A DIAGONAL ON THE ASSOCIAHEDRA

SAMSON SANEBLIDZE AND RONALD UMBLE

To Jim Stasheff on the occasion of his 65th birthday

1. Introduction

An associahedral set is a combinatorial object generated by Stasheff associahedra \( \{K_n\} \) and equipped with appropriate face and degeneracy operators. Associahedral sets are similar in many ways to simplicial or cubical sets. In this paper we give a formal definition of an associahedral set, discuss some naturally occurring examples and construct an explicit geometric diagonal \( \Delta : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n) \) on the cellular chains \( C_*(K_n) \). The diagonal \( \Delta \), which is analogous to the Alexander-Whitney diagonal on the simplices, gives rise to a diagonal on any associahedral set and leads immediately to an explicit diagonal on the \( A_1 \)-operad. As an application of this, we use the diagonal \( \Delta \) to define a tensor product in the \( A_1 \)-category. This tensor product will play a central role in our discussion of \( A_\infty \)-Hopf algebras to appear in the sequel.

We mention that Chapoton [1], [2] constructed a diagonal of the form \( \sum_{i+j=n} C_*(K_i) \otimes C_*(K_j) \) on the direct sum \( \bigoplus_{n \geq 2} C_*(K_n) \), which coincides with the diagonal of Loday and Ronco [8] in dimension zero. Whereas Chapoton’s diagonal is formally defined to be primitive on generators, our diagonal is obtained by a purely geometrical decomposition of the generators and is totally different from his.

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2. The Stasheff Associahedra

In his seminal papers of 1963, J. Stasheff [17] constructs the associahedra \( \{K_{n+2}\}_{n \geq 0} \) as follows: Let \( K_2 = * \); if \( K_{n+1} \) has been constructed, let

\[
L_{n+2} = \bigcup_{1 \leq k \leq n+3} (K_r \times K_s)_k,
\]

and define \( K_{n+2} = CL_{n+2} \), i.e., the cone on \( L_{n+2} \). The associahedron \( K_{n+2} \) is an \( n \)-dimensional polyhedron, which serves as a parameter space for homotopy associativity in \( n + 2 \) variables. The top dimensional face of \( K_{n+2} \) corresponds to a pair of level 1 parentheses enclosing all \( n + 2 \) indeterminants; each component \( (K_{n-\ell+2} \times K_{\ell+1})_{i+1} \) of \( \partial K_{n+2} \) corresponds to a pair of level 2 parentheses enclosing
\( \ell + 1 \) indeterminants beginning with the \((i + 1)\)st. We denote this parenthesization by
\[
d_{(i, \ell)} = (x_1 \cdots (x_{i+1} \cdots x_{i+\ell+1}) \cdots x_{n+2})
\]
and refer to the inner and outer parentheses as the first and last pair, respectively. Note that indices \( i \) and \( \ell \) are constrained by
\[
\begin{aligned}
0 &\leq i \leq n \\
1 &\leq \ell \leq n, \quad i = 0 \\
1 &\leq \ell \leq n + 1 - i, \quad 1 \leq i \leq n
\end{aligned}
\]
Thus, there is a one-to-one correspondence between \((n - 1)\)-faces of \(K_{n+2}\) and parenthesizations \(d_{(i, \ell)}\) of \(n + 2\) indeterminants.

Alternatively, \(K_{n+2}\) can be realized as a subdivision of the standard \(n\)-cube \(I^n\) in the following way: Let \(\epsilon = 0, 1\). Label the endpoints of \(K_3 = [0, 1] \times I\) by \(d(\epsilon, 1)\). For \(1 \leq i \leq n\), let \(e^{n-1}_{i, \epsilon}\) denote the \((n - 1)\)-face \((x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ldots, x_n) \subset I^n\) and obtain \(K_4\) from \(K_3 \times I = I^2\) by subdividing the edge \(e^{1,1}_{1,1}\) as the union of intervals \(1 \times I_{0,1} \cup I_{1,\infty,1}\). Label the edges of \(K_4\) as follows: \(e^{1,0}_{i,0} \leftrightarrow d_{(0,i)}\); \(e^{1,1}_{1,1} \leftrightarrow d_{(2,1)}\); \(1 \times I_{0,1} \leftrightarrow d_{(1,1)}\); and \(1 \times I_{1,\infty} \leftrightarrow d_{(1,2)}\) (see Figure 1).

![Figure 1: \(K_4\) as a subdivision of \(K_3 \times I\).](image)

Now for \(0 \leq i < j \leq \infty\), let \(I_{i,j}\) denote the subinterval \([\left(\frac{2^i - 1}{2^j}\right) / 2^i, \left(\frac{2^j - 1}{2^i}\right) / 2^j]\) \(\subset I\), where \((2^\infty - 1) / 2^\infty\) is defined to be 1. For \(n > 2\), assume that \(K_{n+1}\) has been constructed and obtain \(K_{n+2}\) from \(K_{n+1} \times I \approx I^n\) by subdividing the \((n - 1)\)-faces \(d_{(i,n-i)} \times I\) as unions \(d_{(i,n-i)} \times I_0, i \cup d_{(i,n-i)} \times I_{i,\infty,0} \), \(0 < i < n\). Label the \((n - 1)\)-faces of \(K_{n+2}\) as follows:

<table>
<thead>
<tr>
<th>Face of (K_{n+2})</th>
<th>Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e^{n-1}_{i,0})</td>
<td>(d_{(0,\ell)}),  (1 \leq \ell \leq n)</td>
</tr>
<tr>
<td>(e^{n-1}_{n,1})</td>
<td>(d_{(n,1)})</td>
</tr>
<tr>
<td>(d_{(i,\ell)} \times I)</td>
<td>(d_{(i,\ell)}),  (1 \leq \ell &lt; n - i, \ 0 &lt; i &lt; n - 1)</td>
</tr>
<tr>
<td>(d_{(i,n-i)} \times I_0, i)</td>
<td>(d_{(i,n-i)}),  (0 &lt; i &lt; n)</td>
</tr>
<tr>
<td>(d_{(i,n-i)} \times I_{i,\infty,0})</td>
<td>(d_{(i,n-i+1)}),  (0 &lt; i &lt; n)</td>
</tr>
</tbody>
</table>
In Figure 2 we have labeled the 2-faces of $K_5$ that are visible from the viewpoint of the diagram.

Compositions $d_{(i_m, l_m)} \cdots d_{(i_2, l_2)} d_{(i_1, l_1)}$ denote a successive insertion of $m + 1$ pairs of parentheses into $n + 2$ indeterminants as follows: Given $d_{(i_r, l_r)} \cdots d_{(i_1, l_1)}$, $1 \leq r < m$, regard each pair of level 2 parentheses and its contents as a single indeterminant and apply $d_{(i_{r+1}, l_{r+1})}$. Conclude by inserting a last pair enclosing everything. Note that each parenthesization can be expressed as a unique composition $d_{(i_r+1, l_r+1)} \cdots d_{(i_1, l_1)}$ with $i_{r+1} \leq i_r$ for $1 \leq r < m$, in which case the parentheses inserted by $d_{(i_r+1, l_{r+1})}$ begin at or to the left of the pair inserted by $d_{(i_r, l_r)}$. Such compositions are said to have first fundamental form. Thus for $0 \leq k < n$, the $k$-faces of $K_{n+2}$ lie in one-to-one correspondence with compositions $d_{(i_m, l_{n-k})} \cdots d_{(i_1, l_1)}$ in first fundamental form. The two extremes with $m$ pairs of parentheses inserted as far to the left and right as possible, are respectively denoted by

$$d_{(0, l_m)} \cdots d_{(0, l_1)} \quad \text{and} \quad d_{(i_m, l_{m-1})} \cdots d_{(i_1, l_0)} \quad \text{where} \quad i_0 = n + 1 \quad \text{and} \quad i_{r+1} < i_r, \quad 0 \leq r < m.$$ 

In particular, the $n$-fold compositions $d_{(0, l_m)} \cdots d_{(0, l_1)}$ and $d_{(1, l_n)} \cdots d_{(1, l_1)}$ denote the extreme full parenthesizations of $n + 2$ indeterminants. When $m = 0$, define $d_{(i_m, l_m)} \cdots d_{(i_1, l_1)} = Id$.

![Figure 2: $K_5$ as a subdivision of $K_4 \times I$.](image)

Alternatively, each face of $K_{n+2}$ can be represented as a planar rooted tree (PRT) with $n+2$ leaves; its leaves correspond to indeterminants and its nodes correspond to pairs of parentheses. Let $T_{n+2}$ denote the PRT with $n+2$ leaves attached to the root.

![Figure 3: The planar rooted tree $T_{n+2}$](image)
at a single node \( N_0 \), called the root node (see Figure 3). The leaves correspond to a single pair of parentheses enclosing all \( n + 2 \) indeterminants. Now consider a node \( N \) of valence \( r + 1 \geq 4 \) in an arbitrary PRT \( T \). Choose a neighborhood \( U \) of \( N \) that excludes the other nodes of \( T \) and note that \( T_r \subseteq U \cap T \). Labeling from left to right, index the leaves of \( T_r \) from 1 to \( r \) as in Figure 3. Perform an \((i, \ell)\)-surgery at node \( N \) in the following way: Remove leaves \( i + 1, \ldots, i + \ell + 1 \) of \( T_r \), reattach them at a new node \( N' \neq N \) and graft in a new branch connecting \( N \) to \( N' \) (see Figure 4). Now

\[
i + 1 \quad \cdots \quad i + \ell + 1
\]

\[
1 \quad \cdots \quad i \quad i + \ell + 2 \cdots n + 2
\]

\[
N' \quad N_0
\]

Figure 4: The planar rooted tree \( T_{n+2}^{(i, \ell)} \)

let \( n \geq 1 \). Given a parenthesization \( d_{(i, \ell)} \) of \( n + 2 \) indeterminants, obtain the PRT \( T_{n+2}^{(i, \ell)} \) from \( T_{n+2} \) by performing an \((i, \ell)\)-surgery at the root node \( N_0 \) as shown in Figure 4. Inductively, given a parenthesization \( d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} \) of \( n + 2 \) indeterminants expressed as a composition in first fundamental form, construct the corresponding PRT \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_m, \ell_m)} \) as follows: Assume that \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_r, \ell_r)} \) with nodes \( N_0, \ldots, N_r \) has been constructed for some \( 1 \leq r < m \) and note that the root node \( N_0 \) has valence \( n + 3 - \ell_1 - \cdots - \ell_r \). Perform an \((i_{r+1}, \ell_{r+1})\)-surgery at \( N_0 \) and obtain \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_{r+1}, \ell_{r+1})} \) containing a new node \( N_{r+1} \) and a new branch connecting \( N_0 \) to \( N_{r+1} \). Finally, define \( T_{n+2}^{(i_{r+2}, \ell_{r+2}), \ldots, (i_m, \ell_m)} = T_{n+2} \) when \( m = 0 \) and obtain a one-to-one correspondence between \( k \)-faces of \( K_{n+2} \), \( 0 \leq k \leq n \) and PRT’s \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_{n-k}, \ell_{n-k})} \) consisting of \( n - k + 1 \) nodes and \( n + 2 \) leaves. In particular, each vertex of \( K_{n+2} \) corresponds to a planar binary rooted tree \( T_{n+2}^{(i_1, \ell_1), \ldots, (i_n, \ell_n)} \) (see Figure 5).

