UNSTABLE SPLITTINGS OF CLASSIFYING SPACES OF \( p \)-COMPACT GROUPS

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Abstract. Dwyer and Wilkerson gave a definition of a \( p \)-compact group, which is a loop space with certain properties and a good generalisation of the notion of compact Lie groups in terms of classifying spaces and homotopy theory; e.g. every \( p \)-compact group has a maximal torus, a normalizer of the maximal torus and a Weyl group. The believe or hope that \( p \)-compact groups enjoys most properties of compact Lie groups establishes a program for the classification of these objects. Following the classification of compact connected Lie groups, one step in this program is to show that every simply connected \( p \)-compact group splits into a product of simply connected simple \( p \)-compact groups. The proof of this splitting theorem is based on the fact that every classifying space of a \( p \)-compact group splits into a product if the normalizer of the maximal torus does.

1. Introduction.

A loop space is a triple \( X = (X, BX, e : \Omega BX \xrightarrow{\sim} X) \), where \( X \) and \( BX \) are topological spaces, where \( BX \) is pointed, and where \( e \) is a homotopy equivalence between the loop space \( \Omega BX \) of \( BX \) and \( X \). Such a triple is called a \( p \)-compact group, if \( X \) is \( \mathbb{F}_p \)-finite, i.e. \( H^*(X; \mathbb{F}_p) \) is finite, and if \( BX \) is a \( p \)-complete connected space. This notion was introduced by Dwyer and Wilkerson in [7]. Examples of \( p \)-compact groups are given by the completion of a compact connected Lie group. For a compact connected Lie group \( G \), the triple \( (G^\wedge_p, BG^\wedge_p, \Omega BG^\wedge_p \simeq G^\wedge) \) is a \( p \)-compact group.

In recent work of Dwyer and Wilkerson [7, 8] and of Möller and the author [12] it turned out that \( p \)-compact groups are a good homotopy theoretic generalisation of compact Lie groups. In particular, it was shown that \( p \)-compact groups enjoy quite a lot of the properties of compact Lie groups; e.g. there exist always maximal tori, normalizer of the maximal tori and Weyl groups [7]. The maximal torus of a \( p \)-compact group \( X \) is a map \( BT_X \to BX \) of an Eilenberg–MacLane space \( BT_X \simeq K((\mathbb{Z}_p)^n, 2) \) into the classifying space \( BX \) of \( X \) with certain properties. The Weyl group, denoted by \( W_X \), is a finite group, and the normalizer of the maximal torus is a map \( BN(T_X) \to BX \), where \( BN(T_X) \) fits into a fibration \( BT_X \to BN(T_X) \to BW_X \) such that the triangle

\[
\begin{array}{c}
BT_X \\
\downarrow \\
BX \\
\downarrow \\
BN_X \\
\end{array}
\]


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commutes up to homotopy. For exact definitions of these notions see Section 2. In general, the normalizer, also denoted by $N_X$, does not give a $p$-compact group, because the space $BN_X$ is not $p$-complete (the fundamental group $\pi_1(BN_X)$ might not be a finite $p$-group). Nevertheless, the space $BN_X$ establishes a finite loop space $N_X := (N_X, BN_X, \Omega BN_X \simeq N_X)$, where $N_X$ is $\mathbb{F}_p$-finite. This triple behaves similar as $p$-compact groups. There exists a $p$-compact subgroup of $P_X \subset N_X$, which we construct by restricting the fibration to the classifying space of the $p$-Sylow subgroup of $W_X$. The classifying space $BP_X$ is $p$-complete, because $\pi_1(BP_X)$ is a finite $p$-group and because $BT_X$ is $p$-complete. This follows from [4, II 5.1, 5.2]. The map $BP_X \to BX$ is called the $p$-toral Sylow subgroup of $X$ and plays the same role as the $p$-toral Sylow subgroup of a compact Lie group.

One of the main questions about $p$-compact groups asks for a classification of these objects. The naive approach to believe that the analogy between $p$-compact groups and compact Lie groups is as good as possible produces a lot of ‘theorems’ and conjectures. Because of the lack of an equivalent for the Lie algebra, one cannot translate all the proofs of statements about compact Lie groups. Translation means to express everything in terms of classifying spaces, e.g. a homomorphism $X \to Y$ between $p$-compact groups is a pointed map $BX \to BY$ between the classifying spaces, and a $p$-compact group $X \cong X_1 \times X_2$ splits into a product of $p$-compact groups if there exists a homotopy equivalence $BX \simeq BX_1 \times BX_2$. The task of the naive approach consists of finding ‘new’ proofs (in terms of homotopy theory) for ‘old’ results, which also work for the larger class of $p$-compact groups.

The classification of compact connected Lie groups says that, for every compact connected Lie group, there exists a finite covering, which is a product of simple simply connected Lie groups and a torus. In [12] was shown that the first part of this result is true for connected $p$-compact groups, namely every connected $p$-compact group has a finite covering, which is a product of a simply connected $p$-compact group and a torus. For the second step one has to show that every simply connected $p$-compact group is a product of simple simply connected $p$-compact groups. A ‘new’ proof of this second step is the main purpose of this paper.

We call a $p$-compact group $X$ simply connected if the space $X$ is simply connected. The definition of simple has to wait until Section 2.

1.1 Theorem. Let $p$ be an odd prime. Let $X$ be a simply connected $p$-compact group. Then, $X \cong X_1 \times ... X_n$ splits into a product of simple simply connected $p$-compact groups.

There exists also a notion of a center of a $p$-compact group[8] [12], which is the generalization of the group or Lie group theoretic center. The center is always a $p$-compact group. Again, for details see Section 2. A $p$-compact group $X$ is called centerfree if the center $Z(X)$ of $X$ is the trivial group, i.e. the classifying space $BZ(X)$ is contractible.

1.2 Theorem. Let $p$ be an odd prime. Let $X$ be a centerfree connected $p$-compact group. Then, $X \cong X_1 \times ... X_n$ splits into a product of simple centerfree connected $p$-compact groups.

The proofs of both theorems are based on a general splitting criteria for $p$-compact groups.
1.3 Theorem. A $p$–compact group $X$ splits into product $X \cong X_1 \times X_2$ of $p$–compact groups if and only if the normalizer $N_X$ of the maximal torus $T_X \to X$ splits into a product $N_X \cong N_1 \times N_2$.

For the proof of Theorem 1.3 we have to study maps from classifying spaces into almost $B\mathbb{Z}/p$–local spaces. A space $A$ is called $B\mathbb{Z}/p$–local if the map $A \to \text{map}(B\mathbb{Z}/p, A)$ is an equivalence and called almost $B\mathbb{Z}/p$–local if the map $A \to \text{map}(B\mathbb{Z}/p, A)_{\text{const}}$ into the component of the constant map is an equivalence. Examples of $B\mathbb{Z}/p$–local and almost $B\mathbb{Z}/p$–local spaces are provided by $\mathbb{F}_p$–finite spaces. $\mathbb{F}_p$–finite space $K$ is $B\mathbb{Z}/p$–local. This follows from the Sullivan conjecture [10]. By [15], the space $BHE(K)$ is almost $B\mathbb{Z}/p$–local. Here, $HE(K)$ denotes the monoid of self equivalences of $K$. In general, this is not an $\mathbb{F}_p$–finite space. Moreover, if $K$ is $p$–complete and a loop space, the classifying space $BHE(K)$ is also $p$–complete [15]. These are the main example we apply the next theorem to.

1.4 Theorem. Let $P \to X$ be a $p$–toral Sylow subgroup of a $p$–compact group $X$, and let $f : BX \to A$ be a map into a connected $p$–complete almost $B\mathbb{Z}/p$–local space.

(1) The restriction $f|_{BP}$ is nullhomotopic if and only if $f$ is nullhomotopic.

(2) The map $A \to \text{map}(BX, A)_{\text{const}}$ is an equivalence.


The paper is organized as follows: In Section 2 we recall material about $p$–compact groups, mostly from [7], and prove some auxiliary lemmas necessary for the proof of Theorem 1.3. Section 3 contains a calculation of some low dimensional cohomology groups of pseudo reflection groups. The proof of Theorem 1.4 is carried out in Section 4, and the proof of Theorem 1.3 in Section 5. The final section is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

One remark about references. There is some overlap between the papers [7] and [12]. For most citation referring to one of these papers one could also use the other one. We used the one which was first at hand.

Finally, we would like to point out that, independently of us, Dwyer and Wilkerson also got proofs for similar results.

2. Background.

In this section we recall the basic notions about $p$–compact groups from [7]. Most of the notions are motivated by classical Lie group theory and by passing to classifying spaces. For keeping things short and because the analogy to compact Lie groups is discussed in [7, 8] and [12], we omit motivations.

2.1 Isomorphisms, monomorphisms, subgroups and exact sequences: A homomorphism $Y \to X$ of $p$–compact groups or loop spaces is an isomorphism if $Bf : BY \to BX$ is an equivalence. A sequence $X \to Y \to Z$ of $p$–compact groups is short exact if the associated sequence $BX \to BY \to BZ$ is a fibration up to homotopy. A monomorphism of $p$–compact groups is a map $BX \to BY$ whose homotopy fiber, denoted by $Y/X$, is $\mathbb{F}_p$–finite. A subgroup $Y \to X$ of $p$–compact
group $X$ is a monomorphism of $p$–compact groups.

2.2 p-compact tori, $p$–compact toral groups and finite extensions of $p$–compact tori: A $p$–compact torus is a $p$–compact group $(T, BT, \Omega BT \simeq T)$, where $BT \simeq K((\mathbb{Z}_p)^n, 2)$ is an Eilenberg–MacLane space of degree 2.

A finite group or a finite loop space is a triple $(K, BK, \Omega BK \simeq K)$ such that $BK$ is an Eilenberg–MacLane space of a finite group of degree 1.

A finite extension of a $p$–adic torus $T$ is a triple $(N, BN, \Omega BN \simeq N)$ which fits into a short exact sequence of loop spaces $T \to N \to W \Rightarrow N/T$, where $W$ is a finite loop space. A $p$–compact toral group $P$ is a finite extension of a $p$–compact torus $T$ such that the quotient $P/T$ is a finite $p$–group. In particular, every $p$–compact toral group is a $p$–compact group.

