Equivariant K-homology for some Coxeter groups

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Abstract

We obtain the equivariant K-homology of the classifying space $\underline{E}W$ for W a right-angled or, more generally, an even Coxeter group. The key result is a formula for the relative Bredon homology of $\underline{E}W$ in terms of Coxeter cells. Our calculations amount to the K-theory of the reduced C^*-algebra of W, via the Baum-Connes assembly map.

1 Introduction

Consider a discrete group G. The Baum-Connes conjecture [1] identifies the K-theory of the reduced C^*-algebra of G, $C_r^*(G)$, with the equivariant K-homology of a certain classifying space associated to G. This space is called the classifying space for proper actions, written $\underline{E}G$. The conjecture states that a particular map between these two objects, called the assembly map,

$$\mu_i : K_i^G(\underline{E}G) \longrightarrow K_i(C_r^*(G)) \quad i \geq 0\,,$$

is an isomorphism. Here the left hand side is the equivariant K-homology of $\underline{E}G$ and the right hand side is the K-theory of $C_r^*(G)$. The conjecture can be stated more generally [1, Conjecture 3.15].

The equivariant K-homology and the assembly map are usually defined in terms of Kasparov's KK-theory. For a discrete group G, however, there is a more topological description due to Davis and Lück [4], and Joachim [10] in terms of spectra over the orbit category of G. We will keep in mind this topological viewpoint (cf. Mislin's notes in [14]).

Part of the importance of this conjecture is due to the fact that it is related to many other relevant conjectures [14, §7]. Nevertheless, the conjecture itself allows the computation of the K-theory of $C_r^*(G)$ from the K^G-homology of $\underline{E}G$. In turn, this K-homology can be achieved by means of the Bredon homology of $\underline{E}G$, as we explain later.

In this article we focus our attention on (finitely generated) Coxeter groups. These groups are well-known in geometric group theory as groups generated by reflections (elements of order 2) and only subject to relations in the form $(st)^n = 1$. Coxeter groups have the Haagerup property [2] and therefore satisfy the Baum-Connes conjecture [8].

For W a Coxeter group, we consider a model of the classifying space $\underline{E}W$ called the Davis complex. We obtain a formula for the relative Bredon homology

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of this space in terms of Coxeter cells (Theorem 5.2). From this, we can deduce the Bredon homology of $\underline{E}W$ in same cases; for instance, for right-angled Coxeter groups and, more generally, even Coxeter groups (Theorems 7.1 and 8.1). The equivariant K-homology of $\underline{E}W$ follows immediately. Since Coxeter groups satisfy Baum-Connes, our results also amount to the K-theory of the corresponding reduced C^*-algebra.

These results appear in the author's PhD thesis [17, Chapter 5]. I would like to thank my PhD supervisor Ian Leary for his guidance through this research project, and particularly for suggesting the use of the Coxeter cell structure and relative Bredon homology.

2 Preliminaries

2.1 Classifying spaces

The classifying space $\underline{E}G$ that appears in the Baum-Connes conjecture is a particular case of a more general construction.

Let G be a discrete group. A G-CW-complex is a CW-complex with a G-action permuting the cells and such that if a cell is sent to itself, it is done by the identity map. Let $\mathfrak{F}$ be a non-empty family of subgroups of G closed under conjugation and passing to subgroups. A *model for $E_{\mathfrak{F}}G$* is a G-CW-complex X such that (1) all cell stabilizers are in $\mathfrak{F}$; (2) for any G-CW-complex Y with all cell stabilizers in $\mathfrak{F}$, there is a G-map $Y \to X$, unique up to G-homotopy equivalence. The last condition is equivalent to the statement that for each $H \in \mathfrak{F}$, the fixed point subcomplex X^H is contractible.

For the family with just the trivial subgroup, we obtain EG, a contractible free G-CW-complex whose quotient BG is the classifying space for principal G-bundles; when $\mathfrak{F} = \mathfrak{Fin}(G)$, the family of all finite subgroups of G, this is the definition of $\underline{E}G$.

It can be shown that general classifying spaces $E_{\mathfrak{F}}G$ always exists. They are clearly unique up to G-homotopy. See [1, §2] or [14] for more information and examples of $\underline{E}G$.

2.2 Bredon (co)homology

Given a group G and a family $\mathfrak{F}$ of subgroups, we will write $\mathcal{O}_{\mathfrak{F}}G$ for the *orbit category*. The objects are left cosets G/K, $K \in \mathfrak{F}$, and morphisms the G-maps $\phi : G/K \to G/L$. Such a G-map is uniquely determined by its image $\phi(K) = gL$, and we have $g^{-1}Kg \subset L$. Conversely, such $g \in G$ defines a G-map, which will be written R_g.

A covariant (resp. contravariant) functor $M\colon \mathcal{O}_{\mathfrak{F}}G \to \mathfrak{Ab}$ is called a *left* (resp. *right*) *Bredon module*. The category of left (resp. right) Bredon modules and natural transformations is written G-$\mathrm{Mod}_{\mathfrak{F}}$ (resp. $\mathrm{Mod}_{\mathfrak{F}}$-$G$). It is an abelian category, and we can use homological algebra to define Bredon homology (see [14, pp. 7-10]). Nevertheless, we give now a practical definition.

Consider a G-CW-complex X, and write $\mathfrak{Iso}(X)$ for the family of isotropy subgroups $\{\mathrm{stab}(x), x \in X\}$. Let $\mathfrak{F}$ be a family of subgroups of G containing $\mathfrak{Iso}(X)$, and M a left Bredon module. The *Bredon homology groups* $H_i^{\mathfrak{F}}(X; M)$ can be obtained as the homology of the following chain complex (C_*, ∂_*). Let

$\{e_\alpha\}$ be orbit representatives of the d-cells ($d \geq 0$) and write S_α for $\mathrm{stab}(e_\alpha) \in \mathfrak{F}$. Define

$$C_d = \bigoplus_\alpha M\left(G/S_\alpha\right) .$$

If $g \cdot e'$ is a typical $(d-1)$-cell in the boundary of e_α then $g^{-1} \cdot \mathrm{stab}(e_\alpha) \cdot g \subset \mathrm{stab}(e')$, giving a G-map (write S' for $\mathrm{stab}(e')$)

$$R_g : G/S_\alpha \to G/S' ,$$

which induces a homomorphism $M(\phi)\colon M\left(G/S_\alpha\right) \to M\left(G/S'\right)$, usually written $(R_g)_*$. This yields a differential $\partial_d\colon C_d \to C_{d-1}$, and the Bredon homology groups $H_i^{\mathfrak{F}}(X; M)$ correspond to the homology of (C_*, ∂_*). Observe that the definition is independent of the family $\mathfrak{F}$ as long as it contains $\mathfrak{Iso}(X)$.

Bredon cohomology is defined analogously, for M a right Bredon module (a contravariant M will reverse the arrows $(R_g)^* = M(\phi)\colon M(G/S') \to M(G/S_\alpha)$ so that $\partial : C_{d-1} \to C_d$).

The Bredon homology of a group G with coefficients in $M \in G\text{-Mod}_{\mathfrak{F}}$ can be defined in terms of a Tor functor ([14, Def. 3.12]). If $\mathfrak{F}$ is closed under conjugations and taking subgroups then

$$H_i^{\mathfrak{F}}(G; M) \cong H_i^{\mathfrak{F}}(E_{\mathfrak{F}}G; M) ,$$

which may as well be taken as a definition.

We are interested in the case $X = \underline{E}G$, $\mathfrak{F} = \mathfrak{Fin}(G)$ and $M = \mathcal{R}$ the complex representation ring, considered as a Bredon module as follows. On objects we set

$$\mathcal{R}(G/K) = R_{\mathbb{C}}(K), \quad K \in \mathfrak{Fin}(G)$$

the ring of complex representations of the finite group K (viewed just as an abelian group), and for a G-map $R_g : G/K \to G/L$, we have $g^{-1}Kg \subset L$ so that $(R_g)_* : R_{\mathbb{C}}(K) \to R_{\mathbb{C}}(L)$ is given by induction from $g^{-1}Kg$ to L after identifying $R_{\mathbb{C}}(g^{-1}Kg) \cong R_{\mathbb{C}}(K)$.

