A GLOBAL STRUCTURE THEOREM FOR THE MOD TWO
DICKSON ALGEBRAS, AND UNSTABLE CYCLIC MODULES
OVER THE STEENROD AND KUDO-ARAKI-MAY
ALGEBRAS

DAVID J. PENGELLEY, FRANKLIN P. PETERSON, AND FRANK WILLIAMS

Abstract. The Dickson algebra \( W_{n+1} \) of invariants in a polynomial algebra over \( F_2 \) is an unstable algebra over the mod 2 Steenrod algebra \( A \), or equivalently, over the Kudo-Araki-May algebra \( K \) of “lower” operations. We prove that \( W_{n+1} \) is a free unstable algebra on a certain cyclic module, modulo just one additional relation. To achieve this, we analyze the interplay of actions over \( A \) and \( K \) to characterize unstable cyclic modules with trivial action by the subalgebra \( A_{n-2} \) on a fundamental class in degree \( 2^n - a \), thereby verifying unstable instances of a conjectured basis for \( A/A_{n-2} \). This involves a new family of left ideals \( I_a \) in \( K \), which play the role filled by the ideals \( A/\overline{A}_{n-2} \) in the Steenrod algebra.

December 9, 1999

1. Introduction.

Let \( W_{n+1} \) be the Dickson algebra of invariants in a polynomial algebra over \( \mathbb{F}_2 \) under the action of the general linear group \( GL(n+1, \mathbb{F}_2) \). Many have studied the action of the Steenrod operations on the Dickson invariants [W]. We are interested in a global structure theorem for \( W_{n+1} \) in the category of unstable algebras over the Steenrod algebra \( A \), or equivalently, over the Kudo-Araki-May algebra \( K \) [PW]. The unstable module that arises in this context leads to a consideration of other unstable modules. These modules are cyclic unstable modules over \( A \) (equivalently over \( K \)) that have an especially simple action on their lowest positive class. The results and techniques of this study demonstrate that using the two equivalent actions simultaneously can be more powerful, productive, and descriptive than using either alone.

1991 Mathematics Subject Classification. Primary 55S05; Secondary 16W30, 16W50, 55S10.

Key words and phrases. Steenrod algebra, Kudo-Araki-May algebra, Dickson algebras, unstable cyclic modules and algebras.
Recall that $W_{n+1} = \mathbb{F}_2[x_{i,n+1-i} | 0 \leq i \leq n]$, where $x_{i,n+1-i}$ has degree $2^i (2^{i+1-i} - 1)$ and $x_{n+1,0} = 1$. Let $D = \sum_{i \geq 0} D_i$ be the total “lower” operation from $\mathcal{K}$, and similarly $Sq = \sum_{i \geq 0} Sq_i$ from $\mathcal{A}$. Recall that the governing relationship between these is $Sq^k z_m = D_{m-k} z_m$. Recall also that on the lowest positive class $x_{n,1}$ in dimension $2^n$ of $W_{n+1}$ the module action is given by

$$Dx_{n,1} = Sq x_{n,1} = x_{n,1}^2 + \sum_{0 \leq j \leq n} x_{j,n+1-j}.$$  

Notice that the smallest nontrivial Steenrod operation is $Sq^{2^{n-1}} x_{n,1} = x_{n-1,2}$, and the only other nonzero $2$-power Steenrod square on $x_{n,1}$ is the cup-square $Sq^2 x_{n,1} = x_{n,1}^2$, a particularly interesting situation. On the other hand, rephrasing this instead with $\mathcal{K}$ reveals a very tractable pattern to exploit:

$$Dx_{n,1} = D_{2^n} x_{n,1} + D_{2^{n-1}} x_{n,1} + \cdots + D_1 x_{n,1} + D_0 x_{n,1},$$

with each term nonzero. More generally, we shall see that the admissibles in $K$ (monomials $D_I$ with nondecreasing indices) that act nontrivially (and nonredundantly) on $x_{n,1}$ are exactly those of the form $D_{n_1} D_{2^{n_2}} D_{2^{n_3}} \cdots D_{2^{n_k}}$, where $0 \leq m_1 \leq \cdots \leq m_k \leq n-1$. This beautiful situation is a prime motivation for this paper.

First we study a bi-indexed family of cyclic modules, defining $\mathcal{M}(n,a) = \mathcal{F}_{2^n-a}/\overline{\mathcal{A}_{n-2} x_{2^n-a}}$ for $0 \leq a \leq 2^n$ to be the quotient of the free unstable $\mathcal{A}$-module $\mathcal{F}_{2^n-a}$ on a class of degree $2^n - a$ by the action of the left ideal $\mathcal{A}_{n-2} = \mathcal{A}(Sq_1, \ldots, Sq^{2^n-2})$. In Theorem 2.11 we will first obtain a basis for the special case $\mathcal{M}(n,0)$, by realizing it as the submodule of the Dickson algebra $W_{n+1}$ generated by $x_{n,1}$. analogous to the description above of a $\mathcal{K}$-basis for this submodule, we shall see also for $a > 0$ that regarding the $\mathcal{M}(n,a)$ as $K$-modules provides them with attractively described bases. We shall provide complete descriptions of the $\mathcal{M}(n,a)$ as cyclic $K$-modules (and, as a corollary, also as cyclic $\mathcal{A}$-modules), including the kernels of the projections from the free modules, in terms of the admissible basis for $K$ (and $\mathcal{A}$). To do this we define and obtain information on the structure of an interesting family of left ideals $I_a (a \geq 0)$ in $K$, and also define certain closely associated and easily described $(n,a)$-special admissibles in $\mathcal{F}_{2^n-a}$. These will be the elements $D_I x_{2^n-a}$ for which the multi-index $I$ is nondecreasing and has its entries less than $2^n - a$ and only of the form $2^k - a$ (also allowing $0$ if $a = 0$). Then our first main result (Theorem 2.11) is

**Theorem.** A basis for $\mathcal{M}(n,a)$ is given by the images of the $(n,a)$-special admissibles from $\mathcal{F}_{2^n-a}$. The ideal $I_a$ annihilates the fundamental class in $\mathcal{M}(n,a)$, and

$$\mathcal{M}(n,a) = \mathcal{F}_{2^n-a}/\overline{\mathcal{A}_{n-2} x_{2^n-a}} = \mathcal{F}_{2^n-a}/I_a x_{2^n-a}.$$
Further, the image of any non-\((n,a)\)-special admissible from \(\mathcal{F}_{2^n-a}\) is zero.

