The Chromatic Filtration and
the Steenrod Algebra

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Abstract. Using Steenrod algebra tools (P^t-homology, in particular), we
develop machinery analogous to the chromatic spectral sequence (and asso-
ciated paraphernalia) of Miller, Ravenel, and Wilson in [MRW77]; and for
any p-local spectrum X we use the Steenrod algebra to construct a variant
of the chromatic tower over X.

1. Introduction

Fifteen years ago in [MRW77], Miller, Ravenel, and Wilson introduced the
word “chromatic” to describe structure present in the Adams-Novikov spectral
sequence E_2-term; their ideas have been carried over to the geometric setting,
as by Ravenel in [Rav] and [Rav87] (see also his seminal paper [Rav84]) and
by Mahowald and Sadofsky in [MS]. If one reads too much of this sort of thing,
one gets the impression that the Steenrod algebra is not much help in trying to
understand these structures; the purpose of this paper is to try to correct this
misconception. In particular, we describe how the chromatic structure manifests
itself for modules over the Steenrod algebra, and we use Steenrod algebra tools
to construct a tower of spectra which is related to the chromatic tower.

On the algebraic side, we show that Margolis’ “Postnikov tower” for P^t-
homology in [Mar83, Section 22.2] is actually closely related to the chromatic
resolution in [MRW77]; along these lines, we develop other algebraic tools sim-
ilar to those used in [MRW77] to calculate the E_2-term. We have tried to make
all of the constructions look just like the BP versions; this has been a success
for all of the algebraic structures except the cohomology of the Morava stabilizer
algebras. We give a partial replacement for this last piece.

On the geometric side, Margolis’ construction for spectra gives a (convergent)
filtration of π_*(X) for any finite spectrum X; this is related to the various

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versions of the chromatic filtration, but it is not clear how closely. Indeed, our method of construction will look similar to the “geometric” or “telescope” chromatic filtration using the functors $L_n'$ of [Rav] and [MS], and the map $X \to L_n' X$ factors through our replacement for $L_n X$; hence the same is true of the map $X \to L_n X$, where $L_n$ is the localization functor used in the “algebraic” chromatic filtration of [Rav87].

The structure of the paper is as follows: in Section 2 we recall some facts about the Steenrod algebra, and we construct some modules and spectra for later use. In Section 3 we present the algebraic constructions analogous to the chromatic spectral sequence and associated machinery of [MRW77], including the Margolis chromatic spectral sequence and Bockstein spectral sequences for computing each column, as well as a candidate for a replacement for the cohomology of the Morava stabilizer algebras. In Section 4 we do geometric constructions, related to the chromatic filtrations in [Rav87], [Rav] and [MS]. Many of the results in this paper have appeared elsewhere in some form or other; to preserve continuity, we have relegated the few proofs that we give to Section 5.

The author would like to thank Haynes Miller for providing the original inspiration for this work; as a result, some of these ideas appeared in the author’s thesis. We also thank Hal Sadofsky for numerous helpful conversations about chromatic things and the Steenrod algebra.

2. Recollections

We recall some facts about the Steenrod algebra. Let $p$ be a prime number, and let $A$ be the mod $p$ Steenrod algebra. As an algebra, the dual of $A$ is isomorphic to $\mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$ if $p$ is odd, and to $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$ if $p = 2$—see [Mil58]. Let the Milnor basis of $A$ be the dual to the monomial basis, and let $P_t^s$ be the Milnor basis element dual to $\xi_t^s$, and, when $p$ is odd, let $Q_n$ be dual to $\tau_n$ (when $p = 2$, let $Q_n = P_0^n$). One can check that $(P_t^s)^p = 0$ when $s < t$ and $(Q_n)^2 = 0$ for all $n$, and so we can define homology with respect to these elements; e.g., given an $A$-module $M$, let

$$H(M, P_t^s) = \frac{\ker(P_t^s : M \to M)}{\text{im}((P_t^s)^p : M \to M)}.$$ 

Hence, we will refer to these elements as differentials (see [Mar83], [MW81], or [Pal92], for instance).

Given a differential $x$, define its slope, $s(x)$, by

$$s(P_t^s) = \frac{p|P_t^s|}{2} \quad \text{and} \quad s(Q_n) = |Q_n|.$$ 

(This definition is motivated by the vanishing line theorems in [AD73] and [MW81].)

The differentials are linearly ordered by slope; let $x_0, x_1, x_2, \ldots$ be the differentials in this ordering (at the prime 2 these are $P_1^0, P_2^0, P_2^1, \ldots$; at odd primes
these are \( Q_0, Q_1, P^0_1, Q_2, \ldots \). Given integers \( m \) and \( n \) with \( 0 \leq m \leq n \), say that a module \( M \) is of type \( (x_m, x_n) \) (also written \( M = M(x_m, x_n) \)) if \( H(M, x_k) = 0 \) whenever \( k < m \) or \( k > n \). We have similar notation for spectra, of course.

Let \( C(x_k) \) be the subalgebra of \( A \) generated by \( x_k \), and let \( u_k \) be the polynomial generator in \( \text{Ext}^*_{C(x_k)}(F_p, F_p) \) (so if \( p = 2 \) or \( x_k = Q_n \), then \( u_k \) is in bidegree \( (1, s(x_k)) \); otherwise, \( u_k \) is in bidegree \( (2, 2s(x_k)) \)).

**Remark 2.1.** For convenience of notation, throughout this paper we work at the prime 2; the main consequence of this is that \( u_k \) then always has homological degree 1. The only change necessary at odd primes is that some numbers will need to be adjusted.

