for $0\le k\le d$; thus $d = \dim\rho$. We write $\mu_k := \rho\setminus w_k$ for the maximal faces of $\partial(\rho)$, and abbreviate the morphism p_{ρ,μ_k} to p_k for $0\le k\le d$.

The definition of lim ensures that $L := \lim D^{\partial(\rho)}$ appears in an exact sequence

$$0\longrightarrow L\longrightarrow \prod_{\rho\supset\sigma} D^K(\sigma) \xrightarrow{\;\delta\;} \prod_{\tau\supset\sigma} D^K(\sigma),$$

where $\delta(u)(\tau\supset\sigma)=u(\sigma)-D^K(p_{\tau,\sigma})u(\tau)$ for any $u\in\prod_{\sigma\subset\rho}D^K(\sigma)$. Hence $u\in L$ is determined by the values $u(\mu_k)$. The natural projection $D^K(\rho)\to\prod_{\rho\supset\sigma}D^K(\sigma)$ therefore factors through L, and it remains to find $u(\rho)\in D^K(\rho)$ such that $D^K(p_k)(u(\rho))=u(\mu_k)$ for every $0\leq k\leq d$.

We set $u(\rho):=\sum_{\rho\supset\sigma}(-1)^{|\rho\backslash\sigma|+1}D_K(p_{\rho,\sigma})u(\sigma)$. The fact that D^K and D_K are twins implies that

$$D^K(p_k)D_K(p_{\rho,\sigma})u(\sigma)=\begin{cases}D_K(p_{\mu_k,\sigma\backslash w_k})u\big(\sigma\backslash w_k\big) & \text{if } w_k\in\sigma\\ D_K(p_{\mu_k,\sigma})u(\sigma) & \text{otherwise}\end{cases}$$

for every $0\leq k\leq d$; thus

$$D^K(p_k)u(\rho)=\sum_{\sigma\not\ni w_k}(-1)^{|\rho\backslash\sigma|+1}D_K(p_{\mu_k,\sigma})u(\sigma)+\sum_{\tau\ni w_k}(-1)^{|\rho\backslash\tau|+1}D_K(p_{\mu_k,\tau\backslash w_k})u(\rho\backslash w_k).$$

But we may write $u(\sigma)$ as $u\big((\sigma\cup w_k)\backslash w_k\big)$ for any $\sigma\not\ni w_k$ other than μ_k. So the summands cancel in pairs, leaving $u(\mu_k)$ as required. □

For cohomology, Lemma 3.8 contributes to our analysis of $c(K)$. The homotopy equivalence (2.9) provides a *cohomology decomposition* [8], in the sense that the cohomology algebra $H^*(c(K);R)$ may be computed by the Bousfield-Kan spectral sequence [2]

$$E_2^{i,j}\Longrightarrow H^{i+j}(hc(K);R),$$

where $E_2^{i,j}$ is isomorphic to the ith derived functor $\lim^i H^j(B^K;R)$ for every $i,j\geq 0$. The vertical edge homomorphism coincides with the map h_K of (3.5). Lemma 3.8 is required for our computation of these limits, and Corollary 3.14 will confirm that the cohomology decomposition is *sharp* in Dwyer's language. Our proof uses the calculus of functors and their limits; the appropriate prerequisites may be deduced from Gabriel and Zisman [12, Appendix II §3], by dualising their results for colimits.

In particular, we follow [12] (as expounded in [20], for example) by calculating $\lim^i D^K$ as the ith cohomology group of a certain cochain complex $\big(C^*(D^K),\delta\big)$ of R-modules. The groups are defined by

$$C^n(D^K)\;:=\prod_{\sigma_0\supseteq\dots\supseteq\sigma_n}D^K(\sigma_n)\quad\text{for } n\geq 0\,, \tag{3.9}$$

and the differential $\delta:=\sum_{k=0}^n(-1)^k\delta^k$ is defined on $u\in C^n(D^K)$ by

$$\delta^k(u)(\sigma_0\supseteq\cdots\supseteq\sigma_{n+1}):=\begin{cases}u(\sigma_0\supseteq\cdots\supseteq\widehat{\sigma}_k\supseteq\cdots\supseteq\sigma_{n+1}) & \text{for } k\neq n+1\\ D^K(p_{\sigma_n,\sigma_{n+1}})u(\sigma_0\supseteq\cdots\supseteq\sigma_n) & \text{for } k=n+1,\end{cases} \tag{3.10}$$

We may replace $C^*(D^K)$ by its quotient $N^*(D^K)$ of normalised cochains, in which the faces $\sigma_0,\ldots,\sigma_n$ of (3.9) and (3.10) are required to be distinct.

