P—ADIC LATTICES OF PSEUDO REFLECTION GROUPS

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Abstract. Let U be a vector space over the p–adic rationals, and let $W \longrightarrow Gl(U)$ be faithful representation of a finite group such that W is generated by pseudo reflections. For odd primes we study the p–adic W–sublattice of this representation and achieve a complete classification. Examples of such situations are given by the Weyl group acting on the 1-dimesional homology of the maximal torus of a connected compact Lie group, or of the so called p–compact groups, a homotopy theoretic generalisation of compact Lie groups. The associated lattices are an important algebraic invariant in the study of these geometric object.

Introduction.

Let U be a finite dimensional vector space over the p–adic rationals $\mathbb{Q}_p^\wedge$. For a faithful representation $\rho : W \rightarrow Gl(U)$ of a group W, an element $1 \neq \sigma \in W$ is called a pseudo reflection if σ or $\rho(\sigma)$ has finite order and if the kernel of $\rho(\sigma) - id_U$ has codimension 1. The element σ is called a honest reflection or a reflection if σ has order 2. Because we are working in characteristic 0, the order of σ divides $p-1$ and the linear transformation $\rho(\sigma)$ is diagonalzable. i.e. U has a basis of eigenvectors with respect to $\rho(\sigma)$.

The representation $\rho : W \rightarrow Gl(U)$ represents W as a pseudo reflection group, if W is generated by pseudo reflections. If we say that W is a pseudo reflection group, then we always have a representation in mind.

In this work we are concerned with the classification of all p–adic W–sublattices $L \subset U$ of a given finite pseudo reflection group $W \rightarrow Gl(U)$. Our motivation to study this question comes from homotopy theory. The Weyl group W_G of a connected compact Lie group G acting on the tangent space of the maximal torus T of G or on the 1-dimensional homology $H_1(T;\mathbb{Z})$ provides an example of a honest reflection group and also of an integral sublattice. This action is an important algebraic invariant in the study of connected compact Lie groups. In [4], Dwyer and Wilkerson gave the notion of p–compact groups, which is the homotopy theoretic generalisation of the notion of compact Lie groups. In their work, pseudo reflection groups occured in the same manner as honest reflection groups for connected compact Lie groups, namely as Weyl groups acting on a 'maximal torus'. These p–compact groups provide examples of pseudo reflection groups and associated p–adic sublattices. Besides these geometric and homotopy theoretic aspects we believe that the study of p–adic W–lattices has interest from it's own.

We will use the following notation and definitions in this paper.

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1.1 Notation, Definitions and Remarks.

We always denote by U a finite dimensional vector space over $\mathbb{Q}_p^\wedge$, and by W a finite pseudo reflection group represented by a faithful representation $\rho : W \to Gl(U)$. By a sublattice or a W–sublattice $L \subset U$ we always understand a sublattice of maximal rank, i.e. $L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q} \cong U$ as W–modules. If we say that W is a pseudo reflection group, then this always includes that W is finite.

1.1.1: The representation $W \to Gl(U)$ is called *fixed point free* if the fixed point set $U^W = 0$ is trivial. A sublattice $L \subset U$ is called *fixed point free* if $L^W = 0$.

1.1.2 The pseudo reflection group $W \to Gl(U)$ is called irreducible if the representation is irreducible. By [3], this is equivalent to the fact that the associated complex representation $U \otimes_{\mathbb{Q}_p^\wedge} \mathbb{C}$ is irreducible.

1.1.3: Let $\mathbb{Z}/p^\infty \subset S^1$ denote the subgroup of all elements of p–power order. Then, for any n, the group $T_\infty := (\mathbb{Z}/p^\infty)^n$ is called a *p–discrete torus* and the inclusion $T_\infty := (\mathbb{Z}/p^\infty)^n \subset (S^1)^n =: T$ is a *p–discrete approximation* in the sense of [4], i.e. the map $BT_\infty \to BT$ between the classifying spaces induces an isomorphism $H_2(BT_\infty, \mathbb{Z}_p^\wedge) \cong H_2(BT; \mathbb{Z}_p^\wedge)$ and an equivalence after p–adic completion. The completion $T_p^\wedge$ is called a p–adic torus.

For every W–lattice L, we have a short exact sequence

$$1 \to L \to L_\mathbb{Q} \to L_\mathbb{Q}/L =: T_{L,\infty} \to 1$$

of W–modules. Completing the classifying spaces of the p–discrete torus $T_{L,\infty} \cong (\mathbb{Z}/p^\infty)^n$ and passing to 2-dimesional homology establishes an isomorphism $H_2(BT_{L,\infty}; \mathbb{Z}_p^\wedge) \cong L$ of W–modules.

1.1.4 For a W–sublattice $L \subset U$, the fixed point set $Z(L) := (T_{L,\infty})^W$ is called the *center* of L. The sublattice L is called *centerfree* if $Z(L) = 0$.

1.1.5 For a W–sublattice $L \subset U$, the *covariants* L_W are given by the quotient L/SL, where $SL \subset L$ is the sublattice generated by all elements of the form $l - w(l)$ with $l \in L$ and $w \in W$. The lattice L is called *simply connected* if $L_W = 0$.