2.3 Equivariant K-homology

There is an equivariant version of K-homology, denoted $K_i^G(-)$ and defined in [4] (see also [10]) using spaces and spectra over the orbit category of G. It was originally defined in [1] using Kasparov's KK-theory. We will only recall the properties we need.

Equivariant K-theory satisfies Bott mod-2 periodicity, so we only consider K_0^G and K_1^G. For any subgroup H of G, we have

$$K_i^G\left(G/H\right) = K_i\left(C_r^*(H)\right) ,$$

that is, its value at one-orbit spaces corresponds to the K-theory of the reduced C^*-algebra of the typical stabilizer. If H is a *finite* subgroup then $C_r^*(H) = \mathbb{C}H$ and

$$K_i^G\left(G/H\right) = K_i(\mathbb{C}H) = \begin{cases} R_{\mathbb{C}}(H) & i = 0\,, \\ 0 & i = 1\,. \end{cases}$$

This allows us to view $K_i^G(-)$ as a Bredon module over $\mathcal{O}_{\mathfrak{Fin}}G$.

We can use an equivariant Atiyah-Hirzebruch spectral sequence to compute the K^G-homology of a proper G-CW-complex X from its Bredon homology (see [14, pp. 49-50] for details), as

$$E^2_{p,q} = H^{\mathfrak{Fin}}_p\left(X; K^G_q(-)\right) \Rightarrow K^G_{p+q}(X) \ .$$

In the simple case when Bredon homology concentrates at low degree, it coincides with the equivariant K-homology:

Proposition 2.1. *Write H_i for $H^{\mathfrak{Fin}}_i(X;\mathcal{R})$ and K^G_i for $K^G_i(X)$. If $H_i = 0$ for $i \geq 2$ then $K^G_0 = H_0$ and $K^G_1 = H_1$.*

Proof. The Atiyah-Hirzebruch spectral sequence collapses at the 2-page. $\diamondsuit$

3 Künneth formulas for Bredon homology

We will need Künneth formulas for (relative) Bredon homology. We devote this section to state such formulas, for the product of spaces and the direct product of groups. Theorem 3.1 holds in more generality (see [7]) but we state the result we need and sketch a direct proof; details can be found in [17, Chapter 3]. Results similar to those explained here are treated in [11].

Künneth formula for $X \times Y$

Let X be a G-CW-complex and Y a H-CW-complex. Let $\mathfrak{F}$ (resp. $\mathfrak{F}'$) be a family of subgroups of G (resp. of H) containing $\mathfrak{Iso}(X)$ (resp. $\mathfrak{Iso}(Y)$). Then $X \times Y$ is a $(G \times H)$-CW-complex (with the compactly generated topology) and

$$\mathfrak{Iso}(X \times Y) = \mathfrak{Iso}(X) \times \mathfrak{Iso}(Y) \subset \mathfrak{F} \times \mathfrak{F}' \, .$$

Moreover, the orbit category $\mathcal{O}_{\mathfrak{F}\times\mathfrak{F}'}(G \times H)$ is isomorphic to $\mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}_{\mathfrak{F}'}H$.

Given $M \in G\text{-Mod}_{\mathfrak{F}}$, $N \in H\text{-Mod}_{\mathfrak{F}'}$ we define their tensor product over $\mathbb{Z}$ as the composition of the two functors

$$M \otimes N : \mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}_{\mathfrak{F}'}H \xrightarrow{M \times N} \mathfrak{Ab} \times \mathfrak{Ab} \xrightarrow{\otimes_{\mathbb{Z}}} \mathfrak{Ab}$$

considered a Bredon module over $\mathcal{O}_{\mathfrak{F}\times\mathfrak{F}'}(G \times H) \cong \mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}_{\mathfrak{F}'}H$. We can easily extend this tensor product to chain complexes of Bredon modules.

Theorem 3.1 (Künneth Formula for Bredon Homology)**.** *Consider X, Y, $\mathfrak{F}$, $\mathfrak{F}'$, M, N as above, with the property that $M(G/K)$ and $N(H/K')$ are free for all $K \in \mathfrak{Iso}(X)$, $K' \in \mathfrak{Iso}(Y)$. Then for every $n \geq 0$, there is a split exact sequence*

$$0 \to \bigoplus_{i+j=n} \Big(H^{\mathfrak{F}}_i(X;M) \otimes_{\mathbb{Z}} H^{\mathfrak{F}'}_j(Y;N) \Big) \to H^{\mathfrak{F}\times\mathfrak{F}'}_n(X \times Y; M \otimes N)$$

$$\to \bigoplus_{i+j=n-1} \mathrm{Tor}\Big(H^{\mathfrak{F}}_i(X;M), H^{\mathfrak{F}'}_j(Y;N) \Big) \to 0$$

Proof. *(sketch)* Consider the chain complexes of abelian groups

$$D_* = \underline{C_*(X)} \otimes_{\mathfrak{F}} M \quad \text{and} \quad D'_* = \underline{C_*(Y)} \otimes_{\mathfrak{F}'} N\,,$$

where $\underline{C_*(-)}$ is the chain complex of Bredon modules defined in [14, p. 10-11], and $-\otimes_{\mathfrak{F}}-$ is the tensor product defined in [14, p. 14]. Then, by Definition 3.13 in [14],

$$H_i(D_*) = H_i^{\mathfrak{F}}(X;M)\,, \quad H_j(D'_*) = H_j^{\mathfrak{F}'}(Y;N)\,.$$

We now use the following lemmas, which follow from the definitions of $\underline{C_*(-)}$ and $-\otimes_{\mathfrak{F}}-$.

Lemma 3.2. *Given $\underline{C_*}$ a chain complex in* $\mathrm{Mod}_{\mathfrak{F}}$*-$G$, $\underline{C'_*}$ a chain complex in Mod$_{\mathfrak{F}'}$-H and Bredon modules $M \in G$-*$\mathrm{Mod}_{\mathfrak{F}}$*, $N \in H$-Mod$_{\mathfrak{F}'}$, the following chain complexes of abelian groups are isomorphic*

$$\left(\underline{C_*} \otimes_{\mathfrak{F}} M\right) \otimes \left(\underline{C'_*} \otimes_{\mathfrak{F}'} N\right) \cong \left(\underline{C_*} \otimes \underline{C'_*}\right) \otimes_{\mathfrak{F}\times\mathfrak{F}'} \left(M \otimes N\right).$$

Lemma 3.3. *The following chain complexes in* $\mathrm{Mod}_{\mathfrak{F}\times\mathfrak{F}'}$*-$(G{\times}H)$ are isomorphic*

$$\underline{C_*(X\times Y)} \cong \underline{C_*(X)} \otimes \underline{C_*(Y)}\ .$$

By Lemma 3.2,

$$D_* \otimes_{\mathbb{Z}} D'_* \cong \left(\underline{C_*(X)} \otimes \underline{C_*(Y)}\right) \otimes_{\mathfrak{F}\times\mathfrak{F}'} (M \otimes N)$$

and therefore, by Lemma 3.3,

$$H_n(D_* \otimes_{\mathbb{Z}} D'_*) = H_n^{\mathfrak{F}\times\mathfrak{F}'}(X\times Y; M\otimes N)\,.$$

The result is now a consequence of the ordinary Künneth formula of chain complexes of abelian groups. Observe that the chain complexes D_* and D'_* are free since

$$D_n = \underline{C_n(X)} \otimes_{\mathfrak{F}} M \cong \bigoplus_{\alpha} M(G/S_\alpha)$$

(see [14, p. 14-15]), and similarly for D'_n. ♢

There is also a Künneth formula for *relative* Bredon homology (see [14, p. 12] for definitions).

