TRANSFERS OF CHERN CLASSES IN BP-COHOMOLOGY AND CHOW RINGS

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Abstract. The $BP^*$-module structures of $BP^*(BG)$ for extraspecial 2-groups are studied using transfer and Chern classes. These give rise to $p$-torsion elements in the kernel of the cycle map from the Chow ring to ordinary cohomology first obtained by Totaro.

1. Introduction

Let $G$ be a compact Lie group, e.g. a finite group, and $h^*(BG)$ a good generalized cohomology theory on the classifying space $BG$ of $G$. Here "good" shall mean that $h^*(BU(m))$ is isomorphic to $h^* \otimes H^*(BU(m))$ for the unitary groups $U(m)$. Then we can define Chern classes in $h^*(BG)$ for complex representations of $G$, and also transfer maps. We are interested in the Mackey closure $\overline{Ch}_h(G)$ of the ring of Chern classes in $h^*(BG)$, namely the subring of $h^*(BG)$ generated by transfers of Chern classes.

For ordinary mod $p$ cohomology, Green-Leary [G-L] showed that the inclusion map $i : \overline{Ch}_{HZ/p} \hookrightarrow H^*(BG;\mathbb{Z}/p)$ is an $F$-isomorphism, i.e., the induced map of varieties is a homeomorphism. Green-Minh [G-M] however noticed that $i/\sqrt{p}$ need not be an isomorphism in general. Next consider $h = BP$ or $h = K(n)$, the $n$-the Morava K-theory, at a fixed prime $p$. Following Hopkins-Kuhn-Ravenel [H-K-R], we shall call a group $G$ "good" for $h$-theory if $h^*(BG)$ is generated (as an $h^*$-module) by transferred Euler classes of representations of subgroups of $G$. It is clear that if the Sylow $p$-subgroup of $G$ is good, then so is $G$ and one has an isomorphism $h^*(BG) \cong \overline{Ch}_h(G)$. Furthermore, it follows from [R-W-Y] that $G$ is good for $BP$ if it is good for $K(n)$ for all $n$. Examples for groups that are $K(n)$-good for all $n$ are the finite symmetric groups. Another typical case are $p$-groups of $p$-rank at most 2 and $p \geq 5$: in [Y] it is shown that the Thom map $\rho : BP^*(-) \to H^*(-)(p)$ induces an isomorphism $BP^*(BG) \otimes_{BP^*} Z(p) \cong H^{even}(BG)$. Note however that I. Kriz claimed that $K(n)^{odd}(BG) \neq 0$ for some $p$-groups $G$.

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On the other hand, B. Totaro [T1] found a way to compare $BP$-theory to the Chow ring. For a complex algebraic variety $X$, the groups $CH^i(X)$ of codimension $i$ algebraic cycles modulo rational equivalence assemble to the Chow ring $CH^*(X) = \sum_i CH^i(X)$. Totaro constructed a map $\tilde{\rho} : CH^i(X) \to BP^*(X) \otimes_{BP^*} \mathbb{Z}(p)$ such that the composition

$$\tilde{\rho} : CH^i(X)_{(p)} \xrightarrow{\tilde{\rho}} BP^*(X) \otimes_{BP^*} \mathbb{Z}(p) \xrightarrow{\rho} H^*(X)_{(p)}$$

coincides with the cycle map. One of the main results of [T1] is that there exists a group $G$ for which the kernel of $\tilde{\rho}$ contains $p$-torsion elements. To prove this, Totaro defined the Chow ring of a classifying space $BG$ as $\lim_{m \to \infty} CH^*((\mathbb{C}^m - S)/G)$ where $G$ acts on $\mathbb{C}^m - S$ freely and $\text{codim}(S) > 0$. He then constructed a non-zero element $x$ in $\text{Ker}(\rho)$ such that