1.1.6 A monomorphism $L \to M$ of W–lattices is called a *W–trivial restriction* or *W–trivial extension* if W acts trivially on the quotient M/L and if M/L is finite.

Most of these notions are motivated by an analogy to connected compact Lie groups. Let W_G denote the Weyl group and T_G the maximal torus of a connected compact Lie group G. Then, the action of W_G on $U_G := H_2(BT_G; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ represents W as a finite reflection group, and $L_G := H_2(BT_G; \mathbb{Z}_p^\wedge) \subset U_G$ is a p–adic sublattice. For odd primes, the center $Z(L_G)$ of L_G is a p–discrete approximation of the center of G [7]. Because the fundamental group $\pi_1(G)$ of G is isomorphic to the quotient of $\pi_1(T)$ by the translations of the extended Weyl group, one can also show that, for odd primes, $(L_G)_{W_G}$ is a p–discrete approximation of $\pi_1(G)$, i.e. $(L_G)_{W_G} \otimes \mathbb{Z}_p^\wedge \cong (L_G \otimes \mathbb{Z}_p^\wedge)_{W_G} \cong \pi_1(G) \otimes \mathbb{Z}_p^\wedge$.

1.2 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be a finite fixed–point free pseudo reflection group. Then the following holds:*

(1) *There exists a centerfree sublattice $P \subset U$, unique up to isomorphism.*
(2) *There exists a simply connected sublattice $S \subset U$, unique up to isomorphism.*

(3) *For every W–sublattice $L \subset U$ there exist a composition $S \to L \to P$ of W–trivial restrictions such that $P/L \cong Z(L)$ and $L/S \cong L_W$ as W–modules.*
(4) *Let $S, P \subset U$ be sublattices and $S \to P$ a W–trivial restriction. If every sublattice $L \subset U$ fits between these both, i.e. there exist W–trivial restrictions $S \to L$ and $L \to P$ such that the composition gives $S \to P$, then S is simply connected and P is centerfree.*

It is well known, that, for a finite pseudo reflection group $W \to Gl(U)$, there exists splitting $U \cong U^W \oplus U_1 \oplus ... \oplus U_n$ of U as W–modules and a splitting $W \cong W_1 \times ... \times W_n$ of W such that W_i acts on U_i as an irreducible pseudo reflection group and trivially on every U_j for $j \neq i$.

For the proof of the first two parts of Theorem 1.2, we first study irreducible pseudo reflection groups. After having done this, the general case is a corollary of the following statement;

1.3 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be a finite fixed–point free pseudo reflection group, and let $W \cong \prod_i W_i$ and $U \cong \bigoplus_i U_i$ be the associated splittings into irreducible pseudo reflection groups. Then the following holds:*

(1) *Every centerfree sublattice $P \subset U$ splits into a direct sum $P \cong \bigoplus_i P_i$ of centerfree sublattices $P_i \subset U_i$. The summands P_i are uniquely determined up to order and isomorphisms.*
(2) *Every simply connected sublattice $S \subset U$ splits into a direct sum $S \cong \bigoplus_i S_i$ of simply connected sublattices $S_i \subset U_i$. The summands S_i are uniquely determined up to order and isomorphisms.*

Finally we consider the general case of a pseudo reflection group $W \to Gl(U)$. Let $U \cong U^W \oplus U'$ be the splitting into the fixed–point set and the fixed point free factor U'. Let $S \subset U'$ be the unique simply connected sublattice and $P \subset U'$ the unique centerfree sublattice given by Theorem 1.2. Because W acts trivially on U^W, all sublattices of U^W are isomorphic as W–modules. We choose a sublattice of U^W and denote it by Z.

1.4 Theorem. *Let p be an odd prime. Let $W \to Gl(U)$ be a pseudo reflection group, and let L be a W–sublattice. Then the following holds:*

(1) *There exists a W–trivial restriction $Z \oplus S \to L$ with quotient $L/(Z \oplus S) \cong (L/L^W)_W$.*
(2) *There exists a W–trivial restriction $L \to Z \oplus P$.*

The classification of lattices of pseudo reflection groups can be done analogously as for connected compact Lie groups. Irreducible honest reflection groups correspond to simple connected compact Lie groups. The following corollary is obvious and classifies all lattices of finite pseudo reflection groups.

1.5 Corollary. *Let p be an odd prime. Then, every W–lattice is a W–trivial extension of a trivial W–lattice and a simply connected W–lattice, and every simply connected W–lattice is a direct sum of simply connected lattices of irreducible pseudo reflection groups.*

The analogy is given by the fact that every connected compact Lie group is a qoutient of a product of simply connected simple Lie groups and a torus.

Our main theorems are statements about odd primes. This comes simply from the following lemma, which plays a litle but important role in several proofs.

1.4 Lemma. *Let p be an odd prime, let W be a pseudo reflection group and let M be a $\mathbb{Z}_p^\wedge$–module with trivial W–action. Then, we have $H_1(W;M) = H^1(W;M) =$ 0*

Proof. Without loss of gnerality we can assume that M is finitely generated. As a pseudo reflection group for an odd prime, W is generated by elements of order coprime to p. By the Hurewicz theorem, the first homology group $H_1(W,\mathbb{Z})$ is isomorphic to the abelinization of W, which is a finite abelian group of order coprime to p. Universal coefiicient theorems imply the statement.