Given an \( A \)-module \( V \), we say that \( f \in \text{Ext}^*(A, V) \) is a \( u_k \)-map of \( V \) if \( f \) restricts to \( u_k \otimes 1 \in \text{Ext}^*_C(V, V) \). Similarly, a \( u_k \)-map of a spectrum \( X \) is an element \( g \in [X, X] \) which is represented at the \( E_2 \)-term of the Adams spectral sequence by a \( u_k \)-map of \( H^*(X) \). In both the algebraic and geometric settings, we will denote such maps by \( u_k \).

Recall the following theorem:

**Theorem 2.2 ([AM71], [MP72]).** Let \( M \) be a bounded below module over \( A \). Then \( M \) is free if and only if \( H(M, x_k) = 0 \) for all \( k \).

This provides some motivation for our next definition: since \( x_k \)-homology ignores free \( A \)-modules (and hence mod \( p \) Eilenberg-Mac Lane spectra), so should we.

We work in the “stable” categories of Margolis—see [Mar83], Chapters 14 and 17 for more details. We give a brief overview of the relevant facts. The **stable category of \( A \)-modules** has bounded below modules for objects, and the maps are the following bigraded group: given \( A \)-modules \( L \) and \( M \), set \( \{L, M\}^{0,0} = \text{Hom}^+_A(L, M)/\simeq \), where \( \simeq \) is stable equivalence: \( f \simeq 0 \) if \( f \) factors through a projective module. To define \( \{L, M\}^{s,0} \) in general, we need a bit more notation: for any \( A \)-module \( M \), define \( \Omega M \) by the short exact sequence

\[
0 \rightarrow \Omega M \rightarrow PM \rightarrow M \rightarrow 0
\]

where \( PM \) is a projective module; then one can check that \( \Omega M \) is well-defined up to stable equivalence. Set

\[
\{L, M\}^{s,0} = \begin{cases} \{\Omega^sL, M\}^{0,0} & \text{if } s \geq 0, \\ \{L, \Omega^{s-M}\}^{0,0} & \text{if } s \leq 0. \end{cases}
\]

One has \( \text{Ext}^*_A(L, M) \cong \{L, M\}^{s,0} \) if \( s > 0 \) (see [Mar83, Proposition 14.8]), so an advantage of this viewpoint is that elements of \( \text{Ext}^*_A \) are equivalence classes of maps between actual modules, using \( \Omega \). (We will usually apply this to a \( u_k \)-map; note, for example, that a \( u_k \)-map of a module \( V \) induces an isomorphism \( H(\Sigma^{s+2k}V, x_k) \cong H(V, x_k) \).

Another advantage of working in this category is that one can turn any map into a surjection: given an \( A \)-module map \( f : L \rightarrow M \), one can replace \( f \) by
\( \tilde{f} : L \oplus PM \to M \). This is analogous to turning any map of spaces into a fibration. Let \( K \) be the kernel of \( \tilde{f} \); we say that

\[
0 \to K \to L \to M \to 0
\]

is a \textit{stable short exact sequence}. The module \( K \) is the \textit{stable kernel} of \( f \); it is well-defined up to stable equivalence.

There are analogous constructions for spectra; we won’t give the details. Briefly, we work in the \textit{stable category of p-local spectra}, in which a map is \textit{stably trivial} if it factors through a locally finite mod \( p \) generalized Eilenberg-Mac Lane spectrum; there is an analogous construction to \( \Omega \) which we will denote \( \ell \) (so that for any spectrum \( X \) there is a cofibration

\[
X \xrightarrow{f} KX \to \ell X,
\]

where \( KX \) is a mod \( p \) generalized Eilenberg-Mac Lane spectrum and \( H^*(f) \) is surjective; i.e., up to suspension, \( \ell^k X \) is the \( k \)th stage in an Adams tower for \( X \). We use \( \{\cdot, \cdot\} \) for maps in this category, as well. We shall also use the notion of \textit{stable cofibration sequence} frequently.

Lastly, we define the subcategory of \textit{stably finite} \( A \)-modules to be the smallest category containing all finite \( A \)-modules, which is closed under stable equivalences and taking summands and stable kernels. Similarly, a spectrum is \textit{stably finite} if it is in the smallest subcategory of stable \( p \)-local spectra containing finite spectra, closed under stable equivalences, retracts, and stable cofibers.

We have the following results.

\textbf{Theorem 2.3.} (a) ([Pal92, Theorem 2.2]). If \( M \) is a stably finite \( A \)-module of type \( \langle x_n, \infty \rangle \) with \( H(M, x_n) \neq 0 \), then \( M \) has a non-nilpotent \( u^i_x \)-map for some \( i > 0 \).

(b) ([PS, Proposition 2.1]). If \( X \) is a stably finite spectrum of type \( \langle x_n, \infty \rangle \) with \( H(H^*(X), x_n) \neq 0 \), then \( X \) has a \( u^i_x \)-map for some \( i > 0 \).

We refer to the following as the “killing construction” for \( x_k \)-homology groups.

\textbf{Theorem 2.4 ([Mar83], [Pal92], [PS]).} (a) Given a bounded below \( A \)-module \( M \) and an integer \( n \), then there is a stable short exact sequence of bounded below modules

\[
0 \to M\langle x_0, x_{n-1} \rangle \xrightarrow{f} M \xrightarrow{g} M\langle x_n, \infty \rangle \to 0,
\]

where \( H(M\langle x_0, x_{n-1} \rangle, x_k) = 0 \) if \( k \geq n \) and \( H(f, x_k) \) is an isomorphism if \( k < n \) (and similarly for \( M\langle x_n, \infty \rangle \) and \( g \)).

