On the Cohomology of the Steenrod algebra

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Abstract. Let \( \mathcal{A}^* \) be the dual of the mod \( p \) Steenrod algebra. \( \mathcal{A}^* \cong \mathbb{F}_p[\xi_1, \ldots] \). Let \( \mathcal{A}(n)^* \) be the subalgebra generated by \( \xi_1, \ldots, \xi_n \). We show that there exists a family of finite \( p \)-groups \( G(n, r) \) whose group algebra \( \mathbb{F}_p(G(n, r)) \) is a quotient Hopf algebra of \( \mathcal{A}(n) \) and the restriction map on the cohomology is an isomorphism up to certain level depending on \( r \).

1. Introduction

One of the main goals in homotopy theory is to compute the homotopy group of sphere, \( \pi_*(S^0) \). An important tool is the mod \( p \) Steenrod algebra \( \mathcal{A} \) and its cohomology \( \text{Ext}_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p) \), which forms the \( E_2 \) term of the Adams' spectral sequence converging to the \( p \)-component of \( \pi_*(S^0) \). However, \( \text{Ext}_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p) \) is itself very difficult to compute. The dual algebra, \( \mathcal{A}^* \) is a polynomial algebra \( \mathbb{F}_p[\xi_1, \ldots] \) with comultiplication \( \Delta \xi_n = \sum \xi_n^{\rho} \otimes \xi_i \). It is isomorphic to the algebra of functions on the group scheme \( G = \text{Aut}_x(F_{ad}(x, y)) \) of strict automorphisms of the additive formal group law over \( \mathbb{F}_p \). [1], [3] and [11]. In this paper, we apply some results in the theory of algebraic groups to analyze \( \text{Ext}_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p) \).

We made two approximations for \( \text{Ext}_{\mathcal{A}}^*(\mathbb{Z}/p, \mathbb{Z}/p) \). First, we replace \( \mathcal{A}^* \) by its finitely generated subalgebra \( \mathcal{A}(n)^* \), or equivalently, we replace the group scheme \( G \) by an algebraic quotient group \( G(n) \). We then conjecture that Quillen's theorem on \( F \)-isomorphism holds for certain class of algebraic groups including \( G(n) \). This is done in Section 2. Next, we approximate \( G(n) \) further by considering...
a family of finite p-subgroups $G(n, r)$ of $F_{p^r}$ rational points of $G(n)$. For each
degree $(i, j)$ we have

$$\text{Ext}^{i,j}_A(Z/p, Z/p) \overset{\psi}{\to} H^{i,j}(G(n); Z/p) \overset{\phi}{\to} H^{i,j}(G(n, r); Z/p),$$

where $\bar{j} \equiv j \pmod{p^r - 1}$ and the finite group cohomology $H^i(G(n, r); Z/p)$ is

$\mathbb{Z}/(p^r - 1)$ graded. The analysis of $\psi$ which is neither surjective nor injective and

of $G(n, r)$ are still under investigation. The main result of this paper is about

the map $\phi$. It is a modification of a result of Cline, Parshall, Scott and van der

Kallen [2].

Theorem 1.1. Let $m, j, e$ and $f$ be positive integers with $f \geq \left\lfloor \log_p(j + 1) \right\rfloor + 2$

and $e \geq \left\lfloor \frac{(p^n - 1)m - 1}{p - 1} \right\rfloor$ and for $p > 2$ also assume $e > \left\lfloor \log_p(\frac{n}{2}) \right\rfloor + n - 1$. Then

the restriction map

$$\text{Ext}^{i,j}_A(Z/p, Z/p) = H^{i,j}(G(n); Z/p) \to H^{i,j}(G(n, e + f); Z/p)$$

is an isomorphism for $i \leq m$ and an injection for $i = m + 1$.

We have the following weak form of Quillen’s theorem

Corollary 1.1. Let $m, j, e$ and $f$ be as in Theorem. Then any element $x$ in

$\text{Ext}^{i,j}_A(Z/p, Z/p)$ is nilpotent if and only if it is nilpotent in $H^i(G(n, r); Z/p)$

for large enough $r$.