The paper is organized as follows: In Section 2, we discuss centerfree lattices and prove part (1) of Theorem 1.2 and of Theorem 1.3. In Section 3 we study simply connected lattices and prove part (2) of Theorem 1.2 and of Theorem 1.3. ‘ The last section contains the proof of the rest of Theorem 1.2 and of Theorem 1.4.

Although the nature of this paper is mostly algebriac, sometimes we deal with completed topological spaces. Completion is always meant in the sense of Bousfield and Kan [1].

Independently of us, Dwyer and Wilkerson got also proofs for the main results of this work, which are not published yet.

Finally a warning: We are only dealing with odd primes, i.e. p always denotes an odd prime.

2. Centerfree sublattices.

As mentioned in the introduction $W \to Gl(U)$ is a finite pseudo reflection group.

2.1 Lemma.

(1) *For a W–lattice L, the center $Z(L)$ is a finite abelian p–group if and only if L is fixed point free.*
(2) *If L is fixed–point free, then we have $Z(L) \cong H^1(W;L)$.*

Proof. Taking fixed–points in the short exact sequence

$$0 \to L \to L_{\mathbb{Q}} \to T_{L,\infty} \to 0$$

gives rise to an exact sequence

$$0 \to L^W \to L_{\mathbb{Q}}^W \to (T_{L,\infty})^W \to H^1(W;L) \to H^1(W;L_{\mathbb{Q}}) = 0\ .$$

The first two terms vanishes if L is fixed point free. Otherwise the qoutient $L_{\mathbb{Q}}^W/L^W$ is a p–discrete torus. Inparticular, the qoutient is not finite. Because $H^1(W;L)$ is always finite, both parts follow. □

For a W–lattice L we denote by $L/p := L \otimes_{\mathbb{Z}_p^\wedge} \mathbb{F}_p$ the associated $\mathbb{F}_p[W]$–module.

2.2 Lemma. *A W–lattice P is centerfree if and only if $(P/p)^W = 0$.*

Proof. The multiplication $\mu_p : P \to P$ by p establishes a short exact sequence

$$0 \to P \xrightarrow{\mu_p} P \to P/p \to 0 \ .$$

Passing to fixed points gives an exact sequence

$$0 \to P^W \xrightarrow{\mu_p} P^W \to (P/p)^W \to H^1(W;P) \xrightarrow{\mu_p} H^1(W;P) \ .$$

If $(P/p)^W = 0$ then $\mu_p : P^W \to P^W$ and $\mu_p : H^1(W;P) \xrightarrow{\mu_p} H^1(W;P)$ are isomorphisms. From the first isomorphism follows that P is fixed–point free. Because $H^1(W;P)$ is a finite abelian p–group, the second isomorphism implies that $H^1(W;P) = 0$. The other direction follows from Lemma 2.1. □

Let L be W–lattice. The qoutient $\overline{T}_\infty := T_{L,\infty}/T^W_{L,\infty}$ carries a W–action and is a p–discrete torus which gives rise to a p–adic torus $\overline{T}$. Passing to classifying spaces establishes a fibration

$$BZ(L)^\wedge_p \to BT_L \to B\overline{T} \ .$$

Passing to 2-dimesional homology yields a W–equivariant map

$$L \cong H_2(BT_L;\mathbb{Z}^\wedge_p) \to H_2(B\overline{T};\mathbb{Z}^\wedge_p) =: PL \ .$$

The lattice PL is called the associated centerfree lattice of L.

2.3 Proposition. *Let L be a fixed point free W lattice.*

(1) *There exists an exact sequence*

$$0 \to L \to PL \to Z(L) \to 0 \ ,$$

and hence, L is a W–trivial restriction of PL.

(2) *The lattice PL is centerfree and $H^1(W;PL) = 0$.*

Proof. Because L is fixed point free, the center $Z(L)$ is finite. The Serre spectral sequence of the fibration $BZ(L) \to BT_L \to B\overline{T}$ has a differential $d : PL = H_2(B\overline{T} : \mathbb{Z}^\wedge_p) \to H_1(BZ(L);\mathbb{Z}^\wedge_p) \cong Z(L)$ which is an epimorphism and has kernel $L \cong H_2(BT;\mathbb{Z}^\wedge_p)$. This establishes the desired exact sequence of part (1).

Taking fixed points in the exact sequence of (1) gives rise to the exact sequence

$$0 = PL^W \to Z(L)^W = Z(L) \to H^1(W;L) \to H^1(W;PL) \to H^1(W;Z(L)) \ .$$

By Lemma 2.1, the second arrow is an isomorphism. By Lemma 1.4, the last term vanishes. Hence, we have $H^1(W;PL) = 0$. Again by Lemma 2.1, the lattice PL is centerfree. □

The following statement is part (1) of Theorem 1.2.

2.4 Theorem. *Let $W \to Gl(U)$ be a finite fixed–point free pseudo reflection group. Then, up to isomorphisms, there exists a unique centerfree W–sublattice $P \subset U$.*

The existence of a centerfree sublattice follows from Proposition 2.3. The key for the proof of the uniqueness is the following technical result.