Theorem 3.4 (Künneth Formula for Relative Bredon Homology)**.** *Under the same hypothesis of Theorem 3.1, plus $A \subset X$ a G-CW-subcomplex and $B \subset Y$ a H-CW-subcomplex, we have that for every $n \geq 0$ there is a split exact sequence*

$$\begin{aligned}
0 \longrightarrow &\bigoplus_{i+j=n} \left(H_i^{\mathfrak{F}}(X,A;M) \otimes_{\mathbb{Z}} H_j^{\mathfrak{F}'}(Y,B;N)\right) \\
&\longrightarrow H_n^{\mathfrak{F}\times\mathfrak{F}'}(X\times Y, A\times Y \cup X\times B; M\otimes N) \\
&\longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}\left(H_i^{\mathfrak{F}}(X,A;M), H_j^{\mathfrak{F}'}(Y,B;N)\right) \longrightarrow 0\ .
\end{aligned}$$

The proof is the same as the non-relative formula but using the chain complexes

$$\begin{aligned} D_* &= \underline{C_*(X,A)} \otimes_{\mathfrak{F}} M \\ D'_* &= \underline{C_*(Y,B)} \otimes_{\mathfrak{F}'} M \end{aligned}$$

and a relative version of Lemma 3.3.

Künneth formula for $G\times H$

We will work in general with respect to a class of groups $\mathfrak{F}$ (formally, $\mathfrak{F}$ is a collection of groups closed under isomorphism, or a collection of isomorphism types of groups). Our main example will be $\mathfrak{Fin}$, the class of finite groups.

Given a specific group G, let us denote by $\mathfrak{F}(G)$ the family of subgroups of G which are in $\mathfrak{F}$, and by $H_*^{\mathfrak{F}}(-)$ the Bredon homology with respect to $\mathfrak{F}(G)$. Suppose now that $\mathfrak{F}$ is closed under taking subgroups (note that it is always closed under conjugation). Then so is $\mathfrak{F}(G)$ and we will write $E_{\mathfrak{F}}G$ for $E_{\mathfrak{F}(G)}G$.

We want to know when $E_{\mathfrak{F}}G \times E_{\mathfrak{F}}H$ is a model for $E_{\mathfrak{F}}(G\times H)$.

Proposition 3.5. *Let $\mathfrak{F}$ be a class of groups which is closed under subgroups, finite direct products and homomorphic images. Then, for any groups G and H, and X and Y models for $E_{\mathfrak{F}}G$ respectively $E_{\mathfrak{F}}H$, the space $X\times Y$ is a model for $E_{\mathfrak{F}}(G\times H)$.*

Proof. If $(x,y)\in X\times Y$ then

$$\mathrm{stab}_{G\times H}(x,y)=\mathrm{stab}_G(x)\times\mathrm{stab}_H(y)\in\mathfrak{F}(G)\times\mathfrak{F}(H)\subset\mathfrak{F}(G\times H)\,.$$

Consider the projections $\pi_1 : G\times H\to G$ and $\pi_2 : G\times H\to H$. If $K\in\mathfrak{F}(G\times H)$,

$$(X\times Y)^K = X^{\pi_1(K)}\times Y^{\pi_2(K)}$$

contractible since $\pi_1(K)\in\mathfrak{F}(G)$ and $\pi_2(K)\in\mathfrak{F}(H)$. $\diamondsuit$

Remark 1. The proposition is not true if we remove any of the two extra conditions on $\mathfrak{F}$.

Remark 2. Not every family of subgroups of G can be written as $\mathfrak{F}(G)$ (for instance, $\mathfrak{F}(G)$ is always closed under conjugation). To be more general, one can prove that given two families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of subgroups of G respectively H, $E_{\mathfrak{F}_1}G\times E_{\mathfrak{F}_2}H$ is a model for $E_{\mathfrak{F}}(G\times H)$, where $\mathfrak{F}$ is the smallest family closed under subgroups and containing $\mathfrak{F}_1\times\mathfrak{F}_2$. Nevertheless, we are interested in group theoretic properties (as 'being finite'), so we think of $\mathfrak{F}$ as the class of groups with the required property.

Thus we can apply the Künneth formula (Theorem 3.1) to $E_{\mathfrak{F}}(G\times H)$. We obtain Bredon homology groups with respect to the family $\mathfrak{F}(G)\times\mathfrak{F}(H)$ instead of $\mathfrak{F}(G\times H)$, but both families contain the isotropy groups of $E_{\mathfrak{F}}G\times E_{\mathfrak{F}}H$, so the Bredon homology groups are the same (cf. Section 2.2).

Theorem 3.6 (Künneth formula for $G\times H$)**.** *Let $\mathfrak{F}$ be a class of groups closed under taking subgroups, direct product and homomorphic images. For every $n\geq 0$ there is a split exact sequence*

$$0\to\bigoplus_{i+j=n}\Big(H_i^{\mathfrak{F}}(G;M)\otimes_{\mathbb{Z}}H_j^{\mathfrak{F}}(H;N)\Big)\to H_n^{\mathfrak{F}}(G\times H;M\otimes N)\to$$
$$\bigoplus_{i+j=n-1}\mathrm{Tor}\Big(H_i^{\mathfrak{F}}(G;M),H_j^{\mathfrak{F}}(H;N)\Big)\to 0\,.$$

Remark 3. The analogous result for relative Bredon homology also holds, by Theorem 3.4.

Application to proper actions and coefficients in the representation ring

The original motivation for stating a Künneth formula was the Bredon homology of $G\times H$ for proper actions (i.e. $\mathfrak{F} = \mathfrak{Fin}$) and coefficients in the representation ring. We only need a result on the coefficients.

For finite groups P and Q, the representation ring $R_{\mathbb{C}}(P\times Q)$ is isomorphic to $R_{\mathbb{C}}(P)\otimes R_{\mathbb{C}}(Q)$. The same is true at the level of Bredon coefficient systems. Given groups G and H, consider the left Bredon modules

$$\mathcal{R}^G \in G\text{-Mod}_{\mathfrak{Fin}(G)}\ ,\ \mathcal{R}^H \in H\text{-Mod}_{\mathfrak{Fin}(H)} \ \text{ and } \ \mathcal{R}^{G\times H} \in (G\times H)\text{-Mod}_{\mathfrak{Fin}(G\times H)}\,.$$

Let us denote by $\widetilde{\mathcal{R}}^{G\times H}$ the restriction of $\mathcal{R}^{G\times H}$ to $\mathfrak{Fin}(G)\times\mathfrak{Fin}(H)$.

Proposition 3.7. *These two Bredon modules are naturally isomorphic*

$$\mathcal{R}^G\otimes\mathcal{R}^H \cong \widetilde{\mathcal{R}}^{G\times H}\,.$$

Proof. Consider the isomorphism $\Theta_{P,Q} : R_{\mathbb{C}}(P\times Q)\cong R_{\mathbb{C}}(P)\otimes R_{\mathbb{C}}(Q)$ and the universal property of the tensor product to obtain a map α as

$$\begin{tikzcd}
R_{\mathbb{C}}(P)\times R_{\mathbb{C}}(Q) \arrow[r] \arrow[dr, "\alpha"'] & R_{\mathbb{C}}(P)\otimes R_{\mathbb{C}}(Q) \arrow[d, "\Theta_{P,Q}", "\cong"'] \\
 & R_{\mathbb{C}}(P\times Q)\,.
\end{tikzcd}$$