$$x \in \overline{CH}_{BP}(BG) \cap (BP^*(BG) \otimes_{BP^*} \mathbb{Z}(p)).$$

Since transfers and Chern classes also exist in the Chow ring $CH^*(BG)$, there is an element $\bar{x} \in \overline{CH}(G)$ that also lies in $\text{Ker}(\tilde{\rho})$. The group Totaro uses is $G = \mathbb{Z}/2 \times D^{1+4}_+$, where $D^{1+4}_+ = D(2)$ is the extraspecial 2-group of order 32, which is isomorphic to the central product of two copies of the dihedral group $D_8$ of order 8. He first proves that there exists an element $x \in BP^*(BD(2))$ satisfying (1.1) but which restricts to zero under the map $\rho_{\mathbb{Z}/2} : BP^*(-) \to H^*(-; \mathbb{Z}/2)$, where he uses the computation of $BP^*(BSO(4))$ from [K-Y].

Let $D(n) = 2^{1+2n}_+$ denote the extraspecial 2-group of order $2^{2n+1}$; it is isomorphic to the central product of $n$ copies of $D_8$. In this paper, we construct non-zero elements $x \in BP^*(BD(n))$ satisfying (1.1) but with $\rho_{\mathbb{Z}/2}(x) = 0$ directly for each $n$.

Let $\hat{W}$ be a maximal elementary abelian 2-subgroup and $N$ the center of $D(n)$. For a one-dimensional real representation $e$ of $\hat{W}$ restricting non-trivially to the center, set $\Delta = \text{Ind}_{\hat{W}}^{D(n)}(e)$. This is the unique irreducible representation which acts non-trivially on $N$. Then the $i$-th Stiefel-Whitney class $w_i(\Delta)$ for $i < 2^n$ can be written as a polynomial in variables $w_i(e_j), 1 \leq j \leq 2n$, for 1-dimensional representations $e_j$ of $D(n)/N$ ([Q], Remark 5.13), i.e. $w_i(\Delta) = w_i(e_1), ..., w_i(e_{2n})$. Let $e'_C$ denote the complex representation induced from the real representation $e'$. Then we can prove that

$$x = c_{2n-1}(\Delta_C) - w_{2n-1}(c_1(e_{1C}), ..., c_1(e_{2nC}))$$

satisfies (1.1) together with $\rho_{\mathbb{Z}/2}(x) = 0$, and furthermore conclude $\text{Ker}(\rho) \neq 0$ for $G = \mathbb{Z}/2 \times D(n)$. 


Secondly, we construct a non-nilpotent element

\[ x \in \text{Ker}(\rho) \cap (BP^*(BG) \otimes_{BP} \mathbb{Z}(\rho)) \]  

which is not in \( \text{Ch}_{BP}(BG) \). However we do not know whether \( x \) comes from the Chow ring or not, and we only obtain the result for \( n = 3, 4 \). Set

\[ x = [v_1 \otimes w_2^*(\Delta)] \]  

(1.4)

to be the class represented by \( v_1 \otimes w_{2n}(\Delta) \) in the \( E_\infty \)-page of the Atiyah-Hirzebruch spectral sequence. We can prove that \( d_{2n+1}(w_{2n}(\Delta)) = v_{n-1} \otimes Q_{n-1}(w_{2n}(\Delta)) \neq 0 \) and \( v_1 \otimes Q_{n-1}(w_{2n}(\Delta)) \in \text{Im}(d_3) \). Furthermore, restricting to the center of \( D(n) \) we see that \( x \not\in \text{Ch}_{BP}(BG) \). However, it seems difficult to see that this cycle is permanent. For the case \( n = 3, 4 \), we use the \( BP \)-theory of \( B\text{Spin}(7) \) and \( B\text{Spin}(9) \) computed in [K-Y].

These arguments do not seem to work for other extraspecial 2-groups nor 2-groups that have a cyclic maximal normal subgroup [S].

In Section 2, we recall the mod 2 cohomology of extraspecial 2-groups following [Q]. In particular, \( w_{2n-2}(\Delta) \) is represented by the Dickson invariant \( D_i \), and we study the action of the Milnor primitives \( Q_j \) on \( D_i \).