Lemma 3.11. *Given a diagram $D\colon\mathrm{CAT}^{op}(K)\to\mathrm{MOD}_R$, and a maximal face μ of K, then*

$$\lim{}^iD=\begin{cases}D(\mu) & \text{for } i=0\\ 0 & \text{for } i>0\end{cases}$$

whenever $D(\sigma) = 0$ for all $\sigma \neq \mu$.

Proof. Since μ is maximal, the only morphism $\sigma \supseteq \mu$ is the identity. So the normalised chain complex $N^*(D)$ is $D(\mu)$ in dimension 0, and 0 in higher dimensions, as required. □

Theorem 3.12. *For any fat diagram $F^K \colon \mathrm{CAT}^{op}(K) \to \mathrm{MOD}_R$, we have that $\lim^i F^K = 0$ for all $i > 0$; in particular, $\lim^i D^K = 0$ for every twin D^K.*

Proof. We proceed by induction on the total number of faces $f(K)$; the result obviously holds for the initial example $K = \varnothing$, where $f(K) = 0$. Our inductive hypothesis is that $\lim^i F^K$ vanishes whenever K satisfies $f(K) \leq f$.

We therefore consider an arbitrary complex K with $f(K) = f + 1$, and write $J \subset K$ for the subcomplex obtained by deleting a single maximal face μ, The inclusion of J defines a functor $G \colon \mathrm{CAT}^{op}(J) \to \mathrm{CAT}^{op}(K)$, whose induced functor $G^* \colon [\mathrm{CAT}^{op}(K), \mathrm{MOD}_R] \to [\mathrm{CAT}^{op}(J), \mathrm{MOD}_R]$ acts by restriction, and admits a right adjoint G_*, known as the *right Kan extension* [16]. In particular, $G_* F^J$ is given on $\sigma \in K$ by $\lim F^{\partial(\mu)}$ when $\sigma = \mu$, and F^σ otherwise.

But $F^\mu \to \lim D^{\partial(\mu)}$ is an epimorphism, by Lemma 3.8, so the natural transformation $F^K \to G_* F^J$ is epic on every face of K, and its kernel H is zero on every face except μ. We acquire a short exact sequence of functors

$$0 \longrightarrow H \longrightarrow F^K \longrightarrow G_* F^J \longrightarrow 0,$$

which induces a long exact sequence of higher limits. By Lemma 3.11, this collapses to a sequence of isomorphisms

$$\lim{}^i F^K \;\cong\; \lim{}^i G_* F^J, \tag{3.13}$$

for $i \geq 1$. We now apply the composition of functors spectral sequence [5], [12]

$$\lim{}^n G^i_* F^J \Longrightarrow \lim{}^{n+i} F^J.$$

Here G^i_* denotes the ith derived functor of G_*; it may be evaluated on any face σ of K as $\lim^i F^{\partial(\sigma)}$, and therefore vanishes for $i > 0$, by inductive hypothesis. So the spectral sequence collapses onto the first row of the E^2 page, from which we obtain isomorphisms $\lim^n G_* F^J \cong \lim^n F^J$ for all $n \geq 0$. Since the inductive hypothesis applies to J, we deduce that $\lim^n G_* F^J = 0$ for every $n > 0$. Combining this with (3.13) concludes the proof. □

Corollary 3.14. *The Bousfield-Kan spectral sequence for B^K collapses at the E_2 page; it is concentrated along the vertical axis, and given by*

$$\lim{}^i H^j(B^K; Q) = \begin{cases} \lim H^j(B^K; Q) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The result follows immediately from Theorem 3.12 by letting D^K be $H^j(-; R)$ for every $j \geq 0$. □

Corollary 3.14 confirms that the edge homomorphism h_K is an isomorphism in GCA_R^+. When combined with (3.5) and Proposition 3.6 it implies that the natural map

$$R[K] \;\cong\; H^*(c(K);R) \xrightarrow{\;h\;} \lim H^*(B^K),$$

which is induced by the projections $R[K] \to S_R(\sigma)$, is also an isomorphism. This may be proven directly, by refining the methods of Proposition 3.6.

