UPPER BOUNDS ON HIGHER LIMITS

(Bob Oliver)

**Theorem 1.** Fix $V \cong (\mathbb{Z}/2)^r$, $r \geq 2$. Let $X$ be a finite $V$-complex of dimension $n$. Let $F_X^r : A_2(V) \to Ab$ be the functor $F_X^r(E) = H^i_\ast(X^E)$. Then $$\lim_{\leftarrow A_2(V)} (F^r_X)$$ vanishes if $i \geq r$, or if $i = r - 1$ or $r - 2$ and $j > (2^r - 1)n - r$, or if $1 \leq i \leq r - 3$ and $j > (2^r - 1)n - r + 1$.

The proof of Theorem 1 will use Lemma 2 and Proposition 4 below.

**Lemma 2.** Let $V$ be an elementary abelian 2-group, and let $X \triangleright Y$ be a pair of $V$-complexes such that $X^V = Y^V$. Let $x_1, \ldots, x_s \in V^* \setminus \{0\}$ be any set of elements such that every isotropy subgroup of $X \setminus Y$ is contained in $\text{Ker}(x_i)$ for some $i$. Set $n = \dim(X)$. Then for every $\alpha \in H^i_Y(X)$, $x_1 \cdot \cdots \cdot x_s \alpha \in \text{Im}[H^i_Y(X) \to H^i_Y(Y)]$.

**Proof.** For each $i = 0, \ldots, n$, set $X_i = X^{(i)} \setminus Y$. Then $H^i_Y(X_0)$ surjects onto $H^i_Y(Y)$ (projection onto a direct summand). And for all $i \geq 0$, $x_1 \cdots x_r \cdot H^i_Y(X_{i+1}, x_i) = 0$. \hfill $\square$

The proof of Theorem 1 will be reduced to putting upper bounds on the degrees of quotients of the following form. The following notation will be useful: for any $V$ and any subset $D \subseteq V^* \setminus \{0\}$, set $\pi_D = \prod_{x \in D} x \in F_2[V^*]$.

**Proposition 3.** Fix $V \cong (\mathbb{Z}/2)^k$ $(k \geq 2)$. Assume, for some fixed $N \geq 1$, ideals $I_x \subseteq F_2[(V/x)^*]$ are given, for all $x \in V \setminus \{0\}$, such that

1. $(\pi_{(V/x)^*})^N \in I_x$
2. For all $x \neq y$, $(\pi_{(V/y)^*})^N \cdot I_y \subseteq I_x \cdot F_2[V^*]$. Then the quotient group

$$\left( \bigcap_{x \in V \setminus \{0\}} (I_x \cdot F_2[V^*]) \right) / \langle (\pi_{(V/x)^*})^N \cdot I_x \cdot F_2[V^*] \rangle_{x \in V \setminus \{0\}}$$

vanishes in degrees greater than $(2^k - 1)N - k$.

In order to simplify the notation, Proposition 3 will be proven in the following dualized version.

**Proposition 3’.** Fix $V \cong (\mathbb{Z}/2)^k$ $(k \geq 2)$. Assume, for some fixed $N \geq 1$, that ideals $I_E \subseteq F_2[E]$ are given, for all $E \in S_{k-1}$, such that

1. $(\pi_{E^*})^N \in I_E$ for each $E$, and
2. For any pair $E_1, E_2 \in S_{k-1}$, $(\pi_{E_1 \setminus E_2})^N \cdot I_{E_2} \subseteq I_{E_1} \cdot F_2[V^*]$.

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Then the quotient group
\[
\left( \bigcap_{E \in S_{k-1}} (I_E \cdot \mathbb{F}_2[V^*]) \right) \big/ \left( (\pi_{V \setminus E})^N \cdot I_E \cdot \mathbb{F}_2[V^*] \right)_{E \in S_{k-1}}
\]
vanishes in degrees greater than \((2^k - 1)N - k\).