Now given a \( k \)-face \( a_k \subseteq K_{n+2} \), \( k > 0 \), consider the two vertices of \( a_k \) at which parentheses are shifted as far to the left and right as possible; we refer to these vertices as the minimal and maximal vertices of \( a_k \), and denote them by \( a_k^{\min} \) and \( a_k^{\max} \), respectively. In particular, the minimal and maximal vertices of \( K_{n+2} \) are the origin and the vertex of \( I^n \) diagonally opposite to it, i.e.,

\[
K_{n+2}^{\min} \leftrightarrow (0, 0, \ldots, 0) \quad \text{and} \quad K_{n+2}^{\max} \leftrightarrow (1, 1, \ldots, 1);
\]

the respective binary trees in Figure 5 correspond to \( K_1^{\min} \) and \( K_3^{\max} \).
Given a representation $T_{n+2}^{(i_1, \ell_1), \ldots, (i_{n-k}, \ell_{n-k})}$ of $a_k$, construct the minimal (resp., maximal) tree of $a_k$ by replacing each node of valence $r \geq 4$ with the planar binary rooted tree representing $K_{r-1}^{\text{min}}$ (resp., $K_{r-1}^{\text{max}}$). Note that $a_k^{\text{min}}$ and $a_k^{\text{max}}$ determine $a_k$ since their convex hull is a diagonal of $a_k$. But we can say more.

When a composition of face operators $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ is defined we refer to the sequence of lower indices $I = (i_1, \ell_1), \ldots, (i_m, \ell_m)$ as an admissible sequence of length $m$; if $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ has first fundamental form we refer to the the sequence $I$ as a type I sequence of length $m$. The set of all planar binary rooted trees

$$Y_{n+2} = \{T_n^I \mid I \text{ is type I sequence of length } n\}$$

is a poset with partial ordering defined as follows: Say that $T_{n+2}^I \leq T_{n+2}^J$ if there is an edge-path in $K_{n+2}$ from vertex $T_{n+2}^I$ to vertex $T_{n+2}^J$ along which parentheses shift strictly to the right. This partial ordering can be expressed geometrically in terms of the following operation on trees: Let $N_0$ denote the root node of $T_{n+2}^I$ and let $N$ be a node joining some left branch $L$ and a right branch or leaf $R$ in $T_{n+2}^I$. Let $N_L$ denote the node on $L$ immediately above $N$. A right-shift through node $N$ repositions $N_L$ either at the midpoint of leaf $R$ or midway between $N$ and the node immediately above it. Then $T_{n+2}^I \leq T_{n+2}^J$ if there is a right-shift sequence of planar binary rooted trees $\{T_n^I\}_{p \leq r < q}$, i.e., for each $r < q$, $T_{n+2}^{I_{r+1}}$ is obtained from $T_{n+2}^{I_r}$ by a right-shift through some node in $T_{n+2}^I$ (see Figure 6).

![Figure 6: A right-shift sequence of planar binary trees](image)

Let $I' = (i'_1, \ell'_1), \ldots, (i'_m, \ell'_m)$ be an admissible sequence of length $m > 0$ and consider a node $N'$ distinct from the root node $N_0$ in the PRT $T_{n+2}^I$. Let $N$ denote the node immediately below $N'$ and let $NN'$ denote the branch from $N$ to $N'$; we refer to the quotient space $T_{n+2}^I = T_{n+2}^I/NN'$ as the $(N, N')$-contraction of
Now given a type I sequence $J'$ of length $n$, consider the planar binary tree $T_{n+2}^{J'}$ and let $T_{n+2}^{J}$ be the PRT obtained from $T_{n+2}^{J'}$ by some sequence of $k$ successive $(N,N')$-contractions. The subposet $Y_{n+2}^{J} \subseteq Y_{n+2}$ of all planar binary rooted trees from which $T_{n+2}^{J}$ can be so obtained is exactly the poset of vertices of the $k$-face $\alpha_k \subseteq K_{n+2}$ represented by $T_{n+2}^{J}$. In this way, we may regard $\alpha_k$ as the geometric realization of $Y_{n+2}^{J}$ just as we regard a $k$-face of the standard $n$-simplex as the geometric realization of a $(k+1)$-subset of a linearly ordered $(n+1)$-set. In particular, $K_{n+2}$ is the geometric realization of $Y_{n+2}^{J}$.

We summarize the discussion above as a proposition:

**Proposition 1.** For $0 \leq k \leq n$, there exist one-to-one correspondences

$\{k\text{-faces of } K_{n+2}\} \leftrightarrow \{ (n-k)\text{-fold compositions of face operators in first fundamental form} \}$

$\leftrightarrow \{ \text{Planar rooted trees with } n-k+1 \text{ nodes and } n+2 \text{ leaves} \}$

$\leftrightarrow \{ \text{Subposets of planar binary rooted trees } Y_{n+2}^{J} \text{ where } J \text{ is a type I sequence of length } n-k \}$

3. Associahedral Sets

**Definition 1.** An associahedral set is a graded set

$$K = \left\{ K_{n-k+3}^{n_1,\ldots,n_k} \mid n_j \geq 0 \text{ and } \sum_{j=1}^{k} n_j = n-k+1 \right\}_{n \geq 0; \ k \geq 1}$$

together with face operators

$$\left\{ d_q^{i} : K_{n-k+3}^{n_1,\ldots,n_k} \to K_{n-k+2}^{n_1,\ldots,n_{q-1},\ell_q-1, n_{q}-\ell_q,n_{q+1},\ldots,n_k} \right\}$$

$$\mid 1 \leq q \leq k; \ 0 \leq i_q \leq n_q; \ 1 \leq \ell_q \leq n_q; \ i_q + \ell_q \leq n_q + 1 \right\}_{n \geq 0; \ k \geq 1}$$

and degeneracy operators

$$\left\{ s_q^{j} : K_{n-k+3}^{n_1,\ldots,n_k} \to K_{n-k+4}^{n_1,\ldots,n_{q-1},n_q+1,n_{q+1},\ldots,n_k} \right\}$$

$$\mid 1 \leq q \leq k; \ 1 \leq j \leq n_q + 3 \right\}_{n \geq 0; \ k \geq 1}$$

that satisfy the following relations:

$$d_p^{i} \cdot d_q^{i} = d_{p+1}^{i+1} \cdot d_q^{i} \quad (p < q) \quad (1)$$

$$d_{i+1}^{i} \cdot d_q^{i} = d_q^{i} \cdot d_{i+1}^{i+1} \quad (i_q \leq i \leq i_q + 1 + \ell_q) \quad (2)$$

$$d_{i+1}^{i} \cdot d_q^{i} = d_{i+1}^{i} \cdot d_q^{i} \quad (i_q < i_q + 1) \quad (3)$$

(continued)
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\[ d^q_{(i, \epsilon)} s^q_j = s^{q+1}_j d^p_{(i, \epsilon)}; \quad p < q \]
\[ d^p_{(i, \epsilon)} s^q_j = s^q d^p_{(i, \epsilon)}; \quad p > q \]
\[ d^q_{(i, \epsilon)} s^q_j = s^{q-1}_j d^q_{(i, \epsilon)}; \quad i + \epsilon + 1 < j \]
\[ d^q_{(i, \epsilon)} s^q_j = s^{q-1}_i d^q_{(i, \epsilon-1)}; \quad i < j < i + \epsilon + 2, \quad \epsilon > 1 \]
\[ d^q_{(i, \epsilon)} s^q_j = s^{q+1}_j d^q_{(i-1, \epsilon)}; \quad i \geq j, \quad \epsilon \leq n_q \]
\[ d^q_{(i, \epsilon)} s^q_j = 1; \quad (i, \epsilon) = (j - 1, 1), \quad 1 \leq j < n_q + 3 \]
\[ d^2_{(i, \epsilon)} s^q_j = 1; \quad (i, \epsilon) = (j - 2, 1), \quad 1 \leq j < n_q + 3 \]
\[ d^q_{(i, \epsilon)} s^q_j = 1; \quad (i, \epsilon) = (0, n_q + 1), \quad j = n_q + 3 \]
\[ d^q_{(i, \epsilon)} s^q_j = 1; \quad (i, \epsilon) = (1, n_q + 1), \quad j = 1 \]
\[ s^q_j s^{q'}_{j'} = s^q_{j'+1} s^{q'}_j; \quad p \neq q \]
\[ s^q_j s^{q'}_{j'} = s^q_{j'+1} s^{q'}_j; \quad p = q, \quad j \leq j'. \]

Remark 1. If the compositions \( d^p_{(i_p, \epsilon_p)} d^q_{(i_q, \epsilon_q)} : K^{n_1, \ldots, n_m}_{n-m+3} \to K^{n_1, \ldots, n_m}_{n-m+1} \) and
\( d^{p'}_{(i_{p'}, \epsilon_{p'})} d^{q'}_{(i_{q'}, \epsilon_{q'})} : K^{n_1, \ldots, n_m}_{n-m+3} \to K^{n'_1, \ldots, n'_m}_{n-m+1} \) are equal via relations (1) to (3), then
\( K^{n_1, \ldots, n_m}_{n-m+3} \cap K^{n'_1, \ldots, n'_m}_{n-m+1} \) contains the image of these compositions. When sub-
sequently applying face operators to an element \( a \in K^{n_1, \ldots, n_m}_{n-m+3} \cap K^{n'_1, \ldots, n'_m}_{n-m+1} \),
where \( n_1, \ldots, n_m+2 > n'_1, \ldots, n'_m+2 \) in the lexicographic ordering, think of \( a \in K^{n_1, \ldots, n_m}_{n-m+1} \).

Example 1. Consider the set of all planar rooted trees
\[ K = \left\{ T^{(1, \epsilon_1), \ldots, (m, \epsilon_m)}_{n+2} \in K^{n_1, \ldots, n_m+1}_{n-m+2} \right\}_{n \geq 0; \quad 0 \leq m \leq n}, \]
where \( n_q = \epsilon_q - 1 \) for \( 1 \leq q \leq m \) and \( n_{m+1} = n - \sum_{j=1}^{m} \epsilon_j \), and note that \( K^{n}_{n+2} = \{ T_{n+2} \} \) consists of a single element. When \( n = 2 \), the sets in \( K \) are pairwise disjoint; when \( n = 3 \) however, the sets in \( K \) intersect nontrivially:
\[ K_{3}^{1,0,0} \cap K_{3}^{0,1,0} = \{ ((\bullet \bullet \bullet)) (\bullet \bullet \bullet)), \quad ((\bullet \bullet \bullet)) (\bullet \bullet \bullet) \}. \]
To evaluate the face operator \( d^q_{(i, \epsilon)} \) on the element \( T^{(1, \epsilon_1), \ldots, (m, \epsilon_m)}_{n+2}, 1 \leq q \leq m+1 \),
recall that the composition \( d^m_{(i, \epsilon)} \cdots d^1_{(i, \epsilon)} \) is given in first fundamental form. Repeatedly apply face relations (1) to (3) to the composition \( d^q_{(i, \epsilon)} d^m_{(i, \epsilon)} \cdots d^1_{(i, \epsilon)} \)
to obtain the composition \( d^{m+1}_{(i, \epsilon)} d^m_{(i, \epsilon)} \cdots d^1_{(i, \epsilon)} \) in first fundamental form
and define
\[ d^q_{(i, \epsilon)} \left( T^{(i_1, \epsilon_1), \ldots, (i_m, \epsilon_m)}_{n+2} \right) = T^{(i'_1, \epsilon'_1), \ldots, (i'_{m+1}, \epsilon'_{m+1})}_{n+2}. \]
Conceptually, given a parenthesization \( d_{(i_1, \epsilon_1)} \cdots d_{(i, \epsilon_1)} \) in first fundamental form,
the face operator \( d^q_{(i, \epsilon)} \) inserts a new pair of parentheses inside the \( q^{th} \) pair while
treating the \((q - 1)^{st}\) pair and its contents as a single indeterminant. In terms of the
tree \(T_{n+2}^{(i_1, \ell_1), \ldots, (i_m, \ell_m)}\), apply \(d^1_{(i, \ell)}\) by performing an \((i, \ell)\)-surgery at \(N_q\) and relabel
the nodes as required by the first fundamental form to obtain \(T_{n+2}^{(i'_1, \ell'_1), \ldots, (i'_m, \ell'_m+1)}\).