We use our representation of \(M(n,0)\) as isomorphic to a submodule of \(W_{n+1}\) to give a global structure theorem for \(W_{n+1}\), our second main result (Theorem 4.1).

**Theorem.** Consider the free unstable algebra \(U_{n+1}\) over \(K\), or equivalently over \(A\) (as in [SE, pp. 26–29]), formed from \(M(n,0)\). Then the Dickson algebra \(W_{n+1}\) is isomorphic as a \(K\)-algebra to the quotient \(G_{n+1}\) of \(U_{n+1}\) by the one additional relation
\[
D_{2^{n-1}}D_{2^n-1}(x_{2^n}) = x_{2^n} \cdot D_{2^{n-1}}(x_{2^n}),
\]
or equivalently, as an \(A\)-algebra, by the relation
\[
Sq^{2^n}Sq^{2^{n-1}}(x_{2^n}) = x_{2^n} \cdot Sq^{2^{n-1}}(x_{2^n}).
\]

The following is an immediate corollary.

**Corollary.** Let \(UA\) denote the category of unstable algebras over \(A\). Let \(B\) be in \(UA\). Then
\[
\text{Hom}_{UA}(W_{n+1}, B) = \{b \in B^{2^n} \mid Sq^{2^i}(b) = 0 \text{ for } 0 \leq i < n-1 \text{ and } Sq^{2^n}Sq^{2^{n-1}}(b) = b \cdot Sq^{2^{n-1}}(b) \}.
\]

Returning to the simple \(K\)-module basis for \(M(n,a)\) given above from Theorem 2.11, and translating it into the notation of the \(A\)-action, yields the following result (Corollary 2.12), whose neat formulation in terms of alpha numbers was suggested to the authors by Haynes Miller.

**Corollary.** A basis for \(M(n,a)\) in terms of \(A\) is given by those admissibles \(Sq^{\alpha_1} \cdots Sq^{\alpha_q} x_{2^n-a}\) of excess \(\leq 2^n - a\) such that for each \(j\), \(\alpha(p_j-1) \geq n-1\). (Here \(\alpha(p_j-1)\) denotes, as usual, the number of ones in the binary expansion of \(p_j-1\).)

We will also provide a specific procedure, in Remark 2.13, for producing the \(A\)-admissibles in this basis.

The alpha number condition of the corollary immediately suggests an admissible basis for \(A/A\mathbb{F}_{n-2}\), by simply removing the unstable excess condition. However, R. Kleinberg has recently shown that, while this holds through \(A/A\mathbb{F}_{1}\), it fails for \(A/A\mathbb{F}_k, k \geq 2\).

We have instead considered an alternative conjecture for extending the unstable admissible \(K\)-bases of Theorem 2.11 to the stable setting, based on the stabilization \(K(\infty)\) [PW] of the Kudo-Araki-May algebra \(K\), for which there is a sheared algebra bijection \(\psi_\infty: K(\infty) \to A\). However, the most natural generalization in this context also fails to hold stably. It thus still remains to be seen if one can find a reasonable conjecture for analogous (Cartan-Serre) admissible bases for the stabilized modules \(A/A\mathbb{F}_k\).
Finally, returning again to our first main theorem above, we observe that it provides a sense in which the family of ideals \( \{ I_a \} \) in \( K \) is analogous to the family of ideals \( \{ A_{a-2} \} \) in \( A \). In fact \( I_a \) will be defined as the left ideal generated by the elements \( D_i \) for which \( i \) is not of the form \( 2^k - a \) (nor 0 if \( a = 0 \)), and a very important feature of the ideal \( I_a \) is that it vanishes on the fundamental class of \( M(n, a) \). A curious and eminently useful feature of \( I_a \), given in Theorem 2.9, is that, while it is not a two-sided ideal, any admissible in \( K \) that contains one of the generators of the ideal in its admissible representation automatically belongs to the ideal.

With our theorem on unstable modules in hand, we can provide a similar result for the ideal \( I_a \) in \( K \) itself. First we analogously define \( a \)-special admissibles in \( K \) to be those \( D_I \) for which \( I \) is nondecreasing (i.e., admissible) and has entries only of the form \( 2^k - a \) (also allowing 0 if \( a = 0 \)). Then we have (Theorem 2.15)

**Theorem.** For any \( a \geq 0 \), the non-\( a \)-special admissibles are a basis for \( I_a \), so the \( a \)-special admissibles project to a basis for \( K/I_a \).

Of course this result has no translation in terms of \( A \), since we are not considering an unstable module of any particular degree here.

2. **Cyclic modules and the ideals \( I_a \) in \( K \).**

We begin by recalling the definition and some properties of the Kudo-Araki-May algebra [PW].

**Definition 2.1.** The bialgebra \( K = K_{*,*} \) is the \( \mathbb{F}_2 \)-algebra with identity in bidegree \((0,0)\) and generated by elements \( D_i \in K_{1,i} \) \((i = 0,1,\ldots)\), subject to the (Adem) relations

\[
D_i D_j = \sum_k \binom{k - 1 - j}{2k - i - j} D_{i+2j-2k}D_k, \quad (i > j).
\]

The bidegree of elements in \( K \) is defined inductively by the requirement that the multiplication be a map

\[
K_{m,i} \otimes K_{n,j} \rightarrow K_{m+n,i+2^n j}.
\]

We recall that \( D_I = D_{i_1} \cdots D_{i_q} \) is called an admissible if the multi-index \( I = (i_1,\ldots,i_q) \) is nondecreasing, and that the admissibles form a basis for \( K \).