2.5 Proposition. *Let $W \to Gl(U)$ be an irreducible pseudo reflection group, and let P be a centerfree sublattice. If there exists an exact sequence*

$$0 \to V_0 \to P/p \to V_1 \to 0$$

of $\mathbb{F}_p[W]$–modules, such that $V_0^W = 0 = V_1^W$, then either $V_0 = 0$ or $V_1 = 0$.

Proof. Let assume that V_0 and V_1 are nontrivial vector spaces. We choose a basis for V_0 and extend it to a basis of P/p. Then every element $w \in W$ can be represented by a matrix of the form

$$\begin{pmatrix} A_w & C_w \\ 0 & B_w \end{pmatrix}$$

where A_w decribes the action of w on V_0, B_w the action on V_1 and $C_w : V_1 \to V_0$ the twisting, i.e. the failure to be a direct product. This description establishes a homomorphism $\phi : W \to Gl(V_0) \times Gl(V_1)$ given by $\phi(w) := (A_w, B_w)$. Let W_i be the image of W in the factor $Gl(V_i)$. That is we have a homomorphism $\phi : W \to W_0 \times W_1$. Because V_0 and V_1 have no non trivial fixed point, both groups W_0 and W_1 are non trivial.

The kernel K of ϕ consists of those elements which are desribed by a matrix of the form $\begin{pmatrix} id & C \\ 0 & id \end{pmatrix}$. Therefore, every element of the kernel has order p and the kernel is an elementary abelian p–group and a normal subgroup of W.

Now let $\sigma \in W$ be a p–adic pseudo reflection. The matrix

$$\sigma - id = \begin{pmatrix} A_\sigma - id & C_\sigma \\ 0 & B_\sigma - id \end{pmatrix}$$

has rank 1. That is that all columns and all rows are multiple of one column or one row. We have $A_\sigma - id \neq 0$ if and only if $B_\sigma = id$. The equivalence follows from the fact that the order of σ is coprime to p. Therefore, W_0 and W_1 are generated by p-adic reflections. Let $(w_0, w_1) \in W_0 \times W_1$. We can assume that w_0 is the image of a product of p–adic reflections which are mapped onto the identity in W_1, and similiar for w_1. This shows that ϕ is an epimorphism.

The above considerations show that W allows a short exact sequence

$$(*) \qquad 1 \to K \to W \to W_0 \times W_1 \to 1 \ .$$

where W_0 and W_1 are nontrivial groups, where both are generated by elements coming from pseudo reflections in W, and where $K \subset W$ is an elementary abelian normal subgroup. For abbreviation, we say that W has the property $(*)$.

We want to show that either W_0 or W_1 is the trivial group. This would imply that either $V_0 = 0$ or $V_1 = 0$. The proof of this conclusion splits into two part, the nonmodular case, i.e. $(|W|, p) = 1$, and the modular case. For the modular case we use the classification of the irreducible pseudo reflection groups by Clark and Ewing [3]. We also use there numbering of the different cases.

First let $(|W|, p) = 1$. Then $K = 0$ and $W \cong W_0 \times W_1$ splits into a product of pseudo reflection groups. Because the representation $W \to Gl(U)$ is irreducible, this implies that either W_0 or W_1 is the trivial group.

Now let p divide $|W|$. If $K \subset W$ is a central subgroup, then every element of K establishes a W–equivariant self map $U \to U$ of the irreducible representation $U \otimes_{\mathbb{Q}_p^\wedge} \mathbb{C}$, which therefore is given by a p-adic multiple of the identity. That is to say that there exists a homomorphism $K \to \mathbb{Z}_p^{\wedge *} \cong \mathbb{Z}/p-1 \times \mathbb{Z}_p^\wedge$. Because W is finite and because W acts faithfully on U, this homomorphism is injective, and the kernel K is trivial. We can proceed as in the nonmodular case.

If $K \to W$ is not central, then there exists a pseudo reflection $\sigma \in W$ acting nontrivially on K. Because the order of σ is coprime to p, the representation K of the group $< \sigma >$, generated by σ, splits into 1-dimesional irreducible summands. Let $K' \subset K$ be one of the summands with a nontrivial action of σ and let $x \in K'$ be a generator. The subgroup $D :=< \sigma, x\sigma x^{-1} >=< \sigma, x\sigma >=< \sigma, x >$ of W, generated by two pseudo reflections, fits into a short exact sequence

$$1 \to K' \to D \to < \sigma > \to 1 \ .$$

The order $m = |\sigma|$ of σ is coprime to p. Therefore, the sequence splits and $D \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$ acts on U as a pseudo relection group. As a $\mathbb{Z}/p$–module, $U \cong \bigoplus_i U_i$ splits into a direct sum of irreducible $\mathbb{Z}/p$–modules which are permuted by $\mathbb{Z}/m$. Each factor is either 1–dimensional with trivial $\mathbb{Z}/p$–action ($\mathbb{Q}_p^\wedge$ contains no p–th root of unity) or isomorphic to $U' \cong (\mathbb{Q}_p^\wedge)^{p-1}$ where we consider U' as the kernel of the map $(\mathbb{Q}_p^\wedge)^p \to \mathbb{Q}_p^\wedge$ given by summing up the coordinates and where $\mathbb{Z}/p$ acts via cyclic permutation on $(\mathbb{Q}_p^\wedge)^n$. The factors with trivial $\mathbb{Z}/p$–action does not lead to a faithful representation of D. Every factor isomorphic to U' is fixed under the action of $\mathbb{Z}/m$, and $\mathbb{Z}/m$ acts on U' via permutation associated to the action on $\mathbb{Z}/p$ considered as a set. Therefore, U' represents D as a pseudo reflection group if and only if $m = 2$. That is to say that $D \cong D_{2p}$ is a dieder group. By the classification list of irreducible pseudo reflection groups [3] the only modular cases are given by D_6 and D_{12}. Hence, we have $p = 3$ and $D \cong D_6$.