When \(n = 2\), the following five face operators relate \(T_4 \in \mathcal{K}_4^2\) to the edges of the
pentagon \(K_4\):

\[
\begin{array}{l}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Definition 2. An associahedral complex is a poset $K$ with distinguished finite sub-posets $\{K_\alpha\}$ such that

(a) Each element of $K$ is distinguished;
(b) For each $\alpha$, there is an ordering isomorphism

$$i_\alpha : K_\alpha \xrightarrow{\sim} Y_{n_\alpha}^{(i_1, \ell_1), \ldots, (i_m, \ell_m)};$$

(c) If $K_{\alpha'} \subseteq K_\alpha$ and $i_\alpha(K_{\alpha'}) = Y_{n_{\alpha'}}^{(i'_1, \ell'_1), \ldots, (i'_{m'}, \ell'_{m'})}$, then $K_{\alpha'}$ is distinguished.
(d) Face and degeneracy operators on $K_\alpha$ are determined by the face and degeneracy operators on $i_\alpha(K_\alpha)$ defined in Example 2.

Let $K$ be an associahedral set and let $b = d^{q_1}_{(i_1, \ell_1)} \cdots d^{q_m}_{(i_m, \ell_m)}(a)$ for some $a \in K$. For notational simplicity, we henceforth suppress upper indices $q_2, \ldots, q_m$ when $q_{j+1} = q_j + 1$ for all $j \geq 1$; we suppress all upper indices if, in addition, $q_1 = 1$.

Definition 3. Let $m \geq 2$. A sequence of lower indices $I = (i_1, \ell_1), \ldots, (i_m, \ell_m)$ is admissible whenever the composition of face operators $d^{q_m}_{(i_m, \ell_m)} \cdots d^{q_1}_{(i_1, \ell_1)}$ is defined.

The sequence $I$ is a type I (resp. type II) sequence if $I$ is admissible and $i_k \leq i_{k+1} + \ell_{k+1}$ (resp. $i_k \geq i_{k+1} + \ell_{k+1}$) for $1 \leq k < m$. The empty sequence $(m = 0)$ and sequences of length 1 $(m = 1)$ are both type I and II sequences. A composition of face operators $d^{q_m}_{(i_m, \ell_m)} \cdots d^{q_2}_{(i_2, \ell_2)}d^{q_1}_{(i_1, \ell_1)}$ has first (resp. second) fundamental form if $(i_1, \ell_1), \ldots, (i_m, \ell_m)$ is a type I (resp. type II) sequence. When $m = 0$, the composition $d^{q_m}_{(i_m, \ell_m)} \cdots d^{q_2}_{(i_2, \ell_2)}d^{q_1}_{(i_1, \ell_1)}$ is defined to be the identity. An element $b = d^{q_m}_{(i_m, \ell_m)} \cdots d^{q_2}_{(i_2, \ell_2)}d^{q_1}_{(i_1, \ell_1)}(a)$ is expressed in first (resp. second) fundamental form as a face of $a$ if

$$d^{q_m} \cdots d^{q_2}d^{q_1} = (d \cdots dd^{s_m}) \cdots (d \cdots dd^{s_2})(d \cdots dd^{s_1}),$$

where $s_1 < s_2 < \cdots < s_m$ and each composition $d^{q_j}_{(i_j, \ell_j)} \cdots d^{q_2}_{(i_2, \ell_2)}d^{q_1}_{(i_1, \ell_1)}$ has first (resp. second) fundamental form.

Obviously, every composition of face operators can be uniquely transformed into first or second fundamental form by successive applications of face relations (1) to (3).

Let $C_\ast(K_{n+2})$ denote the cellular chains on $K_{n+2}$. For $0 \leq m \leq n$, let $I_m = (i_1, \ell_1), (i_2, \ell_2), \ldots, (i_m, \ell_m)$ be a type I sequence and let $d_{I_m} = d^{q_m}_{(i_m, \ell_m)} \cdots d^{q_1}_{(i_1, \ell_1)}$. Then

$$T_{n+2}^{I_m} = d_{I_m}(T_{n+2}) \in C_{n-m}(K_{n+2})$$

and

$$\{ T_{n+2}^{I_m} \mid I_m \text{ is a type I sequence} \}_{0 \leq m \leq n}$$

defines a system of generators for $C_\ast(K_{n+2})$. 

$Y_{n+2}^{(i_1, \ell_1), \ldots, (i_m, \ell_m)}$ with the tree $T_{n+2}^{(i_1, \ell_1), \ldots, (i_m, \ell_m)}$ and applying the evaluation rules in Example 1.
Definition 4. For \( m \geq 0 \), let \( I_m = (i_1, \ell_1), (i_2, \ell_2), \ldots, (i_m, \ell_m) \) be a type I sequence. If \( 1 \leq r \leq m + 1 \), the sign of the face \( b = d^r_{(i, \ell)}(T^{d^r_{(i, \ell)}}_{n+2}) \in C_{n-m-1}(K_{n+2}) \) is defined to be
\[
\text{sgn} (b) = -(-1)^{\ell(b)}, \text{ where } \ell(b) = r + (i + 1) \ell + \sum_{j=1}^{r-1} \ell_j.
\]

Remark 2. When applying face relations (1) to (3) in Definition 1 to cellular chains in \( C_*(K_{n+2}) \), only (3) requires the sign \((-1)^{(\ell_q+1)(\ell_{q+1}+1)}\) for commuting factors.

Example 3. The set \( K \) defined in Example 1 generates the associahedral set of cellular chains \( \{C_*(K_{n+2})\}_{n \geq 0} \). According to Definition 4, the cellular boundary of \( T_{n+2} \in C_n(K_{n+2}) \) is given by
\[
\partial T_{n+2} = \sum_{1 \leq i \leq n} (-1)^{i+1} d_{(i, \ell)}(T_{n+2}).
\]

Example 4. Let \( X \) be a topological space. Define the singular associahedral complex \( \text{Sing}^K X \) as follows: Let
\[
\text{Sing}^K X = \frac{\text{Sing}^K X}{\sigma}
\]
denote the orbit set of the symmetric group action on \( \text{Sing}^K X \) given by the permutation isomorphisms
\[
\sigma : K_{n+2} \times \cdots \times K_{n+2} \overset{\cong}{\rightarrow} K_{\sigma^{-1}(n)+2} \times \cdots \times K_{\sigma^{-1}(n)+2},
\]
where \( \sigma \in S_K \), \( k \geq 2 \). Let
\[
\delta^q_{(i_q, \ell_q)} : K_{n+2} \times \cdots \times K_{n+2} \times K_{\ell_q+1} \times K_{n-\ell_q+2} \times K_{n+1} \times K_{n+2} \rightarrow K_{n+2} \times \cdots \times K_{n+2} \times \cdots \times K_{n+2}
\]
be the inclusion and let
\[
\beta^q_i : K_{n+2} \times \cdots \times K_{n+2} \times \cdots \times K_{n+2} \rightarrow K_{n+2} \times \cdots \times K_{n+2} \times \cdots \times K_{n+2}
\]
be the projection (cf. [17]). Then for \( f \in (\text{Sing}^K X)_{n+2}^{n_1, \ldots, n_k} \), define
\[
d^q_{(i_q, \ell_q)} : (\text{Sing}^K X)_{n+2}^{n_1, \ldots, n_k} \rightarrow (\text{Sing}^K X)_{n+1}^{n_1, \ldots, n_q-1, \ell_q-1, n_q-\ell_q, n_{q+1}, \ldots, n_k}
\]
and
\[
s^q_i : (\text{Sing}^K X)_{n+2}^{n_1, \ldots, n_k} \rightarrow (\text{Sing}^K X)_{n+3}^{n_1, \ldots, n_q+1, n_{q+1}, \ldots, n_k}
\]
as compositions
\[
d^q_{(i_q, \ell_q)}(f) = f \circ \delta^q_{(i_q, \ell_q)} \text{ and } s^q_i(f) = \beta^q_i \circ f.
\]
It is easy to check that \((\text{Sing}^K X, d^q_{(i_q, \ell_q)}, s^q_i)\) is an associahedral set.
The singular associahedral complex $\text{Sing}^K X$ determines the singular (co)homology of $X$ in the following way: Form the chain complex $(C_\ast(\text{Sing}^K X), d)$ of $\text{Sing}^K X$ and consider the quotient

$$C_\ast^N(\text{Sing}^K X) = C_\ast(\text{Sing}^K X)/D,$$

where $D$ is the subcomplex of $C_\ast(\text{Sing}^K X)$ generated by the degenerate elements of $\text{Sing}^K X$. The composition of canonical projections

$$K_{n+2} \times \cdots \times K_{n+k} \to I^{n_1} \times \cdots \times I^{n_k} = I^n \to \Delta^n$$

induces a sequence of chain maps

$$C_\ast(\text{Sing} X) \to C_\ast(\text{Sing}^I X) \to C_\ast(\text{Sing}^K X) \to C_\ast^N(\text{Sing}^K X),$$

and consequently, a natural isomorphism $H_\ast(X) \approx H_\ast(C_\ast^N(\text{Sing}^K X), d)$. As in the cubical setting (but unlike the simplicial setting), normalized chains are required to obtain the singular homology of $X$. In the discussion that follows below, we construct a diagonal on the associahedra, which is compatible under the projections above with the Alexander-Whitney diagonal on the standard simplex. Thus $H^\ast(C_\ast^N(\text{Sing}^K X), d)$ determines the singular cohomology ring of $X$ as well.