(b) We can realize this geometrically: given a connective spectrum \( X \) and an integer \( n \), there is stable cofibration sequence of connective spectra

\[
X\langle x_n, \infty \rangle \xrightarrow{f} X \xrightarrow{g} X\langle x_0, x_{n-1} \rangle,
\]
so that if \( M = H^*(X) \), then applying cohomology gives the short exact sequence in (a).

**Remark 2.5.** (a) The modules (and spectra) with \( x_k \)-homology groups killed are unique up to stable equivalence; for instance, for any \( A \)-module \( M \), one can get the stable short exact sequence for \( M \) (up to stable equivalence) by tensoring \( M \) with the stable short exact sequence for \( F_p \). The same goes for the spectrum result; see [Mar83].

(b) If \( M \) is of finite type (resp., \( X \) is locally finite), then so are \( M \langle x_0, x_{n-1} \rangle \) and \( M \langle x_n, \infty \rangle \) (resp., \( X \langle x_0, x_{n-1} \rangle \) and \( X \langle x_n, \infty \rangle \)). Margolis [Mar83] proves this at the prime 2; we prove it in general in Section 5.

**Definition 2.6.** Given integers \( k \leq n \) and an \( A \)-module \( M \), define \( M \langle x_k, x_n \rangle \) to be any \( A \)-module stably equivalent to \( (M \langle x_k, \infty \rangle \langle x_0, x_n \rangle) \); note that this is stably equivalent to \( (M \langle x_0, x_n \rangle)(x_k, \infty) \). We will also write \( M \langle x_n \rangle \) for \( M \langle x_n, x_n \rangle \).

We will use the construction of these modules and spectra below; see Proposition 2.8 for our key result. To describe this construction, and for other reasons, we define certain stably finite \( A \)-modules and spectra; first we need some notation.

**Notation 2.7.** We use capital letters to represent multi-indices; for example, \( I_k \) denotes \((i_0, \ldots, i_{k-1})\) (the subscript on \( I \) indicates the length of the multi-index). Given such a multi-index \( I_k \) of length \( k \) and an integer \( r \leq k \), we let \( I_r = (i_0, \ldots, i_{r-1}) \). Let \( I_0 = \phi \). Also, with \( U_k = (u_0, \ldots, u_{k-1}) \), we let \( U_k^{I_k} = (u_0^{i_0}, \ldots, u_{k-1}^{i_{k-1}}) \).

For each integer \( k \), we partially order the multi-indices \( I_k \) of length \( k \) as follows: set \( I_k \leq J_k \) if \( i_r \leq j_r \) for all \( r \), \( 0 \leq r \leq k - 1 \).

For each \( k \geq 0 \) and “suitable” \( I_k \), we inductively define stably finite \( A \)-modules \( V(U_k^{I_k}) \), so that \( V(U_k^{I_k}) \) is of type \( \langle x_k, \infty \rangle \) with non-zero \( x_\ell \)-homology for \( \ell \geq k \)—see [Pal92, Lemma 3.2]. Start by setting \( V(U_0^{I_0}) = F_p \). Now, assume that we have constructed \( V(U_k^{I_k}) \), stably finite, of type \( \langle x_k, \infty \rangle \), with \( H(V(U_{\ell}^{I_{\ell}}), x_\ell) \neq 0 \) for all \( \ell \geq k \). By Theorem 2.3, \( V(U_k^{I_k}) \) has a \( u_k^{i_k} \)-map for some \( i_k \), so define \( V(U_{k+1}^{I_{k+1}}) \) by the stable short exact sequence

\[
0 \rightarrow V(U_{k+1}^{I_{k+1}}) \rightarrow \Sigma^{-i_k s(x_k)} V(U_k^{I_k}) \xrightarrow{u_k^{i_k}} V(U_k^{I_k}) \rightarrow 0.
\]

Then \( V(U_{k+1}^{I_{k+1}}) \) is stably finite, of type \( \langle x_{k+1}, \infty \rangle \), and with non-zero \( x_\ell \)-homology for all \( \ell \geq k + 1 \).

Note that for each \( k \), there are infinitely many multi-indices \( I_k \) for which we can define such a module \( V(U_k^{I_k}) \), since one can choose different powers of the maps \( u_k^{i_k} \) at each stage.

One has a similar construction for spectra (for which the exponents \( i_k \) may need to be chosen larger); see [PS, Theorem 3.3]. The end result is a collection of stably finite spectra \( W(U_k^{I_k}) \), each indexed by an integer \( k \) and a \( k \)-tuple of
integers $I_k = (i_0, \ldots, i_k-1)$, so that $W(U^I_k)$ has a $u^I_k$-map for some $i_k > 0$ and there are stable cofibration sequences

$$W(U^I_k) \xrightarrow{u^I_k} \Sigma^{-i_k s(x)} \ell^{i_k} W(U^I_k) \to W(U^I_{k+1})$$

Note that for both the module and the spectrum cases, the maps $u^I_k$ can be chosen compatibly for different multi-indices $I_k$, so that, for example, if $I_k$ is smaller than $J_k$, then there is a map $V(U^I_k) \to V(U^J_k)$; for the spectrum case one needs to use the nilpotence theorem to see this—see [PS].