2.3 Conjugation and subconjugation: Two homomorphisms $f, g : Y \to X$ of $p$–compact groups are called conjugate if the induced maps $Bf, Bg : BY \to BX$ are homotopic.

For a homomorphism $f : Y \to X$ of $p$–compact groups and for a $p$–compact toral subgroup $i : P \to X$ we say that $P$ is subconjugate to $Y$ if there exists a homomorphism $j : P \to Y$ such that $fj$ and $i$ are conjugate.

2.4 Centralizers and centers: For a homomorphism $f : Y \to X$ between $p$–compact groups, we define the centralizer $C_X(f(Y))$ to be the loop space given by the triple

$$C_X(f(Y)) := (\Omega \text{map}(BY, BX)_{Bf}, \text{map}(BY, BX)_{Bf}, id).$$

The evaluation at the basepoint $ev : \text{map}(BY, BX)_{Bf} \to BX$ establishes a homomorphism $C_X(f(Y)) \to X$ of loop spaces. If $Y$ is a $p$–compact toral group the centralizer $C_X(f(Y))$ is again a $p$–compact group and the evaluation $C_X(f(Y)) \to X$ is a monomorphism [7, 5.1, 5.2 and 6.1].

A subgroup $Z \to X$ of a $p$–compact group $X$ is called central if the monomorphism $C_X(Z) \to X$ is an isomorphism. The center $Z(X)$ of $X$ is the maximal central subgroup of $X$ [8, 1.2] [12, 4.3, 4.4]. To give an explicit definition we use a result of Dwyer and Wilkerson [8, 1.3]. For every $p$–compact group $X$, the centralizer $C_X(X)$ is a $p$–compact group and $Z(X) := C_X(X) \to X$ is the center of $X$. For every $p$–compact group $X$ there exists a short exact sequence $Z(X) \to X \to X/Z(X) =: PX$ of $p$–compact groups, and, if $X$ is connected, the quotient $PX$ has a trivial center [12, 4.7].

We call a $p$–compact group $X$ centerfree if $Z(X)$ is the trivial group.

2.5 Maximal tori: The maximal torus of a $p$–compact group $X$ is a monomorphism $T_X \to X$ of a $p$–compact torus into $X$ such that the centralizer $C_X(T_X)$ is a $p$–compact toral group, whose component of the unit is given by $T_X$.

2.6 Theorem [7, 8.11, 8.13 and 9.1]. Let $X$ be a $p$–compact group.

(1) The $p$–compact group $X$ has a maximal torus $T_X \to X$.

(2) Any subtorus $T \to X$ of $X$ is subconjugated to the maximal torus $T_X \to X$. 
Any two maximal tori of $X$ are conjugated.

If $X$ is connected then $T_X \to C_X(T_X)$ is an isomorphism.

If $X$ is connected every finite cyclic subgroup $\mathbb{Z}/p^n \to X$ of $X$ is subconjugate to $T_X$.

2.7 Weyl spaces and Weyl groups: Let $T_X \to X$ be a maximal torus of a $p$–compact group. We think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_T(X)$ is defined to be the mapping space of all fiber maps over the identity on $BX$. Then each component of $\mathcal{W}_T(X)$ is contractible and the Weyl group $W_T(X) := \pi_0(\mathcal{W}_T(X))$ is a finite group under composition [D-W, 9.5].

2.8 Theorem [7, 9.5 and 9.7]. Let $T_X \to X$ be the maximal torus of a connected $p$–compact group $X$.

1. The action of $W_X$ on $BT_X$ induces representations

$$W_X \to \text{Aut}(H^2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}) \cong \text{Gl}(n, \mathbb{Q}_p^\wedge)$$

and

$$W_X \to \text{Aut}(H_2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}) \cong \text{Gl}(n, \mathbb{Q}_p^\wedge)$$

which are monomorphisms whose images are generated by pseudoreflections.

2. The map $H^*_\mathbb{Q}_p(BX) \to H^*_\mathbb{Q}_p(BT_X)^{W_X}$ is an isomorphism.

2.9 Normalizers, $p$–normalizers of maximal tori and $p$–toral Sylow subgroups: Let $i : T_X \to X$ be a maximal torus of a $p$–compact group $X$. Again we think of $BT_X \to BX$ as being a fibration. The Weyl space $\mathcal{W}_X$ acts on $BT_X$ via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to HE(BT_X)$ where $HE(BT_X)$ denotes the monoid of all self equivalences of $BT_X$. Passing to classifying spaces establishes a map $BW_X \to BHE(BT_X)$ which can be thought of as being a classifying map of the fibration $BT_X \to BN(T_X) \to BW_X$. The total space gives the classifying space of the normalizer $N(T_X)$ of $T_X$. This is always a finite extension of the $p$-compact torus $T_X$.

Let $W_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a $p$–Sylow subgroup $W_p$ of $W_X$. The restriction of the above construction to $W_p$ gives the classifying space of the $p$–normalizer $N_p(T_X)$.

Since the action of $W_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$. The $p$–normalizer fits into an exact sequence $T_X \to N_p(T_X) \to W_p$ and is therefore a $p$-compact toral group. The restriction $N_p(T_X) \to X$ is a monomorphism [7, 9.9] and is a $p$–toral Sylow subgroup of $X$, i.e. every $p$–toral toral subgroup $P \to X$ of $X$ is subconjugate to $N_p(T_X)$. 
2.10 Proposition. Let \( N_p(T_X) \to X \) be the \( p \)-toral Sylow subgroup of a \( p \)-compact group \( X \). Then, the following holds:

1. Every \( p \)-compact torus subgroup \( P' \to X \) of \( X \) is subconjugated to \( N_p(T_X) \).
2. The induced map \( H^*(BX, \mathbb{F}_p) \to H^*(BN_p(T_X); \mathbb{F}_p) \) is a monomorphism.

Proof. The Euler characteristic of \( X/N_p(T_X) \) is coprime to \( p \) [7, proof of Theorem 2.3]. Therefore, part (1) follows from [7, 2.14] and part (2) from [7, 9.8]. \( \square \)

We also call \( N_p(T_X) \) the \( p \)-normalizer of \( T_X \).

2.11 Simply connected and simple \( p \)-compact groups: A \( p \)-compact group \( X \) is called simply connected, if \( X \) is simply connected, and \( X \) is called simple, if the associated representation \( W_X \to \text{Aut}(H_2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}) \) is irreducible. By [5], this is equivalent to the fact that the associated complex representation is irreducible. The notion of simple is motivated by the classical situation. For a compact connected Lie group \( G \), the representation \( W_G \to H_2(BT_G; \mathbb{C}) \) is irreducible if and only if \( G \) is simple.

2.12 Elementary abelian subgroups

2.13 Let \( X \) be a \( p \)-compact group, and let \( i : P \to X \) be the \( p \)-toral Sylow subgroup. Let \( j : E \to X \) a monomorphism of an elementary abelian group \( E \) into \( X \). By 2.10, the subgroup \( E \) is subconjugate to \( P \) via a homomorphism \( j_0 : E \to P \). Such a subconjugation is called special if \( C_P(j_0(E)) \to C_X(j(E)) \) is a \( p \)-toral Sylow subgroup.

2.13 Lemma.

1. For every monomorphism \( j : E \to X \), there exists a special subconjugation \( j_1 : E \to P \).
2. Let

\[
\begin{array}{c}
E \\
\downarrow j \\
X \end{array} \longrightarrow 
\begin{array}{c}
E_1 \\
\downarrow j_1 \\
\end{array}
\]

be a diagram commuting up to conjugation. and let \( j' : E \to P \) be a special subconjugation of \( j \). Then, there exists a special subconjugation \( j'_1 : E_1 \to P \) such that

\[
\begin{array}{c}
E \\
\downarrow j' \\
P \end{array} \longrightarrow 
\begin{array}{c}
\end{array}
\]

be a diagram commuting up to conjugation.

Proof. Let \( P' \to C_X(j(E)) \) be the \( p \)-toral Sylow subgroup. Because \( j(E) \) is central in \( C_X(j(E)) \), it is also contained in \( P' \) [12, 4.]. By Proposition 2.10, there exists a subconjugation \( k : P' \to P \). The composition \( j' : E \to P' \to P \) is a subconjugation of \( j \) into \( P \) and \( C_P(j'(E)) \cong P' \). Hence, \( j' \) is special. This proves part (1).
Let $E_2 \subset E_1$ be a complement of $E$, i.e. $E_1 \cong E \oplus E_2$. Taking the adjoint of $j_1 : E \oplus E_2 \to X$ establishes map $j_2 : E_2 \to C_X(j(E))$. Because $j'$ was special, $P' := C_P(j'(E)) \to C_X(j(E))$ is a $p$–toral Sylow subgroup. Applying part (1) yields a special subconjugation $j'_2 : E_2 \to P'$, and passing back to maps into $P$ and $X$ proves part (2). \[\square\]

The elementary abelian subgroups of a $p$–compact group $X$ build a category $A_p(X)$, which we will call the Quillen category of $X$. An object of $A_p(X)$ is a monomorphism $E \xrightarrow{i_E} X$, where $E$ is a nontrivial elementary abelian group, and a morphism is a triangle

$$
\begin{array}{ccc}
E_1 & \rightarrow & E_2 \\
\downarrow & & \downarrow \\
X & \rightarrow & \\
\end{array}
$$

which is commutative up to conjugation. For the functor

$$
\phi : A_p(X) \to \mathcal{Top} : (E \to X) \mapsto BC_X(E)
$$

there exists a map

$$
\Phi : \text{hocolim} \phi \to BX.
$$

Let $\Phi^* : A_p(X) \to \text{Ab}$ be the functor given by $\Phi^*(E \to X) := H^*(BC_X(E); \mathbb{F}_p)$. The following theorem is a collection of results of [8, 8.1 and the proof, 8.2].

**2.14 Theorem** [8]. *Let $X$ be a $p$–compact group.*

1. The category $A_p(X)$ is mod–$p$ acyclic, i.e. for the constant functor $F_{\mathbb{Z}/p}$ taking as value $\mathbb{Z}/p$, we have

$$
\lim_{A_p(X)}^i F_{\mathbb{Z}/p} = \begin{cases} 
0 & \text{for } i > 0 \\
\mathbb{Z}/p & \text{for } i = 0
\end{cases}
$$

2. The map $\Phi : \text{hocolim} \phi \to BX$ is a homotopy equivalence.

3. We have

$$
\lim_{A_p(X)}^i \Phi^* = \begin{cases} 
0 & \text{for } i > 0 \\
\mathbb{Z}/p & \text{for } i = 0
\end{cases}
$$

We finish this section with a result, necessary for later purpose.