4. A Diagonal $\Delta$ on $C_\ast(K_n)$

We begin with an overview of the geometric ideas involved. Let $0 \leq q \leq n$ and let $I_{n-q}$ be a type 1 sequence. The $q$-dimensional generator $a_q = T_{n+2}^{I_{n-q}}$ is associated with a face of $K_{n+2}$ corresponding to $n+2$ indeterminants with $n-q+1$ pairs of parentheses. Identify $a_q$ with its associated face of $K_{n+2}$ and consider the minimal and maximal vertices $a_q^{\text{min}}$ and $a_q^{\text{max}}$ of $a_q$. Define the primitive terms of $\Delta T_{n+2}$ to be

$$T_{n+2}^{\text{min}} \otimes T_{n+2} + T_{n+2} \otimes T_{n+2}^{\text{max}}.$$

Let $0 < p < p+q = n$ and consider distinct $p$-faces $b$ and $b'$ of $T_{n+2}$. Say that $b \leq b'$ if there is a path of $p$-faces from $b$ to $b'$ along which parentheses shift strictly to the right. Now given a $q$-dimensional face $a_q$ of $T_{n+2}$ such that $a_q^{\text{min}} \neq T_{n+2}^{\text{min}}$, there is a unique path of $p$-faces $b_1 \leq b_2 \leq \cdots \leq b_r$ with minimal length such that $T_{n+2}^{\text{min}} = b_1^{\text{min}}$ and $a_q^{\text{min}} = b_r^{\text{max}}$. Up to sign, we define the non-primitive terms of $\Delta T_{n+2}$ to be

$$\sum \pm b_j \otimes a_q.$$

To visualize this, consider the edge $d_{(1,2)}(T_4) \in C_1(K_4)$ whose minimal vertex is the point $(1, \frac{1}{2})$ (see Figure 7). The edges $d_{(0,2)}(T_4) \leq d_{(1,1)}(T_4)$ form a path of minimal length from $(0,0)$ to $(1, \frac{1}{2})$. Consequently, $\Delta T_4$ contains the non-primitive terms $\left\{ (\pm d_{(0,2)} \pm d_{(1,1)}) \otimes d_{(1,2)} \right\} (T_4 \otimes T_4)$. 

Figure 7: Edge paths from $T_{4}^{\text{min}}$ to $T_{4}^{\text{max}}$.

Precisely, for $T_{2} \in C_{+}(K_{2})$ define $\Delta T_{2} = T_{2} \otimes T_{2}$; inductively, assume that the map $\Delta : C_{+}(K_{i+2}) \to C_{+}(K_{i+2}) \otimes C_{+}(K_{i+2})$ has been defined for all $i < n$. For $T_{n+2} \in C_{n}(K_{n+2})$ define

$$
\Delta T_{n+2} = \sum_{0 \leq p, q = n} (-1)^{p} d_{(i_{p}, \ell_{p})} \cdots d_{(i_{q}, \ell_{q})} (T_{n+2}) \otimes d_{(i_{p}, \ell_{p})} \cdots d_{(i_{q}, \ell_{q})} (T_{n+2})
$$

where

$$
\epsilon = \sum_{j=1}^{q} \ell_{j}'(\ell_{j}' + 1) + \sum_{k=1}^{p} (i_{k} + k + p + 1)\ell_{k},
$$

and lower indices $((i_{1}, \ell_{1}), \ldots, (i_{p}, \ell_{p}); (i_{1}', \ell_{1}'), \ldots, (i_{q}', \ell_{q}'))$ range over all solutions of the following system of inequalities:

\begin{align}
1 \leq i_{j} < i_{j-1} & \leq n + 1 & (1) \\
1 \leq \ell_{j} \leq n + 1 - i_{j} - \ell_{(j-1)} & (2) \\
0 \leq \ell_{k}' & \leq \min_{\ell_{(t(k-1))} < k} \{\ell_{r}', \ i_{k} - \ell_{(t(k))}\} & (3) \\
1 \leq \ell_{k}' = \epsilon_{k} - \ell_{k}' - \ell_{(k-1)} & (4) \quad 1 \leq j \leq p \quad 1 \leq k \leq q \nonumber
\end{align}

where

\begin{align}
\{\epsilon_{1} < \cdots < \epsilon_{q}\} &= \{1, \ldots, n\} \setminus \{i_{1}, \ldots, i_{p}\} \\
\epsilon_{0} = \ell_{0} = \ell_{0}' = i_{p+1} = i_{q+1}' = 0; \\
i_{0} = i_{0}' = \epsilon_{q+1} = \ell_{(p+1)} = \ell_{(q+1)}' = n + 1; \\
\ell_{(u)} &= \sum_{j=0}^{u} \ell_{j} \text{ for } 0 \leq u \leq p + 1; \\
\ell_{(u)}' &= \sum_{k=0}^{u} \ell_{k}' \text{ for } 0 \leq u \leq q + 1; \\
\epsilon_{u} &= \min \{r \mid i_{r} + \ell_{(r)} - \ell_{(o(u))} > \epsilon_{u} > i_{r}\} \nonumber \\
o_{(u)} &= \max \{r \mid i_{r} \geq \epsilon_{u}\} \text{ and} \\
o_{(u)}' &= \max \{r \mid \epsilon_{r} \leq i_{u}\}. \nonumber
\end{align}
Extend $\Delta$ multiplicatively to all of $C_*(K_{n+2})$, using the fact that the cells of $K_{n+2}$ are products of cells $K_{i+2}$ with $i < n$.

Note that right-hand and left-hand factors in each component of $\Delta T_{n+2}$ are expressed in first and second fundamental form, respectively. In particular, the terms given by the extremes $p = 0$ and $p = n$ are the primitive terms of $\Delta T_{n+2}$:

$$\{ d_{(0,1)} \cdots d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} \cdots d_{(n,1)} \} (T_{n+2} \otimes T_{n+2}).$$

The sign in $\Delta T_{n+2} = \sum (-1)^{\varepsilon} b \otimes a$ is the product of five signs: $(-1)^{\varepsilon} = \text{sgn}(b) \cdot \text{sgn}(b_0) \cdot \text{sgn}(a_0) \cdot \text{sgn}(a_1) \cdot \text{sgn}(a)$, where the face $b_0$ is obtained from $b$ by keeping $\varepsilon$’s but all $i'_k = 0$ (i.e., $\ell'_k$ is replaced by $\ell'_k + i''_k - i''_{k-1}$), the face $a_1$ is obtained from $a$ by keeping $i_r$’s but all $\ell_r = 1$ and $\text{sgn}(b_0, a_1)$ is the sign of the shuffle $\{i_p < \cdots < i_1, \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_q\}$. Geometrically, $b_0$ and $a_1$ lie on orthogonal faces of the cube $I^n$ and are uniquely defined by the property that the canonical cellular projection $K_{n+2} \to I^n$ maps $b_0 \mapsto x_1, \ldots, x_{i_p-1}, 0, \ldots, x_{i_q-1}, 0, \ldots, x_n$ and $a_1 \mapsto x_1, \ldots, x_{i_p-1}, 1, \ldots, x_{i_q-1}, 1, \ldots, x_n$.

**Example 5.** For $T_4 \in C_2(K_4)$ we obtain by direct calculation:

$$\Delta T_4 = \left\{ d_{(0,1)} d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} d_{(2,1)} + d_{(0,2)} \otimes d_{(1,1)} \\
rule\right. + d_{(0,2)} \otimes d_{(1,2)} + d_{(1,1)} \otimes d_{(1,2)} - d_{(0,1)} \otimes d_{(2,1)} \right\} (T_4 \otimes T_4).$$

A proof of our main result, stated as the following theorem, appears in the appendix:

**Theorem 1.** For each $n \geq 0$, the map $\Delta : C_*(K_{n+2}) \to C_*(K_{n+2}) \otimes C_*(K_{n+2})$ defined above is a chain map.

Identify the sequence of cellular chain complexes $\{C_*(K_n)\}_{n \geq 2}$ with the $A_\infty$-operad $A_\infty$ [11]. Since $\Delta$ is extended multiplicatively on decomposable faces, we immediately obtain the following interesting fact:

**Corollary 1.** The sequence of chain maps

$$\{ \Delta : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n) \}_{n \geq 2}$$

induces a morphism of operads

$$A_\infty \to A_\infty \otimes A_\infty.$$

5. Tensor Products in the $A_\infty$ Category

The notion of an $A_\infty$-algebra also appears in Stasheff’s seminal papers [17]. An $A_\infty$-algebra is a dga in which the associative law holds up to homotopy, the homotopies between the various associations are homotopic, the homotopies between these homotopies are homotopic, and so on. Although $A_\infty$-algebras first appeared in topology as the singular chain complex on the loop space of a connected CW-complex, they have since assumed their rightful place as fundamental structures in algebra [10], [15], topology [3], [7], [18], and mathematical physics [4], [5], [12], [13], [20], [21]. Furthermore, Stasheff’s idea carries over to homotopy versions of
coalgebras [14], [16], [19] and Lie algebras [6], and one can deform a classical dga, dgc or dg Lie algebra to the corresponding homotopy version in a standard way.

An $R$-module $A$ equipped simultaneously with an $A_\infty$-algebra and an $A_\infty$-coalgebra structure is an “$A_\infty$-Hopf algebra” if the two $A_\infty$ structures are sufficiently compatible. Our definition of an $A_\infty$-Hopf algebra, which appears in the sequel, requires an independently interesting tensor product in the $A_\infty$ category. In this concluding section, we apply the diagonal $\Delta$ defined above to obtain this tensor product in maximal generality. We note that a special case was given by J. Smith [16] for certain objects with a richer structure than we have here. We also mention that Lada and Markl [6] defined a $A_\infty$ tensor product structure on a construct different from the tensor product of graded $R$-modules.

We adopt the following notation and conventions: The symbol $R$ denotes a commutative ring with unity; all $R$-modules are assumed to be $\mathbb{Z}$-graded. The reduced $R$-module $V/V_0$ of a connected $V$ is denoted by $\overline{V}$. All tensor products and $Hom$’s are defined over $R$ and all maps are $R$-module maps unless indicated otherwise. The symbol $1 : V \to V$ denotes the identity map; the suspension and desuspension maps, which shift dimension by $+1$ and $-1$, are denoted by $\uparrow$ and $\downarrow$, respectively. We let $V^\otimes n = V \otimes \cdots \otimes V$ with $n > 0$ factors and define $V^\otimes 0 = R$; then $TV = \sum_{n \geq 0} V^\otimes n$. If $A$ is an $R$-algebra (resp., $R$-coalgebra) , then $T^eA$ (respectively, $T^cA$) denotes the tensor algebra (resp., the tensor coalgebra) of $A$. Given a $R$-modules $V_1,\ldots,V_n$ and a permutation $\sigma \in S_n$, define the permutation isomorphism $\sigma : V_1 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ by $\sigma(x_1 \cdots x_n) = \pm x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$, where the sign is determined by the Mac Lane commutation rule [9] to which we strictly adhere. In particular, $\sigma_m : (V_1 \otimes V_2)^\otimes m \to V_1^\otimes m \otimes V_2^\otimes m$ is the permutation isomorphism induced by $\sigma_m = (1 \, 3 \, \cdots \, (2m-1) \, 2 \, 4 \, \cdots \, 2m)$. If $f : V^\otimes p \to V^\otimes q$ is a map, we let $f_{i,n-p-i} = 1^\otimes i \otimes f \otimes 1^\otimes n-p-i : V^\otimes n \to V^\otimes n-p+q$, where $0 \leq i \leq n-p$. The abbreviations $dgm$, $dga$,and $dgc$ stand for differential graded $R$-module, dg $R$-algebra and dg $R$-coalgebra, respectively.