To see \( \rho(x) \neq 0 \) in \( H^*(BD(n); \mathbb{Z}) \), we recall the integral cohomology in Section 3. In Section 4, we show that \( x \) satisfies (1.1). In Section 5, we study how elements in \( \text{Ker}(\rho) \) are represented in the Atiyah-Hirzebruch spectral sequence, explaining the easiest case \( h^* = \mathbb{Z}/2[\nu_{n-2}, \nu_{n-1}] \). The element \( x \) in (1.4) is proved not to be in \( \text{Ch}_{BP}(BD(n)) \) in Section 6. In the last section the element \( x \) in (1.4) is proved to be a permanent cycle in the Atiyah-Hirzebruch spectral sequence for \( n = 3, 4 \) by comparing the spectral sequence to the corresponding spectral sequence for \( H^*(B\text{Spin}(2n + 1)) \).

## 2. Extraspecial 2-groups

The extraspecial 2-group \( D(n) = 2^{1+2n}_+ \) is the central product of \( n \) copies of the dihedral group \( D_8 \) of order 8. So there is a central extension

\[ 0 \to N \to D(n) \xrightarrow{\pi} V \to 0 \]  

(2.1)
with \( N \cong \mathbb{Z}/2 \) and \( V \) elementary abelian of rank \( 2n \). Take a set of generators \( c, \tilde{a}_1, \ldots, \tilde{a}_{2n} \) of \( D(n) \) such that \( c \) is a generator of \( N \), the elements \( a_i = \pi(\tilde{a}_i) \) form a \( \mathbb{Z}/2 \)-basis of \( V \), and

\[
[\tilde{a}_j, \tilde{a}_{2i}] = \begin{cases} 
  c & \text{if } j = 2i - 1 \\
  0 & \text{else}
\end{cases}
\]

Using the Hochschild-Serre spectral sequence associated to extension (2.1), Quillen [Q] determined the mod 2 cohomology of \( D(n) \). Let \( e_i \) denote the real 1-dimensional representation of \( D(n) \) given as the projection onto \( \langle a_i \rangle \) followed by the nontrivial character \( \langle a_i \rangle \rightarrow \{ \pm 1 \} \subset \mathbb{R} \), and \( e : \mathbb{V}^{\text{odd}} \rightarrow N \rightarrow \{ \pm 1 \} \subset \mathbb{R} \) where \( \mathbb{V}^{\text{odd}} = \langle c, \tilde{a}_{2i-1} \mid 1 \leq i \leq n \rangle \) is a maximal elementary abelian 2-subgroup of \( D(n) \). Define classes \( x_i \in H^1(D(n); \mathbb{Z}/2) \), \( w_{2n} \in H^{2n}(D(n); \mathbb{Z}/2) \) as the Euler classes of the \( e_i \) and of \( \Delta = \text{Ind}_{\mathbb{V}^{\text{odd}}}^{D(n)}(e) \), respectively. The extension (2.1) is represented by the class \( f = x_1x_2 + \cdots + x_{2n-1}x_{2n} \), and one has

\[
(2.2) \quad H^*(BD(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2n}] \otimes \mathbb{Z}[x_1, \ldots, x_{2n}]/(f, Q_0f, \ldots, Q_{n-2}f)
\]

where the \( Q_i \) are Milnor’s operations recursively defined by \( Q_0 = Sq^1 \) and \( Q_i = [Sq^{2i}, Q_{i-1}] \).

The extension class \( f \) defines a quadratic form \( q : V \rightarrow \mathbb{Z}/2 \) on \( V \). A subspace \( W \subset V \) is said to be \( q \)-isotropic if \( q(x) = 0 \) for all \( x \in W \). The maximal (elementary) abelian subgroups of \( D(n) \) are in one-to-one correspondence with the maximal isotropic subspaces of \( V \). Indeed, if \( W \) is maximal isotropic, then \( \mathbb{V} := \pi^{-1}(W) \cong N \oplus W \) is maximal (elementary) abelian. Quillen also proved that the mod 2 cohomology of \( D(n) \) is detected on maximal elementary abelian subgroups, i.e. the restrictions define an injective map