In [PW] we defined unstable module and algebra structures over \( K \). There, in order to present a single unified unstable setting encompassing both the Dyer-Lashof and Steenrod algebras, \( K \)-modules arising from cohomology were nonpositively graded, whereas here we reverse this, returning to the tradition of grading them nonnegatively. Recall then that there is a natural correspondence.
between the unstable cohomology module (or algebra) structures for a nonnegatively graded vector space over \( K \) and unstable ones over the Steenrod algebra \( A \), via iteration of the relationship \( Sq^k x_m = D_{m-k} x_m \) on a class of degree \( m \).

**Definition 2.2.** Let \( F_m \) denote the free unstable module (equivalently over either \( K \) or \( A \)) on a single generator \( x_m \) of degree \( m \).

Clearly a basis for \( F_m \) is given by

\[
\{ D_I x_m \mid I \text{ is nondecreasing, with largest entry less than } m \},
\]

and we call such \( D_I x_m \) the admissible representation of these elements.

Now we define a simple quotient module we wish to study.

**Definition 2.3.** For any \( n \geq 0 \) and any \( 0 \leq a \leq 2^n \), let \( \mathcal{M}(n,a) \) denote the quotient module of \( F_{2^n-a} \) by the action of \( \overline{A_{n-2}} = A(Sq^1, \ldots, Sq^{2^n-2}) \) on \( x_{2^n-a} \), i.e.,

\[
\mathcal{M}(n,a) = F_{2^n-a} / \overline{A_{n-2}} x_{2^n-a}.
\]

We will describe \( \mathcal{M}(n,a) \) first using the images of certain selected \( K \)-admissibles from \( F_{2^n-a} \).

The following example indicates which admissibles we should select.

**Example 2.4.** We take \( n = 3 \) and \( a = 0 \). Thus we are looking at the cyclic unstable \( A \)-module on a class of degree 8, modulo the action of \( A(Sq^1, Sq^2) \). In low degrees we have basis elements consisting of the generator \( x_8 \) and the images of \( x_8 \) under the Steenrod algebra elements \( Sq^4, Sq^6, Sq^7, Sq^8, Sq^8 Sq^4, Sq^{10} Sq^4, Sq^{11} Sq^4, Sq^{12} Sq^4, Sq^{12} Sq^6, Sq^{13} Sq^6, Sq^{13} Sq^6, \ldots \). It is a little difficult to determine by inspection exactly which admissibles are in this basis. But when we translate these elements into lower notation, they become immediately recognizable from their appearance. As a \( K \)-module, these elements are the images of \( x_8 \) under the elements \( D_1, D_2, D_1, D_0, D_1 D_4, D_2 D_4, D_1 D_4, D_0 D_4, D_2 D_2, D_1 D_2, D_0 D_2, \ldots \), in fact, exactly the admissible monomials in this range of degrees that are formed from \( D_0, D_1, D_2, \) and \( D_4 \).

If we take \( n = 3 \) and \( a = 1 \), so that we are now looking at the cyclic unstable \( A \)-module on a class of degree 7, again modulo the action of \( A(Sq^1, Sq^2) \), we get a basis in low dimensions given by the action on \( x_7 \) of \( Sq^4, Sq^6, Sq^7, Sq^8 Sq^4, Sq^{10} Sq^4, Sq^{11} Sq^4, Sq^{12} Sq^6, Sq^{13} Sq^6, \ldots \). In terms of \( K \), these elements become the images under \( D_3, D_1, D_0, D_3 D_3, D_1 D_3, D_0 D_3, D_1 D_1, D_0 D_1, \ldots \). This time we get the admissibles formed from \( D_0, D_1, \) and \( D_3 \).

The patterns that appear in examples such as these motivate the following definition.

**Definition 2.5.** If \( D_I x_{2^n-a} \) is admissible in \( F_{2^n-a} \) for a pair \( (n,a) \), and if \( I \) consists only of entries of the form \( 2^k - a \) (also allowing 0 if \( a = 0 \)), then we call the admissible \((n,a)\)-special.
Remark 2.6. Note that the \((n;a)\)-special admissibles all occur in distinct degrees (see the form of their degrees in the proof of Lemma 3.6).

We will show that the images of the \((n;a)\)-special admissibles form a basis for \(M(n,a)\). In fact we will do this and more in Theorem 2.11, showing that all other (i.e. non-\((n,a)\)-special) admissibles in \(F_{2^n-a}\) map to zero in \(M(n,a)\). In fact we shall see that, for fixed \(a\), the admissibles in \(K\) corresponding to non-\((n,a)\)-special admissibles in \(F_{2^n-a}\), for any \(n\), form a basis for a left ideal \(I_a\) in \(K\) (which ideal depends on \(a\), but not on \(n\)), even though these particular admissibles do not at first appear, from the definition of special, to form a left ideal of \(K\) at all.

Definition 2.7. For each \(a \geq 0\), let

\[ I_a = K \cdot \{ D_i \mid i \text{ is not of the form } 2^k - a, \]

and, if \(a = 0\), then additionally \(i \neq 0\).

Remark 2.8. In particular, \(D_0 \in I_a\) precisely when \(a \neq 0, 2^k\).

The following theorem (proven in the next section) shows that the admissibles in \(K\) corresponding to non-\((n,a)\)-special admissibles in \(F_{2^n-a}\) for any \(n\) lie in \(I_a\), and later we will show that they actually form a basis for the ideal:

Theorem 2.9. Let \(J = (2^{m_1} - a, \ldots, 2^{m_q} - a)\), where \(0 \leq m_1 \leq \cdots \leq m_q\). Let \(p < 2^{m_1+1} - 2a\) be not of the form \(2^k - a\) (nor 0 when \(a = 0\)). Then \(D_pD_J \in I_a\).

Hence, if \(D_I\) is admissible and any index in \(I\) is not of the form \(2^k - a\) (nor 0 when \(a = 0\)), then \(D_I \in I_a\).