We begin with a discussion of $A_\infty$-(co)algebras paying particular attention to the signs; in structural compatibility considerations, sign issues become critical. Let $A$ be a connected graded $R$-module and consider a sequence of maps $\{\varphi^k \in Hom^{k-2}(A^\otimes k, A)\}_{k \geq 1}$. For each $k$ and $n \geq 1$, linearly extend $\varphi^k$ to $A^\otimes n$ via

$$
\sum_{i=0}^{n-k} \varphi_{i,n-k-i} : A^\otimes n \to A^\otimes n-k+1,
$$

and consider the induced map of degree $-1$ given by

$$
\sum_{i=0}^{n-k} (\uparrow \varphi_{i,n-k-i}) : (\uparrow A)^\otimes n \to (\uparrow A)^\otimes n-k+1.
$$

Let $\overline{BA} = T^c (\uparrow A)$ and define a map $d_{\overline{BA}} : \overline{BA} \to \overline{BA}$ of degree $-1$ by

$$
d_{\overline{BA}} = \sum_{1 \leq k \leq n \atop 0 \leq \ell \leq n-k} (\uparrow \varphi_{i,n-k-i})_{\ell,n-k-i}.
$$

(5.1)
The identities $(-1)^{[n/2]} \uparrow \otimes n \downarrow \otimes n = 1 \otimes n$ and $[n/2] + [(n + k)/2] \equiv nk + [k/2]$ (mod 2) imply that

$$d_{BA} = \sum_{1 \leq k \leq n \atop 0 \leq i \leq n - k} (-1)^{(n-k)/2 + i(k+1)} \uparrow \otimes n-k+1 \varphi_{1,n-k-i} \downarrow \otimes n.$$

**Definition 5.** $(A, \varphi^n)_{n \geq 1}$ is an $A_\infty$-algebra if $d_{BA}^2 = 0$. The maps $\{\varphi^n\}$ in an $A_\infty$-algebra are called higher homotopies.

**Proposition 2.** Let $(A, \varphi^n)_{n \geq 1}$ be an $A_\infty$-algebra. The higher homotopies $\{\varphi^k\}$ satisfy the following quadratic relations for $n \geq 1$:

$$0 = \sum_{0 \leq i \leq n - \ell - 1} (-1)^{\ell(i+1)} \varphi^{n-\ell-\ell+1}\varphi_{1,n-\ell-1-i} = 0.$$

**Proof:** For $n \geq 1$,

$$0 = \sum_{1 \leq k \leq n \atop 0 \leq i \leq n - k} (-1)^{(n-k)/2 + i(k+1)} \uparrow \varphi^{n-k+1} \downarrow \otimes n-k+1 \varphi_{1,n-k-i} \downarrow \otimes n.$$

$$= \sum_{1 \leq k \leq n \atop 0 \leq i \leq n - k} (-1)^{n-k+i(k+1)} \varphi^{n-k-1} \varphi_{1,n-k-i}$$

$$= -(-1)^n \sum_{0 \leq i \leq n - \ell - 1} \sum_{0 \leq i \leq n - \ell - 1} (-1)^{\ell(i+1)} \varphi^{n-\ell-\ell+1} \varphi_{1,n-\ell-1-i}.$$

It is easy to prove that

**Proposition 3.** Given an $A_\infty$-algebra $(A, \varphi^n)_{n \geq 1}$, $(\Bar{BA}, d_{\Bar{BA}})$ is a dgc.

**Definition 6.** Let $(A, \varphi^n)_{n \geq 1}$ be an $A_\infty$-algebra. The tilde bar construction on $A$ is the dgc $(\Bar{BA}, d_{\Bar{BA}})$.

**Definition 7.** Let $A$ and $C$ be $A_\infty$-algebras. A chain map $f = f^1 : A \to C$ is a map of $A_\infty$-algebras if there exists a family of higher homotopies $\{f^k \in Hom^{k-1} (A^\otimes k, C)\}_{k \geq 2}$ such that

$$\Bar{f} = \sum_{n \geq 1} \left( \sum_{k \geq 1} f^k \downarrow \otimes k \right) \uparrow \otimes n : \Bar{BA} \to \Bar{BC}$$

is a dgc map.

Dually, consider a sequence of maps $\{\psi^k \in Hom^{k-2} (A^\otimes k, A)\}_{k \geq 1}$. For each $k$ and $n \geq 1$, linearly extend each $\psi^k$ to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-1} \psi_{1,n-1-i}^k : A^{\otimes n} \to A^{\otimes n+k-1},$$
and consider the induced map of degree $-1$ given by
\[
\sum_{i=0}^{n-1} (\psi^k |_{i,n-1-i}) : (\downarrow A)^{\otimes n} \rightarrow (\downarrow A)^{\otimes n+k-1}.
\]

Let $\bar{\Omega} = T^a (\downarrow A)$ and define a map $d_{\bar{\Omega}A} : \bar{\Omega}A \rightarrow \bar{\Omega}A$ of degree $-1$ by
\[
d_{\bar{\Omega}A} = \sum_{n,k \geq 1} \sum_{0 \leq i \leq n-1} (\psi^k |_{i,n-1-i})^{n-1} \otimes \psi^k |_{i,n-1-i} \otimes \psi^i |_{n-1-i}.
\]
which can be rewritten as
\[
d_{\bar{\Omega}A} = \sum_{n,k \geq 1} \sum_{0 \leq i \leq n-1} (-1)^{(k+1)+i} \psi^k |_{i,n-1-i} \otimes \psi^i |_{n-1-i} \otimes \psi^{n-i}.
\]

\textbf{Definition 8.} $(A, \psi^n)_{n \geq 1}$ is an $A_\infty$-coalgebra if $d_{\bar{\Omega}A}^2 = 0$. The maps $\{\psi^n\}$ in an $A_\infty$-coalgebra are called higher homotopies.

\textbf{Proposition 4.} Let $(A, \psi^n)_{n \geq 1}$ be an $A_\infty$-coalgebra. The higher homotopies $\{\psi^k\}$ satisfy the following quadratic relations for $n \geq 1$:
\[
\sum_{0 \leq \ell \leq n-1} (-1)^{(n/2)+i(k+1)+k(n+1)} \psi^\ell |_{i,n-1-i} \otimes \psi^{n-\ell} = 0.
\]

\textbf{Proof:} The proof is similar to the proof of Proposition 2 and is omitted.

Again, it is easy to prove that

\textbf{Proposition 5.} Given an $A_\infty$-coalgebra $(A, \psi^n)_{n \geq 1}$, $(\bar{\Omega}A, d_{\bar{\Omega}A})$ is a dga.

\textbf{Definition 9.} Let $(A, \psi^n)_{n \geq 1}$ be an $A_\infty$-coalgebra. The tilde cobar construction on $A$ is the dga $(\bar{\Omega}A, d_{\bar{\Omega}A})$.

\textbf{Definition 10.} Let $A$ and $B$ be $A_\infty$-coalgebras. A chain map $g = g^1 : A \rightarrow B$ is a map of $A_\infty$-coalgebras if there exists a family of higher homotopies $g^k \in Hom^{k-1}$ $(A, B^{\otimes k})_{k \geq 2}$ such that
\[
\bar{g} = \sum_{n \geq 1} \left( \sum_{k \geq 1} \psi^k |_{i,n-1-i} \otimes g^k |_{i,n-1-i} \right)^{\otimes n} : \bar{\Omega}A \rightarrow \bar{\Omega}B,
\]
is a dga map.

The essential ingredient in an $A_\infty$-(co)algebra is the given family of higher homotopies. Given such a family, one is free to independently reverse the direction of each homotopy, i.e., its sign, and subsequently alter the signs in the quadratic relations. Nevertheless, the signs in the relations relate directly to the initial given family. The “simplest” signs in the quadratic relations in an $A_\infty$-algebra appear when the tilde bar differential is the sum of the given higher homotopies, i.e., the direction of each higher homotopy is unaltered as in (5.1) above. Consequently, the simplest signs appear in (5.3) above. On the other hand, a straightforward
calculation shows that the “simplest” signs appear in the quadratic relations for an $A_\infty$-coalgebra when $\psi^n$ is replaced by $(-1)^{(n-1)/2} \psi^n$, $n \geq 1$, i.e., the direction of every third and fourth homotopy is reversed. The choices one makes will depend on the application; for us the appropriate choices are as in (5.3) and (5.5).

Let $(A, \varphi^n)_{n \geq 1}$ be an $A_\infty$-algebra with structure relations as in (5.3). For each $n \geq 2$, associate $T_n \in C_{n-2}(K_n)$ with the operation $\varphi^n$ via

$$T_n \mapsto (-1)^n \varphi^n$$

(5.7)

and $d_{(i,\ell)}(T_n) \in C_{n-3}(K_n)$ with the quadratic composition $\varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}$, i.e.,

$$d_{(i,\ell)}(T_n) \mapsto \varphi^{n-\ell}\varphi^{\ell+1}_{i,n-\ell-1-i}.$$  

(5.8)

Then (5.7) and (5.8) induce a chain map

$$\zeta_A : A_\infty \rightarrow \{\text{Hom}^* (A^\otimes n, A) \}_{n \geq 2}$$

(5.9)

representing the $A_\infty$-algebra structure on $A$.

Dually, if $(A, \psi^n)_{n \geq 1}$ is an $A_\infty$-coalgebra with structure relations as in (5.5), the associations

$$T_n \mapsto \psi^n$$

and

$$d_{(i,\ell)}(T_n) \mapsto \psi^{\ell+1}_{i,n-\ell-1-i}\psi^{n-\ell}$$

induce a chain map

$$\xi_A : A_\infty \rightarrow \{\text{Hom}^* (A, A^\otimes n) \}_{n \geq 2}$$

(5.10)

representing the $A_\infty$-coalgebra structure on $A$.

**Definition 11.** Let $A$ and $B$ be $A_\infty$-algebras with respective $A_\infty$-algebra structures $\zeta_A$ and $\zeta_B$. The tensor product $A \otimes B$ is the $A_\infty$-algebra with structure

$$\{\zeta_{A \otimes B}(T_n) = (\sigma_n)^* (\zeta_A \otimes \zeta_B) \Delta T_n \}_{n \geq 2},$$

where $(\sigma_n)^* = \text{Hom}(\sigma_n, A \otimes B)$. Similarly, if $A$ and $B$ are $A_\infty$-coalgebras with respective structures $\xi_A$ and $\xi_B$, $A \otimes B$ is the $A_\infty$-coalgebra with structure

$$\{\xi_{A \otimes B}(T_n) = (\sigma_n^{-1})_* (\xi_A \otimes \xi_B) \Delta T_n \}_{n \geq 2},$$

where $(\sigma_n^{-1})_* = \text{Hom}(A \otimes B, \sigma_n^{-1})$.

