UPPER BOUNDS ON HIGHER LIMITS

(Bob Oliver)

Theorem 1. *Fix $V \cong (\mathbb{Z}/2)^r$, $r \geq 2$. Let X be a finite V-complex of dimension n. Let $F_X^* : \mathcal{A}_2(V) \to \mathcal{A}b$ be the functor $F_X^*(E) = H_V^*(X^E)$. Then $\lim\limits_{\longleftarrow \mathcal{A}_2(V)} (F_X^j)$ vanishes if $i \geq r$, or if $i = r-1$ or $r-2$ and $j > (2^r-1)n - r$, or if $1 \leq i \leq r-3$ and $j > (2^r-1)n - r + 1$.*

The proof of Theorem 1 will use Lemma 2 and Proposition 4 below.

Lemma 2. *Let V be an elementary abelian 2-group, and let $X \supseteq Y$ be a pair of V-complexes such that $X^V = Y^V$. Let $x_1, \dots, x_s \in V^* \smallsetminus 0$ be any set of elements such that every isotropy subgroup of $X \smallsetminus Y$ is contained in $\operatorname{Ker}(x_i)$ for some i. Set $n = \dim(X)$. Then for every $\alpha \in H_V^*(Y)$, $x_1^n \cdots x_s^n \alpha \in \operatorname{Im}[H_V^*(X) \to H_V^*(Y)]$.*

Proof. For each $i = 0, \dots, n$, set $X_i = X^{(i)} \cup Y$. Then $H_V^*(X_0)$ surjects onto $H_V^*(Y)$ (projection onto a direct summand). And for all $i \geq 0$, $x_1 \cdot \cdot \cdot x_r \cdot H_V^*(X_{i+1}, X_i) = 0$. □

The proof of Theorem 1 will be reduced to putting upper bounds on the degrees of quotients of the following form. The following notation will be useful: for any V and any subset $D \subseteq V^* \smallsetminus 0$, set $\pi_D = \prod_{x \in D} x \in \mathbb{F}_2[V^*]$.

Proposition 3. *Fix $V \cong (\mathbb{Z}/2)^k$ $(k \geq 2)$. Assume, for some fixed $N \geq 1$, ideals $I_x \subseteq \mathbb{F}_2[(V/x)^*]$ are given, for all $x \in V \smallsetminus 0$, such that*

(1) $(\pi_{(V/x)^* \smallsetminus 0})^N \in I_x$

(2) For all $x \neq y$, $(\pi_{(V/x)^* \smallsetminus (V/y)^*})^N \cdot I_y \subseteq I_x \cdot \mathbb{F}_2[V^*]$.

Then the quotient group

$$\Big(\bigcap_{x \in V \smallsetminus 0} (I_x \cdot \mathbb{F}_2[V^*]) \Big) \Big/ \Big\langle (\pi_{V^* \smallsetminus (V/x)^*})^N \cdot I_x \cdot \mathbb{F}_2[V^*] \,|\, x \in V \smallsetminus 0 \Big\rangle$$

vanishes in degrees greater than $(2^k - 1)N - k$.

In order to simplify the notation, Proposition 3 will be proven in the following dualized version. For convenience in notation, when $V \cong (\mathbb{Z}/2)^k$ is given, then for any $i \leq k$, $\mathcal{S}_i$ will denote the set of subspaces of V of rank i.

Proposition 3'. *Fix $V \cong (\mathbb{Z}/2)^k$ $(k \geq 2)$. Assume, for some fixed $N \geq 1$, that ideals $I_E \subseteq \mathbb{F}_2[E]$ are given, for all $E \in \mathcal{S}_{k-1}$, such that*

(1) $(\pi_{E \smallsetminus 0})^N \in I_E$ for each E, and

(2) For any pair $E_1, E_2 \in \mathcal{S}_{k-1}$, $(\pi_{E_1 \smallsetminus E_2})^N \cdot I_{E_2} \subseteq I_{E_1} \cdot \mathbb{F}_2[V^*]$.

Typeset by $\mathcal{AMS}$-TEX

Then the quotient group

$$\Big(\bigcap_{E\in\mathcal{S}_{k-1}}(I_E\cdot\mathbb{F}_2[V^*])\Big)\Big/\Big\langle(\pi_{V\smallsetminus E})^N\cdot I_E\cdot\mathbb{F}_2[V^*]\big|E\in\mathcal{S}_{k-1}\Big\rangle$$

vanishes in degrees greater than $(2^k-1)N-k$.