Also, by construction each $V(U^I_k)$ is deloopable $k$ times, and $\mathbf{F}_p$ maps to $\Omega^{-k} V(U^I_k)$; let $\overline{V}(U^I_k)$ be the stable kernel. Similarly, let $\overline{W}(U^I_k)$ be the stable cofiber of the map $\ell^{-k} W(U^I_k) \to S^0$.

We have the following proposition; parts (a) and (b) are the key results in the proof of Theorem 2.4.

**Proposition 2.8 ([Pal92], [PS]).**

(a) Taking the inverse limit over the partial ordering of multi-indices $I_n$ of length $n$ of the stable short exact sequence

$$0 \to \overline{V}(U^I_n) \to \mathbf{F}_p \to \Omega^{-n} V(U^I_n) \to 0$$

gives the stable short exact sequence in Theorem 2.4(a).

(b) Similarly, taking the direct limit over the $I_n$’s of

$$\ell^{-n} W(U^I_n) \to S^0 \to \overline{W}(U^I_n)$$

gives the stable cofibration sequence of Theorem 2.4(b).

(c) If $M$ is a stably finite module of type $(x_n, \infty)$, then

$$M \langle x_n \rangle \simeq \lim_{\leftarrow} (\cdots \xrightarrow{u^I_n} \Sigma^{-i_n s(x_n)} \Omega^{i_n} M \xrightarrow{u^I_n} M).$$

(d) If $X$ is a stably finite spectrum of type $(x_n, \infty)$, then

$$X \langle x_n \rangle \simeq \lim_{\rightarrow} (X \xrightarrow{u^I_n} \Sigma^{-i_n s(x_n)} \ell^{i_n} X \xrightarrow{u^I_n} \cdots).$$

### 3. The algebraic constructions

In this section we describe the analogs in the Steenrod algebra setting to the results in [MRW77]. In particular, we have analogs for the chromatic resolution, for the chromatic spectral sequence, and for the Bockstein spectral sequences used to compute each column of the chromatic spectral sequence; we also have a partial replacement for the role played by the cohomology of the Morava stabilizer algebras.
3.1. The Margolis chromatic resolution. In this subsection we construct a resolution of $F_p$ by $A$-modules analogous to the chromatic resolution of $BP$ by $BP,BP$-comodules. The end result will be the tower constructed in Section 22.2 of [Mar83] (see Theorem 3.1). Our construction and our point of view are the new results here.

We define modules $N^m$ and $M^m$ for $m \geq 0$, which play roles analogous to those of the $BP,BP$-comodules of the same name in [MRW77]. We give two equivalent descriptions.

The quick way to define them is to let $N^m = \Omega^m F_p \langle x_m, \infty \rangle$ and $M^m = \Omega^m F_p \langle x_m \rangle$; then for each $m$ we have a stable short exact sequence

$$0 \to N^{m+1} \to M^m \to N^m \to 0.$$

Splicing these short exact sequences together yields

$$\cdots \to M^2 \to M^1 \to M^0 \to F_p \to 0 \to \cdots \to \Omega^2 F_p \langle x_2 \rangle \to \Omega^1 F_p \langle x_1 \rangle \to F_p \langle x_0 \rangle \to F_p \to 0.$$

As in the $BP,BP$ picture, we have resolved the sphere by "monochromatic" objects. Call this the Margolis chromatic resolution of $F_p$; the terms and maps are well-defined up to stable equivalence. Note that this is the same as the tower constructed in [Mar83, Section 22.2]. Also, given any bounded below $A$-module $K$, we can tensor the tower with $K$ to get a Margolis chromatic resolution for $K$.

Another way to describe this tower bears stronger resemblance to the construction in [MRW77]. We define $N^m$ and $M^m$ inductively, starting with $N^0 = F_p$. We have, for any $i_0 > 0$,

$$0 \to V(\Sigma^{-i_0} \Omega^{i_0} F_p) \to \Sigma^{-i_0} \Omega^{i_0} F_p \to \Sigma^{-2i_0} \Omega^{2i_0} F_p \to F_p \to 0.$$

Since all the modules involved are of finite type, taking inverse limits over $i_0$ gives a short exact sequence, which we use to define $M^0$ and $N^1$:

$$0 \to N^1 \to M^0 \to N^0 \to 0.$$

By Proposition 2.8, we have $N^1 \simeq \Omega F_p \langle x_1, \infty \rangle$ and $M^0 \simeq F_p \langle x_0 \rangle$. 
Now we iterate this, as in [MRW77]: for each \( m \geq 0 \) and suitable \( I_m \), we have

\[
0 \to V(U_{m+1} I_m) \xrightarrow{\Sigma^{-1} \Omega^m V(U_{m+1} I_m)} V(U_m I_m) \to 0
\]

\[
0 \to V(U_{m+1} I_m) \xrightarrow{\Sigma^{-2} \Omega^{2m} V(U_{m+1} I_m)} V(U_m I_m) \to 0
\]

(where \( i'_k = i_k \) for \( k < m \), and \( i'_m = 2i_m \)). Taking inverse limits over the vertical maps yields the same short exact sequence as above,

\[
0 \to \Omega^{m+1} F_p^{\langle x_{m+1}, \infty \rangle} \to \Omega^m F_p^{\langle x_m \rangle} \to \Omega^m F_p^{\langle x_m, \infty \rangle} \to 0.
\]

3.2. The Margolis chromatic spectral sequence. Let \( K \) and \( L \) be \( A \)-modules. Applying \( \{-, L\}^{**} \) to the Margolis chromatic resolution for \( K \) gives a trigraded spectral sequence which we call the Margolis chromatic spectral sequence:

**Theorem 3.1** ([Mar83], Theorem 22.4). Given bounded below \( A \)-modules \( K \) and \( L \), there is a spectral sequence with

\[
E_1^{s,t,u} \cong \{ K \otimes M^s, L \}^{s+t,u} \cong \{ K^{\langle x_s \rangle}, L \}^{s+t,u}
\]

and \( d_r : E_r^{s,t,u} \to E_r^{s+r,t+1-r,u} \), abutting to \( \{ K, L \}^{s+t,u} \). If \( K \) is of finite type, then this spectral sequence converges.