That is, $\Theta_{P,Q}(\rho\otimes\tau) = \alpha(\rho,\tau)$, where the latter denotes the ordinary tensor product of representations and $\rho\otimes\tau$ is their formal tensor product in $R_{\mathbb{C}}(P)\otimes R_{\mathbb{C}}(Q)$. Extend this map to a natural isomorphism $\Theta : \mathcal{R}^G\otimes\mathcal{R}^H \to \widetilde{\mathcal{R}}^{G\times H}$. It only remains to check naturality, that is, the commutativity of

$$\begin{tikzcd}
R_{\mathbb{C}}(P)\otimes R_{\mathbb{C}}(Q) \arrow[r, "\Theta_{P,Q}"] \arrow[d, "(R_g)_*\otimes(R_h)_*"'] & R_{\mathbb{C}}(P\times Q) \arrow[d, "(R_{(g,h)})_*"] \\
R_{\mathbb{C}}(P')\otimes R_{\mathbb{C}}(Q') \arrow[r, "\Theta_{P',Q'}"] & R_{\mathbb{C}}(P'\times Q')\,,
\end{tikzcd}$$

for any $(G\times H)$-map $R_{(g,h)} = R_g\times R_h$. To do this, one can compute both sides of the square and show that the two representations have the same character, using the following two observations. Denote by $\chi(-)$ the character of a representation. Since $\alpha(-,-)$ is the tensor product of representations we have $\chi\left(\alpha(\tau^1,\tau^2)\right) = \chi(\tau^1)\cdot\chi(\tau^2)$. If $R_g : G/K\to G/L$ is a G-map, $\rho\in R_{\mathbb{C}}(K)$ and $t\in G$ then

$$\chi\left((R_g)_*\,\rho\right)(t) = \frac{1}{|K|}\sum_{*}\chi(\rho)(k)\,,$$

where the sum $*$ is taken over the $s\in G$ such that $k = gsts^{-1}g^{-1}\in K$. We leave the details to the reader. $\diamondsuit$

For an arbitrary group G, write $H_i^{\mathfrak{Fin}}(G;\mathcal{R})$ for $H_i^{\mathfrak{Fin}}(G;\mathcal{R}^G)$.

Corollary 3.8. *For every $n \geq 0$ there is a split exact sequence*

$$0 \to \bigoplus_{i+j=n} \Big(H_i^{\mathfrak{Fin}}(G;\mathcal{R}) \otimes_{\mathbb{Z}} H_j^{\mathfrak{Fin}}(H;\mathcal{R}) \Big) \to H_n^{\mathfrak{Fin}}(G \times H;\mathcal{R}) \to$$

$$\bigoplus_{i+j=n-1} \operatorname{Tor}\Big(H_i^{\mathfrak{Fin}}(G;\mathcal{R}), H_j^{\mathfrak{Fin}}(H;\mathcal{R}) \Big) \to 0\,.$$

Remark 4. Again, there is a similar statement for relative Bredon homology.

4 Coxeter groups and the Davis complex

We briefly recall the definitions and basic properties of Coxeter groups, and describe a model of the classifying space for proper actions. Most of the material is well-known and can be found in any book on the subject, as [9].

4.1 Coxeter groups

Suppose W is a group, $S = \{s_1, \dots, s_N\}$ a finite subset of elements of order 2 which generate W. Let m_{ij} denote the order of $s_i s_j$. Then $2 \leq m_{ij} = m_{ji} \leq \infty$ if $i \neq j$ and $m_{ii} = 1$. We call the pair (W, S) a *Coxeter system* and W a *Coxeter group* if W admits a presentation

$$\langle\, s_1, \dots, s_N \mid (s_i s_j)^{m_{ij}} = 1 \,\rangle\ .$$

We call N the *rank* of (W, S). The *Coxeter diagram* is the graph with vertex set S and one edge joining each pair $\{s_i, s_j\}$, $i \neq j$, with label m_{ij}, and the conventions: if $m_{ij} = 2$ we omit the edge; if $m_{ij} = 3$ we omit the label. A Coxeter system is *irreducible* if its Coxeter diagram is connected. Every Coxeter group can be decomposed as a direct product of irreducible ones, corresponding to the connected components of its Coxeter diagram. All *finite* irreducible Coxeter systems have been classified; a list of their Coxeter diagrams can be found, for instance, in [9, p. 32].

Every Coxeter group can be realized as a group generated by reflections in $\mathbb{R}^N$. Namely, there is a faithful *canonical representation* $W \to GL_N(\mathbb{R})$ (see [9, §5.3] for details). The generators in S correspond to reflections with respect to hyperplanes. These hyperplanes bound a *chamber*, which is a strict fundamental domain (examples below).

Coxeter cells

Suppose now that W is finite. Take a point x in the interior of the chamber. The orbit Wx is a finite set of points in $\mathbb{R}^N$. Define the *Coxeter cell* C_W as the convex hull of Wx. It is a complex polytope by definition.

Examples:

(1) Rank 0: $W = \{1\}$ the trivial group and C_W is a point.

(2) Rank 1: $W = C_2$ the cyclic group of order 2, $\mathbb{R}^N = \mathbb{R}$, the W-action consists on reflecting about the origin, and there are two chambers, $(-\infty, 0]$ and $[0, +\infty)$. Take a point x in the interior of, say, $[0, +\infty)$; then $Wx = \{-x, x\}$ and $C_W = [-x, x]$, an interval.

(3) Rank 2: $W = D_n$ dihedral group of order $2n$, $\mathbb{R}^N = \mathbb{R}^2$ and the canonical representation identifies W with the group generated by two reflections respect to lines through the origin and mutual angle π/n. A chamber is the section between the two lines and the W-orbit Wx is the vertex set of a $2n$-gon (see Figure 1). Therefore, C_W is a $2n$-gon.

Figure 1: Coxeter cell for a dihedral group.

(4) If $W = W_1 \times W_2$ then $C_W = C_{W_1} \times C_{W_2}$.

The Davis complex

It is a standard model of $\underline{E}W$ introduced by M. Davis in [5]. Given $T \subset S$, the group $W_T = \langle T \rangle$ generated by T is called a *special subgroup*. It can be shown that (W_T, T) is a Coxeter system, and that $W_T \cap W_{T'} = W_{T \cap T'}$. The subset T is called *spherical* is W_T is finite. We will write $\mathcal{S}$ for the set $\{T \subset S \,|\, T \text{ is spherical}\}$.

A *spherical coset* is a coset of a finite special subgroup, that is, wW_T for some $w \in W$, $T \in \mathcal{S}$. Note that $wW_T = w'W_{T'}$ if and only if $T = T'$ and $w^{-1}w' \in W_T$ (use the so-called *deletion condition* [9, §5.8]). Denote by $W\mathcal{S}$ the set of all spherical cosets in W, that is, the disjoint union

$$W\mathcal{S} = \bigcup_{T \in \mathcal{S}} W/W_T \,.$$

This is a partially ordered set (*poset*) by inclusion, and there is a natural W-action whose quotient space is $\mathcal{S}$. The *Davis complex* Σ is defined as the geometric realization $|W\mathcal{S}|$ of the poset $W\mathcal{S}$, that is, the simplicial complex with one n-simplex for each chain of length n

$$w_1 W_{T_1} \subset \ldots \subset w_n W_{T_n} \,, \quad T_i \in \mathcal{S} \,, \tag{4.1}$$

and obvious identifications. The W-action on $W\mathcal{S}$ induces an action on Σ for which this is a proper space: the stabilizer of the simplex corresponding to the chain (4.1) is $w_1 W_{T_1} w_1^{-1}$, finite. Moreover, Σ admits a CAT(0)-metric [16] and therefore it is a model of $\underline{E}W$, by the following result.

Proposition 4.1. *If a finite group H acts by isometries on a CAT(0)-space X, then the fixed point subspace X^H is contractible.*

Proof. The idea is to find a fixed point for H and then use that if $x, y \in X$ are fixed by H then so is the geodesic between x and y (from which the contractibility of X^H follows). The existence of a fixed point for H is a consequence of the Bruhat-Tits fixed point theorem [3, p. 157]. $\diamondsuit$

Examples:

(1) Finite groups.- Suppose that W is a finite Coxeter group and C is its Coxeter cell, defined as the convex hull of certain orbit Wx. Denote by $\mathcal{F}(C)$ the poset of faces of the convex polytope C. The correspondence

$$w \in W \;\mapsto\; wx \in Wx \;=\; \text{vertex set of C}$$

allows us to identify a subset of W with a subset of vertices of C. In fact, a subset of W corresponds to the vertex set of a face of C if and only if it is a coset of a special subgroup ([3, III]); see Figure 1 for an example. Hence, we have the isomorphism of posets $W\mathcal{S} \cong \mathcal{F}(C)$ and the Davis complex $\Sigma = |W\mathcal{S}|$ is the barycentric subdivision of the Coxeter cell.