Remark 2.10. Despite this theorem, \(I_a\) is not a 2-sided ideal. For instance, if we consider \(a = 0\) and examine \(K_{2,9}\), with basis elements \(D_1D_4\) and \(D_3D_3\), we find the relations \(D_9D_0 = 0\) and \(D_7D_1 = D_5D_2 = D_1D_4\) for the other possible monomials. Thus in this bidegree \(I_0\) is spanned by \(D_3D_3\), so \(D_5D_2 \notin I_0\), despite the fact that \(D_5 \in I_0\).

Our main theorem for unstable modules (also proven in the next section) is

Theorem 2.11. A basis for \(M(n,a)\) is given by the images of the \((n,a)\)-special admissibles from \(F_{2^n-a}\). The ideal \(I_a\) annihilates the fundamental class in \(M(n,a)\), and

\[ M(n,a) = F_{2^n-a}/I_a x_{2^n-a}. \]

Further, the image of any non-\((n,a)\)-special admissible from \(F_{2^n-a}\) is zero.

As mentioned in the introduction, this can be translated to a description over \(\mathcal{A}\):
Corollary 2.12. A basis for $\mathcal{M}(n,a)$ in terms of $\mathcal{A}$ is given by admissibles $Sq^{p_1} \cdots Sq^{p_q}x_{2^n-a}$ of excess $\leq 2^n-a$ such that for each $j$, $\alpha(p_j-1) \geq n-1$. (Here $\alpha(p_j-1)$ denotes, as usual, the number of ones in the binary expansion of $p_j-1$.)

Proof. We shall briefly outline how the corollary follows from the theorem. First, we note that a $K$-admissible $D_{i_1} \cdots D_{i_q}$ acting on a class $x_m$ of degree $m$ translates to $Sq^{p_1} \cdots Sq^{p_q}x_m$, where $p_j = 2^{q-j}m - 2^{q-j-1}i_q - \cdots - 2i_{j+2} - i_{j+1} - i_j$. It is straightforward to check that if $D_{i_1} \cdots D_{i_q}$ is an $(n,a)$-special admissible, then each $(p_j-1)$ is as specified in the corollary, then it translates to an $(n,a)$-special admissible $D_{i_1} \cdots D_{i_q}x_{2^n-a}$. \hfill $\Box$

Remark 2.13. As also mentioned in the introduction, there is an alternative way of expressing the $\mathcal{A}$-admissibles in this basis for $\mathcal{M}(n,a)$. We will explain via two examples of special admissible basis elements in $\mathcal{M}(3,0)$. Consider first the $(3,0)$-special admissible basis element $D_0D_0D_1D_4D_4(D_8)x_8$, which converts to the Cartan-Serre admissible

$$Sq^{142}Sq^{71}Sq^{35}Sq^{16}Sq^{8}(Sq^0)x_8.$$ 

Notice how the excess in the Cartan-Serre admissible, produced from the successive amounts by which each $Sq$ exceeds twice the one to its right, is reflected in the $K$ description precisely as the successive differences between the two-power indices, which we can record as $(0,1,3,0,0,4)$. Moreover, since these differences step only through monotone two-powers, this can be written as $(0,1,1+2,0,0,4)$, where the sum merely reflects that we jumped two steps at once; and in general for any $(3,0)$-special admissible on $x_8$ we must always see these same numbers $1,1,2,4$, in this order (or some right segment thereof, which will produce excess strictly less than 8), perhaps combined by partial sums as in the example, and perhaps separated by some zeros, including possible leading zeros. Notice now that the procedure for producing the equivalent Cartan-Serre admissible directly from our sequence $(0,1,1+2,0,0,4)$ is simply to begin with the ghostly $(Sq^0)$ at the right, and proceed from right to left, always doubling the degree of the most recently added $Sq$ and adding the next number from our sequence to produce the degree of the next $Sq$ to be adjoined.

As a second example, let us take as our sequence

$$(1,0,2+4),$$

which fits the procedure we have described, but uses only the right segment $(1,2,4)$ of the basic $(1,1,2,4)$, and will thus produce a basis element with excess less than 8. Using it to produce differences between two-powers immediately yields $D_1D_2D_2(D_8)x_8$, and our procedure for using doubling and
adding to produce the Cartan-Serre equivalent directly from \((1, 0, 2 + 4)\) yields \(S_8^{25}S_8^{12}S_8^6(S_8^0)x_8\).

In general, for \(M(n, 0)\) we simply follow this procedure using the numbers \(1, 1, 2, 4, \ldots, 2^{n-1}\); and for \(M(n, a)\) with \(a > 0\), use only the admissibles thus produced that have their excess \(\leq 2^n - a\), i.e., if \(2^{k-1} < a \leq 2^k\), then use in the procedure only the proper sub-segment of numbers \(2^k, 2^{k+1}, \ldots, 2^{n-1}\).

We will end this section by deriving from Theorem 2.11 the overall implications we claimed for \(I_a\), and subsequently for the quotient \(K/I_a\).

**Definition 2.14.** For fixed \(a \geq 0\), if \(D_I\) is admissible in \(K\), and if \(I\) consists only of entries of the form \(2^k - a\) (also allowing 0 if \(a = 0\)), then we call the admissible \(a\)-special in \(K\).

**Theorem 2.15.** For any \(a \geq 0\), the non-\(a\)-special admissibles are a basis for \(I_a\), so the \(a\)-special admissibles project to a basis for \(K/I_a\).

**Proof.** By choosing \(n\) sufficiently large, we can faithfully represent \(K\) through any desired range on \(x_{2^n-a}\). In this representation the \(a\)-special admissibles of \(K\) correspond precisely to the \((n; a)\)-special admissibles in \(F_{2^n-a}\). The result follows immediately from Theorem 2.11.

3. **Proofs for cyclic modules.**

The proof of Theorem 2.9 will follow quickly from the following two lemmas, for whose proofs the following observation is useful.

**Remark 3.1.** The possible non-zero terms in the right hand side of the Adem relations of Definition 2.1 occur only for \(k\) in the range \((i+j)/2 \leq k \leq \min\{i-1, (i+2j)/2\}\).