**Proof.** Assume first that \(k = 2\), and set \(V = \langle x, y \rangle\). Then
\[
I_{\langle x \rangle} = x^a \cdot \mathbb{F}_2[x], \quad I_{\langle y \rangle} = y^b \cdot \mathbb{F}_2[x] \quad \text{and} \quad I_{\langle x + y \rangle} = (x + y)^c \cdot \mathbb{F}_2[x]
\]
for some \(a, b, c \geq 0\). The quotient group
\[
\left( x^a y^b (x + y)^c \mathbb{F}_2[x, y] \right) \big/ \left( x^N y^N (x + y)^c, x^N y^b (x + y)^N, x^a y^N (x + y)^N \right)
\]
has Poincaré series
\[
\frac{1}{(1-t)^2} \left( t^{a+b+c} - t^{2N+a} - t^{2N+b} + 2t^{3N} \right),
\]
which is a polynomial of degree \(3N-2\). So the proposition holds in this case.

Now assume \(k > 2\), and assume the proposition holds for rank \(k - 1\). For any \(m : V \setminus \{0, \ldots, N\} \rightarrow \{0, \ldots, N\}\), we say that the system \(\{I_E\}\) is subordinate to \(m\) if \(I_E \subseteq \bigcap_{x \in E^{\setminus \{0\}}} x^m(x) \cdot \mathbb{F}_2[E]\) for all \(E \in S_{k-1}\). Clearly, any \(\{I_E\}\) is subordinate to the zero function. And if \(\{I_E\}\) is subordinate to the constant function \(N\), then \(I_E = (\pi_{V \setminus E})^N \cdot \mathbb{F}_2[E]\) for all \(E\), and the quotient is trivial.

It thus suffices to prove the following. Assume \(m\) is maximal among functions with the property that \(\{I_E\}\) is subordinate to \(m\); and assume that \(m\) is not the constant function \(N\) (otherwise, as noted above, we are done). Fix \(y\) such that \(m(y) < N\), and define \(m' : V \setminus \{0, \ldots, N\} \rightarrow \{0, \ldots, N\}\) by setting \(m'(y) = m(y) + 1\), and \(m'(x) = m(x)\) for \(x \neq y\). Define ideals \(J_E\) by setting
\[
J_E = I_E \cap \left( \bigcap_{x \in E^{\setminus \{0\}}} x^m(x) \right) \cdot \mathbb{F}_2[E]
\]
(all \(E \in S_{k-1}\)).

We claim that the proposition holds for \(\{I_E\}\) if it holds for \(\{J_E\}\).

To see this, it suffices to show that the cokernel of the map
\[
\varphi : \left( \bigcap_{E \in S_{k-1}} J_E \cdot \mathbb{F}_2[V] \right) \big/ \left( (\pi_{V \setminus E})^N \cdot J_E \cdot \mathbb{F}_2[V] \right) \rightarrow \left( \bigcap_{E \in S_{k-1}} I_E \cdot \mathbb{F}_2[V] \right) \big/ \left( (\pi_{V \setminus E})^N \cdot I_E \cdot \mathbb{F}_2[V] \right)
\]
vansishes in dimensions greater than \((2^k - 1)N - k\). Note first that \(I_E = J_E\) if \(y \notin E\), and that \(yI_E \subseteq J_E \subseteq I_E\) if \(y \in E\). Define a system \(\{I_{E/y}\}\) of ideals over \(\mathbb{F}_2[V/y]\) by setting
\[
I_{E/y} = \text{Im} \left[ y^{-m(y)} I_E \rightarrow \mathbb{F}_2[E/y] \right]
\]
for all \(E \in S_{k-1}\) containing \(y\). Then the \(I_{E/y}\) satisfy the hypotheses of Proposition 3’ (with \(N\) replaced by \(2N\)), and
\[
\text{Coker}(\varphi) \cong \sum_{y \in E} \left( \bigcap_{E \in S_{k-1} \setminus \langle E \rangle} I_{E/y} \cdot \mathbb{F}_2[V/y] \right) \big/ \left( (\pi_{V \setminus E') \setminus E})^N \cdot I_{E/y} \cdot \mathbb{F}_2[V/y] \right).
\]
So by the induction hypothesis, \(\text{Coker}(\varphi)\) vanishes in dimensions greater than
\[
m(y) + (2^k - 1) \cdot 2N - (k - 1) \leq (2^k - 1)N - k; \quad (m(y) \leq N - 1)
\]
and this finishes the proof. \(\square\)

