On the rational LS–category of a cartesian product of maps

Hans Scheerer and Don Stanley
July 29, 1998

Abstract
We give an example of a rational map, \( f \), such that \( \text{cat} f = \text{cat} f \times \text{id}_{S^3} = 2 \).

1 Introduction
For a CW complex \( X \) the Lusternik–Schnirelmann category of \( X \), \( \text{cat}(X) \), is one less than the minimum number of open sets contractible in \( X \) that it takes to cover \( X \). This notion was introduced by Lusternik and Schnirelmann [10] where their invariant is one greater than the one above. Ganea’s conjecture states that for every CW–complex \( X \) and integer \( n \geq 1 \), \( \text{cat}X \times S^n = \text{cat}X + 1 \). This was proved for rational \( X \) by Jessup [9] and Hess [5]. Integrally, however, the conjecture is false [8].

For a map \( f : X \rightarrow Y \) of CW complexes the Lusternik–Schnirelmann category of \( f \), \( \text{cat}(f) \), is one less than the minimum number of open subsets of \( X \) whose inclusion composed with \( f \) is contractible that it takes to cover \( X \). This notion was introduced by Fox [4] where again his definition differs from the one given above by normalization. A similar conjecture can be made for maps, too. It would state that for every map \( f \) and integer \( n \geq 1 \) \( \text{cat}(f \times S^n) = \text{cat}(f) + 1 \) (where we use the convention that for a space \( X \), \( \text{id} : X \rightarrow X \) is also denoted by \( X \)). It is to this statement in the rational case that we have constructed a counterexample.

This paper is about rational homotopy theory. We work in the category of commutative differential graded algebras CDGA’s. For basic definitions and properties of these see [6], [12] and [13].
For a CDGA $A$ we also work with the category of $A$-modules. Any map $f : A \to B$ in CDGA gives $B$ the structure of an $A$-module and also makes $f$ an $A$-module map.

All of our CDGA’s will be of finite type and simply connected. Whenever we talk about spaces they will also be of finite type and simply connected. All of our objects in any category will be over the rationals. For a graded vector space $V$, $\Lambda(V)$ denotes the free commutative graded algebra on $V$. It may also be considered to implicitly have a differential. If we wish to make it explicit we will write $(\Lambda V, d)$; $\Lambda V$ is called a Sullivan model. For a graded set $\{a_1, \ldots, a_r\}$, $\Lambda(a_1, \ldots, a_r)$ means $\Lambda W$ where $W$ is the graded vector space with basis $\{a_1, \ldots, a_r\}$. $\Lambda V \otimes \Lambda W$ is just $\Lambda(V \oplus W)$ with the added condition that $dV \subset \Lambda V$. The inclusion

$$\Lambda V \to \Lambda V \otimes \Lambda W$$

is called a KS extension.

A map $f : \Lambda U \to \Lambda V$ is called $n + 1$ connected if $H_i(f)$ is an isomorphism for $i \leq n$ and a surjection for $i = n + 1$. Given $\Lambda V$ and an integer $n$ we let $(\Lambda V)_n$ denote the Sullivan model $\Lambda V \otimes \Lambda W$ such that the KS extension

$$\Lambda V \to \Lambda V \otimes \Lambda W$$

is $n + 1$ connected and $H_i(\Lambda V \otimes \Lambda W) = 0$ if $i \geq n + 1$. Notice that if we have a map $f : \Lambda V \to \Lambda W$ either CDGA’s or $V$ modules then there exists a unique homotopy class of extensions $f_n : (\Lambda V)_n \to (\Lambda W)_n$. This fact will be used in what follows.

**Definition** [2] Given $\Lambda V$, consider the following diagram

$$\begin{tikzcd}
\Lambda V \arrow{r}{i} \arrow[bend left=30]{d}[swap]{q} & \Lambda V \otimes \Lambda W \arrow{d}{p} \\
\Lambda V / \Lambda^{>n} V
\end{tikzcd}$$

such that $q$ is the quotient map, $p$ is a homology equivalence and $i$ is a KS extension.
Then \( \text{cat} \Lambda V \leq n \) if and only if there exists a map of CDGA’s \( r : \Lambda V \otimes \Lambda W \to \Lambda V \) such that \( ri = \text{id} \). Also \( \text{mcat} \Lambda V \leq n \) if and only if there exists a map of \( \Lambda V \) modules \( r : \Lambda V \otimes \Lambda W \to \Lambda V \) such that \( ri = \text{id} \).

Felix and Halperin [2] show for rational simply connected finite type \( X \) that if \( \Lambda V \) is the Sullivan model of \( X \) then \( \text{cat} \Lambda V = \text{cat} X \).

We can make the same definitions for maps.

**Definition** [1] Given a KS extension \( f : \Lambda V \to \Lambda V \otimes \Lambda U \), consider the diagram of the last definition. Then \( \text{cat} f \leq n \) if and only if there exists a map of CDGA’s \( r : \Lambda V \otimes \Lambda W \to \Lambda V \otimes \Lambda U \) such that \( ri = f \). Also \( \text{mcat} f \leq n \) if and only if there exists a map of \( \Lambda V \) modules \( r : \Lambda V \otimes \Lambda W \to \Lambda V \otimes \Lambda U \) such that \( ri = f \). For spaces and maps between spaces the corresponding mcat \( X \) is taken to be mcat of Sullivan models.