(2) Triangle groups.- These are the Coxeter groups of rank 3,

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \,\rangle\,.$$

Suppose that $p, q, r \neq \infty$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. All subsets $T \subsetneq S$ are spherical, giving the poset (where arrows stand for inclusions) of Figure 2. It can be realized as the barycentric subdivision of an euclidean or hyperbolic triangle with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$, and a, b and c acting as reflections through the corresponding sides. The whole model consists on a tessellation of the euclidean or hyperbolic space by these triangles.

Figure 2: Davis complex (quotient) for a triangle group.

A second definition of the Davis complex

There is an alternative description of the Davis complex in terms of Coxeter cells. Given a poset $\mathcal{P}$ and an element $X \in \mathcal{P}$, we denote by $\mathcal{P}_{\leq X}$ the subposet of elements in $\mathcal{P}$ less or equal to X. Consider a spherical subset $T \subset S$ and an element $w \in W$.

Proposition 4.2. *There is an isomorphism of posets* $(W\mathcal{S})_{\leq wW_T} \cong W_T\mathcal{S}$.

Proof. Firstly, observe that the poset $(W\mathcal{S})_{\leq wW_T}$ is equivalent to $(W\mathcal{S})_{\leq W_T}$ via the isomorphism induced by multiplication by w^{-1}. So it suffices to show that $W\mathcal{S}_{\leq W_T} = W_T\mathcal{S}$. A standard element in the right hand side is a coset $wW_{T'}$ with $w \in W_T$ and $T' \subset T$ so we have $W_{T'} \subset W_T$ and $wW_{T'} \subset wW_T = W_T$. On the other hand, if $wW_{T'} \subset W_T$ then $w = w \cdot 1 \in W_T$. Finally, $T' \subset T$: if $s' \in T'$ then $ws' \in W_T$ so $s' \in W_T$ and, using the deletion condition, $s' \in T$. ♢

Now, since W_T is a finite Coxeter group, we can identify the subcomplex

$$|\,(W\mathcal{S})_{\leq wW_T}\,| = |W_T\mathcal{S}|$$

with the barycentric subdivision of the associated Coxeter cell C_{W_T}. From this point of view, the poset $W\mathcal{S}$ is the union of subposets $(W\mathcal{S})_{\leq wW_T}$ with T spherical and $w \in W/W_T$. That is, Σ can be viewed as the union of Coxeter cells

$$\bigcup_{\substack{T\in\mathcal{S}\\ wW_T\in W/W_T}} C_{wW_T}$$

where C_{wW_T} is the copy of C_{W_T} corresponding to the coset wW_T. This union is obviously not disjoint, and the inclusions and intersections among subposets are precisely the following.

Lemma 4.3. *(i)* $wW_T \subset w'W_{T'}$ *if and only if* $T \subset T'$ *and* $w^{-1}w' \in W_{T'}$.

(ii) $wW_T \cap w'W_{T'} = w_0W_{T\cap T'}$ *if there is any* $w_0W_T \in wW_T \cap w'W_{T'}$*, empty otherwise.*

Proof. (i) One implication is obvious, the other uses the deletion condition in the fashion of the proof of Proposition 4.2.

(ii) Straightforward, since $W_T \cap W_{T'} = W_{T\cap T'}$. ♢

Consequently, the intersection of two Coxeter cells is

$$C_{wW_T} \cap C_{w'W_{T'}} = \begin{cases} C_{w_0W_{T\cap T'}} & \text{if } wW_T \cap w'W_{T'} = w_0W_{T\cap T'} \\ \varnothing & \text{otherwise}. \end{cases} \tag{4.2}$$

Denote by ∂C_W the *boundary* of the Coxeter cell, that is, the topological boundary in the ambient space $\mathbb{R}^N$. We have the following explicit description.

Proposition 4.4.

$$\partial C_W = \bigcup_{\substack{T\subsetneq S\\ wW_T\in W/W_T}} wC_{W_T}$$

Proof. C_W is a convex polytope whose faces correspond to cosets of special subgroups (Example on page 10). The barycentric subdivision $sd(C_W) = |W\mathcal{S}|$ is indeed a cone over its center (the vertex corresponding to the coset $W_S = W$). ♢

5 Relative Bredon homology of the Davis complex

The aim of this section is to deduce a formula for the relative Bredon homology of the n-skeleton of the Davis complex Σ with respect to the $(n-1)$-skeleton, $H_*^{\mathfrak{Fin}}(\Sigma_n, \Sigma_{n-1})$, in terms of the Bredon homology of the Coxeter cells.

We will denote by Σ the Davis complex with the Coxeter cell decomposition explained on page 10 and by Σ' the original definition as the nerve of $W\mathcal{S}$. Note that both are W-homeomorphic spaces and $\Sigma' = sd(\Sigma)$ (barycentric subdivision) but Σ is *not* a W-CW-complex while Σ' is (recall that in a G-CW-complex a cell is sent to itself only by the identity map).

As an example, see Figure 3: it is the tessellation of the euclidean plane induced by the triangle group $\Delta(2,4,4)$ together with the dual tessellation given by squares and octagons. The latter is Σ, the Davis complex given as union of Coxeter cells. The skeleton filtration correspond to Coxeter cells of rank 0 (points), rank 1 (intervals) and rank 2 ($2n$-gons).

Figure 3: Tessellation of the euclidean plane given by the triangle group $\Delta(2,4,4)$, and its dual tessellation.

Definition 5.1. *For $n \geq -1$ define $\Sigma_n \subset \Sigma$ as the union of Coxeter cells corresponding to finite W_T with rank$(T) \leq n$. That is,*

$$\Sigma_n = \bigcup_{\substack{T \in \mathcal{S},\, |T| \leq n \\ wW_T \in W/W_T}} C_{wW_T} .$$

Then $\Sigma_{-1} = \varnothing$, Σ_0 is a free orbit of points and Σ_1 is indeed the Cayley graph of W with respect to S (see Figure 3 for an example or compare with Proposition 5.4).

We have that Σ_n is a W-subspace ($w' C_{wW_T} = C_{w'wW_T}$) and, since the dimension of C_{W_T} is rank(T), Σ_n is indeed the n-skeleton of Σ.

Our aim is to prove the following theorem.

Theorem 5.2. *For any dimension $n \geq 0$, any degree i and any Bredon coefficient system M, we have the isomorphism*

$$H_i^{\mathfrak{Fin}}(\Sigma_n, \Sigma_{n-1}; M) \cong \bigoplus_{\substack{T \in \mathcal{S} \\ \operatorname{rank}(T)=n}} H_i^{\mathfrak{Fin}}(C_{W_T}, \partial C_{W_T}; M) \; .$$

Comments on the statement of the theorem:

(i) We have defined Bredon homology on G-CW-complexes so we implicitly assume that we take the barycentric subdivision on the spaces.

(ii) The Bredon homology on the left-hand side is with respect to the W-action and the family $\mathfrak{Fin}(W)$. The Bredon homology on the right-hand side is with respect to the W_T-action and the family $\mathfrak{Fin}(W_T) = \mathfrak{All}(W_T)$ (all subgroups), for each Coxeter cell.

Firstly, we state some properties of the Coxeter cells that we will use.

Lemma 5.3. *(a) Suppose $T \in \mathcal{S}$ with rank$(T) \leq n$. Then*

$$C_{W_T} \cap \Sigma_{n-1} = \begin{cases} C_{W_T} & \textit{if rank}(T) < n \, , \\ \partial C_{W_T} & \textit{if rank}(T) = n \, . \end{cases}$$

(b) The intersection of any two different Coxeter cells in Σ_n is included in Σ_{n-1}.

(c) The Coxeter cell C_{W_T} is a W_T–subspace of Σ_n and $\operatorname{stab}_W(x)$ is contained in W_T for all $x \in C_{W_T}$.