**Lemma 3.2.** If \(p < 2^m - a\), then
\[
D_pD_{2^m-a} = D_{2^{m+1}-2a-p}D_p + \text{terms } D_rD_s
\]
where \(2^m - a < s < 2^{m+1} - 2a - p\).

**Proof.** Immediate from the Adem relation for \(D_{2^{m+1}-2a-p}D_p\).

**Lemma 3.3.** If \(p > 2^m - a\), then
\[
D_pD_{2^m-a} = \text{sum of terms } D_rD_s
\]
where \(2^m - a < s < p\).

**Proof.** Immediate from the Adem relation for \(D_pD_{2^m-a}\).

**Corollary 3.4.** If \(p < 2^{m+1} - 2a\) is not of the form \(2^k - a\) (nor 0 if \(a = 0\)), then
\[
D_pD_{2^m-a} = \text{sum of terms } D_rD_s
\]
where \( s < 2^{m+1} - 2a \) is not of the form \( 2^k - a \) (nor 0 if \( a = 0 \)).

**Proof of Theorem 2.9.** Induction on \( q \). If \( q = 0 \), the theorem is simply the definition of \( \mathcal{I}_a \). So suppose that \( q \geq 1 \). Let

\[
J' = (2^{m_2} - a, \ldots, 2^{m_q} - a).
\]

By Corollary 3.4,

\[
D_p D_J = D_p D_{2^{m_1} - a} D_{J'} = \text{sum of terms } D_r D_s D_{J'},
\]

where \( s < 2^{m_1 + 1} - 2a \) is not of the form \( 2^k - a \) (nor 0 if \( a = 0 \)). Since \( m_1 \leq m_2 \) and the length of \( J' \) is less than \( q \), the induction is complete. \( \square \)

Theorem 2.11 will now follow from Theorem 2.9 and another sequence of lemmas. As a first step we confirm that the ideal \( \mathcal{I}_a \) annihilates the fundamental class in \( \mathcal{M}(n, a) \).

**Lemma 3.5.** In \( \mathcal{M}(n, a) \), \( \mathcal{I}_a x_2^{m_1} - a = 0 \).

**Proof.** Consider \( D_l x_2^{m_1} - a = Sq^{2^k - a - i} x_2^{m_1} - a \) for \( i \neq 2^k - a \) (and, if \( a = 0 \), then additionally \( i \neq 0 \)). Notice that \( 2^n - a - i \neq 2^n - 2^k \) (and, if \( a = 0 \), then additionally \( \neq 2^n \)). But \( 2^n - 2^k \) (and \( 2^n \) if \( a = 0 \)) are the only degrees up to \( 2^n - 1 \) (to \( 2^n \) if \( a = 0 \)) in which \( \mathcal{A}/\mathcal{A}_{n-2} \) is nonzero, since \( (\mathcal{A}/\mathcal{A}_{n-2})^* = \mathbb{F}_2[\zeta_1^{2^{n-1}}, \zeta_2^{2^{n-2}}, \ldots, \zeta_{n-1}^2, \zeta_n, \ldots] \). (This well-known equality does not seem to have a proof in the literature, so we provide one in the appendix.) \( \square \)

**Lemma 3.6.** Let \( y_l \) be an \((n, a)\)-special admissible in \( \mathcal{M}(n, a) \) in degree \( l \). Let \( 2^k \mid l + a \). If \( i \leq k - 2 \), then \( Sq^2 z_p = 0 \).

**Proof.** Induction on the length of \( I = (2^{m_1} - a, \ldots, 2^{m_q} - a) \) [allowing also zeros if \( a = 0 \)], where \( y_l = D_l x_2^{m_1} - a \). If \( I \) is empty, the lemma follows immediately from the definition of \( \mathcal{M}(n, a) \). So write \( y_l = D_{2^{m_1} - a} z_p \) [allowing also \( D_0 z_p \) if \( a = 0 \)], with \( z_p = D_p x_2^{m_1} - a \) of degree \( p \). Since \( l = 2^{n+q} - 2^{m_q} + 1 - \ldots - 2^{m_2+1} - 2^{m_1} - a \), we thus have \( k \leq m_1 \).

Now we compute

\[
Sq^2 y_l = Sq^2 Sq^{2^{m_1} + a} z_p \quad \text{[} Sq^2 z_p \text{ if } a = 0 \text{ and } y_l = D_0 z_p \text{]}
\]

\[
= d \cdot Sq^{p-2^{m_1} + a+2} z_p \quad \text{(for } d \in \mathbb{F}_2 \text{)}
\]

\[
[d \cdot Sq^{p+2^l} z_p \text{ if } a = 0 \text{ and } y_l = D_0 z_p]
\]

\[
+ \text{terms that are zero inductively (since 2-power squares generate } \mathcal{A})
\]

\[
= d \cdot D_{2^{m_1} - 2^{l} - a} z_p = d \cdot D_{2^{m_1} - 2^{l} - a} D_p x_2^{m_1} - a
\]

\[
[0 \text{ if } a = 0 \text{ and } y_l = D_0 z_p],
\]

where \( s < 2^{m+1} - 2a \) is not of the form \( 2^k - a \) (nor 0 if \( a = 0 \)).
where \( D_{2m_1-2^i-a}D_1 \) is admissible because \( D_1 \) was. Moreover, our hypothesis ensures that \( i \leq m_1 - 2 \), and thus \( 2m_1 - 2^i - a \) is not of the form \( 2^m - a \), so the final term is zero by Theorem 2.9 and Lemma 3.5.

**Lemma 3.7.** Let \( D_{2k+1-a}y_1 \) be an \((n,a)\)-special admissible in \( \mathcal{M}(n,a) \). Then \( Sq^{2k} D_{2k+1-a}y_1 = D_{2k-a}y_1 \).