The proof that rationally \( \text{cat}(X \times S^n) = \text{cat}(X) + 1 \) has two steps. Jessup [9] showed that \( \text{mcat}(X \times S^n) = \text{mcat}(X) + 1 \). Later Felix–Halperin–Lemaire [3] showed that in fact \( \text{mcat}(X \times Y) = \text{mcat}(X) + \text{mcat}(Y) \) for every \( X, Y \). Recently Parent [11] has used this technique to show the corresponding statement for maps, \( \text{mcat}(f \times g) = \text{mcat}(f) + \text{mcat}(g) \) for every \( f, g \). The other step was taken by Hess [5] who showed that for every \( X \), \( \text{cat} X = \text{mcat} X \). The corresponding statement for maps, however, is false. Idrissi [7] constructed a map \( f \) such that \( \text{mcat} f = 1 \) but \( \text{cat} f = 2 \). So we see that the same method of proof cannot be used to show \( \text{cat}(f \times S^n) = \text{cat} f + 1 \). Still the question of whether they are equal remained open. Such is the state of affairs in which we present:

**Theorem 4.1.** There exists a map \( f \) such that

\[
\text{cat} f = \text{cat}(f \times S^3) = 2
\]

## 2 A space with four cells.

Let \( Y \) be the graded set

\[
Y = \{a(1), a(2), u, v(1), v(2), w(1), w(2), w(1, 2), x(1), x(2), x(1, 2), x(2, 1)\}
\]

where \(|a(i)| = 3\), \(|u| = 5\), \(|v(i)| = 7\), \(|w(j)| = |w(i, j)| = 9\), and \(|x(j)| = |x(i, j)| = 11\).
Let $W$ be the graded vector space with basis $Y$ and let $(\Lambda W, d)$ be the Sullivan model with differential determined by:

\begin{align*}
    da(i) &= 0 \\
    du &= a(1)a(2) \\
    dv(i) &= a(i)u \\
    dw(i) &= a(i)v(i) \\
    dw(1, 2) &= a(1)v(2) + a(2)v(1) \\
    dx(i) &= a(i)w(i) \\
    dx(1, 2) &= a(1)w(2) + a(2)w(1, 2) \\
    dx(2, 1) &= a(2)w(1) + a(1)w(1, 2)
\end{align*}

We next give a few facts about the homology of $\Lambda W$. $\Lambda W$ corresponds to a space with four cells.

**Lemma 2.1.** $H_0(\Lambda W) \cong \mathbb{Q}$. $H_3(\Lambda W) \cong \mathbb{Q}^2$ and is generated by the image of $a(1)$ and $a(2)$. $H_{12}(\Lambda W) \cong \mathbb{Q}^2$ and generated by the image of $uv(1) - a(2)w(1)$ and $uv(2) + a(1)w(2)$. $H_i(\Lambda W) = 0$ if $i = 1, 2$ or $4 \leq i \leq 11$.

**Proof:** A straightforward calculation. $\square$

Let $W'$ be the graded vector space with basis $\{y(1), y(2)\}$, $|y(i)| = 11$. Let $(\Lambda W \otimes \Lambda W', d)$ be the KS extension of $(\Lambda W, d)$ with differential determined by setting

\begin{align*}
    dy(1) &= uv(1) - a(2)w(1) \\
    dy(2) &= uv(2) + a(1)w(2).
\end{align*}

**Lemma 2.2:** The map $\pi' : (\Lambda W \otimes \Lambda W')_{12} \longrightarrow \Lambda(a(1), a(2))/\Lambda^{>1}(a(1), a(2))$ determined by sending $a(i)$ to $a(i)$ is a weak equivalence.

**Proof:** The lemma follows directly from the definition of the differential and the homology calculations of Lemma 2.1. $\square$
3 A map for which $m\text{cat} \neq \text{cat}$

Consider the map $f : \Lambda(a(1), a(2)) \to (\Lambda W)_{12}$ determined by letting $f(a(i)) = a(i)$.

We will show that $m\text{cat}(f) = 1$ but $\text{cat}(f) = 2$. A different example of this behavior was found by Idrissi [7].