**2.15 Proposition.** *Let $X$ be a $p$–compact group and $i : P \to X$ a $p$–toral Sylow subgroup. Then $BZ(P) \to \text{map}(BP, BP)_{id} \to \text{map}(BP, BX)_{Bi}$ is a sequence of homotopy equivalences.*

*Proof.* The first equivalence follows from [8, 1.2]. Let $i : T \to X$ be a maximal torus of $X$. This inclusion factors over $i : T \to P$. (Confusing notations we denote all inclusions by $i$.) If we think a of the map $BT \to BP$ as a fibration respectively
a covering, then we get a free action of $P := P/T$ on $BT$. By [9, 5.1], there exists a natural transformation of functors $\text{map}(BP, ) \to \text{map}(BT, )^{h\mathcal{P}}$, which, for every target, is an equivalence. The map $\text{map}(BT, BP)_{Bi} \to \text{map}(BT, BX)_{Bi}$ is an equivalence because $T$ is a maximal torus and because $C_X(T)$ is subconjugate to $P$ (Proposition 2.10). Moreover, this map is $\mathcal{P}$-equivariant, and induces therefore an equivalence $\text{map}(BT, BP)^{h\mathcal{P}}_{Bi} \to \text{map}(BT, BX)^{h\mathcal{P}}_{Bi}$. The component of $\text{id} : BP \to BP$ is clearly mapped onto the component $Bi : BP \to BX$, which finishes the proof of the statement. □

For further definitions, background and explanations, in particular for the motivation of these notions, we refer the reader to [7,8] and [12].

3. Low dimensional cohomology groups of pseudo reflection groups.

The main result of this chapter states vanishing results for some cohomology groups of pseudo reflection groups at odd primes. Let $U$ be a finite dimensional vector space over the $p$-adic rationals. A faithful representation $\rho : W \to GL(U)$ of a finite group $W$ is called a pseudo reflection group, if the image $\rho(W)$ is generated by pseudo reflections. The representation is called a honest real reflection group, if $\rho(W)$ is generated by honest reflections and if the representation is already defined over $\mathbb{Q}$. A finite group $W$ is a pseudo reflection group, if there exists a representation making $W$ to a pseudo reflection group. A pseudo reflection group is called irreducible, if the associated representation is irreducible.

3.1 Proposition. Let $W$ be a pseudo reflection group. Then, for an odd prime $p$, the homology and cohomology groups $H_1(W, \mathbb{Z}_p^\wedge)$, $H_1(W; \mathbb{F}_p)$, $H_2(W, \mathbb{Z}_p^\wedge)$, $H_2(W, \mathbb{F}_p)$, $H_1(W, \mathbb{Z}_p^\wedge)$, $H^1(W; \mathbb{Z}_p^\wedge)$, $H^2(W; \mathbb{Z}_p^\wedge)$, $H^3(W, \mathbb{F}_p)$, $H^3(W; \mathbb{Z}_p^\wedge)$ all vanish.

Proof. Because $W$ is a finite group, it is sufficient to look at $H_1(W; \mathbb{Z}_p^\wedge)$ and $H^2(W; \mathbb{Z}/p)$. Then, for the other groups, the statement follows by universal coefficient theorems. Because every pseudo reflection group splits into a product of irreducible pseudo reflection groups and because of the Künneth-formula we also can assume that $W$ is irreducible.

The group $H_1(W, \mathbb{Z})$ is isomorphic to the abelinization of $W$. Because $W$ is generated by elements of order dividing $p - 1$, this is a finite abelian group of order coprime to $p$. Thus, $H_1(W; \mathbb{Z}_p^\wedge) = 0$.

The cohomology group $H^2(W; \mathbb{F}_p)$ classifies central extensions of the form $\mathbb{Z}/p \to N \to W$. We want to show that every central extension of this form splits.

First we consider the case of an honest real reflection group. Let $R \subset W$ denote the set of reflections of $W$ and $R_B \subset R$ a minimal set of generators of $W$. Let $\sigma_1, ..., \sigma_n$ denote the elements of $R_B$. Then there exists integers $m_{i,j}$ such that $W \cong \langle \sigma_1, ..., \sigma_n \rangle / \langle (\sigma_i, \sigma_j)^{m_{i,j}} : 1 \leq i, j \leq n \rangle$ is the quotient of the free group generated by the elements of $R_B$ dividing out only relations of the form $(\sigma_i, \sigma_j)^{m_{i,j}}$ [3, chap. 5]. Let $\hat{s} : W \to N$ be a set theoretic section. Then we define $s : R_B \to W$ by $s\sigma := \hat{s}(\sigma)^p$. Because the extension is central, the map $s$ does not depend on $\hat{s}$. We want to show that $s$ can be extended to group theoretic section $W \to N$. That is to say that $s$ has to satify the relation $(s(\sigma_i)s(\sigma_j))^{m_{i,j}} = 1$ for all $1 \leq i, j \leq n$.\n
Let $W' \subset W$ be the subgroup generated by $\sigma_1$ and $\sigma_2$, and let $N' \subset N$ be the subgroup defined by the pull back diagram

$$
\begin{array}{cccccc}
\mathbb{Z}/p & \longrightarrow & N' & \longrightarrow & W' \\
\text{Z/p} & \longrightarrow & N & \longrightarrow & W
\end{array}
$$

By the classification list of pseudo reflection groups [5] there exist only three reflection groups generated by two honest reflections, namely $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\Sigma_3$ and the dieder group $D_{12}$ of 12 elements. A short calculation shows that in all cases $H^2(W', \mathbb{Z}/p) = 0$. For $\mathbb{Z}/2 \times \mathbb{Z}/2$ this is obvious, for $\Sigma_3$ one uses the fact that $\Sigma_3/\mathbb{Z}/3 \cong \mathbb{Z}/2$, and for $D_{12}$ one observes that $\Sigma_3 \subset D_{12}$ is a subgroup of index 2. Therefore there exists a group thoeretic section $s' : W' \rightarrow N'$. By the independence of $s$, for $i = 1, 2$, we have $s(\sigma_i) = s'(\sigma_i)^p = s'(\sigma_i^{p^2}) = s'(\sigma_i)$. This shows that $s$ satisfies the relation for $\sigma_1$ and $\sigma_2$, and analogously, all relation. Hence, the map $s$ can be extended to a group theoretic section, the sequence $\mathbb{Z}/p \rightarrow N \rightarrow W$ splits, and $H^2(W; \mathbb{Z}/p) = 0$ for honest real reflection groups.

If $W$ is a pseudo reflection group, then by the classification list [5], the order of $W$ is coprime to $p$ or $W$ is one of the groups of number 1, 2b for $p = 3, 28, 35, 36$ or 37, which all describe honest real reflection groups or belongs to one of the numbers 2a, 12, 29, 31 or 34. We refere here to the numbering of [5]. In all the latter cases we only have to consider one prime. The following table indicates this prime and denotes a subgroup of index coprime to $p$. Morover, the subgroup is a honest real reflection group.

<table>
<thead>
<tr>
<th>no.</th>
<th>grouporder</th>
<th>prime</th>
<th>subgroup</th>
<th>index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a</td>
<td>$r \cdot m^{n-1} \cdot n!$</td>
<td>$p \leq n$</td>
<td>$\Sigma_n$</td>
<td>$r \cdot m^{n-1}$</td>
</tr>
<tr>
<td>12</td>
<td>48</td>
<td>$p = 3$</td>
<td>$\Sigma_3$</td>
<td>8</td>
</tr>
<tr>
<td>29</td>
<td>$2^8 \cdot 3 \cdot 5$</td>
<td>$p = 5$</td>
<td>$\Sigma_5$</td>
<td>$2^5$</td>
</tr>
<tr>
<td>31</td>
<td>$64 \cdot 6!$</td>
<td>$p = 5$</td>
<td>$\Sigma_5$</td>
<td>$3 \cdot 2^7$</td>
</tr>
<tr>
<td>34</td>
<td>$108 \cdot 9!$</td>
<td>$p = 7$</td>
<td>$\Sigma_7$</td>
<td>$2^5 \cdot 3^5$.</td>
</tr>
</tbody>
</table>

In no. 2a, the number $m$ divides $p - 1$ and $r$ divides $m$. For no. 2a the information about the subgroup might be found in [17] and for all other numbers in [1]. Therefore, in all these cases, $H^2(W; \mathbb{Z}/p)$ also vanishes. □

The following proposition is needed in Section 6.

3.2 Proposition. Let $p$ be an odd prime. For $i = 1, 2$ let $W_i \rightarrow Gl(U_i)$ be a pseudo reflection group and $L_i \subset U_i$ be a $[W_i]$–sublattice.

(1) The map

$$H^3(W_1; L_1) \rightarrow H^3(W_1; L_1 \times L_2)$$

is an isomorphism.

(2) The map

$$H^3(W_1; L_1) \oplus H^3(W_2; L_2) \rightarrow H^3(W_1 \times W_2; L_1 \times L_2)$$

is an isomorphism.
Proof. Part (2) is a consequence of (1). By Proposition 3.1 we have
\[ H^2(W_1; H^1(W_2; L_1)) = H^1(W_1; H^2(W_2; L_1)) = H^0(W_1; H^3(W_2; L_1)) = 0. \]
Hence, part (1) follows from the Hochschild–Serre spectral sequence for calculating $H^3(W_1 \times W_2; L_1)$. \(\square\)


In [8] is set up a induction principle for proving statements about $p$–compact group, which we will use in this and the next section and which we recall here.

The cohomological dimensio of a $\mathbb{F}_p$–finite space is given by the maximal degree of the nonvanishing mod–$p$ cohomology groups. For two $p$–compact groups $X$ and $Y$, we say that $X < Y$ if the cohomological dimension of $X$ is smaller than the one of $Y$ or if both have the same cohomological dimension but $\pi_0(Y)$ has a smaller order than $\pi_0(X)$.