The map $\phi$ following by Quillen’s F-isomorphism for the finite group cohomology

of $G(n, r)$ yields a result related to that of C. Wilkerson in [12] concerning the ext
group of a finite dimensional graded connected Hopf subalgebra of the Steenrod
algebra at prime 2, and its elementary Hopf subalgebras. More precisely, in

[9] Quillen gave a theorem comparing the mod p cohomology of a finite group

with the limit of the mod p cohomology of its elementary abelian subgroups.
Roughly speaking, the two cohomology groups are isomorphic modulo nilpotent

elements as unstable $\mathcal{A}$-modules. Such theorem does not hold if we replace

finite group by finite dimensional graded connected Hopf algebra and replace

elementary abelian subgroups by any family of abelian Hopf subalgebras. In

[12], Clarence Wilkerson introduced the notion of elementary Hopf algebra and showed

that for every finite dimensional graded connected Hopf subalgebra $B$ of the mod 2 Steenrod algebra, $\text{Ext}^*_B(Z/2, Z/2)$ satisfies the theorem of Quillen

and Venkov [10] with respect to the family of all elementary subalgebras: i.e.
an element $x \in \text{Ext}^*_B(Z/2, Z/2)$ is non nilpotent if and only if there exists an

elementary subalgebra $V$ such that $\text{res}^V_B(x) \neq 0$. The second condition for $F$

isomorphism is however, not satisfied in general. This is because the dual of any

finite dimensional graded connected Hopf algebra is nilpotent and the transfer

argument which is one of the main ingredients for Quillen’s theorem fails.
2. Quillen’s theorem for unipotent algebraic groups

It is by now a well-known fact that the dual of the (reduced) Steenrod algebra $A^*$ is isomorphic to the coordinate algebra of the (infinite dimensional) group scheme $G = \text{Aut}_s(F_a(x,y))$ of the strict automorphism of the additive formal group law defined over $\mathbb{F}_p$. See [1], [11] and [3]. The group scheme $G$ is in a natural way an inverse limit of unipotent algebraic groups $G(n)$, associated to the Hopf subalgebra $A(n) = \mathbb{F}_p[\xi_1, \ldots, \xi_{n-1}] \subset A^*$, see [3]. We have the restriction map on the rational cohomology:

$$Q : H^*(G) \to \lim_{G(n)} H(V).$$

Then $Q$ is an isomorphism modulo nilpotent of unstable modules over the universal Steenrod algebra.

The conjecture is true if $E$ is a finite $p$-group. Recall that, for any positive integer $r$, the finite group $E(\mathbb{F}_p^r)$ is a $p$-group and as a (constant) algebraic group, it is a finite, reduced algebraic subgroup of $E$, denoted by $E(r)$. The $\mathbb{F}_p^r$ rational points of the map $Q$ in (1) is precisely Quillen’s F-isomorphism. Conversely, any finite $p$-group can be realized as a subgroup of $\mathbb{F}_p^r$ rational points of some unipotent algebraic group for some $r$.

The Conjecture is true for any commutative reduced connected unipotent algebraic group. Any such group is a product of Witt groups $W(d)$ at prime $p$ of finite length, so it suffices to consider $W(d)$. $W(d)$ has a unique vector subgroup $V$ of rank 1. $H^*(V)$ and $H^*(W(d))$ are computed in [2] and [8] respectively. Forgetting the higher Bocksteins, both cohomology groups are of the form $\mathbb{F}_p[x_i | 0 \leq i] \otimes E(y_i | 1 \leq i)$, where the symbol $E$ stands for exterior algebra. The cokernel of $Q : H^*(W(d)) \to H^*(V)$ is exterior algebra and the kernel is
generated by exterior generators. Hence $Q$ is clearly an $F$-isomorphism. Let $W(n)$ be the algebraic group of big Witt vectors of length $n - 1$. Then

$$\widetilde{W}(n) \cong \prod_{p \mid i} W(r_i),$$

where $r_i$ is the smallest integer such that $ip^{r_i} \geq n$. Our group $G(n)$ is in some sense the twisted version of $\widetilde{W}(n)$. It is a (semidirect) product of smaller groups and it has a unique vector subgroup of maximal rank $[\frac{n}{2}]$. We believe that the Conjecture holds for $G(n)$.