Margolis proves this at the prime 2; his proof carries over easily to the odd prime case.

This is our analog of the chromatic spectral sequence of [MRW77]; the \( s \)th column \( \{ K^{\langle x_s \rangle}, L \}^{**} \) contains information which should be, in some sense, periodic with respect to \( u_s \) and torsion with respect to all of the periodicity operators \( u_k \) when \( k < s \); the spectral sequence tells us how to assemble these various kinds of periodic information to get, essentially, the Adams \( E_2 \)-term.

**Remark 3.2.** If \( K \) is finite, then \( \{ K, L \}^{s,s} \cong \text{Ext}_A^{s,s}(K, L) \) for all \( s \geq 0 \), and \( \{ K, L \}^{s,s} = 0 \) if \( s < 0 \)—see [Pal92, Proposition 1.9]. So in this case we get precisely the Adams \( E_2 \)-term.

3.3. Bockstein spectral sequences. Another ingredient of the \( BP \) chromatic picture is the following: for each \( n \), there is a collection of Bockstein spectral sequences, in which the \( E_1 \)-term of the first is purely \( v_n \)-periodic and “computable” (as the cohomology of the \( n \)th Morava stabilizer algebra), each converges to the \( E_1 \)-term of the next, and the last converges to the \( n \)th column of the chromatic spectral sequence. We have the same machinery here, constructed in the same fashion as the \( BP \) version.

Fix \( n \) and a suitable \( I_n = (i_0, \ldots, i_{n-1}) \). Recall that for \( k \leq n \) we have \( I_k = (i_0, \ldots, i_{k-1}) \). The \( V(U_k I_n) \)’s with \( k \leq n \) fit together in \( n \) Bockstein short
exact sequences, starting with $V(\phi) = F_p$. Applying $\langle x_n \rangle$ is exact (e.g., tensor with $F_p(x_n)$), so we get new Bockstein short exact sequences:

$$0 \to V(U_{k+1}^l)(x_n) \to \Sigma^{-i(s(x_k))} V(U_k^l)(x_n) \xrightarrow{u_k^{i_k}} V(U_k^l)(x_n) \to 0.$$ 

Applying $\{-, L\}^{**}$ to these gives a collection of Bockstein spectral sequences, in which the $k$th has $E_1$-term $V(U_{n-k+1}^l)(x_n), L\}^{**}$ and abuts to $E_1$ of the next, $V(U_{n-k}^l)(x_n), L\}^{**}$. So we have a cascade of BSS’s, ending with $\{F_p(x_n), L\}^{**}$, as desired. (And, of course, we can tensor the short exact sequences with $K$ to compute $\{K\langle x_n \rangle, L\}^{**}$, instead.)

**Proposition 3.3.** If $L$ is stably finite, then these spectral sequences all converge.

**Proof.** This follows from [MRW77, Remark 3.11] and a vanishing line argument; namely, if we have a spectral sequence with $E_1 \cong \{V(U_{k+1}^l)(x_n), L\}^{**}$, abutting to $V(U_k^l)(x_n), L\}^{**}$, then to show that the spectral sequence converges, it suffices to show that $\{V(U_k^l)(x_n), L\}^{**}$ is torsion under multiplication by $u_k^{i_k}$. This follows from the vanishing line theorem of [MW81]: $V(U_k^l)(x_n), L\}^{**}$ has a vanishing line of slope $s(x_n)$ (see [Pal92] for the vanishing line theorem in this generality), and multiplication by $u_k^{i_k}$ acts at a slope of $s(x_k)$. Since $k < n$, then $s(x_k) < s(x_n)$.

### 3.4. The start of the Bockstein cascade.

The last piece of the chromatic machine for $BP$ is the computation of the $E_1$-term of the first BSS for each $n$, as the cohomology of the $n$th Morava stabilizer algebra (see [MRW77], [MR77]). We don’t have a precisely analogous result, but we have a (partial) replacement.

The salient feature of the $E_1$-term of the first BSS in our cascade is that $V(U_n^l)$ is stably finite of type $\langle x_n, \infty \rangle$, so we have the following corollary of Theorem 2.4 and Proposition 2.8.

**Lemma 3.4 ([Pal92], Theorem 2.2).** If $L$ is stably finite, then

$$\{V(U_n^l)(x_n), L\}^{**} \cong u_n^{-1} \text{Ext}^*_A(V(U_n^l), L).$$

So regardless of whether we can compute this, it is purely $u_n$-periodic, as in the $BP$ case.

There are some tools available for calculating localized Ext groups; we will discuss one here, the Margolis Adams spectral sequence.

**Theorem 3.5.** Fix a differential $x_n \in A$; let $A(x_n)$ be the algebra of operations for $x_n$-homology (defined below). Let $K$ be stably finite of type $\langle x_n, \infty \rangle$, $L$ stably finite; then there is a spectral sequence with

$$E_2 \cong \text{Ext}^*_A(x_n)(H(K, x_n), H(L, x_n)) \otimes F_p(u_n^{-1}),$$

abutting to $u_n^{-1} \text{Ext}^*_A(K, L)$. If $p = 2$ and $x_n = Q_j$, then the spectral sequence converges.
(Note that if $x_n = Q_j$, then $u_n = v_j$.)