4.1 Definition. A class $C\ell$ of $p$–compact groups is called saturated if it satisfies the following 5 conditions:

(1) If $X \in C\ell$ and $Y \cong X$, then $Y \in C\ell$, i.e. $C\ell$ is closed under equivalences.
(2) The trivial $p$–compact group belongs to $C\ell$.
(3) If the identity component $X_0$ of $X$ is in $C\ell$ and if any $p$–compact group $Y$, such that $Y < X$, is in $C\ell$ then $X$ also belongs to $C\ell$.
(4) If $X$ is connected, and if $X/Z(X)$ is in $C\ell$, then $X$ is in $C\ell$.
(5) If $X$ is connected and centerfree, and $Y \in C\ell$ for all $p$–compact groups such that $Y < X$, then $X \in C\ell$.

4.2 Theorem [8, 9.2]. Any saturated class of $p$–compact groups contains all $p$–compact groups.

4.3 Remark. Our definition of a saturated class is not exactly the same as Dwyer and Wilkerson give (there is minor difference in (3)), but their argument also works in our situation to prove Theorem 4.2.

Proof of Theorem 1.4. We want to prove the statement by the induction principle. That is we have to show that the class $C\ell$ of all $p$–compact groups satisfying (1) and (2) is a saturated class.

One direction of part (1) is obvious. Let $f : BX \to A$ be a map from the classifying space of a $p$–compact group into an almost $B\mathbb{Z}/p$–local space. If $f$ is null homotopic, the restriction $f|_{BP}$ is nullhomotopic.

If $X$ and $Y$ are isomorphic $p$–compact groups then they have isomorphic $p$–toral Sylow subgroups $P_X$ and $P_Y$ fitting into a homotopy commutative diagram

\[
\begin{array}{ccc}
BP_X & \longrightarrow & BX \\
\approx & \downarrow & \approx \\
BP_Y & \longrightarrow & BY
\end{array}
\]

Hence, $X$ satisfied the statement if and only if $Y$ does. This is Step (1).
The trivial group obviously satisfies the statement, which is Step 2. Going a little aside trip, we want to prove the statement for \( p \)-compact toral groups. The first part is obvious. Let \( P \) be a \( p \)-compact toral group. The loop space \( \Omega A \) is \( B\mathbb{Z}/p \)-local [15, §1]. The proof of the Sullivan conjecture for \( p \)-compact groups [8, 9.3] does not only apply to \( \mathbb{F}_p \)-finite spaces, but also to \( B\mathbb{Z}/p \)-local spaces without any change of the arguments. Therefore, for the mapping space \( map_*(BP, A)_{const} \) of pointed maps, we have \( \Omega map(BP, A)_{const} \simeq map_*(BP, \Omega A) \) is contractible. Hence, the map \( A \to map(BP, A)_{const} \) is an equivalence.

Now let \( P \to X \) be the \( p \)-toral Sylow subgroup of a \( p \)-compact group \( X \). The composition \( P \to X \to \pi_0(X) \) is an epimorphism [12, 3.8]. This establishes a commutative diagram of fibrations

\[
\begin{array}{ccc}
BP_0 & \longrightarrow & BP \\
\downarrow & & \downarrow \\
BX_0 & \longrightarrow & BX
\end{array}
\]

and \( P_0 \to X_0 \) is a \( p \)-toral Sylow subgroup of \( X_0 \) [12, 3.9]. Let \( f : BX \to A \) be a map. If \( f|_{BP} \simeq \text{const} \) and if \( X_0 \in Cl \), then \( f|_{BX_0} \simeq \text{const} \).

Thinking of \( BX_0 \to BX \) as a fibration, the space \( BX_0 \) carries a free \( \pi := \pi_0(X) \) action. Taking the trivial action on \( A \), the map \( A \to map(BX_0, A)_{const} \) is an equivalence, \( \pi \)-equivariant and establishes equivalences

\[
(* \quad map(BX_0, A)_{const} \simeq (map(BX_0, A)_{const})^{h_\pi} \simeq map(BX, A)_{f|_{BX_0} = \text{const}}.
\]

Here, \( A^{h_\pi} \) denotes the homotopy fixed-point set of the \( \pi \)-action on \( A \) (see [9]). Let \( \overline{f} : B\pi \to A \) be the map corresponding to \( f : BX \to A \). Applying the same trick to \( map(BP, A) \) allows to calculate the map \( \overline{f} \). In this case, \( \overline{f} \) corresponds to the constant map \( \text{const} : BP \to A \), and hence, \( \overline{f} \) is homotopic to the constant map. This shows that \( f \simeq \text{const} \), which is part (1). The second part follows from the equivalences in (*)). Therefore, we have \( X \in Cl \). This is the third step in the induction process.

For any connected \( p \)-compact group \( X \) with \( p \)-toral Sylow subgroup \( P \to X \), there exists a diagram of fibrations

\[
\begin{array}{ccc}
BZ & \longrightarrow & BP \\
\downarrow & & \downarrow \\
BZ & \longrightarrow & BX
\end{array}
\]

where \( Z = Z(X) \) is the center of \( X \) and where \( \overline{P} \to \overline{X} := X/Z \) is a \( p \)-toral Sylow subgroup of the centerfree group \( \overline{X} \). Actually, both fibrations are principal bundles, because \( Z \to P \) and \( Z \to X \) are central [2, 7.2]. Let \( f : BX \to A \) be a map such that \( f|_{BP} \simeq \text{const} \). Then \( f|_{BZ} \simeq \text{const} \) and \( A \to map(BZ, A)_{const} \). In this situation we can apply a lemma of Zabrodsky [19] (see also [10]), which tells
us that in the diagram

\[
\begin{array}{ccc}
\text{map}(BP, A)_{f'}|_{BZ} \simeq \text{const} & \overset{\sim}{\rightarrow} & \text{map}(B\overline{P}, A) \\
\downarrow & & \downarrow \\
\text{map}(BX, A)_{f'}|_{BZ} \simeq \text{const} & \overset{\sim}{\rightarrow} & \text{map}(BX, A)
\end{array}
\]

the horizontal lines are homotopy equivalences. Let \(\overline{f} : BX \rightarrow A\) be the map corresponding to \(f\). The top row implies that \(\overline{f}|_{B\overline{P}} \simeq \text{const}\). Thus, by the assumptions, \(\overline{f}\) is also nullhomotopic as well as \(f\). This finishes the proof of the fourth step in the induction.

Now let \(X\) be a centerfree connected \(p\)-compact group. Then, for an object \(j : E \rightarrow X\), the centralizer \(C_X(j(E))\) is smaller than \(X\). We choose a special subconjugation \(j' : E \rightarrow P\) of \(j\) (Lemma 2.13). Then, \(C_P(j'(E)) \rightarrow C_X(j(E))\) is a \(p\)-toral Sylow subgroup and \(C_P(j'(E))\) is subconjugate to \(P\). Let \(f : BX \rightarrow A\) be a map such that \(f|_{BP} \simeq \text{const}\). Then \(f|_{BC_P(j'(E))}\) and \(f|_{BC_X(j(E))}\) are also null homotopic and \(A \rightarrow \text{map}(BC_X(j(E)), A)_{\text{const}}\) is an equivalence (by induction hypothesis). Moreover, these maps are compatible with all morphisms in the Quillen category \(A_p(X)\). In the sequence

\[
\begin{aligned}
\text{holim}(A) \rightarrow & \text{holim}(\text{map}(BC_X(E), A)_{\text{const}}) \\
& \overset{\sim}{\leftarrow} \text{holim}(\text{map}(BC_X(E), A)_{\text{const}}) \\
& \overset{\sim}{\rightarrow} \text{map}(\text{holim}(BC_X(E)), A)_F \rightarrow \text{map}(BX, A)_F \\
& \overset{\sim}{\rightarrow} \text{map}(BX, A)_{\text{const}}
\end{aligned}
\]

all maps are equivalences; the first because of the above argument, the second by general nonsense and the third because of Theorem 2.14. By \(F\) we denoted the set of all homotopy classes of maps \(f' : BX \rightarrow A\) such that for every object \(E \rightarrow X\) of \(A_p(X)\) the restriction \(f'|_{BC_X(E)}\) is null homotopic. Using the Bousfield–Kan spectral sequence for calculating the first expression, Theorem 2.14 shows that \(\text{holim}_{A_p(X)} A \simeq A\), that \(F = \{\text{const}\}\), that \(f \simeq \text{const}\) and that \(A \rightarrow \text{map}(BX, A)_{\text{const}}\) is a homotopy equivalence. This finishes the proof of the last induction step as well as the proof of the theorem.

**4.4 Corollary.** Let \(X\) be a \(p\)-compact group with \(p\)-toral Sylow subgroup \(P \rightarrow X\). Let \(F \rightarrow E \rightarrow BX\) be a fibration, such that \(F\) is \(\mathbb{F}_p\)-finite, \(p\)-complete and a loop space. If the restriction to \(BP\) induces a fibration, fiber homotopy equivalent to the trivial fibration, then the fibration itself is fiber homotopically trivial.

**Proof.** By [18], the fibration is classified by a map \(BX \rightarrow BHE(F)\). The restriction to \(BP\) is null homotopic. The space \(BHE(F)\) is connected, \(p\)-complete and almost \(B\mathbb{Z}/p\)-local. Thus, the statement follows from Theorem 1.4.

A similar statement as the last corollary is also true if the fiber is the classifying space of a \(p\)-compact torus.
4.5 Proposition. Let $T$ be a $p$-compact torus, let $X$ be a $p$-compact group with $p$-toral Sylow subgroup $P \to X$, and let $BT \to E \to BX$ be a fibration. If the restriction to $BP$ induces a fiber homotopic trivial fibration, then the fibration itself is fiber homotopic trivial.

The proof is analogously as for Corollary 4.4, but based on the lemma

4.6 Lemma. Let $T$ be a $p$-compact torus, and let $X$ be a $p$-compact group with $p$-toral Sylow subgroup $P \to X$. If, for a map $f : BX \to BHE(BT)$, the restriction $f|_{BP}$ is null homotopic, then $f$ is null homotopic.