4. Integral Formality

In this section we study the formality of $c(K)$ over our arbitrary commutative ring R, and construct a zig-zag of quasi-isomorphisms between the singular cochain algebra $C^*(c(K);R)$ and its cohomology ring.

We work in the model category DGA_R^+ of differential graded R-algebras with augmentation. The model structure arises by interpreting DGA_R^+ as the category of monoids in the monoidal model category DGMOD_R of unbounded cochain complexes over R. The latter is isomorphic to Hovey's category of unbounded chain complexes [14], and the model structure is induced on DGA_R^+ by checking that it satisfies the monoid axiom of [22]. As Schwede and Shipley confirm, weak equivalences are zig-zags of quasi-isomorphisms and fibrations are epimorphisms. Cofibrations are defined by the appropriate lifting property, and are necessarily degreewise split injections. We emphasise that the objects of DGA_R^+ are not necessarily commutative.

A differential graded R-algebra C^* is *formal in* DGA_R^+ whenever there is a zig-zag of quasi-isomorphisms

$$H(C^*) \xrightarrow{\;\sim\;} \cdots \xleftarrow{\;\sim\;} C^*, \tag{4.1}$$

where we follow the convention of assigning the zero differential to the cohomology algebra $H(C^*)$. Our aim is to show that the cochain algebra $C^*(c(K);R)$ is always formal in DGA_R^+. This extends Franz's result [11], which only applies to complexes arising from smooth fans.

We begin by choosing D to be the j-dimensional cochain functor $C^j(-;R)$ in (3.1), thus creating twin diagrams $\big(C^j(B^K;R), C^j(B_K;R)\big)$ for each $j \geq 0$. As in (3.4), we may consider the graded version $C^*(-;R)$ in DGMOD_R. In fact its values are always R-algebras, with respect to the cup product of cochains. This product is no longer commutative, but the procedure for forming the limit of a DGA_R^+-diagram remains the same; work in DGMOD_R^+, and superimpose the induced multiplicative structure.

For the Eilenberg-Mac Lane space $B = H(\mathbb{Z},2)$, we let v denote a generator of $H^2(B;R)$, which is isomorphic to R. We choose a cocycle ψ_v representing v in $C^2(B;R)$, and define a homomorphism $\psi\colon H^*(B;R) \to C^*(B;R)$ by $\psi(v^k) = \psi_v^k$, for all $k \geq 0$. By construction, ψ is multiplicative, and is a quasi-isomorphism in DGA_R^+. We extend this procedure to $H^*(B^V;R)$ via the zig-zag

$$H^*(B^V;R) \xrightarrow{\;\cong\;} H^*(B;R)^{\otimes V} \xrightarrow{\;\psi^{\otimes}\;} C^*(B;R)^{\otimes V} \xleftarrow{\;ez\;} C^*(B^V;R) \tag{4.2}$$

of quasi-isomorphisms in DGA_R^+. Here the leftmost map is the Künneth isomorphism, and ez is the Eilenberg-Zilber map.

By restriction, we may also interpret the ψ_j as cocycles in $C^2(B^\sigma;R)$ for every face σ; they represent v_j in $H^2(B^\sigma;R)$ when σ contains v_j, and 0 otherwise. Because the restrictions are compatible, we obtain homomorphisms

$$H^*(B^K;R) \xrightarrow{\cong} H^*(B;R)^{\otimes K} \xrightarrow{\psi^\otimes} C^*(B;R)^{\otimes K} \xleftarrow{ez} C^*(B^K;R)$$

of $\mathrm{CAT}^{op}(K)$-diagrams, whose components are quasi-isomorphisms on each B^σ. Taking limits in DGA_R^+ yields homomorphisms

$$\text{(4.3)} \quad \lim H^*(B^K;R) \xrightarrow{\cong} \lim H^*(B;R)^{\otimes K} \xrightarrow{\psi^\otimes} \lim C^*(B;R)^{\otimes K} \xleftarrow{ez} \lim C^*(B^K;R).$$

Lemma 4.4. *The leftmost homomorphism in* (4.3) *is an isomorphism in* DGA_R^+*, and the remaining two are quasi-isomorphisms.*

Proof. The limit of the Künneth isomorphisms is automatically an isomorphism.