By the above arguments it is only left to consider modular cases for $p = 3$. We will finish the proof by a case by case checking following the list of [3]. We only have to discuss the numbers 2a, 2b, 12, 28, 35, 36 and 37.

Case number 2a. In this case $\Sigma_n \subset W \subset \mathbb{Z}/l \wr \Sigma_n$ where l divides $p-1$. Inparticular, the subgroup $\pi \subset W$ is a normal subgroup of Σ_n as well as of $A_n \subset \Sigma_n$. Here, A_n denotes the group of permutations of positive sign. For $n \geq 5$, the group A_n is simple. For $n = 4$, we have $A_4 \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/3$. Therefore, in these cases there exists no normal elementary abelian 3–subgroup, and we can proceed as in the nonmodular case.

For $n = 3$, the representation U is 2–dimensional and is irreducible even considered as a $\Sigma_3 \cong D_6$–module. Let $\sigma, \tau \in \Sigma_3$ be two tanspositions generating Σ_3. Let $L \subset U$ be the standard sublattice with the action given by $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $\tau = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$. A straight forward calculation shows that $L \otimes \mathbb{F}_p$ is an irreducible $\mathbb{F}_p[\Sigma_3]$–module. Therefore, every other sublattice of U also produces a mod–p irreducible module. This shows that either V_0 or V_1 is trivial.

For later purpose we note the following observation: For $n \geq 5$, the above argument shows that, if W has the property $(*)$, every pseudo reflection representation of W splits into two summands, where both factors carry a nontrivial W–action.

Case number 2b. In this case, we have $W = D_6$ or $W = D_{12}$ and U is 2–dimensional. In particular, U is an irreducible D_6–module. We can argue as in the above case.

Case number 12. In this case we have $dim_{\mathbb{Q}_p^\wedge} U = 2$. By the above arguments, we have a subgroup $D_6 \subset W$. Moreover, U is irreducible as D_6–module. Again we can procedd as above to show that either V_0 or V_1 is trivial.

Case number 28. In this case, we have $W = W_{F_4} \cong ((\mathbb{Z}/2)^3 \rtimes \Sigma_4) \rtimes \Sigma_3$. The last isomorphism may be found in [5, p. 45]. A straight forward calculations shows that $K = 0$. We can proceed as in the nonmodular case.

Case number 35, 36, 37. In this case we have $W = W_{E_6}$, $W = W_{E_7}$ or $W = W_{E_8}$. We describe two maximal subgroups of maximal rank for each of these connected compact Lie groups.

G	H'	H''
E_6	$S^1 \times_{\mathbb{Z}/2} Spin(10)$	$SU(2) \times_{\mathbb{Z}/2} SU(6)$
E_7	$S^1 \times_{\mathbb{Z}/2} Spin(12)$	$S^1 \times_{\mathbb{Z}/3} E_6$
E_8	$SSpin(16)$	$SU(2) \times_{\mathbb{Z}/2} E_7$

A list of all maiximal subgroups of maximal rank may be found in [6]. This establishes subgroups of W as follows:

W	W'	W''
W_{E_6}	$W_{H'} \cong (\mathbb{Z}/2)^5 \rtimes \Sigma_5$	$W_{SU(6)} \cong \Sigma_6$
W_{E_7}	$W_{H'} \cong (\mathbb{Z}/2)^5 \rtimes \Sigma_6$	W_{E_6}
W_{E_8}	$W_{H'} \cong (\mathbb{Z}/2)^7 \rtimes \Sigma_8$	W_{E_7}

In all cases, the two groups W' and W'' generate W. This follows because $H' \subset G$ is maximal of maximal rank. Moreover, the intersection $W' \cap W''$ is nonempty. We want to show that there exists no epimorphism $W \longrightarrow W_0 \times W_1$ as in $(*)$ with kernel given by an elementary abelian p–group.

Let us look at the case $W = W_{E_6}$. By the observation at the end of case number 2a, the W'–module U splits into a direct sum of nontrivial W'–modules, if W' has the property $(*)$. The same is true for W''. But by the choice of the groups, both belong to case 2a with $n \geq 5$, we only can split of a trivial summand of U considered as a W' or W''–module. Therefore, an epimorphism $W_{E_6} \longrightarrow W_0 \times W_1$ maps W' and W'' only into one factor. Because $W' \cap W''$ is nonempty, both are mapped into the same factor, let us say into W_0. Because W_{E_6} is generated by W' and W'', the group W is only mapped into W_0, too. Hence, W_1 is trivial, This proves the statement in this case. In particular, this argument also shows that there exists no epimorphism of the form $(*)$ with kernel given by an elementary abelian p–group.