Proof. Let $P_0 \to X_0$ be the $p$-toral Sylow subgroup of the component of the unit. In the diagram

\[
\begin{array}{c}
\text{BSHE}(BT) \simeq B^2T \\
BP \xrightarrow{f} BX \xrightarrow{BHE(BT)} \\
BP/P_0 \simeq B\pi_0(BX) \xrightarrow{\bar{f}} B\pi_0(HE(BT))
\end{array}
\]

the map $\bar{f}$ is null homotopic. This follows because

$\pi := \pi_0(HE(BT)) \simeq \text{Gl}(H_2(BT; \mathbb{Z}_p))$ is a discrete group, because therefore $B\pi$ is almost $B\mathbb{Z}/p$-local, and because we can apply the Zabrodsky lemma in this case as in the proof of Theorem 1.4. Homotopy classes of lifts of the composition $BP \to B\pi_0(X) \to B\pi$ and $BX \to B\pi_0(X) \to B\pi$ to $BHE(BT)$ are classified by the obstruction groups $H^3(BP, \pi_3(BT))$ or $H^3(BX, \pi_3(BT))$. The existence of a transfer as a stable map [6] shows that the inclusion $BP \to BX$ induces a monomorphism $H^*(BP, M) \to H^*(BX, M)$ for any systems of coefficients. Hence, the map $f$ is also null homotopic, which proves the statement.

5. Splittings of $p$-compact groups.

In this chapter we want to prove the main "technical" theorem which allows several interesting applications to $p$-compact groups. Let $X$ be a $p$-compact group with normalizer $N_X$ of the maximal torus $T_X \to X$ of $X$. Let $N_X \cong N_1 \times N_2$ be a splitting into two factors. For $i = 1, 2$, we want to construct a subgroup $Y_i \to X$ of $X$ such that this inclusion induces an isomorphism $N_{Y_i} \cong N_i$ in a way which we will make precise working out the construction.

For any $p$-compact group $X$ there exists a fibration

$BX_0 \to BX \to B\pi$

where $X_0$ denotes the connected component of the unit and $\pi$ the group of the components. The composition $BN_X \to BX \to B\pi$ factors over $BW_X \to B\pi$, which is induced by a homomorphism $W_X \to \pi$. The kernel is given by the Weyl group $W_{X_0}$ [12, 3.8], which is a pseudo reflection group. If $N_X \cong N_1 \times N_2$, the
Weyl group $W_X \cong W_1 \times W_2$ also splits into two factors as well as the maximal torus $T_X \cong T_1 \times T_2$. Every pseudo reflection $\sigma \in W_{X_0}$ is contained either in $W_1$ or in $W_2$. Let $W'_i \subset W_{X_0}$ be the subgroup generated by all pseudo reflection contained in $W_i$, and let $N'_i$ be the counterimage of $W'_i$ in $N_i$. Then, we have splittings $W_{X_0} \cong W'_1 \times W'_2$, $N_{X_0} \cong N'_1 \times N'_2$ and $\pi \cong \pi_1 \times \pi_2$. Now we define $Y' =: C_X(T_2)$. This is a subgroup of maximal rank. For later use we note the following lemma:

5.1 Lemma. $N_{Y'} = C_{N_X}(T_2) = N_1 \times C_{N_2}(T_2)$.

Proof. The Weyl group $W_{Y'}$ is given by all elements of $W_X$ acting trivially on $T_2$ [8, 7.6]. Hence, $W_{Y'} = W_1 \times W_2$, where $\tilde{W}_2 = W_{Y'} \cap W_2 \subset W_2$. Let $q : N_X \to W_X$ be the projection. Then, $N'_{Y'} = q^{-1}(W_{Y'}) = C_{N_X}(T_2) = N_1 \times C_{N_2}(T_2)$. For the last equivalence, there is a remark in order. The loops spaces $N_X$ and $N_2$ are not $p$–compact groups. But the types of centralizers in question are studied in [14, 3.7]. That result implies that $C_{N_X}(T_2)$ as well as $C_{N_2}(T_2)$ are finite extensions of a $p$–compact tori. \hfill \Box

By the above lemma, $N_1 \subset N_{Y'}$, and $\pi_1 \subset \pi_0(Y')$. Let $Y''$ be defined by the pull back diagram

\[
\begin{array}{ccc}
BY'' & \rightarrow & BY' \\
\downarrow & & \downarrow \\
B\pi_1 & \rightarrow & B\pi_1 \times B\pi_2
\end{array}
\]

By construction, we have $N_{Y''} = N_1 \times T_2$. Applying the functor $map(BT_2, )$ shows that $T_2 \subset Y''$ is central, and dividing out this central subgroup gives a diagram of fibrations

\[
\begin{array}{ccc}
BT_2 & \rightarrow & BN_1 \times BT_2 \\
\downarrow & & \downarrow \\
BT_2 & \rightarrow & BY'' \\
\downarrow & & \downarrow \\
BN_1 & \rightarrow & B(Y''/T_2) =: BY
\end{array}
\]

Both lines are principal fibration [2, 7.2]. Thus, the lower fibration is classified by a map $BY \to B^2T_2$, which is determined by a cohomology class in $H^3(BY; \pi_3(B^2T_2))$. The restriction of this map to $BN_1$ is homotopically trivial, and the induced map $H^3(BY; \pi_3(B^2T_2)) \to H^3((BN_1; \pi_3(B^2T_2))$ is a monomorphism (Proposition 2.10). Hence, the lower row is equivalent to the trivial fibration and we can split $BY'' \simeq BY \times BT_2$ in such a way that we get a commutative diagram

\[
\begin{array}{ccc}
BN_1 \times BT_2 \\
BY \times BT_2 & \rightarrow & BY''
\end{array}
\]

The composition $BY \rightarrow BY'' \rightarrow BY$ is an equivalence. Hence, $N_1 \cong N_{Y'} \subset Y$ is the normalizer of the maximal torus. The $p$–compact subgroup $Y \subset Y'' \subset X$ is the one which we associate with $N_1$. 

5.2 Lemma. Let $X$ be a connected $p$–compact group. Let $N_X \cong N_1 \times N_2$ split into two factors, and let $Y_1 \subset X$ be the subgroup associated to $N_1$.

1) We have $Y_1 \times T_2 \cong C_X(T_2)$.

2) There exists an isomorphism $Z(X) \cong Z(Y_1) \times Z(Y_2)$ making the diagram

\[
\begin{array}{ccc}
Z(X) & \cong & Z(Y_1) \times Z(Y_2) \\
\downarrow & & \downarrow \\
(T_X)^W & \cong & (T_1)^W \times (T_2)^W
\end{array}
\]

commutative.

Proof. The first part follows directly from the above construction. By [8, 7.5] the center can be calculated, only knowing the normalizer of the maximal torus. This proves part (2). $\square$

Theorem 1.3 is now contained in the following statement.

5.3 Theorem. Let $X$ be a $p$–compact group such that $N_X \cong N_1 \times N_2$. For $i = 1, 2$, let $Y_i \rightarrow X$ be the subgroup associated to $N_i$. Then there exists a homotopy equivalence $j : Y_1 \times Y_2 \rightarrow X$ making the following diagram commutative up to homotopy for $i = 1, 2$

\[
\begin{array}{ccc}
N_1 \times N_2 & \rightarrow & N_X \\
\downarrow & & \downarrow \\
Y_1 \times Y_2 & \rightarrow & X \\
\downarrow_{j_i} & & \downarrow \\
Y_i & \rightarrow & X
\end{array}
\]

Proof. The proof is carried out in several steps and based on the induction principle of Dwyer and Wilkerson explained in Section 4. The proof takes the rest of this chapter. For abbreviation, we collect the assumptions of the theorem in condition (S). Let $Cl$ be the class of all $p$–compact groups satifying the statement. We want to show that $Cl$ is a saturated class.

Step 1: Let $X$ be a $p$–compact group such that $N_X \cong N_1 \times N_2$. Let $Y \rightarrow X$ be an isomorphism of $p$–compact groups. Then, the normalizer $N_Y \cong N_X \cong N_1 \times N_2$ also splits. Let $Y_i \rightarrow Y$ be the subgroup associated to $N_i$. If $Y$ satifies the theorem, the composition $Y_1 \times Y_2 \cong Y \cong X$ establishes the desired splitting of $X$. Hence, the class $Cl$ is closed under equivalences.

Step 2: Theorem 5.3 is satified by $p$–toral groups. This is obvious.

Step 3: Let us assume that the theorem is true for connected $p$–compact groups. We want to show that it is true in general.
Let $X$ be a $p$-compact group satisfying condition (S), and let $X_0$ be the component of the unit. If $T_i \to Y_i$ is a central subgroup for $i = 1, 2$, the Weyl group $W_X$ acts trivially on $T_X$, and the inclusion $C_X(T_X) \to X$ is a subgroup of maximal rank and induces an isomorphism between the Weyl groups. Moreover, both spaces fit into a diagram

\[
\begin{array}{cccc}
C_0 := C_{X_0}(T_X) & \longrightarrow & C := C_X(T_X) & \longrightarrow & \pi_0(C) \\
\downarrow & & \downarrow & & \downarrow \\
X_0 & \longrightarrow & X & \longrightarrow & W_X/W_{X_0} \cong \pi_0(X)
\end{array}
\]

Because $C_0$ is connected [12, ??], the top row is an exact sequence of $p$-compact groups and therefore $C_0$ is the connected component of the unit of $C$. The left column describes a subgroup of maximal rank and $W_C \cong W_X$. By [12, 3.11] follows that the left column is an isomorphism of $p$-compact groups. The isomorphism in the lower right corner follows from [12, 3.8], which also implies that $\pi_0(C) \cong W_C/W_{C_0}$. Thus the right column is an isomorphism of a $p$-compact groups as well as the middle vertical arrow. By 2.5, the $p$-compact group $C$ is a $p$-compact toral group and so is $X$. This case we already considered in Step 2. Hence, we can assume that $T_1 \to Y_1$ is not a central subgroup, that $C_{Y_1}(T_1)$ is smaller than $Y_1$, that the Weyl group of $C_X(T_1)$ is a proper subgroup of $W_X$ and that $C_X(T_1)$ is smaller than $X$ (if both have the same cohomological dimension, then $C_X(T_1)$ has less components than $X$).