Margolis constructed this spectral sequence in [Mar]; independently, Eisen used it to compute certain $v_j^{-1}\text{Ext}$ groups in his thesis [Eis87]. We prove Theorem 3.5 in [Pal]; we give a brief description here, for completeness.

First of all, we need to define the algebra of operations for $H(-, x_n)$; to do this we use the following lemma.

**Lemma 3.6** ([Mar83], 19.3). For any bounded below $A$-module $K$, we have

$$H_i(K, x_n) \cong \{A/Ax_n, K\}^0_{-i}.$$

Define the algebra of operations for $x_n$-homology, $A(x_n)$, to be the opposite algebra to $H(A/Ax_n, x_n) = \{A/Ax_n, A/Ax_n\}^0_{-*}$. Then for any $A$-module $K$, $H(K, x_n)$ is a left $A(x_n)$-module. The $A(x_n)$’s can be computed when $x_n = Q_j$ (e.g., for $p = 2$ see [Mar83, 19.25]).

The spectral sequence is constructed in a standard fashion: one takes an $A/Ax_n$-resolution of $K$ so that applying $H(-, x_n)$ gives an $A(x_n)$-free resolution of $H(K, x_n)$; then one applies $u_n^{-1}\text{Ext}^*_A(-, L)$. One uses Lemma 3.6 to identify the $E_2$-term. One can show that convergence follows from a certain generic condition; convergence then follows by exhibiting a nice enough module which satisfies this condition (for $p = 2$ and $x_n = Q_j$, this module is $\frac{1}{2}A_j = A_j/\overline{A}_jQ_j$; here $A_j$ is the sub-Hopf algebra of $A$ generated as an algebra by the set $\{P_{s+t} : s + t \leq j + 1\}$, with Mitchell’s $A$-module structure—see [Mit85]).

4. The geometric construction

In this section we give the Steenrod algebra analog of the chromatic filtration for spectra; as for the Margolis chromatic resolution of $F_p$, this first appeared for $p = 2$ in [Mar83, Section 22.2]. As above, our construction and our point of view are the new results here. This filtration bears some relation to the chromatic filtrations of [Rav84], [Rav] and [MS], but it is not clear exactly what this relation is.

Our main tool here is the self-map theorem for spectra, Theorem 2.3(b). As in the algebraic case, we give two descriptions: according to [Mar83, Section 22.2], we let $N_m = \ell^m S^0(x_m, \infty)$ and $M_m = \ell^m S^0(x_m)$; then we have stable cofibration sequences (as in Theorem 2.4)

$$N_m \to M_m \to N_{m+1}.$$ 

Splicing these together gives a “resolution” of $S^0$; smashing this with a connective spectrum $Z$ gives a tower over $Z$:

$$
\begin{array}{cccccc}
\quad & \quad & \quad & \quad & \quad & \quad \\
Z & \longrightarrow & Z \wedge M_0 & \longrightarrow & Z \wedge M_1 & \longrightarrow & \cdots \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
Z & \longrightarrow & Z(x_0) & \longrightarrow & \ell Z(x_1) & \longrightarrow & \cdots \\
\end{array}
$$
We can also rewrite this as
\[ Z \to \cdots \to Z\langle x_0, x_k \rangle \to \cdots \to Z\langle x_0, x_1 \rangle \to Z\langle x_0 \rangle, \]
as in [Mar83, Section 22.2] (where the maps are the ones given in Theorem 2.4). Call this the Margolis chromatic tower for \( Z \).

Instead we can construct the \( M_m \)'s and \( N_m \)'s using direct limits, as in the algebraic case. As before, start with \( N_0 = S^0 \). For any \( i_0 > 0 \), we have this diagram of stable cofibration sequences:
\[
\begin{array}{c}
S^0 \xrightarrow{u_0^{i_0}} \Sigma^{-i_0 s(x_0)} \ell^{i_0} S^0 & \longrightarrow & W(u_0^{i_0}) \\
\downarrow & & \downarrow \\
S^0 \xrightarrow{u_0^{2i_0}} \Sigma^{-2i_0 s(x_0)} \ell^{2i_0} S^0 & \longrightarrow & W(u_0^{2i_0}).
\end{array}
\]

Taking direct limits gives
\[ N_0 \to M_0 \to N_1, \]
with \( M_0 \approx S^0\langle x_0 \rangle \) and \( N_1 \approx \ell S^0\langle x_1, \infty \rangle \).

Now iterate this: for each \( m \geq 0 \) and suitable \( I_m \), we have
\[
\begin{array}{c}
W(U^I_m) \xrightarrow{u_m^{i_m}} \Sigma^{-i_m s(x_m)} \ell^{i_m} W(U^I_m) & \longrightarrow & W(U^I_{m+1}) \\
\downarrow & & \downarrow \\
W(U^I_m) \xrightarrow{u_m^{2i_m}} \Sigma^{-2i_m s(x_m)} \ell^{2i_m} W(U^I_m) & \longrightarrow & W(U^I_{m+1})
\end{array}
\]
(where \( i'_m = 2i_m \)). As in the algebraic case, we take limits (direct, this time), and end up with stable cofibration sequences
\[ \ell^m S^0\langle x_m, \infty \rangle \to \ell^m S^0\langle x_m \rangle \to \ell^{m+1} S^0\langle x_{m+1}, \infty \rangle, \]
as desired. This construction clearly bears some resemblance to that of the geometric chromatic filtration of [MS] and [Rav87]; in fact, Mahowald and Sadofsky use a modification of this construction to construct \( L'_n X \) and to study it via a localized Adams spectral sequence.