*Proof.* We begin by noting that, as in the proof of Lemma 3.6, the form of the \((n,a)\)-special admissible in \( \mathcal{M}(n,a) \) ensures in this case that \( 2^{k+1} \mid l + a \). We compute

\[
Sq^{2k} D_{2k+1-a}y_1 = Sq^{2k} Sq^{l-2k+1+a} y_1 \\
= \left(l + a - 2^{k+1} - 1\right) D_{2k-a}y_1 \\
+ \text{terms that are zero by Lemma 3.6} \\
= Sq^{l-2k+a} y_1 = D_{2k-a}y_1.
\]

**Lemma 3.8.** Let \( D_{2k}y_1 \) be an \((n,a)\)-special admissible in \( \mathcal{M}(n,0) \) and \( k \geq 1 \). Then \( Sq^{2k} D_1 D_{2k}y_1 = D_1 D_{2k-1}y_1 \).

*Proof.* We compute:

\[
Sq^{2k} D_1 D_{2k}y_1 = Sq^{2k} Sq^{2l-2k-1} D_{2k}y_1 \\
= \left(2l - 2^k - 2\right) Sq^{2l-1} D_{2k}y_1 \\
+ \text{terms that are zero by Lemma 3.6} \\
= D_1 D_{2k-1}y_1.
\]

*Proof of Theorem 2.11.* We begin by noting that, by Lemma 3.5 and Theorem 2.9, all admissibles in \( K \) not of the specified form map to zero on the fundamental class in \( \mathcal{M}(n,a) \), so the ones of the specified form span. So it remains to show, since the \((n,a)\)-special admissibles all lie in distinct degrees (notice the form of their degrees given in the proof of Lemma 3.6), that they are all nonzero. First we prove this for \( a = 0 \), by representing \( \mathcal{M}(n,0) \) as a submodule of the Dickson algebra \( W_{n+1} \). Since the action of \( A(Sq^1, \ldots, Sq^{n-2}) \) is trivial on the lowest dimensional generator \( x_{n,1} \) of the Dickson algebra, there is a unique map of \( A \)-modules (and hence of \( K \)-modules) \( \eta : \mathcal{M}(n,0) \to W_{n+1} \) specified by taking
We shall show that $\eta$ maps each $(n,0)$-special admissible in $\mathcal{M}(n,0)$ nontrivially. We recall from the introduction that the generators $x_{j,n+1-j}$ of the Dickson algebra satisfy $D_jx_{n,1} = x_{j,n+1-j}$ for $0 \leq j \leq n-1$, and note also that $D_0x_{n,1} = x_{n,n}^2 \neq 0$, and thus the result holds for $(n,0)$-special admissibles $D_jx_{2^n}$ where $I$ has length one. We proceed now by simultaneous induction on length and degree, and note first that since left multiplication by $D_0$ is the squaring operation in the polynomial algebra $W_{n+1}$, and thus is monic, we need only consider admissibles not involving $D_0$. If $D_jx_{2^n}$ is any $(n,0)$-special admissible in $\mathcal{M}(n,0)$ with $I$ of length at least two, and containing only two-power indices, we may apply Lemma 3.7 as many times as necessary to obtain an image in $W_{n+1}$ under the Steenrod action on this element of the form $D_jD_1x_{n,1}$. Next, we apply Lemma 3.8 to obtain an image of the form $D_1D_1D_Kx_{n,1}$. Finally we note that $Sq^1D_1D_1D_Kx_{n,1} = D_0D_1D_Kx_{n,1}$, which is nonzero by the inductive assumption and the nontriviality of the squaring map $D_0$ in $W_{n+1}$. Hence $D_1x_{n,1}$ is nonzero, and the proof is complete for $a = 0$.

Now inductively, assume the theorem true for a fixed $a \geq 0$. Define $f : \mathcal{F}_{2^n-a} \rightarrow \mathcal{F}_{2^n-a-1}$ to be the $A$-map specified by $f(x_{2^n-a}) = x_{2^n-a-1}$. As $K$-modules, $f(D_jx_{2^n-a}) = \alpha(D_1)x_{2^n-a-1}$, where $\alpha : K \rightarrow K$ is the algebra map given on generators by $\alpha(D_1) = D_{i-1}$ (and $\alpha(D_0) = 0$) [PW].

Let $D_jx_{2^n-a}$ be an $(n,a+1)$-special admissible in $\mathcal{F}_{2^n-a-1}$. Our goal is to show that $D_jx_{2^n-a-1} \neq 0$ in $\mathcal{M}(n,a+1)$. So assume that $D_jx_{2^n-a-1} = 0$ in $\mathcal{M}(n,a+1)$. Thus in $\mathcal{F}_{2^n-a-1}$, $D_jx_{2^n-a-1} = \gamma x_{2^n-a-1}$ for some $\gamma \in \mathcal{A}_{n-2}$. Form a multi-index $I$ by adding one to each entry in $J$. Then $D_jx_{2^n-a}$ is an $(n,a)$-special admissible and $f(D_jx_{2^n-a}) = D_jx_{2^n-a-1}$. Hence $D_jx_{2^n-a} - \gamma x_{2^n-a} \in \ker(f) = D_0 \cdot \mathcal{F}_{2^n-a}$, so we can write $D_jx_{2^n-a} - \gamma x_{2^n-a} = D_0 \cdot y$ in $\mathcal{F}_{2^n-a}$, with $y$ a sum of $(n,a)$-special admissibles there. Now we project this equation to $\mathcal{M}(n,a)$, where the term $\gamma x_{2^n-a}$ becomes zero and $D_0 \cdot y$ can now be represented as a sum of images of $(n,a)$-special admissibles with leading operation $D_0$. But recall that all the $(n,a)$-special admissibles are in distinct degrees, so none of these can be in the same degree as our $D_jx_{2^n-a}$, since $I$ contains no zeros. Thus the right side of the equation is zero, and we have that $D_jx_{2^n-a} = 0$ in $\mathcal{M}(n,a)$, contradicting the inductive hypothesis on $a$. Hence $D_jx_{2^n-a-1} \neq 0$ in $\mathcal{M}(n,a+1)$, as desired.