Proof. Assume first that $k=2$, and set $V=\langle x,y\rangle$. Then

$$I_{\langle x\rangle}=x^a\cdot\mathbb{F}_2[x],I_{\langle y\rangle}=y^b\cdot\mathbb{F}_2[x]\quad\text{and}\quad I_{\langle x+y\rangle}=(x+y)^c\cdot\mathbb{F}_2[x]$$

for some $a,b,c\geq 0$. The quotient group

$$\Big(x^ay^b(x+y)^c\mathbb{F}_2[x,y]\Big)\Big/\Big\langle x^Ny^N(x+y)^c,x^Ny^b(x+y)^N,x^ay^N(x+y)^N\Big\rangle$$

has Poincaré series

$$\frac{1}{(1-t)^2}\left(t^{a+b+c}-t^{2N+a}-t^{2N+b}-t^{2N+c}+2t^{3N}\right),$$

which is a polynomial of degree 3N-2. So the proposition holds in this case.

Now assume $k>2$, and assume the proposition holds for rank $k-1$. For any $m:V\smallsetminus 0\to\{0,\dots,N\}$, we say that the system $\{I_E\}$ is *subordinate to* m if $I_E\subseteq\prod_{x\in E\smallsetminus 0}x^{m(x)}\cdot\mathbb{F}_2[E]$ for all $E\in\mathcal{S}_{k-1}$. Clearly, any $\{I_E\}$ is subordinate to the zero function. And if $\{I_E\}$ is subordinate to the constant function N, then $I_E=(\pi_{E\smallsetminus 0})^N\cdot\mathbb{F}_2[E]$ for all E, and the quotient is trivial.

It thus suffices to prove the following. Assume m is maximal among functions with the property that $\{I_E\}$ is subordinate to m; and assume that m is not the constant function N (otherwise, as noted above, we are done). Fix y such that $m(y)<N$, and define $m':V\smallsetminus 0\to\{0,\dots,N\}$ by setting $m'(y)=m(y)+1$, and $m'(x)=m(x)$ for $x{\neq}y$. Define ideals J_E by setting

$$J_E=I_E\cap\Big(\prod_{x\in E\smallsetminus 0}x^{m'(x)}\Big)\cdot\mathbb{F}_2[E]\qquad\qquad(\text{all }E\in\mathcal{S}_{k-1}.)$$

We claim that the proposition holds for $\{I_E\}$ if it holds for $\{J_E\}$.

To see this, it suffices to show that the cokernel of the map

$$\varphi:\frac{\bigcap_{E\in\mathcal{S}_{k-1}}J_E\cdot\mathbb{F}_2[V]}{\langle(\pi_{V\smallsetminus E})^N\cdot J_E\cdot\mathbb{F}_2[V]\rangle}\longrightarrow\frac{\bigcap_{E\in\mathcal{S}_{k-1}}I_E\cdot\mathbb{F}_2[V]}{\langle(\pi_{V\smallsetminus E})^N\cdot I_E\cdot\mathbb{F}_2[V]\rangle}$$

vanishes in dimensions greater than $(2^k-1)N-k$. Note first that $I_E=J_E$ if $y\notin E$, and that $yI_E{\subseteq}J_E{\subseteq}I_E$ if $y{\in}E$. Define a system $\{\bar{I}_{E/y}\}$ of ideals over $\mathbb{F}_2[V/y]$ by setting

$$\bar{I}_{E/y}=\operatorname{Im}\Big[y^{-m(y)}I_E\to\mathbb{F}_2[E/y]\Big]$$

for all $E\in\mathcal{S}_{k-1}$ containing y. Then the $\bar{I}_{E/y}$ satisfy the hypotheses of Proposition 3' (with N replaced by $2N$), and

$$\operatorname{Coker}(\varphi)\cong$$

$$\Sigma^{m(y)}\bigg(\Big(\bigcap_{E/y\in\mathcal{S}_{k-2}(V/y)}\bar{I}_{E/y}\cdot\mathbb{F}_2[V/y]\Big)\Big/\Big\langle(\pi_{V/y\smallsetminus E/y})^{2N}\cdot\bar{I}_{E/y}\cdot\mathbb{F}_2[V/y]\Big\rangle\bigg).$$

So by the induction hypothesis, Coker(φ) vanishes in dimensions greater than

$$m(y)+(2^{k-1}-1)\cdot 2N-(k-1)\leq(2^k-1)N-k;\qquad\qquad(m(y)\leq N-1)$$

and this finishes the proof. $\square$

Let $\bar{\mathcal{A}}_2(V)$ denote the category of *all* subgroups of V (including the trivial subgroup).