Proof. (a) If rank$(T) < n$ then $C_{W_T} \subset \Sigma_{n-1}$. If rank$(T) = n$ then (Proposition 4.4)

$$\partial C_{W_T} = \bigcup_{\substack{T' \subsetneq T \\ wW_{T'} \in W_T/W_{T'}}} wC_{W_{T'}} \subset \Sigma_{n-1} \; .$$

On the other hand, if $x \in C_{W_T} \cap \Sigma_{n-1}$ then $x \in C_{wW_{T'}}$ with rank$(T') \leq n-1$ so $x \in C_{W_T} \cap wC_{W_{T'}}$ which, by equation (4.2), equals to either empty or $w_0 C_{W_{T \cap T'}} \subset \partial C_{W_T}$.

(b) In general, the intersection of two different Coxeter cells of rank n and m is either empty or another Coxeter cell of rank strictly less than $\min\{n, m\}$ — see equation (4.2).

(c) The first part is obvious: $wC_{W_T} = C_{wW_T} = C_{W_T}$ if $w \in W_T$. Suppose now that $x \in C_{W_T}$ and $wx = x$. We want $w \in W_T$. Identify $sd(C_{W_T}) = |W_T \mathcal{S}|$. Suppose that x belongs to a cell corresponding to the chain $W_{T_0} \subsetneq \ldots \subsetneq W_{T_k}$, $T_k \subset T$. If w fixes x, it has to fix at least one of the vertices of the cell, i.e., there is an i such that $wW_{T_i} = W_{T_i}$ so that $w \in W_{T_i} \subset W_T$. $\diamondsuit$

Secondly, we observe that the copies of a Coxeter cell in Σ admit an interpretation as induced spaces. If H is a subgroup of G and X is an H-space, the associated *induced* G-space is

$$\operatorname{Ind}_H^G X = G \times_H X \; ,$$

the orbit space for the H-action $h \cdot (g, x) = (gh^{-1}, hx)$. The (left) G-action on the induced space is given by $g \cdot \overline{(k, x)} = \overline{(gk, x)}$. This definition carries on to pairs of H-spaces.

Proposition 5.4. *For each spherical $T \subset S$, there is a W-homeomorphism*

$$\bigcup_{wW_T \in W/W_T} C_{wW_T} \cong_W W \times_{W_T} C_{W_T}\,.$$

Proof. If $x \in C_{wW_T} = w \cdot C_{W_T}$, write $x = wx_0$ and send x to $\overline{(w, x_0)} \in W \times_{W_T} C_{W_T}$. This is well-defined by equation (4.2): $x = wx_0 = w'y_0$ then there is $w'' \in W_T$ such that $w = w'w''$. This defines a continuous W-map, bijective, with continuous inverse

$$\overline{(w, x)} \mapsto w \cdot x \in C_{wW_T}\,. \qquad \Diamond$$

Consequently, we may write Σ_n, $n \geq 1$, as a union of induced spaces

$$\Sigma_n \cong \bigcup_{|T| \leq n} W \times_{W_T} C_{W_T}\,.$$

Next, we will need the following easy consequence of a relative Mayer-Vietoris sequence.

Proposition 5.5. *Let X be a G-CW-complex and $Y, A_1, \ldots, A_n$ G-subcomplexes such that $X = A_1 \cup \ldots \cup A_n$ and $A_i \cap A_j \subset Y$ for all $i \neq j$. Write $H_n(-)$ for Bredon homology with some fixed coefficients and with respect to a family $\mathfrak{F} \supset \mathfrak{Iso}(X)$ or, more generally, any G-homology theory. Then*

$$H_n(X, Y) \cong \bigoplus_{i=1}^{n} H_n(A_i, A_i \cap Y)\,.$$

Proof. By induction on n. For $n = 1$ it is a tautology. For $n > 1$ call $A = A_1$, $B = A_2 \cup \ldots \cup A_n$, $C = A \cap Y$ and $D = B \cap Y$. The relative Mayer-Vietoris of the CW-pairs (A, C) and (B, D) is

$$\ldots \to H_i(A \cap B, C \cap D) \to H_i(A, C) \oplus H_i(B, D) \to H_i(A \cup B, C \cup D) \to \ldots \quad (5.1)$$

Observe that

$$\begin{array}{rcl} A \cup B & = & X \\ C \cup D & = & (A \cup B) \cap Y = Y \\ A \cap B & \subset & Y \\ C \cap D & = & A \cap B \cap Y = A \cap B \end{array}$$

Therefore $H_i(A \cap B, C \cap D) = 0$ for all i and the sequence (5.1) gives isomorphisms

$$H_i(X, Y) \cong H_i(A, C) \oplus H_i(B, D)$$

Now apply induction to $X' = B$, $Y' = B \cap Y$ and the result follows. $\Diamond$

Finally, recall that if $h_*^?$ is an *equivariant homology theory* (see, for instance, [13, §1]), it has an *induction structure*. Hence, for a pair of H-spaces (X, A),

$$h_n^G(\mathrm{Ind}_H^G(X, A)) \cong h_n^H(X, A)\,.$$

Bredon homology has an induction structure [15]; in particular,

$$H_n^{\mathfrak{Fin}(W)}\left(\mathrm{Ind}_{W_T}^W(C_{W_T}, \partial C_{W_T})\right) \cong H_n^{\mathfrak{Fin}(W_T)}\left(C_{W_T}, \partial C_{W_T}\right)\,.$$

Proof. (*Theorem 5.2*) Write $H_n(-)$ for Bredon homology of proper G-spaces with coefficients in the representation ring. Let $T_1, \ldots, T_m$ be the spherical subsets of generators up to rank n. Define

$$A_i = W \times_{W_{T_i}} C_{W_{T_i}} \quad 0 \le i \le m\,.$$

Then $\Sigma_n = \bigcup A_i$ and $A_i \cap A_j \subset \Sigma_{n-1}$ for all $i \neq j$, by Lemma 5.3(b). By Proposition 5.5

$$H_n(\Sigma_n, \Sigma_{n-1}) \cong \bigoplus_{i=1}^{n} H_n(A_i, A_i \cap \Sigma_{n-1})\,.$$

If $\operatorname{rank}(T_i) < n$ then $A_i \subset \Sigma_{n-1}$ and the corresponding term is zero. If $\operatorname{rank}(T_i) = n$ then $A_i \cap \Sigma_{n-1} = \partial A_i$ (Lemma 5.3(a)) and, by the induction structure,

$$H_n(A_i, \partial A_i) \cong H_n(C_{W_{T_i}}, \partial C_{W_{T_i}})\,.$$

$\diamondsuit$

Corollary 5.6. *Write $H_i(-)$ for Bredon homology with respect to finite subgroups and some fixed Bredon coefficient system. For each $n \geq 1$, there is a long exact sequence*

$$\ldots \to H_i^{\mathfrak{Fin}}(\Sigma_{n-1}) \to H_i^{\mathfrak{Fin}}(\Sigma_n) \to \bigoplus_{\substack{T \in \mathcal{S} \\ \operatorname{rank}(T)=n}} H_i^{\mathfrak{Fin}}(C_{W_T}, \partial C_{W_T}) \to \ldots$$

Remark 5. Theorem 5.2 also holds for any (proper) equivariant homology theory, in particular, for equivariant K-homology.

6 Relative Bredon homology of some Coxeter cells

As an application of Theorem 5.2, we compute the relative Bredon homology of (Σ_n, Σ_{n-1}), with coefficients in the representation ring, for the first cases $n = 0, 1, 2$. To do so, we need the Bredon homology of $(C_{W_T}, \partial C_{W_T})$ when $T \subset S$ spherical with $\operatorname{rank}(T) = 0, 1, 2$.

$\boxed{\operatorname{rank}(T) = 0}$ $T = \varnothing$, $W_T = \{1\}$ and C_{W_T} is a point. So the homology is $R_{\mathbb{C}}(\{1\}) \cong \mathbb{Z}$ at degree 0 and vanishes elsewhere.