**Remark 3.9.** A description of $\eta(\mathcal{M}(n,0))$ is given in [HP], based on results in [A], as follows (with notation adjusted to match our situation). The Dickson algebra $W_{n+1}$ is defined as a subalgebra of $P_{n+1} = \mathbb{F}_2[t_1, \ldots, t_{n+1}]$. In $P_{n+1}$, define elements $\omega(k) = \sum t_1^{i_1} \cdots t_{n+1}^{i_{n+1}}$, where the sum is over all sequences $i_1, \ldots, i_{n+1}$ with each $i_j$ either 0 or a power of 2 and $\sum i_j = k$. Then $\{\omega(k) \mid 2^{n+1-\alpha(k)} \text{ divides } k\}$ is a basis for $\eta(\mathcal{M}(n,0))$. We could appeal to this divisibility criterion to provide an alternative proof for the main claim in the first paragraph of the proof of Theorem 2.11, that certain $D_jx_{n,1}$ are nonzero.
One can achieve this by verifying the divisibility criterion whenever \( k \) is the
topological degree of a \((n,0)\)-special admissible \( D_1 x_{2^n} \).


We now apply Theorem 2.11 to prove our global structure theorem for the
Dickson algebras.

**Theorem 4.1.** Consider the free unstable algebra \( U_{n+1} \) over \( K \), or equivalently
over \( A \) (as in [SE, pp. 26–29]), formed from \( M(n,0) \). Then the Dickson algebra
\( W_{n+1} \) is isomorphic as a \( K \)-algebra to the quotient \( G_{n+1} \) of \( U_{n+1} \) by the one
additional relation

\[
D_{2n-1} D_{2n-1}(x_{2^n}) = x_{2^n} \cdot D_{2n-1}(x_{2^n}),
\]
or equivalently, as an \( A \)-algebra, by the relation

\[
Sq^{2^n} Sq_0^{2^n-1}(x_{2^n}) = x_{2^n} \cdot Sq_0^{2^n-1}(x_{2^n}).
\]

**Proof.** The map \( \eta : M(n,0) \to W_{n+1} \) (see the proof of Theorem 2.11) deter-
dined by \( \eta(x_{2^n}) = x_{n,1} \) extends to a \( K \)-algebra map from \( U_{n+1} \) to \( W_{n+1} \). Recall
[PW] that the full \( K \)-action on \( W_{n+1} \) is determined by

\[
D x_{n+1-m,m} = x_{n+1-m,m}^2 + \sum_{j<m \leq k} x_{n+1-k,k} x_{n+1-j,j}.
\]

Thus since \( D_{2n-1} D_{2n-1}(x_{n,1}) = x_{n,1} \cdot D_{2n-1}(x_{n,1}) \), we obtain an induced map
\( \eta : G_{n+1} \to W_{n+1} \). This map is clearly onto since \( D_{2j} x_{n,1} = x_{n,1-j} \) for \( 0 \leq j \leq n-1 \) shows that it hits all the polynomial generators of \( W_{n+1} \).

To show that \( \eta \) is one-to-one, we will show that the corresponding algebra
generators \( x_{2^n} \) and \( D_{2j} x_{2^n} \) of \( G_{n+1} \) (which are precisely its algebra generators
that use only \((n,0)\)-special admissibles of length zero and one from \( M(n,0) \))
are in fact the only algebra generators of \( G_{n+1} \), i.e., that all the other algebra
generators of \( U_{n+1} \) become decomposable in \( G_{n+1} \). Thus since \( W_{n+1} \) is a free
commutative algebra, \( \eta \) must be monic.

To show that all other algebra generators of \( U_{n+1} \) become decomposable in
\( G_{n+1} \), we shall first consider those from \((n,0)\)-special admissibles of length two
in \( M(n,0) \). We shall prove that if \( D_{2n} D_{2k} x_{2^n} \) is an \((n,0)\)-special admissible
in \( M(n,0) \), then \( D_{2n} D_{2k} x_{2^n} \in A \cdot D_{2n-1} D_{2n-1} x_{2^n} \), which by the Cartan formula
consists entirely of algebra decomposables since \( D_{2n-1} D_{2n-1} x_{2^n} \) was made
decomposable by definition of the quotient \( G_{n+1} \). Then algebra generators of \( U_{n+1} \)
coming from \((n,0)\)-special admissibles of length longer than two in \( M(n,0) \) will
be decomposable in \( G_{n+1} \) by induction using the Cartan formula.

By Lemma 3.7, \( D_{2n} D_{2k} x_{2^n} \in A \cdot D_{2k} D_{2k} x_{2^n} \). So the result will follow from
showing that

\[
Sq^{2^k} D_{2k-1} D_{2k} x_{2^n} = D_{2k-1} D_{2k-1} x_{2^n} \quad \text{for} \; k < n.
\]
We compute
\[ Sq^{2k} D_{2k-1} D_{2k} x_2^n = Sq^{2k} Sq^{2n+1 - 2k - 2k - 1} D_{2k} x_2^n \]
\[ = \left( 2^{n+1 - 2k - 2k - 1} - 2k \right) Sq^{2n+1 - 2k - 1} D_{2k} x_2^n \]
\[ + A(Sq^1, \ldots, Sq^{2k-2}) D_{2k} x_2^n \]
\[ + Sq^{2n+1 - 2k} D_{2k} x_2^n \]
\[ = 0 + 0 \text{ (Lemma 3.6)} \]
\[ + Sq^{2n+1 - 2k} D_{2k-1} x_2^n \text{ (Lemma 3.7)} \]
\[ = D_{2k-1} D_{2k-1} x_2^n. \]

\[ \square \]

5. Appendix

Theorem 5.1. Consider the Hopf subalgebra \( A_n = \langle Sq^{2i} : 0 \leq i \leq n \rangle \subset A \) for \( n \geq -1 \).