Conjecture 2.1 can be stated in terms of reductive groups as follows. Let $B$ be a Borel subgroup of a reductive group $G$ with maximal torus $T$ and unipotent radical $U$. Then for any $G$-module $M$, $H^*(G; M) \cong H^*(B; M) \cong H^*(U; M)^T$. In particular, if $M$ is the induced module $\text{Ind}_{G}^{B}(G_1)$ associated to a dominant weight, then $H^i(G; \text{Ind}_{G}^{B}(G_1)) = H^i(G; Z/p) = 0$ for $i > 0$. We have the following consequence of the Conjecture.

**Precorollary 2.1.** Let $G$ be a connected reductive group or any parabolic subgroup. Then for any $G$-module $M$, the restriction map $Q : H^*(G; M) \to \lim_{\text{coker}(U)} H^*(V; M|_V)^T$ is an $F$-isomorphism.

### 3. Some preliminaries

We assume some familiarity with the theory of reductive algebraic groups. Let $G(n)$ denote the algebraic group represented by the finitely generated Steenrod algebra $A(n^*) = \mathbb{F}_p[\xi_1, \xi_2, \ldots, \xi_{n-1}]$ with $\Delta \xi_m = \sum \xi_{m-i} \otimes \xi_i$, where $\xi_0 = 1$. We may impose an action of a rank one torus on $G(n)$ as follows. Let $B(n)$ be the group of upper triangular $n \times n$ matrices and let $U(n)$ be the subgroup of strict upper triangular matrices. Their associated algebras are

$$\mathbb{F}_p[B(n)] = \mathbb{F}_p[a_{i,j}, a_{ii}^{\pm 1} | 1 \leq i \leq j \leq n],$$

$$\mathbb{F}_p[U(n)] = \mathbb{F}_p[a_{i,j} | 1 \leq i < j \leq n]$$

$$\Delta(a_{ij}) = \sum_{j-i} a_{i,j+k} \otimes a_{i+k,j},$$

$$\Delta(a_{ii}^\pm) = a_{ii}^\pm \otimes a_{ii}^\pm,$$

where $a_{i,i} = 1$ in $\mathbb{F}_p[U(n)]$. There is a natural inclusion $G(n) \subset U(n) \subset B(n)$ given by the surjection of Hopf algebras defined by $j(a_{ij}) = \xi_{j-i}^{p^{r_i}}$. Let $T^n$ be the maximal torus in $B(n)$. As a subgroup of $B(n)$, $G(n)$ is stable under the conjugation of a rank 1 torus $f : T^1 \to T^n; \quad f(a) = (a, a^{p}, \ldots, a^{p^{n-1}})$. We have
the following injective map of group extensions:

\[
\begin{array}{cccccc}
1 & \to & G(n) & \to & \hat{G}(n) & \to & T^1 & \to & 1 \\
\downarrow i & & \downarrow j & & \downarrow f & & \\
1 & \to & U(n) & \to & B(n) & \to & T^n & \to & 1 \\
\end{array}
\]

where \( \hat{G}(n) \) is represented by the Hopf algebra \( \mathbb{F}_p[\xi_0^{\pm 1}, \xi_1, \ldots, \xi_{n-1}] \) with \( \Delta \xi_d = \sum \xi_{d-i} \otimes \xi_i \). We note here that this algebra is closely related to the dual of the unstable Steenrod algebra referred to in section 5.

The character groups \( X(T^1) \) and \( X(T^n) \) are free Abelian groups of rank 1 and \( n \) respectively. Let \( \delta \) (resp. \( \{ e_i | 1 \leq i \leq n \} \)) be the canonical basis for \( X(T^1) \) (resp. \( X(T^n) \)). The homomorphism \( f \) induces a map of Abelian groups \( f^* : X(T^n) \to X(T^1) \) which is the projection \( f^* (e_1 + \cdots + p^{n-1} e_n) = \delta \). The conjugation action of \( T^1 \) provides a pseudo root system of type \( A_1 \) for \( G(n) \).