For W_{E_7} and W_{E_8}, we can argue analogously using the result for W_{E_6} or W_{E_7}. This finishes the discusion of all possible cases and the proof of the statement. □

Remark. The last proposition as well as the proof originates in a discussion with C.Broto and J.Aguadé on a similar question.

2.6 Lemma. *Let $P \to L$ be a monomorphism between W–sublattices of U. If P is centerfree, then we have $(L/P)^W = 0$.*

Proof. Because P is centerfree, every sublattice of $U \cong P \otimes_{\mathbb{Z}} \mathbb{Q}$ is fixed–point free. The short exact sequence $P \to L \to L/P$ gives rise to an exact sequence $L^W = 0 \to (L/P)^W \to H^1(W;P) = 0$. Thus, the quotient L/P has no fixed points. □

Proof of Theorem 1.7 for irreducible pseudo reflection groups.

Let P and Q be two centerfree sublattices of an irreducible pseudo reflection group $W \to Gl(U)$. Then, there exists a W–equivariant monomorphism $\alpha : P \to Q$ such that $rk(Q/P) < rk(Q) = rk(P)$. Here, $rk(M)$ denotes the rank of a module, which we define to be the dimension of M/p over $\mathbb{F}_p$. Otherwise we have $P \subset pQ := \{px : x \in Q\}$ and we can replace Q by pQ. Because P is centerfree we know that $(Q/P)^W = 0$ (Lemma 2.6). Because U is irreducible we know that $P \xrightarrow{\alpha} Q$ is rationally an isomorphism and that Q/P is finite.

Applying the functor $\otimes \mathbb{F}_p$ yields an exact sequence

$$0 \to Tor(Q/P, \mathbb{F}_p) \to P/p \xrightarrow{\overline{\alpha}} Q/p \to Q/P \otimes \mathbb{F}_p \to 0$$

of W–modules. Let $V_0 := Tor(Q/P; \mathbb{F}_p)$ and let $V_1 := Im(\overline{\alpha})$ be the image of α which is isomorphic to the kernel of $Q/p \to Q/P \otimes \mathbb{F}_p$. Because P and Q are centerfree we have $V_0^W = 0 = V_1^W$ (Lemma 2.2). Applying Proposition 2.5 shows that either V_0 or V_1 are trivial vector spaces. If $V_1 = 0$ then $rk(Q/P) = rk(Tor(Q/P; \mathbb{F}_p) = rk(P)$, which is a contradiction. Thus, $V_0 = 0$ and $Q/P = 0$. That is to say that $\alpha : P \to Q$ is an isomorphism. This proves the statement for irreducible pseudo reflection groups. □

Next we consider the case of a reducible fixed–point free pseudo reflection group W, i.e. $W \cong W_1 \times W_2$ splits into a nontrivial product of pseudo reflection groups. Moreover, $U \cong U_1 \times U_2$ also splits into a direct sum where $U_1 = U^{W_2}$ and $U_2 = U^{W_1}$.

2.7 Lemma. *Let $W \to Gl(U)$ be a reducible pseudo reflection group, and let P be a centerfree W–sublattice of $U = U_1 \oplus U_2$. Then, the fixed–point set P^{W_1} is centerfree with respect to the W_2–action and $P \cong P^{W_1} \oplus P^{W_2}$ as W–modules.*

Proof. The qoutient P/P^{W_1} is torsionfree. Hence, the sequence of W–modules

$$0 \to P^{W_1}/p \to P/p \to (P/p)/(P^{W_1}/p) \to 0$$

is short exact . Taking fixed–points yields an exact sequence

$$0 \to (P^{W_1}/p)^W \cong (P^{W_1}/p)^{W_2} \to (P/p)^W = 0 \ .$$

The last fixed point set vanishes because P is centerfree and because of Lemma 2.2. Again by Lemma 2.2, the fixed–point set P^{W_1} is centerfree with respect to the W_2–action.

Applying the functor $\otimes\mathbb{Q}$ establishes an exact sequence

$$0 \to P^{W_1} \otimes \mathbb{Q} \to P \otimes \mathbb{Q} \to (P/P^{W_1}) \otimes \mathbb{Q} \to 0 \ .$$

Because $P^{W_1} \otimes \mathbb{Q} \cong (P \otimes \mathbb{Q})^{W_1}$, this sequence splits and shows that $(P/P^{W_1}) \otimes \mathbb{Q}$ as well as P/P^{W_1} are trivial W_2–module. Taking W_2–fixed points establishes the exact sequence

$$0 = P^W = (P^{W_1})^{W_2} \to P^{W_2} \to (P/P^{W_1})^{W_2} = P/P^{W_1} \to H^1(W_2; P^{W_1}) = 0 \ .$$

The last identity follows from Lemma 2.1 since P^{W_1} is W_2–centerfree. This implies that the middle arrow is an isomorphism, and that $P^{W_1} \oplus P^{W_2} \to P$ is an isomorphism of W–modules. □