a) Let \(-1 \leq n < m \leq \infty \). Then \( (A_m/A_mA_n)^* \) is the subquotient
\[ P[\zeta_1, \ldots, \zeta_i, \ldots, \zeta_n + 1, \zeta_n + 2, \ldots]/(\zeta_1^{2m+1}, \ldots, \zeta_i^{2m-i+2}, \ldots, \zeta_{n+1}^{2m}, \zeta_{n+2}, \ldots) \]
of the dual Steenrod algebra \( A^* = P[\zeta_1, \ldots, \zeta_i, \ldots] \), where \( \zeta_i \) is the conjugate of Milnor’s \( \xi_i \) [M]. For \( m = \infty \) we mean \( A_m = A_i \) and the ideal in the zetas is zero, i.e.,
\[ (A/AA_n)^* = P[\zeta_1^{2n+1}, \ldots, \zeta_i^{2n-i+2}, \ldots, \zeta_{n+1}^{2n}, \zeta_{n+2}, \ldots]. \]

b) If \( n = -1 \), then \( \zeta_i \) may be replaced by \( \xi_i \), i.e.,
\[ A_m^* = A^*/(\xi_1^{2m+1}, \ldots, \xi_i^{2m-i+2}, \ldots, \xi_{n+1}^{2m}, \xi_{n+2}, \ldots) \]
\[ = A^*/(\xi_1^{2m+1}, \ldots, \xi_i^{2m-i+2}, \ldots, \xi_{n+1}^{2m}, \xi_{n+2}, \ldots) \]

Proof.

a) First we consider \( m = \infty \). We let \( B_n = P[\zeta_1^{2n+1}, \ldots, \zeta_i^{2n-i+2}, \ldots, \zeta_n^{2n+1}, \zeta_{n+2}, \ldots] \)
for \( n \geq -1 \), and proceed by induction on \( n \). Consider the exact sequence
\[ A \xrightarrow{Sq^n} A/AA_{n-1} \rightarrow A/AA_n \rightarrow 0 \]
for \( n \geq 0 \). We will show that the kernel of
\[ f: B_{n-1} = P[\zeta_1^{2n}, \ldots, \zeta_i^{2n-i+2}, \ldots, \zeta_n^{2n}, \zeta_{n+2}, \ldots] = (A/AA_{n-1})^*(Sq^n)^* A^* \]
is \( B_n \).
Since $Sq^{2n}$ is a coalgebra primitive in $A/\mathcal{A}_{m-1}$, $f$ obeys the formula $f(xy) = f(x)y + xf(y)$ for a derivation, where both domain and range of $f$ are considered within $A^*$. From $\Delta \zeta_i = \sum \zeta_k \otimes \zeta_{t-j}^i$ (with $\zeta_0 = 1$) we have $f(\zeta_{t-j}^{2n-i+1}) = \zeta_t^{2n-i+1}$, for $1 \leq i \leq n + 1$, and $f(\zeta_i) = 0$ for $i \geq n + 2$. Clearly $B_n \subseteq \ker f$.

To see that the kernel is not larger than this, we examine the form of $f$ in terms of the natural factorizations

$$B_{n-1} \cong E(\zeta_1^{2n}, \zeta_2^{2n-1}, \ldots, \zeta_{n+1}^{2n} \otimes B_n$$

and

$$A^* \cong P[\zeta_1, \ldots, \zeta_{n+1}]/(\zeta_1^{2n+1}, \ldots, \zeta_{n+1}^{2n-1}) = B_n$$

of its domain and range. Then the derivation $f$ takes the form $f \otimes \text{identity}$, where $f$ is simply the natural inclusion into $A^*$, followed by $f$, and then projection. The crucial fact for this to hold is that $f(\zeta_1^{2n-i+1}) = \zeta_t^{2n-i+1}$, whose exponent is less than $2n-i+2$, so this term (and all others as well) is represented in the pure form $\zeta_t^{2n-i+1} \otimes 1$ in the factorization. It is also clear that $f$ has kernel $\{1\}$, since the other elements of $E(\zeta_1^{2n}, \ldots, \zeta_{n+1}^{2n-1}, \zeta_1^2, \ldots, \zeta_{n+1}^2)$ are in distinct degrees and clearly have nonzero images. Thus the kernel of $f$ is $B_n$, as claimed.

Next we consider the case where $m$ is finite and $n = -1$, i.e., we look at the Hopf algebra quotient map $A^* \to A_m^*$. Since the composite $A_m \to A \to A/\mathcal{A}_m$ is trivial in positive degrees, the description already proven above for $(A/\mathcal{A}_m)^*$ shows that $(\zeta_1^{2m+1}, \ldots, \zeta_t^{m-1}, \ldots, \zeta_{m+1}^2, \zeta_{m+2}, \ldots)$ is in the kernel of the quotient map, so it factors as

$$A^* \to A^*/(\zeta_1^{2m+1}, \ldots, \zeta_t^{2m-1}, \ldots, \zeta_{m+1}^2, \zeta_{m+2}, \ldots) \to A_m^*,$$

and we need only verify that the last map is monic. We know from [MM, Thm. 4.4] that as graded vector spaces, $(A \otimes A_m \mathbb{F}_2) \otimes \mathbb{F}_2 \cong A_m$, and thus dually

$$P[\zeta_1^{2m+1}, \ldots, \zeta_{m+1}^2, \zeta_{m+2}, \ldots] \otimes \mathbb{F}_2 \cong A_m^* \cong A^* = \{P[\zeta_1, \ldots, \zeta_t, \ldots]\}.$$

So the map in question can have no kernel.

Finally, for any finite $m$, we note that the proof we gave above by induction on $n$ applies, simply replacing $A$ with $A_m$ throughout, and using the identification of $A_m^*$ we have just obtained.

b) For $n = -1$, the quotient $A_m^*$ is invariant under conjugation since $A_m$ is a Hopf subalgebra of $A$.

REFERENCES


New Mexico State University, Las Cruces, NM 88003
E-mail address: davidp@nmsu.edu

Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail address: fpp@math.mit.edu

New Mexico State University, Las Cruces, NM 88003
E-mail address: frank@nmsu.edu