**Lemma 3.1.** The action of \( T^1 \) on \( G(n) \) can be described in terms of Hopf algebras as the algebra homomorphism

\[
\rho : \mathbb{F}_p[\xi_1, \ldots, \xi_{n-1}] \to \mathbb{F}_p[\xi_1, \ldots, \xi_{n-1}] \otimes \mathbb{F}_p[t, t^{-1}]
\]

with \( \rho(\xi_i) = \xi_i \otimes t^{p^{i-1}}. \)

**Proof.** Check the conjugation action of \( T^1 \) on \( G(n) \) as subgroups of \( B(n) \). Then translate it to coaction of Hopf algebras. □

This action gives \( A(n)^* \) a weight decomposition which can be identified with the topological grading in \( A^* \). Moreover, it extends naturally to an action on the rational cohomology \( H^*(G(n)) \) and make it a bigraded object. To avoid possible confusion with the cohomological grading, we refer to the degree of this “internal” grading algebraically as weight. More generally, any \( G(n) \) module \( M \) becomes \( X(T^1) = \delta \mathbb{Z} \) graded; \( M = \oplus M_d \), where we write \( M_d \) as \( M_d \) and call \( M_d \) the weight space of \( M \) of weight \( d \) or simply \( d \).

For any positive integer \( r \), let \( q = p^r \). The ideal \( J_r \subset A(n)^* \) generated by \( \xi_i^{p^r} - \xi_i \) for \( 1 \leq i < n \) is a Hopf ideal and the quotient \( A(n, r)^* = A(n)^*/J_r \) is a reduced finite dimensional Hopf algebra. It represents the subgroup of \( \mathbb{F}_q \) rational points \( G(n, r) = G(n)(\mathbb{F}_q) \). As an abstract group, \( G(n, r) \) is a finite \( p \)-group and the dual \( A(n, r) \) is isomorphic to the group algebra of \( G(n, r) \). Similarly, we write \( T^1(r) \) for \( T^1(\mathbb{F}_q) \). As an abstract group \( T^1(r) \) is isomorphic to the cyclic group \( \mathbb{Z}/(q-1) \). The action of \( T^1 \) on \( G(n) \) restricts to an action of \( T^1(r) \) on \( G(n, r) \) and gives any \( G(n, r) \)-module a \( \mathbb{Z}/(q-1) \) graded structure.

**Definition 3.1.** Let \( G_a \) be the additive algebraic group. The associated algebra of \( G_a \) is \( \mathbb{F}_p[x] \) with \( \Delta(x) = x \otimes 1 + 1 \otimes x \). An action of a 1-torus on \( G_a \) is given
by the coaction

\[ \phi : \mathbb{F}_p[x] \to \mathbb{F}_p[x] \otimes \mathbb{F}_p[t, t^{-1}]; \quad \phi(x) = x \otimes t^i. \]

We said \( T^1 \) acts on \( G_a \) with weight \( i \) and denote \( G_a \) with this action by \( G_{ai} \).

The cohomology of \( G_a \) is computed by Cline, Parshall, Scott and van der Kallen in [2]. We quote two of their results here for reference.

**Theorem 3.1.** Let \( T^1 \) acts on \( G_a \) with weight \( \mu \in X(T^1) \). Let \( V \) (resp. \( W \)) be the vector space spanned by elements \( a_i = -p^i \mu \), \( 0 \leq i \) (resp. \( b_i = -p^{i+1} \mu \), \( 0 < i \)). Then

\[ H^*(G_a, \mathbb{Z}/p) = \begin{cases} S^*(V), & \text{for } p = 2 \\ S^*(W) \otimes E^*(V), & \text{for } p \text{ odd,} \end{cases} \]

where \( S^* \) and \( E^* \) denote the symmetric algebra and the exterior algebra respectively. For \( q = p^r \), \( H^*(G_a(\mathbb{F}_q), k) \) is as in (2) but with \( V \) (resp. \( W \)) replaced by the subspace \( V(p) \) (resp. \( W(p) \)) spanned by \( a_i \) (resp. \( b_i+1 \)) for \( 0 \leq i < r \).

### 4. Main results

In what follows, we fix a power \( q = p^r \) of \( p \) and a weight \( \lambda = d \delta \in X(T^1) \). \( G(n) \) admits a filtration by normal subgroups:

\[ G_a = G(2) \hookrightarrow G(3) \hookrightarrow \cdots \hookrightarrow G(n) \]

**Lemma 4.1.** The conjugation action of \( T^1 \) on \( G(n) \) induces and action of \( T^1 \) on \( G(i)/G(i-1) \cong G_a \) with weight \( (p^{n-i+1} - 1) \).