A diagram $D^K\colon \mathrm{CAT}^{op}(K) \to \mathrm{DGA}_R^+$ is fibrant whenever the projection onto the constant diagram 0 is a fibration. By (2.5) this occurs precisely when D^K is fat, and therefore holds for $H^*(B^K;R)$ and $C^*(B^K;R)$ by Lemma 3.8; it follows for $H^*(B;R)^{\otimes K}$ by the Künneth isomorphism. So far as $C^*(B;R)^{\otimes K}$ is concerned, we note that singular cochains determine a pair of diagram twins $(C^{\otimes K}, C_{\otimes K})$ in DGA_R^+. Both functors assign $C^*(B;R)^{\otimes\sigma}$ to the face σ. The value of $C^{\otimes K}$ on $p_{\tau,\sigma}$ is the projection $C^*(B;R)^{\otimes\tau} \to C^*(B;R)^{\otimes\sigma}$, and the value of C_K on $i_{\sigma,\tau}$ is the inclusion $C^*(B;R)^{\otimes\sigma} \to C^*(B;R)^{\otimes\tau}$; these require the augmentation. Hence $C^{\otimes K}$ is also fat, and $C^*(B;R)^{\otimes K}$ is fibrant. So the remaining two homomorphisms are objectwise equivalences of fibrant diagrams, and therefore induces weak equivalences of limits by [14]. □

Implicit in this proof is the remark that $C^*(B^n;R)$ is formal in DGA_R^+, for every $n \geq 1$.

Lemma 4.5. *The natural homomorphism* $g\colon C^*(c(K);R) \to \lim C^*(B^K;R)$ *is a quasi-isomorphism in* DGA_R^+.

Proof. The edge isomorphism h_K of Corollary 3.14 factorises as

$$H(C^*(c(K);R)) \xrightarrow{H(g)} H(\lim C^*(B^K;R)) \xrightarrow{l} \lim H^*(B^K;R),$$

in DGA_R^+, where l is induced by the compatible homomorphisms $H(\lim C^*(B^K;R)) \to H^*(B^\sigma;R)$.

Now let d be the differential on $C^j(-;R)$ for every $j \geq 0$, and define the cycle and boundary functors Z^j, $I^j\colon \mathrm{TOP} \to \mathrm{MOD}_R$ as the kernel and image of d respectively. They determine diagram twins, and therefore fat functors Z^K, $I^K\colon \mathrm{CAT}^{op}(K) \to \mathrm{MOD}_R$. Theorem 3.12 then applies to confirm that $\lim^i Z^j(B^K;R) = \lim^i I^j(B^K;R) = 0$ for all $i > 0$ and $j \geq 0$. It follows immediately that l is an isomorphism, and therefore that $H(g)$ is an isomorphism, as sought. □

We may now complete our analysis of $C^*(c(K);R)$.

Theorem 4.6. *The differential graded R-algebras* $C^*(c(K);R)$ *is formal in* DGA_R^+.

Proof. Combining Corollary 3.14 with Lemmas 4.4 and 4.5 yields a zig-zag

$$H^*(c(K);R) \xrightarrow{h} \lim H^*(B^K;R) \longrightarrow \dots \longleftarrow \lim C^*(B^K;R) \xleftarrow{g} C^*(c(K);R)$$

of quasi-isomorphisms, as required by (4.1). □

Remark 4.7. The proof of Theorem 4.6 extends to exponential diagrams X^K for which $C^*(X;R)$ is formal in DGA_R^+ and the Künneth isomorphism $H^*(X^V;R) \cong H^*(X;R)^{\otimes V}$ holds. We replace ψ in (4.2) by the corresponding zig-zag $H^*(X;R) \xrightarrow{\sim} \cdots \xleftarrow{\sim} C^*(X;R)$ of quasi-isomorphisms, and repeat the remainder of the argument above.