Proposition 4. *Fix $V=(\mathbb{Z}/2)^k$ and $N\geq 1$. Let*

$$F:\bar{\mathcal{A}}_2(V)\to(\text{ideals in }\mathbb{F}_2[V^*])$$

be a functor, which satisfies the following three conditions:

(1) $F(V)=\mathbb{F}_2[V^*]$.

(2) For all $E\subseteq V$, $F(E)=I_E\cdot\mathbb{F}_2[V^*]$ for some ideal $I_E\subseteq\mathbb{F}_2[(V/E)^*]$. (Regard $\mathbb{F}_2[(V/E)^*]$ as a subring of $\mathbb{F}_2[V^*]$.)

(3) For all $E\subsetneqq E'\subseteq V$,

$$\left(\pi_{(V/E)^*\smallsetminus(V/E')^*}\right)^N\cdot F(E')\subseteq F(E)\quad(\subseteq F(E'))$$

Then $\varprojlim^i(F)$ for $i>0$, and $\operatorname{Coker}[F(1)\to\varprojlim^0(F)]$, all vanish in degrees greater than $(2^k-1)N-k$.

Proof. We say that F satisfies the *rank t intersection condition* if for any $E\subseteq V$ with $\operatorname{rk}(E)\leq t$,

$$F(E)=\bigcap_{E'\supsetneqq E}F(E').$$

By Proposition 3,

$$\operatorname{Coker}\big[F(1)\to\varprojlim_{\mathcal{A}_2(V)}(F)\big]\cong\Big(\bigcap_{E\in\mathcal{S}_1}F(E)\Big)\Big/F(1)$$

vanishes in degrees greater than $(2^k-1)N-k$. So we can replace $F(1)$ by $\bigcap_{E\in\mathcal{S}_1}F(E)$, and assume that the rank 0 intersection property holds.

Assume now that F satisfies the rank $t-1$ intersection property, but not the rank t property, for some $1\leq t\leq k$. We may assume inductively that the proposition holds for functors which satisfy the rank t intersection property; and for functors on groups of lower rank. If $t=k$, then F is constant, and $\varprojlim^i(F)=0$ for all $i>0$. So we may assume $t\leq k-1$.

Fix $W\subseteq V$ of rank t such that $F(W)\subsetneqq\bigcap_{E\supsetneqq W}F(E)$. Choose ideals

$$I_W=J_0\subseteq J_1\subseteq\cdots\subseteq J_s=\bigcap_{E\supsetneqq W}I_E\cdot\mathbb{F}_2[(V/W)^*],$$

such that for each i, $J_i/J_{i-1}\cong\Sigma^{m_i}\mathbb{F}_2$ for some $m_i\geq 0$. By Proposition 3 again,

$$m_i\leq(2^{k-t}-1)\cdot N-(k-t)$$

for each i. Define functors $F_0=F\subseteq F_1\subseteq\ldots\subseteq F_s$ by inductively setting

$$F_i(E)=\begin{cases}F(E) & \text{if } E\not\subseteq W\\ J_i\cdot\mathbb{F}_2[V^*] & \text{if } E=W\\ \bigcap_{E'\supsetneqq E}F_i(E') & \text{if } E\subsetneqq W.\end{cases}$$

Then for each $1\leq i\leq s$, $(F_i/F_{i-1})(W)\cong\Sigma^{m_i}\mathbb{F}_2[W^*]$; and as a functor on $\mathcal{A}_2(W)$ F_i/F_{i-1} satisfies the hypotheses of Proposition 3 (but with N replaced by $2^{k-t}N$). So for $j\geq 1$, the limit $\varprojlim_{\mathcal{A}_2(W)}(F_i/F_{i-1})$ vanishes in degrees above

$$m_i+(2^t-1)\cdot 2^{k-t}N-t\leq(2^{k-t}-1)N-k+t+(2^t-1)\cdot 2^{k-t}N-t=(2^k-1)N-k.$$

In particular, the higher limits of F_s/F vanish above degree $(2^k-1)N-k$.