We identify \( G(i)/G(i-1) \) with \( G_{aj} \), where \( j = n - i + 1 \). By Theorem 3.1,

\[ H^*(G_{aj}) \cong \begin{cases} \mathbb{F}_p[x_{0j}, x_{1j}, \ldots], & \text{for } p = 2 \\ \mathbb{F}_p[x_{0j}, x_{1j}, \ldots] \otimes E(y_{1j}, y_{2j}, \ldots), & \text{for } p > 2, \end{cases} \]

where the weight of \( x_{ij} \) and \( y_{ij} \) is \( p^i(p^j - 1) \) and \( p^{i+1}(p^j - 1) \) respectively.

The \( E_2 \) term of Serre’s spectral sequence for the extension \( 1 \to G_{an-1} \to G(n) \to G(n-1) \to 1 \) is

\[ E_2^{p,q} = H^p(G(n-1)) \otimes H^q(G_{an-1}) \Rightarrow H^{p+q}(G(n)). \]

The action of \( T^1 \) induces the third grading by weight on \( E_2 \). Since the differential preserves weights, the \( \lambda = d \delta \) weight space of \( E_2 \) also forms a spectral sequence

\[ E_2^{i,j} = \left( H^i(G(n-1), k) \otimes H^j(G_{an-1}, k) \right)^d \Rightarrow H^{i+j,d}(G(n), k). \]
$E_{2,d}^{i,j}$ could be refined using the filtration (3), to

$$E_{2,d}^{s_1,\ldots,s_{n-1}} = (H^{s_1}(G_{a_1}) \otimes \cdots \otimes H^{s_{n-1}}(G_{a_{n-1}}))^d \Rightarrow H^{\Sigma s_i,d}(G(n)).$$

The homogeneous subspace of $E_{2,d}^{s,a}$ of elements of total degree $m$ and weight $d$

$$L(m,d) = \bigoplus_{\Sigma s_i=m} E_{2,d}^{s_1,\ldots,s_{n-1}} \cong H^{m,d}(G_{a_1} \times \cdots \times G_{a_{n-1}}),$$

is spanned by monomials $\prod_{k=1}^{m} a_{i_k,j_k}$, with multiplicity, where $a_{i_k,j_k}$ stands for $x_{i_k}$ for $p = 2$ and either $x_{i_k}$ or $y_{i_k}$ for $p > 2$, such that $i_k \geq 0$, $1 \leq j_k < n$. Let $f = \prod_{k=1}^{m} a_{i_k,j_k} \in L(m,d)$, then the weight of $f$, for $p = 2$ and for $p$ odd are respectively

$$d = \sum_{k=1}^{m} p^{i_k}(p^{j_k} - 1),$$

$$d = \sum_{k=1}^{m_1} p^{i_k}(p^{j_k} - 1) + \sum_{l=1}^{m_2} p^{i_l}(p^{j_l} - 1)$$

where $i_k \geq 0$, $1 \leq j_k < n$, $i'_k > 0$ and $m_1 + 2m_2 = m$.

As for the finite subgroup $G(n,r)$, there exists a similar spectral sequence

$$E^{s_1,\ldots,s_{n-1}}(F_q)_d = (H^{s_1}(G_{a_1}(F_q)) \otimes \cdots \otimes H^{s_{n-1}}(G_{a_{n-1}}(F_q))))^d \Rightarrow H^{\Sigma s_i,d}(G_{r}(n)),$$

where $d$ is the congruent class of $d$ in $X(T_r^1) = \mathbb{Z}/(q - 1)$. Write $R(m,d)$ for the homogeneous subspace of $E^{s_1,\ldots,s_{n-1}}(F_q)_d$ of degree $m$ and weight $d$. $R(m,d)$ has a basis consisting of monomials $\prod_{k=1}^{m} a_{i_k,j_k}$ with the same notation as for $L(m,d)$, but with $0 \leq t_k < r$ and $\sum_{k=1}^{m} p^{i_k}(p^{j_k} - 1) \equiv d \pmod{q - 1}$. The inclusion $G(n,r) \subset G(n)$ induces the (degree 0) restriction map of spectral sequences

$$\phi^m_d : L(m,d) \to R(m,d); \quad \phi^m_d(x_{i_k,j_k}) = x_{i_k,j_k}.$$

If $\phi^m_d$ is an isomorphism for $i \leq m$ and an injection for $i = m + 1$, then the induced map $H^{i,d}(G(n)) \to H^{i,d}(G(n,r))$ is also an isomorphism when $i \leq m$ and an injection when $i = m + 1$. To show this, one has to consider what $\phi^m_d$ does to the basis elements. For each monomial in $L(m,*)$, we may rewrite its weight as

$$d = \sum_{k=1}^{m} p^{i_k}(p^{j_k} - 1) = \sum_{i=1}^{t} p^{l_i},$$

where $l_i = i_k$ for some $k$ and $t \leq m(p^{n-1} - 1)$.