\[
(2.3) \quad H^*(BD(n); \mathbb{Z}/2) \hookrightarrow \prod H^*(\mathbb{V}; \mathbb{Z}/2)
\]

where the product ranges over conjugacy classes of maximal elementary abelian subgroups. Since the restriction of \( \Delta \) to any such \( \mathbb{V} \) is the real regular representation (see [Q], Section 5), we have

\[
(2.4) \quad \text{Res}_{\mathbb{V}}(w_{2n}) = \prod_{x \in H^2(W; \mathbb{Z}/2)} (z + x)
\]

where \( z \) denotes the generator of \( H^*(N; \mathbb{Z}/2) \) dual to \( c \). For simplicity, write \( w' = \text{Res}_{\mathbb{V}} w_{2n} \), and choose generators of \( H^*(W; \mathbb{Z}/2) \cong \mathbb{Z}/2[x'_1, \ldots, x'_n] \). It is well-known that the right hand side of (2.4) can be written in terms of Dickson invariants,

\[
(2.5) \quad w' = z^{2n} + D_1z^{2n-1} + \cdots + D_nz
\]
where \(D_i\) has degree \(2^n - 2^{n-i}\) and \(H^*(W; \mathbb{Z}/2)^{GL_2(\mathbb{Z}/2)} \cong \mathbb{Z}/2[D_1, \ldots, D_n]\). Using that the product of all the \(x'_i\)'s is clearly invariant and that the Milnor primitives are derivations, it is easy to see that the Dickson invariants may be written in terms of the \(Q_i\) as follows:

\[
\begin{align*}
D_n &= Q_0Q_1 \cdots Q_{n-2}(x'_1 \cdots x'_n) \\
D_i &= (Q_0 \cdots Q_{n-i-1}Q_{n-1}(x'_1 \cdots x'_n))/D_n
\end{align*}
\]

**Lemma 2.1.** The Milnor operations act by

1. \(Q_{n-1}D_i = D_nD_i\);
2. \(Q_{n-j-1}D_j = D_n\);
3. \(Q_iD_j = 0\) for \(i < n - 1\) and \(i \neq n - j - 1\).

**Proof.** First note that from (2.6) and \(Q_k^2 = 0\) we immediately get \(Q_k(D_n) = 0\) for \(k \neq n - 1\) and \(Q_{n-1}D_n = Q_0 \cdots Q_{n-1}(x'_1 \cdots x'_n) = D_n^2\). Thus, for each \(1 \leq i \leq n - 1\),

\[
\begin{align*}
0 &= Q_{n-1}(Q_0 \cdots Q_{n-i-1} \cdots Q_{n-1}) (x'_1 \cdots x'_n) = Q_{n-1}(D_iD_n) \\
&= (Q_{n-1}D_i)D_n + D_iQ_{n-1}D_n = (Q_{n-1}D_i)D_n + D_iD_n^2
\end{align*}
\]

whence (1). Similarly, (2) is implied by

\[
\begin{align*}
D_n^2 &= Q_0 \cdots Q_0(x'_1 \cdots x'_n) = Q_{n-i-1}(D_iD_n) \\
&= (Q_{n-i-1}D_i)D_n + D_iQ_{n-i-1}D_n = (Q_{n-i-1}D_i)D_n.
\end{align*}
\]

Finally, for \(k \neq n - i - 1\) we get \(0 = Q_k(D_iD_n) = (Q_kD_i)D_n + D_iQ_kD_n = (Q_kD_i)D_n\).

**Corollary 2.2.** \(Q_{n-1}w' = D_nw'\) and \(Q_kw' = 0\) for \(k < n - 1\).

**Proof.** For \(j \neq n - 1\), we have \(Q_jw' = \sum_{i=1}^{n-1}(Q_jD_i)z^{2^{n-i}} + Q_j(D_nz) = D_nz^{2^{j+1}} + D_nz^{2^{j+1}} = 0\). For \(j = n - 1\), we get \(Q_{n-1}w' = 0 + D_nD_1z^{2^{n-1}} + \cdots + D_nD_{n-1}z^2 + Q_{n-1}(D_nz)\). The last term equals \(D_n^2z + D_nz^{2^n}\); the claim follows.