$\boxed{\operatorname{rank}(T) = 1}$ $T = \{s_i\}$, W_T is cyclic of order two and C_{W_T} is an interval. For the relative homology, we consider one 0-cell and one 1-cell, with stabilizers cyclic order two and trivial respectively (see Figure 4). The associated Bredon chain complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{\mathbb{C}}(\{1\}) & \xrightarrow{\text{Ind}} & R_{\mathbb{C}}(C_2) & \longrightarrow & 0 \\ & & 1 & \mapsto & (1,1) & & \end{array}$$

gives

$$H_i\left(C_{W_T}, \partial C_{W_T}; \mathcal{R}\right) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}. \qquad (6.1)$$

Figure 4: Coxeter cell of rank 1

$\boxed{\mathrm{rank}(T) = 2}$ $T = \{s_i, s_j\}$, W_T is dihedral of order $2n$ ($n = m_{ij} \neq \infty$) and C_{W_T} is a $2n$-gon. The orbit space is a sector of the polygon (see Figure 5) and the relative Bredon chain complex is

Figure 5: Coxeter cell of rank 2

$$0 \longrightarrow R_{\mathbb{C}}\left(\{1\}\right) \xrightarrow{d_2} R_{\mathbb{C}}\left(C_2\right) \oplus R_{\mathbb{C}}\left(C_2\right) \xrightarrow{d_1} R_{\mathbb{C}}\left(D_n\right) \longrightarrow 0\,,$$

where the differentials are given by induction among representation rings. Inducing the trivial representation always yields the regular one, so d_2 in matrix form is

$$d_2 \equiv \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\,,$$

the right-hand side being its Smith normal form. For d_1, recall the character tables of the cyclic group of order two and dihedral group (with Coxeter generators a and b):

C_2	1	a
ρ_1	1	1
ρ_2	1	-1

D_n	$(ab)^k$	$b(ab)^k$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	$(-1)^k$	$(-1)^k$
$\widehat{\chi_4}$	$(-1)^k$	$(-1)^{k+1}$
ϕ_l	$2\cos\left(2\pi lk/n\right)$	0

where $0 \leq k \leq n-1$, l varies from 1 to $n/2$ (n even) or $(n-1)/2$ (n odd) and the hat $\widehat{\ }$ denotes a character which only appears when n is even. The induced representations corresponding to the inclusions $\langle a \rangle \subset \langle a, b \rangle$ and $\langle b \rangle \subset \langle a, b \rangle$ are

$$\begin{aligned} \mathrm{Ind}\,(\rho_1) &= \chi_1 + \widehat{\chi_3} + \sum \phi_l\,, \\ \mathrm{Ind}\,(\rho_2) &= \chi_2 + \widehat{\chi_4} + \sum \phi_l\,. \end{aligned}$$

Consequently, the differential d_1 is written in matrix form as

$$d_1 \equiv \begin{pmatrix} 1 & 0 & \widehat{1} & \widehat{0} & 1 & \dots & 1 \\ 0 & 1 & \widehat{0} & \widehat{1} & 1 & \dots & 1 \\ 1 & 0 & \widehat{0} & \widehat{1} & 1 & \dots & 1 \\ 0 & 1 & \widehat{1} & \widehat{0} & 1 & \dots & 1 \end{pmatrix},$$

which reduces by elementary operations to

$$\left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right) \text{ if } n \text{ is even}\,, \quad \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right) \text{ if } n \text{ is odd}\,.$$

Therefore, the relative Bredon homology of a Coxeter cell of rank two and $n = m_{ij} \neq \infty$ is

$$\begin{array}{llll} n \text{ even:} & H_0 = \mathbb{Z}^{c(D_n)-3} = \mathbb{Z}^{n/2}, & & H_i = 0 \;\; \forall\, i \geq 1\,; \\ n \text{ odd:} & H_0 = \mathbb{Z}^{c(D_n)-2} = \mathbb{Z}^{(n-1)/2}, & H_1 = \mathbb{Z}\,, & H_i = 0 \;\; \forall\, i \geq 2\,. \end{array}$$

Here we write $c(H)$ for the number of conjugacy classes in a finite group H, and $\mathbb{Z}^n$, or sometimes $n \cdot \mathbb{Z}$, for $\oplus_{i=1}^{n} \mathbb{Z}$.

We can now use Theorem 5.2 to deduce the lower-rank relative Bredon homology of Σ. We omit the family $\mathfrak{Fin}$ and coefficients $\mathcal{R}$ for clarity.

Proposition 6.1. *Let (W, S) be a Coxeter system and Σ its associated Coxeter complex. Then*

$$H_i\,(\Sigma_0, \Sigma_{-1}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

$$H_i\,(\Sigma_1, \Sigma_0) = \begin{cases} |S| \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

$$H_0\,(\Sigma_2, \Sigma_1) = \frac{1}{2}\left(\sum_{even,\; i<j} m_{ij} + \sum_{odd,\; i<j} (m_{ij} - 1) \right) \cdot \mathbb{Z}\,,$$

$$H_1\,(\Sigma_2, \Sigma_1) = |\{m_{ij} \;\; odd, i<j\}| \cdot \mathbb{Z}\,, \quad H_i\,(\Sigma_2, \Sigma_1) = 0 \;\; (i \geq 2)\,.$$

7 Bredon homology of the right-angled Coxeter groups

A Coxeter system (W, S) is *right-angled* if $m_{ij} \in \{2, \infty\}$ for all i, j. Thus, the only spherical subgroups are direct products of cyclic groups of order 2.

Recall that $C_{W_1 \times W_2} = C_{W_1} \times C_{W_2}$. Hence, if W_T is a product of cyclic groups of order 2, we can use a relative Künneth formula (Section 3) and equation (6.1) to deduce

$$H_i\left(C_{W_T}, \partial C_{W_T}\right) = \begin{cases} \mathbb{Z} & i = 0\,, \\ 0 & i \neq 0\,. \end{cases} \tag{7.1}$$

Call $s(n)$ the number of distinct spherical subsets $T \subset S$ of rank n ($n \geq 0$). For instance, $s(0) = 1$, $s(1) = |S|$ and $s(2) = |\{m_{ij} \neq \infty,\, i < j\}|$. By Theorem 5.2 and (7.1)

$$H_i\left(\Sigma_n, \Sigma_{n-1}\right) = \begin{cases} s(n) \cdot \mathbb{Z} & i = 0\,, \\ 0 & i \neq 0\,, \end{cases}$$

for all $n \geq 0$. The long exact sequence of the pair (Σ_n, Σ_{n-1}) gives $H_i(\Sigma_n) \cong H_i(\Sigma_{n-1})$ for $i > 0$ and the split exact sequence

$$0 \longrightarrow H_0(\Sigma_{n-1}) \longrightarrow H_0(\Sigma_n) \longrightarrow H_0(\Sigma_n, \Sigma_{n-1}) \longrightarrow 0\,.$$

Using induction on n,

$$\begin{aligned} &H_i(\Sigma_n) = H_i(\Sigma_0) = 0 \quad \forall\, i > 0\,, \\ &H_0(\Sigma_n) = H_0(\Sigma_{n-1}) \oplus H_0(\Sigma_n, \Sigma_{n-1}) = (s(0) + \ldots + s(n)) \cdot \mathbb{Z}\,. \end{aligned}$$

This gives the Bredon homology of (W, S) when $n = |S|$.

Theorem 7.1. *The Bredon homology of a right-angled Coxeter group (W, S) with respect to the family $\mathfrak{Fin}(W)$ and coefficients in the representation ring is*

$$H_i^{\mathfrak{Fin}}(W; \mathcal{R}) = \begin{cases} s \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

where $s = s(0) + \ldots + s(|S|)$, the number of spherical subsets $T \subset S$.

Finally, since the Bredon homology concentrates at degree 0, it coincides with the equivariant K-homology (Proposition 2.1). Also, the Baum-Connes conjecture holds for Coxeter groups, so we have the following corollary.