**Proposition 3.1.** $\text{cat}(f) = 2$.

**Proof:** Since $\text{cat}\Lambda(a(1), a(2)) = 2$, we know that $\text{cat}(f) \leq 2$. So it follows from Lemma 2.2 that to prove the proposition it is enough to show there does not exist a map

$$h : \Lambda W \otimes \Lambda W' \to \Lambda W$$

such that $h(a(i)) = a(i)$. But this is clear since any map $\Lambda W \to \Lambda W$ sending $a(i)$ to $a(i)$ is homotopic to the identity and then there is no choice of extension over the $y(i)$ that is compatible with the differential. \qed

**Proposition 3.2.** $m\text{cat}(f) = 1$.

**Proof:** First observe that there exists a unique homotopy class of maps of differential $\Lambda(a(1), a(2))$–modules

$$h : (\Lambda W \otimes \Lambda W')_{12} \to (\Lambda W)_{12}$$

such that $h|_W = \text{id}$, $h(uv(1)) = a(2)w(1)$, $h(uv(2)) = -a(1)w(2)$ and $h(y(1)) = h(y(2)) = 0$. This fact follows since $h$ has been defined in a way compatible with the differential on all differential $\Lambda(a(1), a(2))$–module generators in dimensions $\leq 12$. But then the proposition follows from Lemma 2.2. \qed

4 A counterexample to the rational Ganea conjecture for maps

In this section we show that the category of the map $f : \Lambda(a(1), a(2)) \to (\Lambda W)_{12}$ of the previous section is the same as the category of $f$ produced with the model of the identity map of $S^3$. 
Let $X$ be the graded set

$$X = \{ a(i), \alpha, \beta(i), \gamma(i), \gamma(i,j), \delta(i), \delta(i,k), \delta(1,2,3) \}$$

where $|a(i)| = 3$, $|\alpha| = 8$, $|\beta(i)| = 10$, $|\gamma(i)| = |\gamma(i,j)| = 12$ and $|\delta(i)| = |\delta(i,k)| = |\delta(1,2,3)| = 14$.

Let $V$ be the graded vector space with basis $X$ and let $(\Lambda V, d)$ be the Sullivan model with differential determined by:

$$
\begin{align*}
    da(i) &= 0 \\
    d\alpha &= a(1)a(2)a(3) \\
    d\beta(i) &= a(i)\alpha \\
    d\gamma(i) &= a(i)\beta(i) \\
    d\gamma(i,j) &= a(i)\beta(j) + a(j)\beta(i) \\
    d\delta(i) &= a(i)\gamma(i) \\
    d\delta(i,j) &= a(i)\gamma(j) + a(j)\gamma(i,j) \text{ where } \gamma(\{i,j\}) = \begin{cases} 
\gamma(i,j) & \text{if } j > i \\
\gamma(j,i) & \text{if } i > j 
\end{cases} \\
    d\delta(1,2,3) &= a(1)\gamma(2,3) + a(2)\gamma(1,3) + a(3)\gamma(1,2)
\end{align*}
$$

**Lemma 4.1.** The map $\pi: (\Lambda V)_{15} \longrightarrow \Lambda(a(1), a(2), a(3))/\Lambda^2(a(1), a(2), a(3))$ determined by sending $a(i)$ to $a(i)$ is a weak equivalence.

**Proof:** Straightforward.

We define a map of algebras

$$g: \Lambda V \longrightarrow \Lambda W \otimes \Lambda(a(3))$$

on generators in the following way:
\[ga(i) = a(i)\]
\[g\alpha = u a(3)\]
\[g\beta(1) = v(1)a(3)\]
\[g\beta(2) = v(2)a(3)\]
\[g\beta(3) = 0\]
\[g\gamma(1) = w(1)a(3)\]
\[g\gamma(2) = w(2)a(3)\]
\[g\gamma(3) = 0\]
\[g\gamma(1, 2) = w(1, 2)a(3)\]
\[g\gamma(2, 3) = 0\]
\[g\gamma(1, 3) = 0\]
\[g\delta(1) = x(1)a(3)\]
\[g\delta(2) = x(2)a(3)\]
\[g\delta(3) = 0\]
\[g\delta(1, 2) = x(1, 2)a(3)\]
\[g\delta(2, 1) = x(2, 1)a(3)\]
\[g\delta(1, 3) = 0\]
\[g\delta(3, 1) = 0\]
\[g\delta(2, 3) = 0\]
\[g\delta(3, 2) = 0\]
\[g\delta(1, 2, 3) = 0\]

**Lemma 4.2.** The map \(g\) commutes with the differential and is therefore a map in CDGA.

**Proof:** A straightforward calculation.

**Theorem 4.1.** Let \(|a(3)| = 3\) and consider \(f \otimes \text{id} : \Lambda(a(1), a(2)) \otimes \Lambda(a(3)) \rightarrow (AW)_{12} \otimes \Lambda(a(3))\) then \(\text{cat}(f) = \text{cat}(f \times \text{id}) = 2\).

**Proof:** That \(\text{cat}(f) = 2\) is Proposition 3.1. This also implies \(\text{cat}(f \otimes \text{id}) \geq 2\). So we proceed to show \(\text{cat}(f \otimes \text{id}) \leq 2\). In light of Lemma 4.1, we then need only to exhibit a commutative diagram.
But the map $g : AV \to AW \otimes \Lambda(a(3))$ determines a map

$$AV \to AW \otimes \Lambda(a(3)) \to (AW)_{12} \otimes \Lambda(a(3)).$$

This then gives the required map $\phi$ since $(AW)_{12} \otimes \Lambda(a(3)) \cong ((AW)_{12} \otimes \Lambda(a(3)))_{15}$. □

**References**