As explained at the beginning of this section, the normalizer $N_{X_0} \cong N_1^i \times N_2^i$ also splits as well as the group $\pi := \pi_0(X) \cong \pi_1 \times \pi_2$ of the components. For the associated subgroups we have fibrations

\[BY_{i,0} \to BY_i \to B\pi_i\]

Moreover, by the hypothesis, we have a commutative diagram

\[
\begin{array}{ccc}
N_{1,0} \times N_{2,0} & \longrightarrow & N_{X_0} \\
\downarrow & & \downarrow \\
Y_{1,0} \times Y_{2,0} & \longrightarrow & X_0 \\
\downarrow & & \downarrow \ j_i \\
Y_{i,0} & \longrightarrow & 
\end{array}
\]

Here, $Y_{i,0}$ denote the component of the unit of $Y_i$ and $N_{i,0}$ the normalizer of the maximal torus of $Y_{i,0}$. We want to show that there exists an extension $Y_1 \times Y_2 \to X$ of the lower horizontal arrow.

Looking at homogeneous spaces we have a commutative diagram of fibrations

\[
\begin{array}{ccc}
Y_{2,0} & \longrightarrow & Y_2 & \longrightarrow & \pi_2 \\
\downarrow & \cong & \downarrow & \longrightarrow & \downarrow \\
X_0/Y_{1,0} & \longrightarrow & X/Y_1 & \longrightarrow & 
\end{array}
\]
which implies that the vertical middle arrow is a homotopy equivalence. Moreover, this map is given by the composition \( Y_2 \to X \to X/Y_1 \).

Let \( P_i \to Y_i \) be the the \( p \)-toral Sylow subgroup of \( Y_i \). Taking pullbacks induces a diagram of fibrations

\[
\begin{array}{ccc}
Y_2 & \longrightarrow & Y_2 \\
\downarrow & & \downarrow \\
E_P & \longrightarrow & E \\
\downarrow & & \downarrow \\
BP_1 & \longrightarrow & BY_1 \\
& & \downarrow \\
& & BX
\end{array}
\]

\((*)\)

The left column establishes a proxy action of \( P_1 \) on \( Y_2 \). For the definition of a proxy action we refer the reader to [7, §10]. Obviously, the induced fibrations have sections \( s_P : BP_1 \to E_P \) and \( s : BY_1 \to E \) which come from the lift \( BY_1 \to BY_1 \). Applying the functor \( \text{map}(BP, \ ) \) to the left and right fibration yields a pull back diagram

\[
\begin{array}{ccc}
(Y_2^{hP_1})_{s_P} & \longrightarrow & (Y_2^{hP_1})_s \\
\downarrow & & \downarrow \\
\text{map}(BP_1, E_P)_{s_P} & \longrightarrow & \text{map}(BP_1, BY_1)_{Bk_1} \\
\downarrow & & \downarrow \\
\text{map}(BP_1, BY_1)_{Bk_1} & \longrightarrow & \text{map}(BP_1, BX)_{Bj_1Bk_1}
\end{array}
\]

\((**)*\)

where \( Y_2^{hP_1} \) denotes the homotopy fixed point set of the proxy action of \( P_1 \) on \( Y_2 \) (again for definitions see [7, §10]. In the fibers we only take those components of the homotopy fixed–point sets which belong to the associated sections. The map \( BZ(P_1) \to \text{map}(BP_1, BY_1)_{Bk_1} \) is an equivalence (Proposition 2.15). Now, let \( \overline{P}_1 \) denote the group of components of \( P_1 \). Replacing \( BT_1 \) by \( \overline{BT}_1 \) to transform \( BT_1 \to BP_1 \) into a fibration, i.e into a covering, we get an action of \( \overline{P}_1 \) on \( \overline{BT}_1 \). By [9], we have an equivalence of functors \( \text{map}(BP_1, \ ) \simeq \text{map}(\overline{BT}_1, )^{h\overline{P}_1} \). This allows to calculate the mapping space in the lower right corner of the above pull back diagram. By Lemma 5.1, we have \( N_{C_X(T_1)} = \hat{N}_1 \times \hat{N}_2 \), where \( \hat{N}_1 := C_{N_{Y_1}}(T_1) \simeq C_{Y_1}(T_1) \) is a \( p \)-compact group. The isomorphism follows because \( T_1 \to Y_1 \) is a maximal torus. Moreover, the centralizer \( C_X(T_1) \) is smaller than \( X \). Hence we can apply the induction hypothesis to establish a commutative diagram

\[
\begin{array}{ccc}
B\hat{N}_1 \times BN_{Y_2} & \cong & BN_{C_X(T_1)} \\
\downarrow & & \downarrow \\
B\hat{N}_1 \times BY_2 & \cong & BC_X(T_1)
\end{array}
\]
The inclusions $Y_1 \times Y_2 \hookrightarrow \hat{N}_1 \times Y_2 \cong C_X(T_1) \to X$ induce equivalences

$$M_2 := \text{map}((\hat{B}T_1, B\hat{N}_1 \times BY_2)_{B_i} \xrightarrow{\simeq} M_1 := \text{map}((\hat{B}T_1, BY_1 \times BY_2)_{B_i}$$

which are also $\overline{\mathcal{P}}_1$–equivariant. Therefore, these equivalences induce equivalences between the homotopy fixed–point sets. The component of $M_1^{h\overline{\mathcal{P}}_1}$ related to the inclusion $P_1 \to Y_1 \to Y_2$ corresponds to the component of $M_3^{h\overline{\mathcal{P}}_1}$ related to the inclusion $P_1 \to Y_1 \to X$. Therefore, we get an equivalence

$$BZ(P_1) \times BY_2 \simeq \text{map}(BP_1, BY_1 \times BY_2)_{B_i} \simeq \text{map}(BP_1, BX)_{B_i}.$$ 

The pull back digram (**) translates now to

\[
\begin{array}{ccc}
(Y_2^{h\overline{\mathcal{P}}_1})_{sp} & \xrightarrow{\text{map}(BP_1, E_P)_{sp}} & BZ(P_1) \\
\downarrow & & \downarrow \\
BZ(P_1) & \xrightarrow{\text{map}(BP_1, BY_1)_{id}} & BZ(P_1) \times BY_2 \\
\end{array}
\]

This shows that

$$(Y_2^{h\overline{\mathcal{P}}_i})_{sp} \simeq (Y_2^{h\overline{\mathcal{P}}_i})_{s} \simeq Y_2.$$ 

Taking the adjoint we can construct a map $Y_2 \times BP_1 \to E_P$ which is a homotopy equivalence and gives a trivialization of the fibration $Y_2 \to E_P \to BP_1$. By Theorem 1.4, the fibration $Y_2 \to E \to BY_1$ is also trivial, i.e. we have a fiber homotopy equivalence $Y_2 \times BY_1 \to E$.

Applying the functor $\text{map}(BY_1, \_)$ to the middle and right columns in the diagram (*) establishes another pull back diagram

\[
\begin{array}{ccc}
\text{map}(BY_1, E)_{s} \simeq BZ_1 \times Y_2 & \xrightarrow{\text{map}(BY_1, BY_1)_{id}} & BZ_1 \\
\downarrow & & \downarrow \\
\text{map}(BY_1, BY_1)_{id} \simeq BZ_1 & \xrightarrow{\text{map}(BY_1, BX)_{Bj_i}} & \text{map}(BY_1, BX)_{Bj_i} \\
\end{array}
\]

where $Z_1 := Z(Y_1)$ denotes the center of $Y_1$. The equivalences follow from the above fiber homotopy equivalence and from [8, 1.3]. Passing to loop spaces shows that $\text{map}(BY_1, BX)_{Bj_i} =: BY_2'$ is a $p$–compact group.

The pull back diagram

\[
\begin{array}{ccc}
BZ_1 \simeq \text{map}(BY_1, BY_1)_{id} & \xrightarrow{\text{map}(BY_1, BY_1)_{id}} & BZ(P_1) \simeq \text{map}(BP_1, BY_1)_{B_i} \\
\downarrow & & \downarrow \\
BY_2' \simeq \text{map}(BY_1, BX)_{B_i} & \xrightarrow{\text{map}(BY_1, BX)_{B_i}} & BZ(P_1) \times BY_2 \simeq \text{map}(BP_1, BX)_{B_i} \\
\end{array}
\]
shows that $BZ_1 \to BY'_2$ has a section $BY'_2 \to BZ_1$ and that the homotopy fiber is equivalent to $BY_2$ (the right column allows a section). Moreover, the diagram also shows that the composition $BY_2 \to BY'_2 \to BX$ equals the inclusion $BY_2 \to BX$. Passing to adjoints shows the existence of a map $BY_1 \times BY_2 \to BX$, which also extends the map $BN_1 \times BN_2 \to BX$. This completes the proof of the third step.

**5.4 Remark.** The arguments for calculating $\text{map}(BP_1, BX)_{Bi} \simeq BZ(P_1) \times BY_2$ always work when $X$ satisfies the condition (S). We only have to assume the induction hypothesis, and that $T_1 \to Y_1$ is not central. The $p$–toral Sylow subgroup $P = P_1 \times P_2$ splits into a product. Hence, we always get a map $BP_1 \times BY_2 \to BX$, extending the composition $BP_1 \times BP_2 \to BN_1 \times BN_2 \to BX$. For the calculation of the mapping space $\text{map}(BY_1, BX)_{Bi}$, we only needed in addition that the composition $Y_2 \to X \to X/Y_1$ is a homotopy equivalence.