Continuing this procedure with the other rank t subgroups, we embed $F\subseteq\bar F$ so that $\bar F$ satisfies the rank $t+1$ intersection property, and so that the higher limits of $\bar F/F$ vanish in degrees above $(2^k-1)N-k$. Since the same holds for $\bar F$ by the induction hypothesis, this finishes the proof of the proposition. $\square$

Proof of Theorem 1. Recall first that for $i\geq r=\mathrm{rk}(V)$, $\varprojlim^i(F)=0$ for *any* functor on $\mathcal{A}_2(V)$.

The theorem clearly holds if the action of V fixes X, or if X is the union of X^V and a finite V-set. So we may consider the case where $X=Y\cup_\varphi(V/W\times D^m)$, with $\mathrm{rk}(W)=k<r$ and $0<m\leq n$, and where the theorem holds for Y.

Consider the following short exact sequences of functors on $\mathcal{A}_2(V)$:

$$0\to K\to F_X\to F_{\mathrm{Im}}\to 0,\qquad 0\to F_{\mathrm{Im}}\to F_Y\to C\to 0,\tag{1}$$

and

$$0\to\Sigma C\to F_{X,Y}\to K\to 0.\tag{2}$$

Here, K, F_{Im}, and C are the kernel, image, and cokernel, respectively, of the restriction map from $F_X\to F_Y$. The relative functor $F_{X,Y}$ has the form

$$F_{X,Y}(E)=H^*_V(X^E,Y^E)\cong\begin{cases}0 & \text{if } E\not\subseteq W\\ H^*_V(X^W,Y^W)\cong\Sigma^m\mathbb{F}_2[W^*] & \text{if } E\subseteq W,\end{cases}$$

and thus can be regarded as a constant functor on $\mathcal{A}_2(W)$. Note also that

$$C(E)\cong\begin{cases}\mathrm{Im}[H^*_V(Y^E)\to H^{*+1}_V(X^E,Y^E)] & \text{if } E\subseteq W\\ 0 & \text{if } E\not\subseteq W;\end{cases}$$

and that C and K can also be regarded as functors on $\mathcal{A}_2(W)$. In addition,

$$\begin{aligned}C(W)&\cong\mathrm{Im}\big[H^*_V(Y^W)\to H^{*+1}_V(X^W,Y^W)\big]\\&\cong\mathrm{Im}\big[H^*_{V/W}(Y^W)\to H^{*+1}_{V/W}(X^W,Y^W)\cong\Sigma^m\mathbb{F}_2\big]\otimes\mathbb{F}_2[W^*];\end{aligned}$$

so that either $C(W)\cong\Sigma^m\mathbb{F}_2[W^*]$, or $C=0$ (as a constant functor). In the second case the result is clear, since $\varprojlim^i(F_{X,Y})=0$ for $i>0$ ($F_{X,Y}$ being a constant functor).

Now, for any $E\subseteq W$, set

$$I_E=\mathrm{Im}\Big[H^*_{V/E}(Y^E)\to H^{*+1}_{V/E}(X^W,Y^W)\cong\Sigma^m\mathbb{F}_2[(V/E)^*]\Big];$$

regarded as an ideal in $\mathbb{F}_2[(V/E)^*]$. Then $C(E)=I_E\cdot\mathbb{F}_2[W^*]$. Also, for all $E\subsetneqq E'\subseteq W$,

$$\big(\pi_{(W/E)^*\smallsetminus(W/E')^*}\big)^{n\cdot 2^{r-1}}C(E')\subseteq C(E)$$

by Lemma 2. Hence, by Proposition 4,

$$\varprojlim{}^i(C^*)\quad(1\leq i<r)\qquad\text{and}\qquad\mathrm{Coker}\big[H^*_V(Y)\to\varprojlim{}^0(F^*_Y)\to\varprojlim{}^0(C^*)\big]$$

vanish in degrees greater than

$$(m-1)+(2^k-1)(n\cdot 2^{r-k})-k=(2^r-2^{r-k}+1)n-(k+1)-(n-m)\leq(2^r-1)n-r.$$

It follows that $\mathrm{Ker}[\varprojlim^i(F_{\mathrm{Im}})\to\varprojlim^i(F_Y)]$, for $1\leq i<r$, also vanishes in degrees greater than $(2^r-1)n-r$.

Finally, since $\varprojlim^i(F_{X,Y})=0$ for $i>0$, we see that $\varprojlim^{k-1}(K)=0$; and that $\varprojlim^i(K)$ for $1\leq i\leq k-2$ ($\leq r-3$) vanishes in degrees greater than $(2^r-1)n-r+1$. $\square$