**Proposition 6.** Let $A$ and $B$ be $A_\infty$-algebras with respective $A_\infty$-algebra structures $\zeta_A$ and $\zeta_B$; let $A \otimes B$ be the $A_\infty$-algebra with structure $\zeta_{A \otimes B}$. Then the following diagram commutes for each $n \geq 2$:

$$\begin{array}{ccc}
C_\ast (K_n) & \xrightarrow{\zeta_{A \otimes B}} & \text{Hom}((A \otimes B)^\otimes n, A \otimes B) \\
\Delta \downarrow & & \uparrow (\sigma_n)^* \\
C_\ast (K_n) \otimes C_\ast (K_n) & \xrightarrow{\zeta_A \otimes \zeta_B} & \text{Hom}(A^\otimes n, A) \otimes \text{Hom}(B^\otimes n, B)
\end{array}$$
Dually, let $A$ and $B$ be $A_\infty$-coalgebras with respective $A_\infty$-coalgebra structures $\xi_A$ and $\xi_B$; let $A \otimes B$ be the $A_\infty$-coalgebra with structure $\xi_{A \otimes B}$. Then the following diagram commutes for each $n \geq 2$:

$$
\begin{array}{ccc}
C_*(K_n) & \xrightarrow{\xi_{A \otimes B}} & \text{Hom}(A \otimes B, (A \otimes B)^{\otimes n}) \\
\Delta & \downarrow & \uparrow (\sigma_n^{-1}) \\
C_*(K_n) \otimes C_*(K_n) & \xrightarrow{\xi_A \otimes \xi_B} & \text{Hom}(A, A^{\otimes n}) \otimes \text{Hom}(B, B^{\otimes n})
\end{array}
$$

Proof: This is an immediate consequence of Definition 11 and the canonical isomorphisms $\text{Hom}(A^{\otimes n}, A) \otimes \text{Hom}(B^{\otimes n}, B) \approx \text{Hom}(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B)$ and $\text{Hom}(A, A^{\otimes n}) \otimes \text{Hom}(B, B^{\otimes n}) \approx \text{Hom}(A \otimes B, A^{\otimes n} \otimes B^{\otimes n})$.

Example 6. Given an $A_\infty$-coalgebra $A$ with operations $\{\psi_i\}_{n \geq 1}$, we have the following operations on $A \otimes A$:

- $\psi^1_{A \otimes A} = \psi^1 \otimes 1 + 1 \otimes \psi^1$
- $\psi^2_{A \otimes A} = \psi^2 \otimes \psi^2$
- $\psi^3_{A \otimes A} = \psi^3 \otimes \psi^2 \psi^2$
- $\psi^4_{A \otimes A} = \psi^4 \otimes \psi^2 \psi^2 \psi^2 + \psi^3 \psi^2 \psi^2 \psi^2$
- $\vdots$

The stage is now set for the sequel in which we introduce the notion of an $A_\infty$-Hopf algebra in full generality. We shall observe that the homology of a loop space and the double cobar construction admit canonical $A_\infty$-Hopf algebra structures.

6. Appendix

In this appendix we prove that the diagonal $\Delta$ defined in Section 4 above is a chain map. We begin with some preliminaries. It will be convenient to rewrite face relation (3) as follows:

$$
d^{q+1}_{(i_{q+1},i_{q+1})} d^q_{(i_{q+1},i_q)} = d^{q+1}_{(i_{q+1},i_{q+1})} d^q_{(i_{q+1},i_{q+1})}; \quad i_q > i_{q+1} + \ell_{q+1}. \quad (3')
$$

Definition 12. For $1 \leq k \leq m$, the $k^{th}$ left-transfer $\tau^k_\ell$ of a composition $d_{(i_m,\ell_m)} \cdots d_{(i_2,\ell_2)} d_{(i_1,\ell_1)}$ in first fundamental form is one of the following compositions:

(a) If $i_{k+1} + \ell_{k+1} \geq i_k$, apply face relation (2) to $d^{k+1} d^k$ and obtain

$$
\cdots d_{(i_{k+1}+\ell_k)} d_{(i_{k+1}+\ell_k+\ell_k)} \cdots ;
$$

then successively apply face relation (1) to $d^{j+2} d^j$ for $j = k, k+1, \ldots, m-2$, and obtain

$$
\tau^k_{\ell} = d_{(i_{k+1}+\ell_k)} d_{(i_{m+1}+\ell_m)} \cdots d_{(i_{k+1}+\ell_k+\ell_k)} \cdots d_{(i_1,\ell_1)}. 
$$
(b) If \( i_k + 1 < i_k + 1 + \ell(q+1) - \ell(k) \) for some smallest integer \( q > k \), successively apply face relation (3') to \( d^{j+1}d^j \) for \( j = k, k+1, \ldots, q \), and obtain
\[
\cdots d_{(i_{q+2}, \ell_{q+2})}^1 d_{(i_{k+1} + \ell_k, \ell_{k+1})}^2 \cdots d_{(i_{k+1}, \ell_{k+1})}^j d(i_k, \ell_{k+1}) \cdots .
\]
Apply face relation (2) to \( d^{j+1}d^j \); then successively apply face relation (1) to \( d^{j+2}d^j \) for \( j = q, q+1, \ldots, m - 2 \), and obtain
\[
\tau_{i_k}^k = d_{(i, \ell)}^m d_{(i_{k+1}, \ell_{k+1})}^j d(i_k, \ell_{k+1}) \cdots d_{(i, \ell_1)}^1,
\]
where \( i = i_k - i_{q+1} + \ell(k) - \ell(q) \).

(c) Otherwise, successively apply face relation (3') to \( d^{j+1}d^j \) for \( j = k, k+1, \ldots, m - 1 \), and obtain
\[
\tau_{i_k}^k = d_{(i, \ell)}^m d_{(i_{k+1}, \ell_{k+1})}^j d(i_k, \ell_{k+1}) \cdots d_{(i, \ell_1)}^1,
\]
where \( i = i_k + \ell(k) - \ell(m) \).

**Definition 13.** For \( 1 \leq k \leq m \), the \( k \)th right-transfer \( \tau_{i_k}^k \) of a composition \( d_{(i, \ell_k)}^m \cdots d_{(i_2, \ell_2)}^2 d_{(i, \ell_1)}^1 \) in first fundamental form is one of the following compositions:

(a) If \( i_p + \ell(p) \leq i_k + \ell(k) < i_p - 1 + \ell(p-1) \) for some greatest integer \( 1 < p \leq k \), successively apply face relation (3') to \( d^{j+1}d^j \) for \( j = p - 1, \ldots, 2, 1 \); \( j = p, \ldots, 3, 2, \ldots, j = k-1, k-2, \ldots, k - (p-1) \). Then apply face relation (2) to \( d^{j+1}d^j \) for \( j = k - p, \ldots, 2, 1 \) and obtain
\[
\tau_{i_k}^k = d_{(i, \ell)}^m d_{(i_{k+1}, \ell_{k+1})}^j d_{(i_1, \ell_1)}^1 \cdots d_{(i_{k-p+1}, \ell_{k-p+1})}^1 \cdots d_{(i_{k-p}, \ell_{k-p})}^1 \cdots d_{(i_{1-p}, \ell_{1-p})}^1 \cdots d_{(i, \ell_1)}^1,
\]
where \( \ell = \ell(k) - \ell(p-1) \).

(b) Otherwise, successively apply face relation (2) to \( d^{j+1}d^j \) for \( j = k - 1, k-2, \ldots, 2, 1 \), and obtain
\[
\tau_{i_k}^k = d_{(i, \ell_k)}^m d_{(i_{k+1}, \ell_{k+1})}^j d_{(i_1, \ell_1)}^1 \cdots d_{(i_{k-1}, \ell_{k-1})}^1 \cdots d_{(i, \ell_1)}^1.
\]

Note that if \( I = (i_1, \ell_1), \ldots, (i_m, \ell_m) \) is a type I sequence and \( a = d_I (T_{n+2}) \in C_{n-m} (K_{n+2}) \), then for each \( k \), the \( k \)th right-transfer \( \tau_{i_k}^k \) \( (T_{n+2}) \) expresses \( a \) in first fundamental form as a face of some \((n-1)\)-face \( b \) of \( T_{n+2} \). The expressions \( \tau_{i_k}^1 \) \( (T_{n+2}) \), \ldots, \( \tau_{i_k}^m \) \( (T_{n+2}) \) determine the \( m \) distinct \((n-1)\)-faces \( b \) containing \( a \).

There are analogous left and right transfers for compositions in second fundamental form.

**Definition 14.** For \( 1 \leq k \leq m \), the \( k \)th left-transfer \( \tau_{i_k}^k \) of a composition \( d_{(i, \ell_k)}^m \cdots d_{(i_2, \ell_2)}^2 d_{(i, \ell_1)}^1 \) in second fundamental form is one of the following compositions:

(a) If \( i_{k+1} \leq i_k \), apply face relation (2) to \( d^{k+1}d_k \), then successively apply face relation (1) to \( d^{k+2}d_k \) for \( j = k, k+1, \ldots, m - 2 \), and obtain
\[
\tau_{i_k}^k = d_{(i_{k+1}, \ell_{k+1})}^1 d_{(i_1, \ell_1)}^1 \cdots d_{(i, \ell_1)}^1.
\]
Definition 15. For a common face operator \( \tau^k \) as a face of some \( \tau \) fundamental form as a face of some \( \tau \) contained in the same \( \tau \).

Lemma 1. For \( i_k+1 > i_k \geq q > k \), successively apply face relation (3) to \( d^i+1d^j \) for \( j = k, k+1, \ldots, q \). Apply face relation (2) to \( d^i+1d^j \); then successively apply face relation (1) to \( d^i+2d^j \) for \( j = q, q+1, \ldots, m-2 \), and obtain

\[
\tau^k = d_{(i_k-i_{q+1}, \ell_k)}d_{(i_{q+1}, \ell_{q+2})} \cdots d_{(i_q, \ell_{q+2})} d_{(i_q, \ell_{q+1}+\ell_k)} d_{(i_{q+1}, \ell_{q+1}+\ell_k)} \cdots d_{(i_1, \ell_1)}.
\]

(c) Otherwise, successively apply face relation (3) to \( d^i+1d^j \) for \( j = k, k+1, \ldots, m-1 \), and obtain

\[
\tau^k = d_{(i_k, \ell_k)}d_{(i_{k+1}, \ell_{k+1})} \cdots d_{(i_{k-1}, \ell_{k-1})} d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}.
\]

Definition 15. For \( 1 \leq k \leq m \), the \( k \) right-transfer \( \tau^k \) of a composition \( d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} \) in second fundamental form is one of the following compositions:

(a) If \( i_p-1 < i_k \leq i_p \) for some greatest integer \( 1 < p < k \), successively apply face relation (3) to \( d^i+1d^j \) for \( j = p-1, \ldots, 2, 1 \); \( j = k-1, k-2, \ldots, k-(p-1) \). Then apply face relation (2) to \( d^i+j \) for \( j = k-1, \ldots, 2, 1 \) and obtain

\[
\tau^k = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}.
\]

(b) Otherwise, successively apply face relation (2) to \( d^i+1d^j \) for \( j = k-1, \ldots, 2, 1 \), and obtain:

\[
\tau^k = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}.
\]