Proof of Theorem 1.7 in the general case. Let P and Q be centerfree $W = W_1 \times W_2$ lattice. Let $P^{W_i} =: P_i$ and $Q^{W_i} =: Q_i$. By Lemma 2.7, we know that $P \cong P_1 \oplus P_2$ and that $Q \cong Q_1 \oplus Q_2$. Because P_1 and Q_1 are both W_1–centerfree, they are isomorphic as W_1–modules by induction over the order of W. Analogously, we have $P_2 \cong Q_2$ as W_2–modules. Putting this together gives the desired W–module isomorphism $P \cong Q$. □

Proof of Theorem 1.3 (1). Let $W \to Gl(U)$ be a reducible pseudo reflection group. Using an induction over the number of irredicble summands of U, the statement follows from Lemma 2.7 and Theorem 1.7 for irreducible pseudo reflection groups. □

3. Simply connected sublattices.

Again, $W \to Gl(U)$ denotes a finite pseudo reflection group. The situation for simply connected lattices is somehow dual to the case of centerfree lattices (see Proposition 4.1).

3.1 Lemma.

(1) *For a W–lattice L, the group L_W of covariants is finite if and only if L is fixed point free.*
(2) *If L is fixed point free, then we have $L_W \cong H_1(W, T_{L,\infty})$.*

Proof. Passing to covariants and using the fact that $L_W \cong H_0(W, L)$, the short exact sequence

$$0 \to L \to L \otimes \mathbb{Q} := L_{\mathbb{Q}} \to T_{L,\infty} \to 0$$

gives rise to the exact sequence

$$0 = H_1(W; L_{\mathbb{Q}}) \to H_1(W; T_{L,\infty}) \to L_W \to (L_{\mathbb{Q}})_W \to (T_{L,\infty})_W \to 0 \ .$$

Because every exact sequence of W–modules over $\mathbb{Q}_p^\wedge$ splits, we have $(L_{\mathbb{Q}})_W \cong L_{\mathbb{Q}}^W$. The cohomology group $H_1(W; T_{L,\infty})$ is finite. Thus, L_W is finite if and only if L is fixed–point free. The second part is obvious. □

In the introduction, for a W–lattice L, we defined SL to be the kernel of $L \to L_W$.

3.2 Propostion. *Let L be a fixed–point free W–lattice.*

(1) *There exists an exact sequence*

$$0 \to SL \to L \to L_W \to 0 ,$$

and L is a W–trivial extension of SL.

(2) *The lattice SL is simply connected.*

Proof. The first part follows from the definition of L_W and SL. Passing to coinvariants, the short exact sequence of (1) establishes the exact sequence

$$H_1(W, L_W) \to SL_W \to L_W \to L_W \to 0 .$$

The first term vanishes (Lemma 3.1) and the second last arrow is an isomorphism. □

The next results connects simply connected and centerfree lattices.

3.3 Proposition. *Let S be a simply connected W–lattice. Let $P := PS$ be the associated centerfree lattice. Then, we have $SP \cong S$ and $Z(S) \cong P/W$.*

Proof. By construction there exists an short exact sequence

$$0 \to S \to P \xrightarrow{q_S} Z(S) \to 0 .$$

Because $Z(S)$ is a trivial W–module, the map q_S factors over the covariants P_W. This establishes a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SP & \longrightarrow & P & \longrightarrow & P_W \cong Z(SP) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & P & \longrightarrow & Z(S) & \longrightarrow & 0 \end{array}$$

where the cokernel S/SP of the monomorphism $SP \to S$ is a W-equivariant quotient of P_W. Therefore, the quotient S/SP is a module with trivial W–action, and the epimorphism $S \to S/SP$ factors over $S_W = 0$. This shows that all vertical arrows are isomorphisms. □

We finish this section with proofs of Part (2) of Theorem 1.2 and Part (2) of Theorem 1.3.

Proof of Theorem 1.2 (2). The existence of a simply connected sublattice follows from Proposition 3.2. Let S and S' be two simply connected sublattices. Let P and P' be the associated centerfree lattices. By Theorem 1.7 we know that $P \cong P'$, and by Proposition 3.3 follows that $S \cong SP \cong SP' \cong S'$. □

Proof of Theorem 1.3 (2). Let $S \subset U$ be a simply connected sublattices and let $P := PS \subset U$ be the associated centerfree sublattice. By Theorem 1.3 (1), we have a splitting $P \cong \bigoplus_i P_i$ of P into centerfree sublattices $P_i \subset U_i$ of the irreducible pseudo reflection groups $W_i \to Gl(U_i)$ and the summands are uniquely determined up to order and isomorphisms. The sublattices $S_i := SP_i \subset U_i$ are simply connected and the sequence $S \cong SP \cong \bigoplus_i SP_i = \bigoplus_i S_i$ proves the statement. The first isomorphism follows from Proposition 3.3, and the second from Lemma 2.7 and Lemma 3.2. □

4. Proof of the main theorems.

In this section we want to prove the last two parts of Theorem 1.2 and Theorem 1.4.

Proof of Theorem 1.2.

Part (1) and part (2) are already proved in the previous sections. Let $L \subset U$ be a W–sublattice, and let P and S the centerfree and the simply connected sublattices of U. The composition $SL \to L \to PL$ consist of W–trivial restrictions with $L/SL \cong L_W$ and $PL/L \cong Z(L)$ (Propositions 2.3 and 3.2). By (1) and (2) follows that $SL \cong S$ and that $PL \cong P$. This proves the third part.