Let \(\mathcal{A}_2(V)\) denote the category of all subgroups of \(V\) (including the trivial subgroup).
Proposition 4. Fix $V = (\mathbb{Z}/2)^k$ and $N \geq 1$. Let
\[ F : \mathcal{A}_2(V) \to (\text{ideals in } \mathbb{F}_2[V^*]) \]
be a functor, which satisfies the following three conditions:

1. $F(V) = \mathbb{F}_2[V^*].$

2. For all $E \subseteq V$, $F(E) = I_E \cdot \mathbb{F}_2[V^*]$ for some ideal $I_E \subseteq \mathbb{F}_2[(V/E)^*]$. (Regard $\mathbb{F}_2[(V/E)^*]$ as a subring of $\mathbb{F}_2[V^*]$.)

3. For all $E' \subseteq V$,
\[ (\pi_{(V/E)^*} \cdot (V/E)^*)^N \cdot F(E') \subseteq F(E) \quad (\subseteq F(E')) \]
Then $\lim_{\leftarrow}^i(F)$ for $i > 0$, and $\text{Coker}[F(1) \to \lim_{\leftarrow}^0(F)]$, all vanish in degrees greater than $(2^k - 1)N - k$.

Proof. We say that $F$ satisfies the rank $t$ intersection condition if for any $E \subseteq V$ with $\text{rk}(E) \leq t$,
\[ F(E) = \bigcap_{E' \supseteq E} F(E'). \]
By Proposition 3,
\[ \text{Coker}[F(1) \to \lim_{\leftarrow}^i(F)] \cong \left( \bigcap_{E \in S_1} F(E) \right) / F(1) \]
vanishes in degrees greater than $(2^k - 1)N - k$. So we can replace $F(1)$ by $\bigcap_{E \in S_1} F(E)$, and assume that the rank 0 intersection property holds.

Assume now that $F$ satisfies the rank $t - 1$ intersection property, but not the rank $t$ property, for some $1 \leq t \leq k$. We may assume inductively that the proposition holds for functors which satisfy the rank $t$ intersection property; and for functors on groups of lower rank. If $t = k$, then $F$ is constant, and $\lim_{\leftarrow}^i(F) = 0$ for all $i > 0$. So we may assume $t \leq k - 1$.

Fix $W \subseteq V$ of rank $t$ such that $F(W) \not\subseteq \bigcap_{E \supseteq W} F(E)$. Choose ideals
\[ I_W = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_s = \bigcap_{E \supseteq W} I_E \cdot \mathbb{F}_2[(W/W)^*], \]
such that for each $i$, $J_i/J_{i-1} \cong \Sigma^{m_i} \mathbb{F}_2$ for some $m_i \geq 0$. By Proposition 3 again,
\[ m_i \leq (2^k - 1) \cdot N - (k - t) \]
for each $i$. Define functors $F_0 = F \subseteq F_1 \subseteq \ldots \subseteq F_s$ by inductively setting
\[ F_i(E) = \begin{cases} F(E) & \text{if } E \not\subseteq W \\ J_i \cdot \mathbb{F}_2[V^*] & \text{if } E = W \\ \bigcap_{E \supseteq W} F_i(E') & \text{if } E \supseteq W. \end{cases} \]
Then for each $1 \leq i \leq s$, $(F_i/F_{i-1})(W) \cong \Sigma^{m_i} \mathbb{F}_2[W^*]$; and as a functor on $\mathcal{A}_2(W)$ $F_i/F_{i-1}$ satisfies the hypotheses of Proposition 3 (but with $N$ replaced by $2^{k-t}N$). So for $j \geq 1$, the limit $\lim_{\leftarrow}^j(F_i/F_{i-1})$ vanishes in degrees above
\[ m_i + (2^t - 1) \cdot 2^{k-t}N - t \leq (2^k - 1)N - k + t + (2^t - 1) \cdot 2^{k-t}N - t = (2^k - 1)N - k. \]
In particular, the higher limits of $F_*/F$ vanish above degree $(2^{k - 1})N - k$.

Continuing this procedure with the other rank $t$ subgroups, we embed $F \subseteq \bar{F}$ so that $\bar{F}$ satisfies the rank $t + 1$ intersection property, and so that the higher limits of $\bar{F}/F$ vanish in degrees above $(2^{k - 1})N - k$. Since the same holds for $\bar{F}$ by the induction hypothesis, this finishes the proof of the proposition. \qed

**Proof of Theorem 1.** Recall first that for $i \geq r = \text{rk}(V)$, $\lim_{i \to -}^{i}(F) = 0$ for any functor on $A_2(V)$.