**Step 4:** We have to show that the theorem is true for a connected $p$–compact group if it is true for the associated centerfree $p$–compact group. Let $X$ be a connected $p$–compact group, let $Z := Z(X)$ be the center of $X$, and let $\overline{X} := X/Z(X)$ be the associated centerfree group. Let $N_X \cong N_1 \times N_2$, let $Y_1, Y_2 \to X$ be the associated subgroups, let $P_i \subset N_i \subset Y_i$ be a $p$–toral Sylow subgroup, and let $Z_i := Z(Y_i)$ denote the center of $Y_i$. The center $Z_i$ is a subgroup of $P_i$ [12, 4.3]. Because $W_i$ acts trivially on $Z_i$, the inclusion $Z_i \subset N_i$ is central [14, 3.7] By [2, 7.2], we can construct a diagram of principal bundles

$$
\begin{array}{ccccccc}
BZ_1 \times BZ_2 & \longrightarrow & BN_1 \times BN_2 & \longrightarrow & B\overline{N}_1 \times B\overline{N}_2 \\
\downarrow & & \downarrow & & \downarrow \\
BZ & \longrightarrow & BX & \longrightarrow & BX \\
\uparrow & & \uparrow & & \uparrow \\
BZ_i & \longrightarrow & BY_i & \longrightarrow & B\overline{Y}_i
\end{array}
$$

By the construction, the subgroup $\overline{Y}_i \to \overline{X}$ is associated to $\overline{N}_i$. Thus, by the hypothesis, there exists an equivalence $B\overline{Y}_1 \times B\overline{Y}_2 \to BX$ extending the map on the normalizers. The diagram of the classifying maps of the principle bundles

$$
\begin{array}{ccc}
B\overline{Y}_1 \times B\overline{Y}_2 & \longrightarrow & B^2 Z_1 \times B^2 Z_2 \\
\downarrow \simeq & & \downarrow \\
B\overline{X} & \longrightarrow & B^2 Z
\end{array}
$$

commutes up to homotopy, because the homotopy classes of the horizontal arrows are determined by cohomology classes and because the restrictions of both maps to the normalizers are equal. Taking fibers establishes the desired equivalence $BY_1 \times BY_2 \to BX$. This finishes the proof of Step 4.

**Step 5:** We have to show that the statement is true for a centerfree connected $p$–compact group, if it is satified by all smaller $p$–compact groups. The proof of
this step is even more complicate than the one of Step 2 and divided into several claims. The outline is as follows: The first major step is the construction of an equivalence between the Quillen categories of $X$ and $Y_1 \times Y_2$ (Claim 3). Let $F_X : A_p(X) \to \text{Ab}$ be the functor given by $F_X(E \to X) := H^*(BC(E); \mathbb{F}_p)$. Analogously, we define a functor $F_Y : A_p(Y_1 \times Y_2) \to \text{Ab}$. Using the equivalence of the Quillen categories, we then show that there exists a natural transformation $F_X \to F_Y$, which is an isomorphism on the objects (Claim 6). The mod-$p$ decomposition theorem of $p$-compact groups (Theorem 2.14) establishes an isomorphism $H^*(BX; \mathbb{F}_p) \cong H^*(BY_1 \times BY_2; \mathbb{F}_p)$ which is compatible with the equivalence $BN_X \cong BN_1 \times BN_2$ (Claim 7). An Eilenberg–Moore spectral sequence argument allows to calculate the fiber of $BY_1 \to BX$ which turns out to be equivalent to $Y_2$ (Claim 8). Now similar arguments as in the proof of Step 2 are applicable and establish the desired equivalence $BY_1 \times BY_2 \tilde{\to} BX$ (Claim 9).

**Claim 1:** The spaces $Y_i$ are connected.

*Proof.* This follows from Lemma 5.2 and from [12, 3.11], which says that, for connected $p$-compact groups, centralizers of tori are always connected. □

Let $i_X : P \to N_X \to X$ be the $p$–toral Sylow subgroup. Then $P = P_1 \times P_2$ splits and $i_Y : P_1 \times P_2 \to N_1 \times N_2 \to Y_1 \times Y_2$ is also the $p$–toral Sylow subgroup of $Y_1 \times Y_2$.

**Claim 2:** Let $j_1, j_2 : E \to P$ be two monomorphisms of an elementary abelian subgroup $E$ into $P$. Then $Bi_Y B_j$ and $Bi_Y B_{j_2}$ are homotopic if and only if $Bi_X B_{j_1} \simeq Bi_X B_{j_2}$.

*Proof.* We prove the statement via induction over the rank of $E$. Let us assume that $E = \mathbb{Z}/p$. Because $X$ is centerfree, the actions of $W_1$ on $T_1$ and $W_2$ on $T_2$ are nontrivial. In particular, $T_1 \to Y_1$ is not a central subgroup. The composition $Bi_Y B_j$ as well as $Bi_X B_{j_1}$ factor over $BP_1 \times BY_2$. This follows from Remark 5.4. Because $Y_2$ is connected (Claim 1), the second coordinate, as a cyclic subgroup, of both maps is subconjugate to $T_2$ (Theorem 2.6). Hence, we can assume that the second coordinate of $j_i : E \to P_1 \times P_2$ takes image in $T_2$. Analogously, we can assume that the first coordinate also takes image in $T_1$. That is to say that $j_i : E \to P_1 \times P_2$ represents $E$ as a subgroup of the torus. By [13, 4.2], the two maps $Bi_Y B_{j_1}$ and $Bi_Y B_{j_2}$ are homotopic if and only if $Bj_1$ and $Bj_2$ differ by a Weyl group element if and only if $Bi_X B_{j_1} \simeq Bi_X B_{j_2}$. This proves Claim 2 for $E = \mathbb{Z}/p$.

Now let us assume that the rank of $E$ is bigger than 1. Let $E_1 \subset E$ denote the first coordinate of $E$ and $E_2 \subset E$ a complement of $E_1$, i.e. $E \cong E_1 \oplus E_2$. By what we already proved we can assume that, for $i = 1, 2$, the restriction $j_i|_{E_1} = j_2|_{E_1} : E_1 \to P_1 \times P_2$ is a subgroup of the maximal torus. Let $j'_1 : E_2 \to C_{P_1 \times P_2}(E_1)$ be the adjoint of $j_1$. The centralizers $X' := C_X(E_1)$ and $C_Y \times Y_2(E_1)$ are subgroups of maximal rank, smaller than the centerfree group $X'$, and the normalizers of the maximal tori are given by $C_{N_X}(E_1) = N'_1 \times N'_2$ which again splits as well as $C_{P_1 \times P_2}(E_1) = P'_1 \times P'_2$. The latter centralizer is a $p$–toral Sylow subgroup of $X'$ and $Y'_1 \times Y'_2$. Moreover, $Y'_i := C_Y((i_Y \cdot j)(E_1)) \to Y_i$ is the subgroup associated to $N'_i$. Therefore, by induction hypothesis, there exist an equivalence
Y_1' \times Y_2' \rightarrow X'$ making the diagram

\[
\begin{array}{ccc}
BY_1' \times BY_2' & \rightarrow & BN_1' \times BN_2' \\
\downarrow & & \downarrow \\
BX' & & \\
\end{array}
\]

commutative up to homotopy.

The two maps $Bi_XBj_1, Bi_XBj_2 : BE_1 \times BE_2 \rightarrow BX$ are homotopic if the adjoints $BE_2 \rightarrow BX'$ are homotopic. The same is true for $Y_1 \times Y_2$. The above homotopy commutative diagram shows that the claim is true for the adjoints. This finishes the induction and the proof of Claim 2.

**Claim 3:** The Quillen categories of $X$ and $Y_1 \times Y_2$ are isomorphic, i.e. there exist a functor $A_p(Y_1 \times Y_2) \rightarrow A_p(X)$, which is an isomorphism of categories.

**Proof.** By Proposition 2.10, every elementary abelian subgroup is subconjugate to $P_1 \times P_2$. Hence the statement follows from Claim 2.

**Claim 4:** Let $j : E \rightarrow P := P_1 \times P_2$ be a monomorphism. Then, the map $j$ is a special subconjugation of $i_Y j$ if and only if it is a special subconjugation for $i_X j$.

**Proof.** Let $E \rightarrow P$ be a special subconjugation of $i_Y j$ and $E \rightarrow j' P$ a special subconjugation of $i_X j$. By Claim 2, the composition $i_Y j'$ is conjugate to $i_Y j$. Hence, the centralizer $P_X = C_P(j'(E))$, which is the $p$-toral Sylow subgroup of $C_X(i_X j(E))$, is subconjugated to the $p$-toral Sylow subgroup $P_Y = C_P(j(E))$ of $C_Y(i_Y j(E))$ and vice versa. Therefore $P_Y$ and $P_X$ are isomorphic. This proves the statement. 

**Claim 5:** Let $E \rightarrow P$ be a monomorphism, which is a special subconjugation of $i_X j$ and $i_Y j$. Then, there exists an equivalence $f_E : BC_Y(E) \rightarrow BC_X(E)$ making the diagram

\[
\begin{array}{ccc}
BC_P(E) & \rightarrow & BC_X(E) \\
\downarrow & & \downarrow \\
BC_Y(E) & \sim & BC_X(E) \\
\end{array}
\]

commutative up to homotopy.

**Proof.** Let $E_0 \cong \mathbb{Z}/p \subset E$ denote the first coordinate of $E$. By Claim 2 and Lemma 2.13 we can assume that $E_0 \rightarrow P$ is a toral subgroup and a special subconjugation of $E_0 \rightarrow P \rightarrow X$ and $E_0 \rightarrow P \rightarrow Y$. As shown in the proof of Claim 2 (using the induction hypothesis), there exists a homotopy commutative diagram

\[
\begin{array}{ccc}
BC_P(E_0) & \rightarrow & BC_X(E_0) \\
\downarrow & & \downarrow \\
BC_Y(E_0) & \sim & BC_X(E_0) \\
\end{array}
\]
Passing to adjoints and taking centralizers finishes the proof of Claim 5. □

**Claim 6:** There exists a natural transformation $F_Y \to F_X$, which is an isomorphism on objects.

**Proof.** Let $E \to P$ be a special subconjugation of $j_Y := i_Y j$ and $j_X := i_X j$. Then we define

$$
\phi_E := H^*(f_E; \mathbb{F}_p) : H^*(BC_Y(E); \mathbb{F}_p) \to H^*(BC_X(E); \mathbb{F}_p) .
$$

Let $E_1 \to E_2$ be a morphism of $A_p(Y) \cong A_p(X)$. By Lemma 2.13, for $i = 1, 2$, there exist monomorphisms $j_i : E_i \to P$, which are special lifts of $j_i,Y$ and $j_i,X$ such that the diagram

$$
\begin{array}{ccc}
BE_1 & \to & BE_2 \\
\downarrow & & \downarrow \\
BP & \to & BP
\end{array}
$$

commutes up to homotopy. Passing to centralizers gives another diagram, namely

$$
\begin{array}{ccc}
BC_Y(E_2) & \to & BC_Y(E_1) \\
\downarrow & & \downarrow \\
BC_P(E_2) & \to & BC_P(E_1) \\
\downarrow f_{E_2} & & \downarrow f_{E_1} \\
BC_X(E_2) & \to & BC_X(E_1)
\end{array}
$$

The two triangle commute up to homotopy (Claim 5) as well as the upper and the lower parallelogram. For any $p$–compact group $U$ with $p$–toral Sylow subgroup $P_U \to U$, the map $H^*(BU; \mathbb{F}_p) \to H^*(BP_U; \mathbb{F}_p)$ is an injection (Proposition 2.10). The diagonal arrows induce injections in mod-$p$ cohomology, because we always chose special subconjugation. Therefore, the square commutes at least in mod-$p$ cohomology, and the maps $\Phi_E$ establish a natural transformation $F_Y \to F_X$, which is an isomorphism on the objects. □

**Claim 7:** There exists an isomorphism

$$
\phi : H^*(BX; \mathbb{F}_p) \xrightarrow{\cong} H^*(BY; \mathbb{F}_p)
$$

of algebras over the Steenrod algebra such that the composition $H^*(BX; \mathbb{F}_p) \xrightarrow{\phi} H^*(BY; \mathbb{F}_p) \to H^*(BY_i; \mathbb{F}_p)$ equals the map $H^*(BX; \mathbb{F}_p) \to H^*(BY_i; \mathbb{F}_p)$ induced by the inclusion $BY_1 \to BX$.