Once again, if \( I = (i_1, \ell_1), \ldots, (i_m, \ell_m) \) is a type II sequence and \( a = d_I(T_{n+2}) \in C_{m-n}(R_{n+2}) \), then for each \( k \), the \( k \) right-transfer \( \tau^k \) \( T_{n+2} \) expresses a in second fundamental form as a face of some \( (n-1) \)-face \( b \); the expressions \( \tau^1 \) \( T_{n+2} \), \ldots, \( \tau^m \) \( T_{n+2} \) determine the \( m \) distinct \( (n-1) \)-faces \( b \) containing \( a \). Thus the two \( (n-1) \)-faces \( a_1 \) and \( a_2 \) expressed in first and second fundamental form, respectively, are contained in the same \( (n-1) \)-face \( b \) if and only if there exist right transfers \( \tau^{k_j} \) such that \( a_j = \tau^{k_j} \) \( T_{n+2} \), \( j = 1, 2 \), and \( b = d_{(i, \ell)}(T_{n+2}) \), where \( d_{(i, \ell)} \) is the rightmost face operator common to \( \tau^{k_1} \) and \( \tau^{k_2} \). We state this formally in the following lemma:

Lemma 1. Let \( 0 \leq m_1, m_2 \leq n \) and assume that \( a_1 = d_{(i_{m_1}, \ell_{m_1})} \cdots d_{(i_1, \ell_1)} \) \( T_{n+2} \) and \( a_2 = d_{(i_0, \ell_0)} \cdots d_{(i_1, \ell_1)} \) \( T_{n+2} \) are expressed in first and second fundamental form, respectively. Then \( a_1 \) and \( a_2 \) are contained in the same \( (n-1) \)-face \( d_{(i, \ell)}(T_{n+2}) \) if and only if there exist integers \( k_j \leq m_j \) and greatest integers \( 1 \leq p_j < k_j \) for \( j = 1, 2 \), such that

\[
i_{k_j} + \ell_{(k_j)} - \ell_{(p_j)} < i_{p_j}
\]
and
\[
\begin{align*}
i &= i_{k_1} = i_{k_2} + l'_{(p_2)}, \\
\ell &= \ell_{(k_1)} - \ell_{(p_1)} = \ell'_{(k_2)} - \ell'_{(p_2)}.
\end{align*}
\]

Consider a solution \(((i_1, \ell_1), \ldots, (i_p, \ell_p); (i'_1, \ell'_1), \ldots, (i'_q, \ell'_q))\) of system (4.1) and its related \((p, q)\)-shuffle \(\{i_p < \cdots < i_1, \epsilon_1 < \epsilon_2 < \cdots < \epsilon_q\}\). Given \(1 \leq k \leq q + 1\), let \(k_1 = k - 1, k_{j+1} = o'(t_{k_j})\) for \(j \geq 1\) and note that \(k_{j+1} < k_j\) for all \(j\). For \(0 \leq m < \ell'_k\), consider the following selection algorithm:

```plaintext
if \(k = 1\) then \(z = p + 1 - i'_1 - m\)
else \(z = n + 2; j = 0\)
repeat
\(j \leftarrow j + 1\)
if \(i_{k_j} < i'_k + m\) then \(i_z = \epsilon_{k_j} - i'_k + i'_k + m\)
if \(i_{k_j} - \ell'_{(k_{j+1})} = i'_k + m\) then \(z = t_{k_j}\)
until \(z < n + 2\)
endif
```

It will be clear from the proof of Lemma 2 below that the selection algorithm eventually terminates.

**Example 7.** Let \(p = q = 4\) and consider the following solution of system (4.1):
\(((7, 1), (6, 1), (4, 2), (2, 3); (0, 1), (1, 1), (1, 2), (0, 4))\).

Then \(t_1 = 5, t_2 = 4, t_3 = 3, t_4 = 4\) and the selection algorithm produces the following:

| \(k\) | 1 | 2 | 3 | 3 | 4 | 4 | 4 | 4 |
| \(m\) | 0 | 0 | 0 | 1 | 0 | 1 | 2 | 3 |
| \(z\) | 5 | 4 | 3 | 5 | 4 | 2 | 1 |

The key to our proof that \(\Delta\) is a chain map is given by our next lemma.

**Lemma 2.** Given \(1 \leq k \leq q + 1\) and \(0 \leq m < \ell'_k\), let \(z\) be the integer given by the selection algorithm.

(a) \(z > o(k)\)
(b) \(i_z + \ell(z) - \ell(o(k)) \geq \epsilon_k\).
(c) \(i'_k + m = i_z - \ell'_{o'(z)}\);
(d) If \(o'(z) < r < k\), then \(i'_k + m \leq i'_r\);
(e) If \(i'_k + m > \min_{o'(t_k) < r < k} \left\{i'_r, i_{t_k} - \ell'_{o'(t_k)}\right\}\), then

\(\text{(e.1)}\) \(z < t_k\) and \(i_z + \ell(z) - \ell(o(k)) = \epsilon_k\);
\(\text{(e.2)}\) \(o(k) = \max_{r < z} \left\{r \mid i_r > i_z + \ell(z) - \ell(r)\right\}\);
\[ (e.3) \quad \alpha'(z) = \max_{r < k} \left\{ r \mid i'_r < i'_k + m \right\}. \]

**Proof:**

(a) When \( k = 1 \), \( z = (p + 1 - \epsilon_1) + (\ell'_1 - m) > p + 1 - \epsilon_1 = a(1) \). For \( k > 1 \), first note that \( i_{t_k} < \epsilon_k \leq i_{o(k)} \) for all \( j \geq 1 \), by the definition of \( t_k \) in (4.2). So the result follows whenever \( i_z \leq i_{t_k} \) for some \( j \geq 1 \). If \( i'_k < i'_k + m \), then
\[ i_z = \epsilon_k - i'_k + m - i'_{k-1} < \epsilon_k - i'_{k-1} + \ell'_k - i'_{k-1} = \epsilon_k. \]
If \( i'_k < i'_k + m \) for some \( j \geq 2 \) and \( i'_k + m \leq i'_k \leq \ell'_{t_{k+1}} \) for all \( r < j \), where the later follows from inequality (3) of system (4.1), then \( i_z = \epsilon_k + i'_k + m - i'_{k-1} = i_{t_k} - i'_k = i_{t_{k-1}}. \)

(b) When \( k = 1 \), indices \( 0 = i_{p+1} < i_p < \cdots < i_{p+2-\epsilon_1} \) are consecutive; hence \( i_z + \ell'(r) = i_{z-1} + \ell'(r-1) + \ell - 1 \geq i_{z-1} + \ell'(r-1) \) for \( p + 3 - \epsilon_1 \leq r \leq p + 1 \). Since \( z \geq p + 2 - \epsilon_1 \) we have \( i_z + \ell'(z) - \ell'(o(1)) \geq i_{z-1} + \ell'(p+2-\epsilon_1) = \epsilon_1 - 1 \). So consider \( k > 1 \). If \( i'_k < i'_k + m \), then \( i'_{t_k} < i'_{t_k} + m \) and \( z = t_k = 1 \), then \( o(1) - o(k) = \epsilon_k - \epsilon_k - 1 \). So consider \( k > 1 \): \( i'_k < i'_k + m \) then \( i'_{t_k} < i'_{t_k} < \epsilon_k \) so that \( \ell'(z) - \ell'(t_k) = \geq s - t_k = i_{t_k} - i', \) where equality holds iff \( i_{t_k} = \cdots = i_z = 1 \); hence \( i_z + \ell'(z) - \ell'(o(k)) = i_z + \ell'(z) - \ell'(t_k) + \ell'(t_k) - \ell'(o(k)) = \). From the previous calculation.

(c) This result follows immediately from the choice of \( z \).

(d) Note that \( k > 1 \). If \( i'_k < i'_k + m \), then \( o'(z) = k - 1 \) and the case is vacuous. So assume that \( m \leq i'_k \). If either \( i'_k + m = i_{t_k} - \ell'(t_k) \) or \( i'_k < i'_k + m \), then \( o'(z) = o'(t_k) \) and \( i'_k \leq i'_r \) for \( o'(z) < r < k \). Otherwise, \( i'_k + m \leq \min \{ i'_k, i'_k \} \). If either \( i'_k + m = i_{t_k} - \ell'(t_k) \) or \( i'_k < i'_k + m \), then \( o'(z) = o'(t_k) \) and \( i'_k < i'_k \) for \( o'(z) < k \). So the desired inequality holds for \( o'(z) \).

(e) If \( k = 1 \), \( o'(z) = 0 \) so that \( i'_k + m = i_z \) and \( i_z + \ell'(z) \geq \epsilon_1 \). Now \( i'_k + m > \ell'(t_k) \) by assumption, hence \( z < t_k \). But \( t_k \) is the smallest integer \( r \) such that \( i_r + \ell'(r) > \epsilon_1 \), therefore \( i_z + \ell'(z) = \epsilon_1 \), by (b). Let \( k > 1 \) and suppose that \( z \geq t_k \). Then \( i_z < t_k \), in which case \( o'(z) = o'(t_k) \). But by (e), \( i'_k + m = i_z - \ell'(o(z)) < i_{t_k} - \ell'(o(t_k)) \) and \( o'(z) < t_k \). But \( t_k \) is the smallest possible integer such that \( i_{t_k} + \ell'(t_k) - \ell'(o(k)) > \epsilon_k \), so (e.1) follows from (b). Results (e.2) and (e.3) are obvious.

**Proof of Theorem 1:**
Let $n \geq 0$. We show that if $u \otimes v$ is a component of $d^{o+2} \Delta(T_{n+2})$ or $\Delta d(T_{n+2})$, there is a corresponding component that cancels it, in which case $(d^{o+2} \Delta - \Delta d)(T_{n+2}) = 0$. Let $a' \otimes a$ be a component of $\Delta T_{n+2}$. We consider various cases.

**Case I.** Consider a component $d'_{i,(\ell)} \otimes 1$ of $\otimes 1$, where $i + \ell \leq \ell'$, $i \geq 0$, $1 \leq \ell \leq \ell'$, and $1 \leq r \leq q + 1$. Reduce $b' = d'_{i,(\ell)}(a')$ to an expression in second fundamental form, i.e., if $1 \leq r \leq q$, successively apply relation (1) to $d^{r+1} d^{i'}$ for $j = q, \ldots, r + 1$; apply (2) to replace $d'_{i,(\ell)}(a')$ by $d'_{(\ell')} \otimes 1$; and then successively apply (3) to $d^{r+1} d^{i'}$ for either $j = r - 1, \ldots, \beta$ if $i + \ell < \ell$ or for $j = q, \ldots, \beta$ if $r = q + 1$, where $\beta$ is the greatest integer such that $1 \leq \beta \leq r$ and $i + \ell > i'_{(\beta - 1)}$. Then for $i + \ell < \ell'$, $1 \leq r \leq q + 1$ we have

$$b' = d^{r+1}_{i,(\ell)} \cdots d^{r+1}_{(\ell'),(l')} \cdot d^{r+1}_{(\ell'),(\ell - t)} \cdots d^{r+1}_{(\ell'),(\ell')} d^{r+1}_{(\ell),(\ell')} (T_{n+2}),$$

and for $i + \ell = \ell'$, $1 \leq r \leq q + 1$ we have

$$b' = d^{r+1}_{i,(\ell)} \cdots d^{r+1}_{(\ell'),(l')} \cdot d^{r+1}_{(\ell'),(\ell - t)} \cdots d^{r+1}_{(\ell'),(\ell')} d^{r+1}_{(\ell),(\ell')} (T_{n+2}).$$

**Case Ia.** Let $i + \ell < \ell'$ and $i'_{(\beta - 1)} + i + \ell > i'_{(\beta - 1)}$. As in the case of $a'$ above, the inequalities for lower indices in an expression of $b'$ as a face of $T_{n+2}$ in first fundamental form are strict, but whose number now is increased by one. Obviously we will have that

$$c_{(\beta)} = i'_{(\beta - 1)} + i + \ell = i_{k},$$

for certain $1 \leq k \leq p$. Apply the $k^{th}$ left-transfer to obtain

$$a = \tau_{(\ell)}^k (T_{n+2}) = d_{(i,(\ell))}^k (b'),$$

where $\tilde{k} = \alpha - 1, i = i_k - i_\alpha - \ell_{(\alpha - 1)} - \ell_{(k)}, \tilde{\ell} = \ell_k$ and $\tilde{\alpha}$ is the smallest integer $k \leq \alpha \leq p + 1$ with $i_{(\alpha)} + \ell_{(\alpha)} - \ell_{(k)} \geq i_k$. Then $b' \otimes b$ is a component of $\Delta T_{n+2}$ as well.