5. Rational Formality

In our final section we turn to the rational case $R = \mathbb{Q}$, and confirm the formality of Sullivan's algebra of rational cochains on $c(K)$ in the commutative setting. This involves stricter conditions than those for general R, and has deeper topological consequences; in particular, it means that the rational homotopy type of $c(K)$ is uniquely determined by K. In other words, the existence of an isomorphism $H^*(Y;\mathbb{Q}) \cong \mathbb{Q}[K]$ implies that there is a rational homotopy equivalence $Y \simeq c(K)$, for any nilpotent space Y. We refer readers to Bousfield and Gugenheim [1] for details of the model category of differential graded commutative $\mathbb{Q}$-algebras, and to Félix, Halperin and Thomas [10] for background information on rational homotopy theory.

We begin by replacing $C^*(X;R)$ with Sullivan's rational algebra $A_{PL}(X)$ of polynomial forms[10]. The commutativity of the latter is crucial, and suggests we work in the category $\mathrm{DGCA}_{\mathbb{Q}}^+$ of differential graded commutative $\mathbb{Q}$-algebras [1]. The existence of a model structure is assured by working over a field; as before, weak equivalences are zig-zags of quasi-isomorphisms, fibrations are epimorphisms, and cofibrations are defined by the appropriate lifting properties.

For each $s \geq 0$, we write the differential algebra of rational polynomial forms on the standard s-simplex as $\nabla_s(*)$. It is an object of $\mathrm{DGCA}_{\mathbb{Q}}^+$. For each $t \geq 0$, the forms of dimension t define a simplicial vector space $\nabla_\bullet(t)$ over $\mathbb{Q}$, and $\nabla_\bullet(*)$ becomes a simplicial object in $\mathrm{DGCA}_{\mathbb{Q}}^+$. So

$$A^*(Y_\bullet) := \mathrm{SSET}(Y_\bullet, \nabla_\bullet(*))$$

is also an object of $\mathrm{DGCA}_{\mathbb{Q}}^+$, which is weakly equivalent to the normalised cochain complex $N^*(Y_\bullet;\mathbb{Q})$. Then $A_{PL}(X)$ is defined as $A^*(S_\bullet X)$, where $S_\bullet$ denotes the total singular complex functor $\mathrm{SSET} \to \mathrm{TOP}$. The PL de Rham Theorem [1] asserts that the cohomology algebra $H(A_{PL}(X))$ is naturally isomorphic to $H^*(X;\mathbb{Q})$. As usual, we consider $H^*(X;\mathbb{Q})$ to be an object of $\mathrm{DGCA}_{\mathbb{Q}}^+$ by investing it with the zero differential.

A differential graded commutative $\mathbb{Q}$-algebras A^* is *formal in* $\mathrm{DGCA}_{\mathbb{Q}}^+$ whenever there is a zig-zag of quasi-isomorphisms

$$H(A^*) \xrightarrow{\sim} \cdots \xleftarrow{\sim} A^* \tag{5.1}$$

in $\mathrm{DGCA}_{\mathbb{Q}}^+$. A topological space X is *rationally formal* whenever $A_{PL}(X))$ is formal in $\mathrm{DGCA}_{\mathbb{Q}}^+$. One of the basic results of rational homotopy theory states that a minimal model for a rationally formal space X may be constructed directly from its rational cohomology algebra, and therefore that the rational homotopy type of X is uniquely determined. Our final goal in this section is to explain why $c(K)$ is rationally formal. The proof is parallel to that for general R, but the need to respect commutativity forces several changes of detail.

We choose D to be A_{PL} in (3.1), creating diagram twins $(A_{PL}(B^K), A_{PL}(B_K))$ in $\text{DGCA}^+_{\mathbb{Q}}$. As before, we form limits by working in DGMOD^+_Q, and superimposing the induced multiplicative structure. Applying cohomology yields the twins $(H^*(B^K;\mathbb{Q}), H^*(B_K;\mathbb{Q}))$, whose value on each face σ is $S_{\mathbb{Q}}(\sigma)$ in DGMOD^+_Q. Both $A_{PL}(B^K)$ and $H^*(B^K;\mathbb{Q})$ are fat, by Lemma 3.8.