**Proof.** We have the following sequence of isomorphisms:

$$
H^*(BX; \mathbb{F}_p) \cong \lim_{\to} F_X \cong \lim_{\to} F_Y \cong H^*(BY; \mathbb{F}_p) .
$$
The first and last isomorphism follow from Theorem 2.14 and the middle isomorphism from Claim 6. The second part of the statement follows from the construction of the natural transformation $F_Y \to F_X$. □

**Claim 8:** The composition $Y_2 \to X \to X/Y_1$ is a homotopy equivalence.

*Proof.* In the pull back diagram

$$
\begin{array}{ccc}
E & \to & BY_1 \\
\downarrow & & \downarrow \\
BY_2 & \to & BX \\
\end{array}
$$

we want to calculate the mod–$p$ cohomology of $E$ via the Eilenberg–Moore spectral sequence. The $E_2$ is given by

$$
E_2 = \text{Tor}_H^*(H^*(BY_1; \mathbb{F}_p), H^*(BY_2; \mathbb{F}_p))
$$

$$
\cong \text{Tor}_H^*(H^*(BY_1; \mathbb{F}_p), \mathbb{F}_p)
$$

$$
\cong \text{Tor}_H^p(\mathbb{F}_p, \mathbb{F}_p).
$$

The functor $\otimes_{\mathbb{F}_p} H^*(BY_i; \mathbb{F}_p)$ is exact. This implies the second isomorphism and also, together with Claim 7, the first isomorphism. Therefore, $E$ is mod–$p$ acyclic. Moreover $E$ is $p$–complete, because all the other involved spaces are $p$–complete. Hence, $E \cong \ast$. The map between the fibers is obviously given by the composition $Y_2 \to X \to X/Y_1$. This proves the statement. □

**Claim 9:** There exists an equivalence $BY_1 \times BY_2 \sim BX$, such that the diagram

$$
\begin{array}{ccc}
BN_1 \times BN_2 & \sim & BN_X \\
\downarrow & & \downarrow \\
BY_1 \times BY_2 & \to & BX \\
\end{array}
$$

commutes up to homotopy.

*Proof.* Using Claim 8, the same arguments as in the proof of Step 2 are applicable (see Remark 5.4). This establishes an equivalence $BY_1 \times BY_2 \to BX$ with the desired properties. In this case the proof even simplifies a little, because $BY_1$ and $BY_2$ are centerfree connected groups, which is to say that, for $i = 1, 2$, the mapping spaces $\text{map}(BY_i, BY_i)_{id} \sim BZ(Y_i)$ are contractible. □

Claim 9 finishes the induction for the proof of Step 5, which was the last step to complete the proof of Theorem 5.3. □

### 6. Simply connected and centerfree connected $p$–compact groups.

In this section we want to prove Theorem 1.2 and Theorem 1.3. We call a finite extension $N$ of a $p$–compact torus $T$ *simple* if the associated representation $N/T \to \text{Gl}(H_2(BT; \mathbb{Z}_p^\lambda) \otimes \mathbb{Q}$ is irreducible.
6.1 Lemma. Let $N$ be a finite extension of a $p$-compact torus $T$, such that $N/T$ acts as a pseudo reflection group on $H_2(BT; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$. Then, up to order, there exists at most one splitting $N \cong N_1 \times \cdots \times N_n$ into simple finite extensions of $p$-compact tori.

Proof. The existence of a splitting into simple factors shows that we have splittings $W := N/T \cong W_1 \times \cdots \times W_n$ for the quotient, $L_T := H_2(BT; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \cong L_1 \oplus \cdots \oplus L_n$ for the 2-dimensional homology of $BT$ and $T \cong T_1 \times \cdots \times T_n$ for the torus $T$ itself. By standard representation theory of pseudo reflection groups, these splittings are unique up to order. Thus, the splitting of $N$ is also unique up to order. \qed

Theorem 1.2 is contained in the following statement.

6.2 Theorem. Let $p$ be an odd prime. Let $X$ be a centerfree connected $p$-compact group. Then, the following holds:

1. The normalizer $N_X \cong N_1 \times \cdots \times N_n$ splits, up to order, uniquely into a product of simple factors. For each factor, the quotient $W_i := N_i/T_i$ acts on $H_2(BT; \mathbb{Z}_p^\wedge)$ as a pseudo reflection group.
2. The $p$-compact group $X \cong X_1 \times \cdots \times X_n$ splits into a product of simple centerfree connected $p$-compact groups.
3. For a splitting $X \cong X_1 \times \cdots \times X_n$ into simple centerfree connected $p$-compact groups, the normalizers of the maximal tori of the factors are determined by the factors of $N_X$, i.e., after reordering the factors, there exist isomorphisms $N_{X_i} \cong N_i$ such that the diagrams

$$
\begin{array}{ccc}
BN_{X_i} & \cong & BN_i \\
\downarrow & & \downarrow \\
BX_i & \rightarrow & BX
\end{array}
$$

commute up to homotopy.

Proof. The action of $W_X$ on $L := H_2(BT_X; \mathbb{Z}_p^\wedge)$ represents $W$ as a pseudo reflection group, and the lattice $L$ is centerfree. Therefore, by [16, 1.3], the lattice $L \cong L_1 \oplus \cdots \oplus L_n$ splits into a direct sum of sublattices, such that $W_i$ acts trivially on $L_j$ for $i \neq j$. Let $T_i := T_{L_i}$ be the p-adic torus associated to $L_i$. Moreover, $T \cong T_L \cong T_1 \oplus \cdots \oplus T_n$ is the torus associated to $L$.

The fibration $BT_X \rightarrow BN_X \rightarrow BW_X$ is classified by a map $c_N : BW_X \simeq \prod_i BW_i \rightarrow \prod_i BHE(BT_i)$. Let $c_0 : BW_X \rightarrow \prod_i BHE(BT_i)$ be the classifying map of the product of the fibrations $BT_i \rightarrow BN' \rightarrow BW_i$ given by the semi direct product. Since $BHE(BT_i) \simeq B^2T_i$, the difference between $c_N$ and $c_0$ is measured by a cohomology class in

$$
H^3(BW_X; \prod_i \pi_3(B^2T_i)) \cong H^3(BW_X; \prod_i L_i) \cong \prod_i H^3(BW_i; L_i).
$$

The last isomorphism follows from Lemma 2.2. This cohomology class describes also a product of fibrations of the desired form which is fiber homotopy equivalent to
$BT_X \to BN_X \to BW_X$. This establishes a splitting into simple factors. The uniqueness follows from Lemma 6.1, which proves (1).

Now we can apply Theorem 1.3, which proves the existence of a splitting. By Lemma 5.2, each factor is centerfree. This is part (2).

Every splitting of $X$ establishes a splitting of $N_X$ into simple factors, which are uniquely determined up to order (Lemma 6.1). This proves part (3). $\square$

Theorem 1.1 is contained in the following result.

**6.3 Theorem.** Let $p$ be an odd prime. Let $X$ be a simply connected $p$–compact group. Then, the following holds:

1. The normalizer $N_X \cong N_1 \times \ldots \times N_n$ splits, up to order, uniquely into a product of simple factors. For each factor, the quotient $W_i := N_i/T_i$ acts on $H_2(BT_i; \mathbb{Z}_p)$ as a pseudo reflection group.
2. The $p$–compact group $X \cong X_1 \times \ldots \times X_n$ splits into a product of simple simply connected $p$–compact groups.
3. For a splitting $X \cong X_1 \times \ldots \times X_n$ into simple simply connected $p$–compact groups, the normalizers of the maximal tori of the factors are determined by the factors of $N_X$, i.e., after reordering the factors, there exist isomorphisms $N_{X_i} \cong N_i$ such that the diagrams

$$
\begin{array}{ccc}
BN_{X_i} & \cong & BN_i \\
\downarrow & & \downarrow \\
BX_i & \to & BX
\end{array}
$$

commute up to homotopy.

**Proof.** Let $\overline{X} := X/Z(X)$ be the associated centerfree connected $p$–compact group. The center $Z := Z(X)$ of $X$ is a finite group [12, 5.3]. All these spaces fit into a principal fibration

$$
BZ \to BX \to B\overline{X}.
$$

with classifying map $B\overline{X} \to B^2Z$ [2, 7.2]. Hence, there exists an isomorphism $\pi_2(B\overline{X}) \cong \pi_1(BZ) \cong Z$.

By Theorem 6.2, there exist splittings $N_X \cong \prod_i N_i$ of $N_X$ and $\overline{X} \cong \prod_i \overline{X}_i$ of $X$ into simple centerfree factors. Let $Z \cong \prod_i Z_i$ be the associated splitting. The above classifying map also splits into a product of maps $B\overline{X}_i \to B^2Z_i$. The product of the fibers $X_i$ gives a splitting of $X \cong \prod_i X_i$ into simple simply connected $p$–compact groups. This also establishes a splitting of $N_X \cong \prod_i N_{X_i}$ into simple factors. The uniqueness properties for these splittings follow on the one hand from Lemma 6.1, on the other hand analogously as in the proof of Theorem 6.2. $\square$
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