Even if we are in a different situation from that of [2], the proof of the main theorem is basically the same as that given in [2]. We may apply their injectivity,
isomorphism conditions and their digit counting lemma, which we quote here with minor change.

**CONDITION 4.1.** \( \phi_m \) is injective if any monomial \( \prod_{k=1}^m a_{i_k j_k} \) in \( L(m, d) \) satisfies, for \( p = 2 \) and for \( p \) odd respectively

\[
d = \sum_{k=1}^m p^{i_k} (p^{j_k} - 1),
\]

\[
d = \sum_{k=1}^{m_1} p^{i_k} (p^{j_k} - 1) + \sum_{l=1}^{m_2} p^{i_l} (p^{j_l} - 1),
\]

with \( m_1 + m_2 = m \) and \( 1 \leq j_k < n \). Then \( 0 \leq i_k < m, 0 < i'_l \leq m \) for all \( k \).

**CONDITION 4.2.** \( \phi_m \) is an isomorphism if any monomial \( \prod_{k=1}^m a_{i_k j_k} \) in \( L(m, *) \) congruent to a monomial in \( R(m, d) \) must be in \( L(m, d) \). More precisely, every congruent relation, for \( p = 2 \) and for \( p \) odd respectively,

\[
\sum_{k=0}^m p^{i_k} (p^{j_k} - 1) \equiv d \pmod{p^r - 1},
\]

\[
\sum_{k=0}^{m_1} p^{i_k} (p^{j_k} - 1) + \sum_{l=0}^{m_2} p^{i_l} (p^{j_l} - 1) \equiv d \pmod{p^r - 1},
\]

where \( 0 \leq i_k < r, 1 \leq j_k < n, 0 < i'_l \leq r \) and \( m_1 + m_2 = m \), is an equality.

**Proposition 4.1.** [2] Proposition 6.5. Let \( t \) be a positive integer. Consider an expression

\[
\sum_{k=1}^t p^{i_k} = L + (p^{e+f} - 1)M
\]

for integers \( L, M, e \geq \left\lceil \frac{t + 1}{p-1} \right\rceil, f \geq \left\lceil \log_p(|L| + 1) \right\rceil + 2 \) and \( 0 \leq i_k < e + f \), where \( \lfloor \cdot \rfloor \) denotes the largest integer function. Then there are no term with \( L < p^{i_k} < p^f \) and the terms on the left hand side with \( p^{i_k} \leq L \) sum to \( L - M \).

**Proposition 4.2.** Let \( m, a \) and \( b \) be non-negative integers. Let \( e \) and \( f \) be integers satisfying \( e \geq \left\lceil \frac{(p^{n-1})^{m-1} - 1}{p - 1} \right\rceil \) and \( f \geq \left\lceil \log_p(a + 1) \right\rceil + 2 \). Suppose

\[
\sum_{k=1}^m p^{i_k} (p^{j_k} - 1) = a + (p^{e+f} - 1)b,
\]

where \( 0 \leq i_k < e + f \), and \( 1 \leq j_k < n \). Then \( a \geq b \geq 0 \), and the terms in the sum with \( p^{i_k} \leq a \) sum to \( a - b \).

**Proof.** Write \( \sum_{k=1}^m p^{i_k} (p^{j_k} - 1) = \sum_{i=1}^t p^{i_i} = a + (p^{e+f} - 1)b \), with \( t \leq (p^{n-1} - 1)m, i_k = i_k < e + f \) for some \( k \). Hence \( e \geq \left\lceil \frac{t - 1}{p - 1} \right\rceil \). Lemma 4 now finishes the proof. \( \square \)
Theorem 4.1. Let $m$ be a non-negative integer and $\lambda = d \delta$ a weight in $X(T^1)$. Let $e, f$ be integers with $e \geq (k^\alpha - 1)m - 1$, $f \geq \log_d (d + 1) + 2$. For $p$ odd assume also that $e > \log_p (m/n) + n - 1$. Then the restriction map

$$\phi_{p^d}^i : H^{p^d}(G(n), k) \to H^{p^d}(G_{e+f}(n), k)$$

is an isomorphism for $i \leq m$ and an injection for $i = m + 1$.