In any case, given a spectrum \( Z \), we get a tower over \( Z \); now we can apply \( \{S^0, -\} \).

**Theorem 4.1 ([Mar83], Theorem 22.5, Proposition 17.2).** Let \( Z \) be a connective \( p \)-local spectrum. \( Z \) is the limit (in the stable category of spectra) of its Margolis chromatic tower; hence, applying \( \{S^0, -\} \) to the Margolis chromatic tower for \( Z \) yields a Hausdorff, complete filtration of \( \{S^0, Z\} \); furthermore, if \( Z \) is finite, then \( \{S^0, Z\} \cong \pi_*(Z) \).

As in the module case, Margolis proves this at the prime 2; his proof works at odd primes, too. Note that this theorem says that \( Z \) is stably equivalent to the limit of its Margolis chromatic tower; in other words, the limit of the Margolis...
chromatic tower is $Z \vee K$ where $K$ is some mod $p$ generalized Eilenberg-Mac Lane spectrum.

Recall that $L_n$ is localization with respect to the spectrum $E(n)$; equivalently, it is localization with respect to

$$K(0) \vee \cdots \vee K(n).$$

$L'_n$ is localization with respect to

$$T(0) \vee \cdots \vee T(n),$$

where $T(k)$ is the telescope of a $v_k$-map on a finite spectrum of type $k$; $L'_n$ is independent of the choice of these finite spectra and the choice of the self-maps. (Note that $L'_n$ is also known as $L^f_n$—see [Mil].) One has the factorization

$$X \to L'_n X \to L_n X$$

for all $X$. See [MS], [Mil], [Rav], [Rav84], and [Rav87] for more information on these functors.

We have the following observations; the second implies the first (recall that $x_0 = Q_0$):

**Proposition 4.2** ([Mar83], p. 434). Fix $p = 2$; then $X \to L_n X$ factors through $X(Q_0, Q_n)$.

**Proposition 4.3.** For any $p$, $X \to L'_n X$ factors through $X(Q_0, Q_n)$.

(We prove this in Section 5.)

Indeed, we get a commutative diagram of towers

\[\begin{array}{ccc}
\vdots & \rightarrow & \vdots \\
X(Q_0, Q_n) & \rightarrow & L'_n X \\
\downarrow & & \downarrow \\
L'_n X & \rightarrow & L_n X
\end{array}\]

\[\begin{array}{ccc}
\vdots & \rightarrow & \vdots \\
X(Q_0, Q_1) & \rightarrow & L'_1 X \\
\downarrow & & \downarrow \\
L'_1 X & \rightarrow & L_1 X
\end{array}\]

\[\begin{array}{ccc}
\vdots & \rightarrow & \vdots \\
X(Q_0) & \rightarrow & L'_0 X \\
\downarrow & & \downarrow \\
L'_0 X & \rightarrow & L_0 X
\end{array}\]

Note that the tower on the left has intermediate terms from the differentials between the $Q_n$'s. This reflects the fact that $X \to L'_n X$ factors through $X \to X(Q_0, x_k)$ whenever $s(x_k) \geq s(Q_n)$.
Assume $X$ is finite. The tower on the right converges, in the sense that $X$ is the homotopy inverse limit of $L_n X$; this is due to Hopkins and Ravenel in [HR]; see also [Rav92]. The tower on the left converges up to stable equivalence; this is the gist of Theorem 4.1. It is not known whether the middle tower converges.

In any case, the Margolis chromatic filtration for spectra is clearly related to these chromatic filtrations; this connection should be of some use in studying chromatic phenomena, as evidenced by the work in [MS].

5. A few proofs

5.1. If $M$ is of finite type, so is $M \langle x_m, x_n \rangle$. In [Pal92] we gave a proof of Margolis’ killing construction; here we prove that if $M$ is an $A$-module of finite type (and bounded below, of course), then $M \langle x_m, x_n \rangle$ is of finite type, as well. As in the proof of the killing construction, [Pal92, Theorem 3.3], it suffices to prove this for $F_p \langle x_n, \infty \rangle$ for each differential $x \in A$.

**Proposition 5.1.** Fix a differential $x_n$ in $A$. The module $F_p \langle x_n, \infty \rangle$ is stably equivalent to a module of finite type.

**Corollary 5.2.** Fix a differential $x_n$ in $A$. The spectrum $S^0 \langle x_n, \infty \rangle$ is stably equivalent to a locally finite spectrum.

**Proof.** Recall (Proposition 2.8) that $F_p \langle x_n, \infty \rangle$ is constructed as an inverse limit of modules $\Omega^{-n} V(U^{I_n})$. We get an inverse system of the $\Omega^{-n} V(U^{I_n})$’s as we increase $I_n$; we want to check that we get an isomorphism through an initial range of dimensions, increasing with $I_n$. Since the $\Omega^{-n} V(U^{I_n})$’s are of finite type, this is good enough.

Proposition 5.1 is a consequence of the following.

**Lemma 5.3.** Given $d$, for $J_n$ large enough (i.e., each $j_k \gg 0$) and $I_n > J_n$, $\Omega^{-n} V(U^{I_n})$ and $\Omega^{-n} V(U^{J_n})$ may be chosen so that the induced map

$$\Omega^{-n} V(U^{I_n}) \to \Omega^{-n} V(U^{J_n})$$

is an isomorphism through degree $d$.