Corollary 7.2. *If W is a right-angled Coxeter group, the equivariant K-homology of $\underline{E}W$ coincides with its Bredon homology at degree 0 and 1 respectively (given in the previous theorem). This corresponds to the topological K-theory of the reduced C^*-algebra of W, via the Baum-Connes assembly map.*

8 Bredon homology of even Coxeter groups

A Coxeter system (W, S) is *even* if m_{ij} is even or infinite for all i, j. Right-angled Coxeter groups are then a particular case. Again, the only spherical subgroups are direct products of cyclic (order two) and dihedral, since any irreducible finite Coxeter group with more than two generators have at least one odd m_{ij}.

Suppose that $W_T = D_{m_1} \times \ldots \times D_{m_r} \times (C_2)^k$, with m_i ($1 \leq i \leq r$) even numbers and $k \geq 0$. We know that the relative Bredon homology of the Coxeter cells corresponding to dihedral D_{m_i} and C_2 concentrates at degree 0 and

$$H_0 = (m_i/2) \cdot \mathbb{Z}\,, \quad \text{respectively} \quad H_0 = \mathbb{Z}\,.$$

By the relative Künneth formula we have

$$H_i\left(C_{W_T}, \partial C_{W_T}\right) = \begin{cases} \left(m_1 \cdot \ldots \cdot m_r/2^r\right) \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases} .$$

Alternatively, if $T = \{t_1, \ldots, t_n\}$ is a spherical subset of rank n,

$$H_0\left(C_{W_T}, \partial C_{W_T}\right) = \prod_{i<j} m_{ij}/2 \cdot \mathbb{Z}\,, \tag{8.1}$$

and 0 at any other degree. This expression is valid for any $n \geq 0$, with the convention that an empty product equals 1.

Now we can compute the relative homology groups of the skeleton filtration of Σ. They concentrate at degree zero, since so does the homology of the Coxeter cells. We already know that

$$H_0\left(\Sigma_0, \Sigma_{-1}\right) = \mathbb{Z}, \quad H_0\left(\Sigma_1, \Sigma_0\right) = |S| \cdot \mathbb{Z}, \quad H_0\left(\Sigma_2, \Sigma_1\right) = \sum_{\substack{i<j \\ m_{ij}\neq\infty}} m_{ij}/2 \ \cdot \mathbb{Z}\,.$$

In general, using Theorem 5.2 and equation (8.1), we have that $H_0\left(\Sigma_n, \Sigma_{n-1}\right)$, $n \geq 0$ is a free abelian group of rank

$$r(n) = \sum_{\substack{T\in\mathcal{S} \\ \operatorname{rank}(T)=n}} \prod_{\substack{s_i,s_j\in T \\ i<j}} m_{ij}/2\,,$$

with the convention that an empty sum is 0 and an empty product is 1. As before, the long exact sequences of (Σ_n, Σ_{n-1}) and induction on n gives

$$\begin{aligned} &H_i\left(\Sigma_n\right) = H_i\left(\Sigma_0\right) = 0 \quad \forall\, i \geq 1 \ \text{ and} \\ &H_0\left(\Sigma_n\right) = H_0\left(\Sigma_{n-1}\right) \oplus H_0\left(\Sigma_n, \Sigma_{n-1}\right) . \end{aligned}$$

Define

$$r = r(0) + \ldots + r(|S|) = \sum_{T\in\mathcal{S}} \prod_{\substack{s_i,s_j\in T \\ i<j}} m_{ij}/2\,.$$

Theorem 8.1. *The Bredon homology of an even Coxeter group is the free abelian group of rank r at degree 0 and vanishes at any other degree.*

Remark 6. This theorem yields Theorem 7.1 for a right-angled Coxeter group. It agrees as well with the Bredon homology of the infinite dihedral group and the triangle groups with p, q and r even (Section 9).

As before, the Bredon homology coincide with the equivariant K-homology.

Corollary 8.2. *If W is an even Coxeter group, the equivariant K-homology of $\underline{E}W$ coincides with its Bredon homology at degree 0 and 1 respectively (given in the previous theorem). This corresponds to the topological K-theory of the reduced C^*-algebra of W via the Baum-Connes assembly map.*

9 Bredon homology of low-rank Coxeter groups

We briefly recall, for completeness, the Bredon homology of the Coxeter groups of rank up to three. If a group H is finite, a one-point space is a model of $\underline{E}G$ so its Bredon homology reduces to the representation ring $R_{\mathbb{C}}(H)$ at degree 0 and vanishes elsewhere, and so its equivariant K-homology. The first infinite Coxeter groups are the infinite dihedral

$$D_\infty = \langle a, b \,|\, a^2 = b^2 = 1 \rangle \ ,$$

and the euclidean and hyperbolic triangle groups

$$\langle a, b, c \,|\, a^2 = b^2 = c^2 = 1,\ (ab)^p = (bc)^q = (ca)^r = 1 \rangle \ ,$$

where $2 \leq p, q, r \leq \infty$ and $1/p + 1/q + 1/r \leq 1$. All cases can be done by direct computation from the correponding orbit spaces $\underline{B}G$ ([17, Section 5.2]). The resulting Bredon homology concentrates at degree $i \leq 1$ so it coincides with their K^G-homology. The results are the following (write $H_i(G)$ for $H_i^{\mathfrak{Fin}}(\underline{E}G; \mathcal{R})$, $n \cdot \mathbb{Z}$ for $\oplus_{i=1}^n \mathbb{Z}$, and $c(H)$ for the number of conjugacy classes on a finite group H).

$$H_i(D_\infty) = \begin{cases} 3 \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_i(\Delta(\infty, \infty, \infty)) = \begin{cases} 4 \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_i(\Delta(p, \infty, \infty)) = \begin{cases} (c(D_p) + 1) \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_i(\Delta(p, q, \infty)) = \begin{cases} (c(D_p) + c(D_q) - 2) \cdot \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$H_0(\Delta(p, q, r)) = \begin{cases} (c(D_p) + c(D_q) + c(D_r) - 4) \cdot \mathbb{Z} & p, q, r \text{ odd} \\ (c(D_p) + c(D_q) + c(D_r) - 5) \cdot \mathbb{Z} & \text{otherwise} \end{cases}$$

$$H_1(\Delta(p, q, r)) = \begin{cases} \mathbb{Z} & p, q \text{ and } r \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(\Delta(p, q, r)) = 0 \qquad \text{for } i \geq 2\,.$$

(Recall that $c(D_n)$ is $n/2 + 3$ if n if even and $(n-1)/2 + 2$ if n is odd.)

Remark 7. Lück and Stamm [12] studied the equivariant K-homology of cocompact (= compact quotient) planar groups. Our groups above, except those triangle groups with p, q or r being ∞, are examples of cocompact planar groups and our results also follow from Part (c) of Theorem 4.31 in their article.

Finally, observe that the results above agree with Theorems 7.1 and 8.1 when p, q and r are even or infinity.

10 A GAP routine for Coxeter groups

We have implemented a GAP [6] program to obtain the Bredon homology with coefficients in the representation ring of virtually any Coxeter group, from its Coxeter matrix.

The GAP routine works as follows. Firstly, we generate the cellular decomposition, cell stabilizers and boundary of the corresponding Davis complex. To do that, we have written functions to find the irreducible components of a Coxeter group and to decide whether the group it is finite (comparing each irreducible component with the finite Coxeter groups of the same rank). We generate a multidimensional list `CHAINS` such that `CHAINS[i]` is the list of all chains $T_1 < \ldots < T_i$ of length i ($<$ means strict inclusion) of subsets of $\{1, \ldots, N\}$ where $N = \mathrm{rank}(W, S)$. Note that a cell σ in the quotient space Σ/W corresponding to $\{T_1 < \ldots < T_n\}$ has dimension $n-1$, stabilizer W_{T_1} and boundary

$$\partial\sigma = \sum_{k=1}^{n} (-1)^k \{T_1 < \ldots < \widehat{T}_i < \ldots T_n\}\,.$$

Secondly, we use this information as initial data to another procedure which computes the Bredon homology with coefficients in the representation ring of a finite proper G-CW-complex. Note that, however, the computing time increases exponentially with the number of generators.

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