Proof. For monomorphism, we check Condition 4.1. Let $f$ be a monomial in $L(i, d)$ of weight $\sum_{k=1}^{i'} p^{j_k} (p^{j_k} - 1) + \sum_{l=1}^{i''} p^{j_l} (p^{j_l} - 1) = p^e d$, with $i' + 2i'' = i$, $0 \leq i_k, 0 < i_l$ and $0 < j_k < n$, where for $p = 2$ we set $i'' = 0$. Need to show $i_k < e + f$ for all $0 \leq k \leq i$. Assume otherwise, for $p = 2$ we have

$$p^e + f(p - 1) = p^e + f \leq p^e d$$

and hence $p^e \leq d < p^l$, which is a contradiction. For $p$ odd, we have

$$p^e + f(p - 1) \leq p^e d$$

and hence $p^e + 1 \leq d < p^{e+1}$, again a contradiction. Note that this argument does not depend on $i$.

For isomorphism part we check condition 4.2. Given a congruent relation

$$\sum_{k=1}^{i'} p^{j_k} (p^{j_k} - 1) + \sum_{k=1}^{i''} p^{j_k} (p^{j_k} - 1) = p^e d + (p^{e+f} - 1)b$$


$$\equiv p^e d \mod (p^{e+f} - 1),$$

with $i' + 2i'' = i$ and set $i'' = 0$. Let $i''$ be the smallest number such that $i' < e + f$, we have

$$\sum_{k=1}^{i'} p^{j_k} (p^{j_k} - 1) + \sum_{k=i''+1}^{i'} (p^{j_k} - 1) = p^e d,$$

We may assume both $m$ and $d$ are greater than zero. By the digit counting lemma of [2], $p^{n-1}m > (p^{n-1} - 1)m \geq t \geq dp^e$. Thus $p^{e-n+1} < \frac{m}{d}$ which is a contradiction. □

Corollary 4.1. Let $V$ be a finite dimensional $G(n)$ module and let $m, e$ and $f$ be positive integers satisfying the conditions given in the theorem. The restriction map

$$H^i(G(n), V^{(e)}) \to H^i(G(n), V^{(e)})$$

is an isomorphism for $i \leq m$ and an injection for $i = m + 1$. 


It is clear that the notion of generic cohomology makes sense. Since a $G(n;r)$-module is cyclically graded by $\mathbb{Z}/(p^r - 1)$, there is an isomorphism $H^i(\tilde{G}(n,r), V) \to H^i(\tilde{G}(n,r), V^*)$ for each $i, r, e$ and finite dimensional $\tilde{G}(n)$ module $V$, and thus $H^i(\tilde{G}(n,r), V)$ is stable for large $r$. By definition, $H^*_{\text{gen}}(\tilde{G}(n), V)$ is this group in the stable range. The restriction map now becomes

$$H^*(\tilde{G}(n), V) \to H^*_{\text{gen}}(\tilde{G}(n), V)$$

and similarly,

$$H^{*,*}(G(n), k) \to H^{*,*}_{\text{gen}}(G(n), k)$$

which is an isomorphism under the conditions of Theorem 4.1 or Corollary 4.1.

Remark 4.1. The bound for $e$ and $f$ are minimal for $n = 2$. This follows directly from Cline et. al’s result. However, for $n > 2$ it is not minimal, in particular, the extra assumption for $e$ when $p > 2$ can be improved.

Remark 4.2. Corollary 4.1 applies to any finite dimensional module $V$ over the universal Steenrod algebra, on which $P_0$ acts invertibly. In particular, if $P_0$ acts as identity, in which case $V$ is just a module over the Steenrod algebra.