**Case Ib.** Let $i + \ell < \ell'$ and $i'_{(\beta - 1)} + i + \ell = i'_{(\beta - 1)}$. Apply the $(\beta - 1)^{th}$ left-transfer to obtain

$$b' = \tau_{(\ell)}^{\beta - 1} (T_{n+2}) = d_{(i,(\ell))}^{\beta - 1} (c'),$$

where $\tilde{i} = \ell$ and $\tilde{\ell} = \ell_{(\beta - 1)}$. Then $c' \otimes a$ is also a component of $\Delta T_{n+2}$.

**Case Ic.** Let $i + \ell = \ell'$; there are two subcases:

**Subcase i.** If $1 \leq r \leq q$ and $i'_{(\beta - 1)} + i \leq \min(i_{(r)} - \ell_{(\alpha(r))}, i'_{(r)}), \alpha'(r) < r < r$, apply the $(r + 1)^{th}$ left-transfer to obtain

$$b' = \tau_{(\ell)}^{r + 1} (T_{n+2}) = d_{(i,(\ell))}^{r + 1} (c'),$$

where $\tilde{r} = \alpha, \tilde{i} = i'_{(r)} - i_\alpha, \tilde{\ell} = \ell_{(r)} - \ell$ and $\tilde{\alpha}$ is the smallest integer $r \leq \alpha \leq p + 1$ with $i'_{(\alpha)} \leq i'_{(\beta - 1)}$. Although certain $i'_{(\alpha)}$‘s are increased by $i$, while $\ell_{(r)}$ is reduced by $i$ to obtain $\ell$, it is straightforward to check that $c' \otimes a$ is also a component of $\Delta T_{n+2}$.

**Subcase ii.** Suppose that $1 \leq r \leq q + 1$ and $i'_{(\beta - 1)} + i > \min(i_{(r)} - \ell_{(\alpha(r))}, i'_{(r)}), \alpha'(r) < j < r$. In view of Lemma 2 (with $k = r$ and $m = i$) we have

$$c_{(r)} = i_z + \ell_{(r)} = \ell_{(\alpha(r))}$$

and $i'_{(r)} + i = i_z - \ell_{(\alpha'(r))}$, from which we also establish the equality

$$\ell_{(r)} - \ell_{(\alpha(r))} = \ell_{(r - 1)} + \ell - \ell_{(\alpha'(r))}. $$
The hypotheses of Lemma 1 are satisfied by setting \( k_1 = z \), \( p_1 = o(r) \) and \( k_2 = r \), \( p_2 = o'(z) \). Hence, \( b' \otimes a \) is a component of \( \Delta d_{(i, \ell)}(T_{n+2}) \) with \( i = i_z \) and \( \ell = \ell(z) - \ell(o(r)) \).

**Case II.** Consider a component \( 1 \otimes d_{(i, \ell)}^r \) of \( 1 \otimes d \), where \( i + \ell \leq \ell \), \( i \geq 0 \), \( 1 \leq \ell < \ell \), and \( 1 \leq r \leq p + 1 \). Reduce \( b = d_{(i, \ell)}^r(a) \) to the first fundamental form, i.e., if \( 1 \leq r \leq p \), successively apply relation (1) to \( d_{(i, \ell)}^r d_{(i, \ell)}^r \) for \( j = p, \ldots, r + 1 \), and apply (2) to replace \( d_{(i, \ell)}^r d_{(i, \ell)}^r \) by \( d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r \). Then if \( i > 0 \), successively apply (3) to \( d_{(i, \ell)}^r d_{(i, \ell)}^r \), for \( j = r - 1, \ldots, \beta \); or if \( r = p + 1 \), successively apply (3) to \( d_{(i, \ell)}^r d_{(i, \ell)}^r \) for \( j = p, \ldots, \beta \), where \( \beta \) is the greatest integer with \( 1 \leq \beta < r \) and \( i_r + i_r + \ldots + \ell_{\beta - 1} \). Then for \( \beta = r \), we have \( i_r + i_r \leq i_r - 1 \); and for \( i > 0 \), \( 1 \leq r \leq p + 1 \),

\[
b = d_{(i, \ell)}^r \cdots d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r \cdots d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r (T_{n+2}) \]

where \( \tilde{i} = i_r + \ell_{(r-1)} - \ell_{(\beta-1)} \), and

\[
b = d_{(i, \ell)}^r \cdots d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r d_{(i, \ell)}^r \cdots d_{(i, \ell)}^r (T_{n+2}) \]

for \( i = 0, 1 \leq r \leq p + 1 \).

**Case IIa.** Let \( i = 0 \) and \( i_r + i_r + \ell_{(r-1)} - \ell_{(\beta-1)} < i_{\beta-1} \). Once again, the inequalities for the lower indices in the expression of \( b \) in first fundamental are strict inequalities as they were for \( a \) but whose number now is increased by one. Obviously we will have that

\[
i_{\beta} = i_r + i_r + \ell_{(r-1)} - \ell_{(\beta-1)} = \epsilon_k
\]

for certain \( 1 \leq k \leq q \). Apply the \( k^{th} \) left-transfer to obtain

\[
a' = \tau_{(i, \ell)}^k(T_{n+2}) = d_{(i, \ell)}^k(b'),
\]

where \( \tilde{k} = \alpha - 1 \), \( \tilde{i} = i'_k - i'_a \), \( \tilde{\ell} = \ell'_k \) and \( \alpha \) is the smallest integer \( k \leq \alpha \leq q + 1 \) with \( i'_k \leq i'_a \). Then \( b' \otimes b \) is a component of \( \Delta T_{n+2} \) as well.

**Case IIb.** Let \( i = 0 \) and \( i_r + i_r + \ell_{(r-1)} - \ell_{(\beta-1)} = i_{\beta-1} \). Apply the \( (\beta - 1)^{th} \) left-transfer to obtain

\[
b = \tau_{(i, \ell)}^{\beta-1}(T_{n+2}) = d_{(i, \ell)}^{\beta-1}(c),
\]

where \( \tilde{i} = 0 \) and \( \tilde{\ell} = \ell_{\beta-1} \). Then \( a' \otimes c \) is also a component of \( \Delta T_{n+2} \).

**Case IIc.** Let \( i = 0 \); there are two subcases:

**Subcase i.** If \( 1 \leq r \leq p \) and no integer \( 1 \leq k \leq q \) exists with \( r = t_k \), then

\[
\epsilon_k = i_r + \ell + \ell_{(r-1)} - \ell_{(o(k))}
\]

and \( i'_k = i_r - \ell_{(o'(r))} \).

Apply the \( (r + 1)^{th} \) left-transfer to obtain

\[
b = \tau_{(i, \ell)}^{r+1}(T_{n+2}) = d_{(i, \ell)}^{r+1}(c),
\]

where \( \tilde{r} = \alpha \), \( \tilde{i} = i_r - i_a + \ell_{(a-1)} - \ell_{(r)} \), \( \tilde{\ell} = \ell_r - \ell \) and \( \alpha \) is the smallest integer \( r < \alpha \leq p + 1 \) such that \( i_a + \ell_{(a)} - \ell_{(r)} \geq i_r \); namely, \( \alpha = r + 1 \) or \( \alpha = t_{o'(r)} \). Now the required inequality for the \( i'_k \)'s could conceivably be violated for \( k > o'(r) \), but this is not so since it is easy to see that: (a) if \( r \) is not realized as \( t_k \) for some \( k > o'(r) \), then each \( z \) with \( \alpha < z < r \) is so; and (b) if \( r = t_k \) for some \( k > o'(r) \), while \( \epsilon_k = i_r + \ell + \ell_{(r-1)} - \ell_{(o(k))} \), then \( \alpha \) (and not \( r \) serves as \( t_k \) for indices of face
operators in expressions of $a$ and $c$; moreover, for either $\alpha = r + 1$ or $\alpha = t_{o'(r)}$ (in which case $i_k' \leq i_{o'(r)}'$), one has $i_k' \leq \min_{o'(\alpha) < j < k} \{ i_j', i_{\alpha} - \ell_{o'(\alpha)} \}$ so that $a' \otimes c$ is also a component of $\Delta T_{n+2}$.

**Subcase ii.** If $1 \leq r \leq p + 1$ and $r = t_k$ for some $1 \leq k \leq q$, then

$$
\epsilon_k = i_r + \ell + \ell_{o(k)} - \ell_{o'(k)} \quad \text{and} \quad i_k' = i_r - \ell_{o'(r)},
$$

from which we establish the equality

$$
\ell + \ell_{(r-1)} - \ell_{o(k)} = \ell_{o'(k)}'
$$

The hypotheses of Lemma 1 are satisfied by putting $k_1 = r = t_k$, $p_1 = o(k)$ and $k_2 = k$, $p_2 = o'(r)$ so that $a' \otimes c$ is a component of $\Delta d_{(i,\ell)}(T_{n+2})$ with $i = i_r$, and

**Case III.** Let $c' \otimes c$ be a component of $\Delta d_{(i,\ell)}(T_{n+2})$. Reduce $c$ and $c'$ to the first and second fundamental forms, respectively. According to Lemma 1, we have either $i = i_k$, or $i = i_{r_1} + i_{r_2} + \ldots + i_{r_k} = i_{r_1} + r_{r_2+k}+1$ for certain integers $r_1$, $r_2$, $k$, i.e., $i_{r_1 + 1} = i_{r_1}$ or $i_{r_2+1} = i_{r_2} + 1$. Note that the shuffles under consideration prevent both cases from occurring simultaneously, and we obtain the situation dual to either Subcase ii of Case IIc, or to Subcase ii of Case Ic. Thus we obtain components $c' \otimes a$ or $a' \otimes c$ of $\Delta T_{n+2}$ with $d_{(i,\ell)}^1(a) = c$ or $d_{(i,\ell)}^2(a') = c'$, respectively.

**References**


A. Razmadze Mathematical Institute, Georgian Academy of Sciences, M. Aleksidze st., 1, 380093 Tbilisi, Georgia
E-mail address: SANE@rmi.acnet.ge

Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA. 17551
E-mail address: Ron.Umble@Millersville.edu