Using the fact that $H(A_{PL}(B^V))$ is isomorphic to $S_{\mathbb{Q}}(V)$, we choose cocycles ϕ_j in $A_{PL}(B^V)$ representing v_j for every $1 \le j \le m$. We may then define a homomorphism

$$\phi\colon H^*(B^V;\mathbb{Q}) \longrightarrow A_{PL}(B^V) \tag{5.2}$$

by $\phi(v_j) = \phi_j$, because $A_{PL}(B^V)$ is commutative. Moreover, ϕ is a quasi-isomorphism, reflecting the well-known rational formality of the Eilenberg-Mac Lane space $H(\mathbb{Q}^V;2)$. By restriction, we interpret the ϕ_j as cocycles in $A_{PL}(B^\sigma)$ for every face σ. They then represent v_j in $S_{\mathbb{Q}}(\sigma)$ when σ contains v_j, and 0 otherwise. We obtain compatible quasi-isomorphisms on each B^σ, which combine to create a map

$$\phi\colon H^*(B^K;\mathbb{Q}) \longrightarrow A_{PL}(B^K)$$

of $\text{CAT}^{op}(K)$-diagrams in $\text{DGCA}^+_{\mathbb{Q}}$. It is an objectwise weak equivalence. Taking limits yields a homomorphism

$$l(\phi)\colon \lim(H^*(B^K;\mathbb{Q}) \longrightarrow \lim A_{PL}(B^K)$$

of differential graded commutative algebras over $\mathbb{Q}$.

Lemma 5.3. *The homomorphism $l(\phi)$ is a quasi-isomorphism in $\text{DGCA}^+_{\mathbb{Q}}$.*

Proof. Both diagrams are fat, and therefore fibrant by (2.5). So ϕ induces a weak equivalence of limits. □

Because the B^σ are Eilenberg-Mac Lane spaces, it is convenient to complete our proof of Theorem 5.5 in terms of simplicial sets. We may then take advantage of the fact that A^* converts colimits in SSET to limits in $\text{DGCA}^+_{\mathbb{Q}}$, for reasons which are purely set-theoretic.

We denote the realisation functor SSET → TOP by $|\ |$. Given an arbitrary simplicial set $Y_\bullet$, we have

$$A_{PL}(|Y_\bullet|) := A^*(S_\bullet|Y_\bullet|) \cong A^*(Y_\bullet), \tag{5.4}$$

where the second isomorphism is induced by the natural equivalence $Y_\bullet \to S_\bullet|Y_\bullet|)$. For each face σ of K, we choose $|H_\bullet(\mathbb{Z}^\sigma;2)|$ as our model for B^σ; it is well-pointed, by the cofibration induced by the inclusion of the trivial subgroup $\{0\} \to \mathbb{Z}^\sigma$. We write $H^K_\bullet$ for the corresponding diagram of simplicial sets, which takes the value $H_\bullet(\mathbb{Q}^\sigma;2)$ on σ.

Theorem 5.5. *The space $c(K)$ is rationally formal.*

Proof. Since realisation is left adjoint to $S_\bullet$, it commutes with colimits. So we may write

$$c(K) = \operatorname{colim}|H^K_\bullet| \cong |\operatorname{colim} H^K_\bullet|,$$

where the second colimit is taken in SSET. Applying (5.4) yields

$$\lim A_{PL}(B^K) \cong \lim A^*(H^K_\bullet) \cong A^*(\operatorname{colim} H^K_\bullet) \cong A_{PL}(c(K)),$$

where the first and third isomorphisms define A_{PL}, and the second follows from the property of A^* described above. Combining with Corollary 3.14 and Lemma 5.3 yields homomorphisms

$$H^*(c(K);\mathbb{Q}) \xrightarrow{h} \lim H^*(B^K;\mathbb{Q}) \xrightarrow{l(\phi)} \lim A_{PL}(B^K) \xrightarrow{\cong} A_{PL}(c(K)),$$

whose composition is a quasi-isomorphism. The result follows from (5.1). □

Remark 5.6. By analogy with Remark 4.7, the proof of Theorem 5.5 extends to exponential diagrams X^K for which X is rationally formal; the Künneth isomorphism holds automatically, because we are working over $\mathbb{Q}$. The product $A_{PL}(X)^{\otimes V} \to A_{PL}(X^V)$ of the projection maps is a quasi-isomorphism, and is natural with respect to projection and inclusion of coordinates. So we may replace ϕ in (5.2) by the corresponding zig-zag

$$H^*(X^V;\mathbb{Q}) \xrightarrow{\cong} H^*(X;\mathbb{Q})^{\otimes V} \xrightarrow{\sim} \cdots \xleftarrow{\sim} A_{PL}(X)^{\otimes V} \xrightarrow{\cong} A_{PL}(X^V)$$

of quasi-isomorphisms, and proceed with the remainder of the argument above.