5. Application to the unstable extension groups

Let $G$ be a finite group, or more generally a finite v.c.d. group or a compact Lie group, an let $O_G$ be the category of finite elementary abelian subgroups of $G$, then, by a theorem of Quillen, the induced map on cohomology $H^*(G, \mathbb{F}_p) \to \lim_{\longrightarrow} H^*(V, \mathbb{F}_p)$ is an $\mathbb{F}$-isomorphism, in the sense that $\text{Ker}(F)$ is the submodule of the nilpotent elements and $\text{Coker}(F)$ is nilpotent as unstable module over the Steenrod algebra.

Theorem 5.1. With the same notations and conditions as in Theorem 4.1, an element $x \in \text{Ext}_A^{i,j}(k, k)$ is nilpotent if and only if it is nilpotent in $H^j(G(n, r))$.

Conjecture 2.1 or Theorem 4.1 might be useful, for example, in the “algebraic homotopy theory”, a program introduced by John Palmeiri in [5, 6], in which he applied Wilkerson’s result to obtain a family of nilpotent “algebraic self maps”, i.e. nilpotent elements in $\text{Ext}_A^{*,*}(M, M)$, for some (stable) $A$-module $M$.

We end this subsection with an application of Theorem 4.1 to the extension group of unstable modules. The link between the theory of algebraic groups and stable mod-p theory exists in the unstable theory as well. The algebra of (reduced) mod $p$ unstable cohomology cooperations is a polynomial algebra $B^* = \mathbb{F}_p[\xi_0, \xi_1, \ldots]$ with comultiplication $\Delta \xi_i = \sum \xi_{n-i} \otimes \xi_i$. $B^*$ represents the multiplicative monoid (scheme) $\text{End}_{\mathbb{F}_p}(F_{ad}(x, y))$. This connection occurs in any
complex oriented cohomology theory, for more detail about this see [7]. We have an extension of monoids

$$1 \to \text{Aut}_s(F_{ad}(x, y)) \to \text{End}(F_{ad}(x, y)) \to L \to 1,$$

where $L$ is the algebraic monoid represented by $\mathbb{F}_p[x]$ with $\triangle(x) = x \otimes x$. The inclusion $\text{Aut}_s(F_{ad}(x, y)) \subset \text{End}(F_{ad}(x, y))$ induces the full embedding $\mathcal{U} \cong \mathcal{B}-\text{Mod} \subset \mathcal{A}-\text{Mod}$. Now, the higher cohomology $H^i(L; M)$ vanishes for any unstable module $M$, for the same reason as the vanishing of the cohomology of the torus. Therefore, the Serre’s spectral sequence for the extension (6) yields an isomorphism

$$\text{Ext}^*_U(k, M) = H^*(\text{End}(F_{ad}(x, y); M) \cong H^*(G; M)^L,$$

for any unstable module $M$ and where the action of $L$ on $H^*(G; M)$ is induced by the conjugation action of $L$ on $G$ in $\text{End}(F_{ad}(x, y))$.

For any positive integer $n$, let $\mathcal{B}^{n*}$ be the sub bialgebra of $\mathcal{B}^*$ generated by $\xi_0, \ldots, \xi_{n-1}$. $\mathcal{B}^{n*}$ represents an algebraic monoid $G(n) \times L = \tilde{G}(n)$. A $\mathcal{B}^{n*}$-module becomes an unstable module by restriction. Moreover, the notion of a submonoid $\tilde{G}_r(n)$ of $\mathbb{F}_p$ rational points, makes sense.

**Corollary 5.1.** Let $m, e$ be given as in Theorem 4.1. Let $M$ be a finite dimensional $\mathcal{B}^n$ module with $d$ the largest integer such that $M_d \neq 0$ and let $f$ be a positive integer satisfying $f \geq \lceil \log_p(d+1) \rceil + 2$. Then the restriction map

$$\text{Ext}^*_U(k, \Sigma^e M) \to H^i(G_{e+f}(n), \Sigma^e M)^L_r$$

is an isomorphism for $i \leq m$ and injection for $i = m + 1$, where $L_r \subset L$ is the sub monoid of $\mathbb{F}_p$ rational points.

**Remark 5.1.** If $E$ is a complex oriented cohomology theory, then the algebra of $E$-cohomology cooperations can be realized as the algebra of functions on an appropriate groupoid scheme depending on the associated formal group law of $E$ [4]. It is likely that, one can find algebraic finite subgroups of such groupoid such that there is a similar results as in Theorem 4.1).

**References**


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