The theorem clearly holds if the action of $V$ fixes $X$, or if $X$ is the union of $X^V$ and a finite $V$-set. So we may consider the case where $X = Y \cup_e (V/W \times D^m)$, with $\text{rk}(W) = k < r$ and $0 < m \leq n$, and where the theorem holds for $Y$.

Consider the following short exact sequences of functors on $A_2(V)$:

\[ 0 \to K \to F_X \to F_{im} \to 0, \quad 0 \to F_{im} \to F_Y \to C \to 0, \tag{1} \]

and

\[ 0 \to \Sigma C \to F_{X,Y} \to K \to 0. \tag{2} \]

Here, $K$, $F_{im}$, and $C$ are the kernel, image, and cokernel, respectively, of the restriction map from $F_X \to F_Y$. The relative functor $F_{X,Y}$ has the form

\[ F_{X,Y}(E) = H^*_V(X^E, Y^E) \cong \begin{cases} 0 & \text{if } E \not\subseteq W \\ H^*_V(X^{W'}, Y^{W'}) \cong \Sigma^m \mathbb{F}_2[W^*] & \text{if } E \subseteq W, \end{cases} \]

and thus can be regarded as a constant functor on $A_2(W)$. Note also that

\[ C(E) \cong \begin{cases} \text{Im}[H^*_V(Y^E) \to H^*_{V+1}(X^E, Y^E)] & \text{if } E \subseteq W \\ 0 & \text{if } E \not\subseteq W; \end{cases} \]

and that $C$ and $K$ can also be regarded as functors on $A_2(W)$. In addition,

\[ C(W) \cong \text{Im}[H^*_V(Y^W) \to H^*_{V+1}(X^W, Y^W)] \cong \text{Im}[H^*_V(Y^W) \to H^*_{V+1}(X^W, Y^W)] \cong \Sigma^m \mathbb{F}_2 \otimes \mathbb{F}_2[W^*]; \]

so that either $C(W) \cong \Sigma^m \mathbb{F}_2[W^*]$, or $C = 0$ (as a constant functor). In the second case the result is clear, since $\lim_{i \to -}^{i}(F_{X,Y}) = 0$ for $i > 0$ ($F_{X,Y}$ being a constant functor).

Now, for any $E \subseteq W$, set

\[ I_E = \text{Im}\left[H^*_V(Y^E) \to H^*_{V+1}(X^W, Y^W) \cong \Sigma^m \mathbb{F}_2[(V/E^*)]\right]; \]

regarded as an ideal in $\mathbb{F}_2 [(V/E^*)]$. Then $C(E) = I_E \cdot \mathbb{F}_2[W^*]$. Also, for all $E \subseteq E' \subseteq W$,

\[ \left(\pi_{(W/E)} \cdot (W/E')^*\right)^{n-2r-1} C(E') \subseteq C(E) \]

by Lemma 2. Hence, by Proposition 4,

\[ \lim_{i \to -}^{i}(C^*) \quad (1 \leq i < r) \quad \text{and} \quad \text{Coker}[H^*_V(Y) \to \lim_{i \to -}^{i}(F_Y^*) \to \lim_{i \to -}^{i}(C^*)] \]

vanish in degrees greater than

\[ (m - 1) + (2^{k - 1})(n \cdot 2^{r-k}) - k = (2^r - 2^{r-k} + 1)n - (k + 1) - (n - m) \leq (2^r - 1)n - r. \]

It follows that $\text{Ker}[\lim_{i \to -}^{i}(F_{im}) \to \lim_{i \to -}^{i}(F_Y)]$, for $1 \leq i < r$, also vanishes in degrees greater than $(2^r - 1)n - r$.

Finally, since $\lim_{i \to -}^{i}(F_{X,Y}) = 0$ for $i > 0$, we see that $\lim_{i \to -}^{i}(F_Y) = 0$; and that $\lim_{i \to -}^{i}(K)$ for $1 \leq i \leq k - 2 \leq r - 3$ vanishes in degrees greater than $(2^r - 1)n - r + 1$. \qed