**Proof.** The proof is by induction on $k$. Assume that we have stably finite modules $V$ and $W$, both of type $\langle x_k, \infty \rangle$ with non-zero $x_k$-homology. Furthermore, assume that we have $g : V \to W$ which is an isomorphism through degree $d$. Then for suitable $i$ and $j$ with $i > j$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \Sigma^{-is(x_k)} \Omega^i V & \xrightarrow{u_k} & V & \xrightarrow{g} & V' & \to & 0 \\
\downarrow{\Omega^i g} & & \downarrow{g} & & \downarrow{g'} & & \downarrow{g'} & & \\
0 & \to & \Sigma^{-is(x_k)} \Omega^i W & \xrightarrow{u_k} & W & \xrightarrow{g} & W' & \to & 0 \\
\downarrow{\Omega^i u_k} & & \downarrow{u_k} & & \downarrow{u_k} & & \downarrow{u_k} & & \\
0 & \to & \Sigma^{-js(x_k)} \Omega^j W & \xrightarrow{u_k} & W & \xrightarrow{g} & W' & \to & 0.
\end{array}
$$
The map $\Omega^d g$ is an isomorphism through degree $d$; by a relative vanishing line argument (as in [Pal92, Theorem 2.2(d)]), if $j \gg 0$ then $u^{i-j}_k$ is an isomorphism through degree $d$; hence the induced map $g' : V' \to W'$ is, also.

Now apply the above argument to the case $V = \Omega^{-(k-1)}V(U_{k-1}^{I_k})$ and $W = \Omega^{-(k-1)}V(U_{k-1}^{J_k})$ (so $V' = \Omega^{-k}V(U_k^{I_k})$ and $W' = \Omega^{-k}V(U_k^{J_k})$).

This finishes the proof of Proposition 5.1.

5.2. $X \to L'_nX$ factors through $X \to X(Q_0, Q_n)$.

Proof. Let $x_{k-1} = Q_n$; let $U = (u_0, u_1, \ldots, u_{k-1}, \ldots)$. We have a cofibration

$$Z \to W(U_k^{I_k}) \to M(U_k^{I_k}),$$

where $W(U_k^{I_k})$ is a stably finite spectrum, $M(U_k^{I_k})$ is finite, and $Z$ has a finite Adams resolution. In particular, $M(U_k^{I_k})$ is defined inductively by cofibrations

$$M(U_k^{I_k}) \xrightarrow{u^r_k} \Sigma^{-i_r} M(U_k^{I_k}) \to M(U_{k+1}^{I_k}),$$

for $r = 0, 1, \ldots, k-1$ (see [PS]). Note that $M(U_k^{I_k})$ is constructed so that its top cell is in dimension $k$, and as with $W(U_k^{I_k})$, there is a map $\Sigma^{-k} M(U_k^{I_k}) \to S^0$.

We define $\overline{M}(U_k^{I_k})$ by the cofibration

$$\Sigma^{-k} M(U_k^{I_k}) \to S^0 \to \overline{M}(U_k^{I_k}),$$

and we define $\overline{W}(U_k^{I_k})$ by the stable cofibration

$$\ell^{-k} W(U_k^{I_k}) \to S^0 \to \overline{W}(U_k^{I_k}).$$

In [MS], Mahowald and Sadofsky show that

$$\lim_{\to} \overline{M}(U_k^{I_k}) = L'_n S^0;$$

in [PS], Sadofsky and the author show that

$$\lim_{\to} \overline{W}(U_k^{I_k}) = S^0 \langle Q_0, Q_n \rangle.$$

We want to construct a map $S^0 \langle Q_0, Q_n \rangle \to L'_n S^0$ that factors the $L'_n$ localization map.

Given a spectrum $X$, let $F_k X$ denote the $k$th term in an Adams tower for $X$. Observe that for any $k$, $\ell^k X \simeq \Sigma^k F_k X$.

We need to do two things: we need to show that the map $W(U_k^{I_k}) \to M(U_k^{I_k})$ factors through $\Sigma^k \ell^{-k} W(U_k^{I_k})$, and we need to show that the resulting maps in the two direct systems are compatible.

We’re given a map $W(U_k^{I_k}) \to M(U_k^{I_k})$. Now, for any finite spectrum $Y$ and any connective spectrum $X$, any map $\ell X \to Y$ is zero on cohomology (cf. [Pal92, Proposition 1.9]); hence it lifts to a map $\ell X \to FY$, and so to a map $\Sigma X \to Y$. Apply this observation $k$ times to the map $W(U_k^{I_k}) \to M(U_k^{I_k})$, using $W(U_k^{I_k}) = \ell^k(\ell^{-k} W(U_k^{I_k}))$, to get the desired factorization.
By applying \([PS, \text{Lemma } 3.1]\), we see that (if \(I_k\) is sufficiently larger than \(J_k\)) the map \(\tau^{-k}W(U_1^{I_k}) \to \tau^{-k}W(U_1^{J_k})\) induces a map \(\Sigma^{-k}M(U_1^{I_k}) \to \Sigma^{-k}M(U_1^{J_k})\), so we get a map of direct systems if we use these maps between the \(M\)’s instead of the ones in \([MS]\). But then the nilpotence theorem says that the two direct systems of \(M\)’s have the same direct limit; this finishes the proof. \(\square\)

**References**


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