Now let $S \to P$ be a W–trivial restriction of sublattices of U, satisfying the assumptions of (4). Let L be a W–sublattice of U. Then, by assumption, there exists a W–trivial restriction $PL \to P$. Inparticular, the qoutient P/PL is finite with trivial W–action. Passing to cohomology gives an exact sequence

$$H^1(W; PL) = 0 \to H^1(W; P) \to H^1(W; P/PL) = 0 \ .$$

The first term vanishes because PL is centerfree and because of Lemma 2.1, the last term vanishes by Lemma 1.4. Thus $H^1(W; P) = 0$ and P is centerfree (Lemma 2.1). On the other hand there exists a W–trivial restriction $S \to SL$. Taking covariants gives an exact sequence $H_1(W; SL/S) = 0 \to S_W \to SL_W = 0$. The first term vanishes because of Lemma 1.4 and the last term because SL is simply connected and because of Proposition 3.2 . Thus, we have $S_W = 0$ and S is simply connected. □

Before we discuss the case of general finite pseudo reflection groups, i.e. before we prove Theorem 1.4, we need some informations about the dual representations. Let $W \to Gl(U)$ be a pseudo reflection group. We consider U as a left $\mathbb{Q}_p^\wedge[W]$–module. Then the set $Hom_{\mathbb{Q}_p^\wedge}(U, \mathbb{Q}_p^\wedge)$ becomes a left $\mathbb{Q}_p^\wedge[W]$–module by defininig $w(x^*) := x^* w^{-1}$ for $x^* \in U^*$ and $w \in W$. The vector space U^* again represents W as a pseudo reflection group. For a W–sublattice $L \subset U$ we define $L^* := Hom_{\mathbb{Z}_p^\wedge}(L; \mathbb{Z}_p^\wedge)$ which becomes analogously as above a left $\mathbb{Z}_p^\wedge[W]$–module. Because $L^* \otimes \mathbb{Q} \cong (L \otimes \mathbb{Q})^*$ as $\mathbb{Q}_p^\wedge[W]$–modules, the lattice L^* is a sublattice of U^*.

4.1 Proposition. *Let $W \to Gl(U)$ be a finite fixed point free pseudo reflection group.*

(1) *A sublattice $P \subset U$ is centerfree if and only if $P^* \subset U^*$ is simply connected.*
(2) *A sublattice $S \subset U$ is simply connected if and only if $S^* \subset U^*$ is centerfree.*

Proof. Let $S \to P$ be the W–trivial restriction between the simply connected sublattice S and the centerfree sublattice P. For any W–lattice L, there exists a natural W-equivariant isomorphism $L^{**} \cong L$. Let M be a W–sublattice of U^*. By Theorem 1.2 (3) the dual M^* fits between S and P. Therefore, the lattice M sits between P^* and S^*, which shows that P^* is simply connected and that S^* is centerfree (Theorem 1.2 (4)). This proves one direction of both parts. The other follows by dualizing again. □

Proof of Theorem 1.4. Let $L \subset U$ be a W–sublattice. The quotient $L/L^W =: \overline{L}$ is a fixed point free W–sublattice of U', where $U \cong U^W \oplus U'$ splits into the direct sum of the fixed points U^W and a fixed point free part U'. Let $S, P \subset U'$ be the simply connected and centerfree sublattices. By Theorem 1.2, there exists a W–trivial restriction $S \to \overline{L}$. Using pullbacks establishes a commutaive diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^W \cong Z & \longrightarrow & L' & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^W \cong Z & \longrightarrow & L & \longrightarrow & \overline{L} & \longrightarrow & 0\ . \end{array}$$

The top row describes an element of the group $Ext_{\mathbb{Z}_p^\wedge[W]}(S, Z)$ of extensions. We have the following sequence of isomorphisms:

$$\begin{aligned} Ext_{\mathbb{Z}_p^\wedge[W]}(S, Z) &\cong H^1(W; Hom_{\mathbb{Z}_p^\wedge}(S, Z)) \\ &\cong H^1(W, S^* \otimes Z) \\ &\cong H^1(W; S^*) \otimes Z \\ &= 0\ . \end{aligned}$$

The first identity follows, because S and Z are free modules over $\mathbb{Z}_p^\wedge$ [2, III; 2.2], the second from the isomorphism between the coefficients, the third because W acts trivially on Z and because of Lemma 1.4, and the last because S^* is centerfree (Proposition 4.1 and Lemma 2.1). That is to say that $L' \cong Z \oplus S$. Moreover, we have an isomorphism $L/(Z \oplus S) \cong \overline{L}/S \cong \overline{L}_W$ which shows that $Z \oplus S \to L$ is a W–trivial restriction. This proves part (1).

For the second statement we dualize the above argument. There exists a W–trivial restriction $Z^* \oplus P^* \to L^*$ (Proposition 4.1 and part (1)). Dualizing again gives a short exact sequence

$$0 \to L \to Z \oplus P \to Ext(L^*/(Z^* \oplus P^*); \mathbb{Z}_p^\wedge) \to 0\ ,$$

which shows that the first arrow is a W–trivial restriction. $\square$

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