Theorem 5.5 confirms that a minimal Sullivan model for $c(K)$ may be constructed directly from the Stanley-Reisner algebra of K. It consists of an acyclic fibration

$$\eta^K\colon (S_{\mathbb{Q}}(W^K), d) \longrightarrow H^*(B^K;\mathbb{Q}),$$

where W^K is an appropriately graded set of generators (necessarily exterior in odd dimensions), and provides a cofibrant replacement for $H^*(B^K;\mathbb{Q})$ in $\mathrm{DGCA}^+_{\mathbb{Q}}$. In general, W^K is not easy to describe, although special cases such as Example 5.8 below are straightforward. The properties of W^K are linked to those of the loop space $\Omega c(K)$, whose study was begun in [21]; we expect to return to this relationship elsewhere.

As described by Bousfield and Gugenheim [1], the fundamental result of rational homotopy theory is the Sullivan-de Rham equivalence

$$ho\,\mathrm{SSET}_{\mathbb{Q}} \rightleftarrows ho\,\mathrm{DGCA}^+_{\mathbb{Q}}$$

of homotopy categories, which identifies homotopy classes of maps $[c(K)_0, c(L)_0]$ with homotopy classes of morphisms $[S_{\mathbb{Q}}(W^L), S_{\mathbb{Q}}(W^K)]$. Since every object of $\mathrm{DGCA}^+_{\mathbb{Q}}$ is fibrant, it actually suffices to consider the homotopy classes $[S_{\mathbb{Q}}(W^L), H^*(c(K);\mathbb{Q})]$; of course $S_{\mathbb{Q}}(W^L)$ cannot be substituted similarly, because $H^*(c(L);\mathbb{Q})$ is not usually cofibrant. Nevertheless, the function

$$[S_{\mathbb{Q}}(W^L), H^*(c(K);\mathbb{Q})] \longrightarrow \mathrm{DGCA}^+_{\mathbb{Q}}(H^*(c(L);\mathbb{Q}), H^*(c(K);\mathbb{Q})) \tag{5.7}$$

induced by taking cohomology is always a surjection, and it would be of interest to understand its kernel.

Example 5.8. Let $(\lambda(k) : 1 \leq k \leq t)$ be a sequence of disjoint subsets of V, where $\lambda(k)$ has cardinality $n(k)$, and define L to be the subcomplex of $\Delta(V)$ obtained by deleting all faces containing one or more of the $\lambda(k)$. We write $\lambda := \cup_k \lambda(k)$, and $|\lambda| = n$. The generating set W^L consists of V in dimension 2, and elements $w(k)$ in dimension $2n(k)-1$, for $1 \leq k \leq t$; the differential is given by $dv_j = 0$ for all j, and $dw(k) = v_{\lambda(k)}$. The fibration η^L identifies the vertices V in dimension 2, and annihilates every $w(k)$. In this situation, any $\mathrm{DGCA}^+_{\mathbb{Q}}$-morphism $S_{\mathbb{Q}}(W^L) \to H^*(c(K); \mathbb{Q})$ is determined by its effect on V, because the $w(k)$ are odd dimensional. So the function (5.7) is bijective.

It follows that $\mathrm{Aut}_{ho}(c(K)_0)$ is isomorphic to the group of automorphisms $\mathrm{Aut}(\mathbb{Q}[K])$ of the algebra $\mathbb{Q}[K]$ under pcomposition, and is therefore a subgroup of $GL(\mathbb{Q}, m)$. It contains all matrices of the form $\left(\begin{smallmatrix} L & 0 \\ M & \Sigma \end{smallmatrix}\right)$, where $L \in GL(\mathbb{Q}, m-n)$ acts on $\mathbb{Q}^{V \setminus \lambda}$, and Σ permutes the elements of λ. The permutations act on the elements of each individual $\lambda(k)$, and interchange those $\lambda(k)$ which are of common cardinality.

We conjecture that *every* element of $\mathrm{Aut}(\mathbb{Q}[K])$ has the form $\left(\begin{smallmatrix} L & 0 \\ M & \Sigma \end{smallmatrix}\right)$.

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