**Corollary 2.3.** \(Q_kw_{2^n} = 0\) for \(0 \leq k \leq n - 2\), but \(Q_{n-1}w_{2^n} \neq 0\).

### 3. The integral cohomology

The integral cohomology of \(D(n)\) is studied by Harada-Kono ([H-K], also see [B-C]) by means of the Bockstein spectral sequence

\[
E_1 = H^*(BG; \mathbb{Z}/2) \implies \mathbb{Z}/2 \otimes H^*(BG)/(2\text{-torsion}).
\]
Harada-Kono computed the $E_2$-page for $D(n)$ as follows:

$$H(H^*(BD(n);\mathbb{Z}/2);Q_n) \cong \Lambda(a,b_1,\ldots,b_{n-1}) \otimes \mathbb{Z}/2[w_2^n]$$

where $|a|=3$ and $|b_i|=2^i$. Since $E_\infty \cong \mathbb{Z}/2$, the first non-trivial differential must be $da = b_1$, and there have to be subsequent differentials $d(ab_i) = b_{i+1}$. Thus there appear exactly $n$ non-zero differentials in this spectral sequence. On the other hand, using corestriction arguments it is easy to see that the exponent of $H^*(BD(n))$ is at most $n+1$. Based on these facts, Harada-Kono proved the following.

**Theorem 3.1.** [H-K] Let $C(n)^* = H^*(BD(n))/H^*(BV)$. Then $C(n)^* \subset H^*(BD(n))$, and there is an additive isomorphism

$$C(n)^k = \begin{cases} \mathbb{Z}/2^{\nu_2(k)} & \text{if } \nu_2(k) \leq n-1, \\ \mathbb{Z}/2^{n+1} & \text{if } \nu_2(k) = n \end{cases}$$

where $\nu_2(k)$ denotes the 2-adic valuation of $k$. \hfill $\square$

Let $c_k(n)$ denote a $\mathbb{Z}/(2)$-module generator of $C(n)^{2^k}$. Then $c_n(n)$ reduces to $w_2^n$ modulo $H^*(BV;\mathbb{Z}/2)$. Consider the restriction map $i: C(n)^* \to C(n-1)^*$. Now $c_{n-1}(n-1) = w_2^{n-1}$ mod $H^*(BV;\mathbb{Z}/2)$ implies $i^*c_n(n) = c_{n-1}(n-1)^2$. Since the order of $c_{n-1}(n)$ is $2^{n-1}$ and the order of $c_{n-1}(n-1)$ is $2^n$, we know that $i^*c_n(n) = 2^s c_{n-1}(n-1)$ for some $s>0$. A corestriction argument now implies $s=1$, since the index of $D(n-1)$ in $D(n)$ is 2.

The elements $a$ and $b_i$ are natural in the sense that $i^*(a) = a$ and $i^*(b_j) = b_j$ for $1 \leq j \leq n-2$, abusing notation. Thus $i^*c_j(n) = c_j(n-1)$ for $j < n-1$, and we obtain

**Corollary 3.2.** If $n \geq 2$, there is an additive isomorphism

$$C(n)^* \cong \mathbb{Z}\{1,2\bar{w}_2^{2^i}\cdots\bar{w}_2^{2n-1} | \epsilon_i = 0 \text{ or } 1\}/(2^{i+1}\bar{w}_2, = 0 | 2 \leq i \leq n)$$

where the $\bar{w}_2$, are the reductions of the elements $w_2$ in $H^{2^i}(BD(i))$. \hfill $\square$

**Remark.** When $n = 1$, the element $w_2 \in H^*(BD_8;\mathbb{Z}/2)$ does not lift to the integral cohomology and $C(1)^* \cong \mathbb{Z}[^2\bar{w}_2]/(4\bar{w}_2^2)$. 
4. Brown-Peterson Cohomology of $BD(n)$

Let $BP^*(-;\mathbb{Z}/2)$ denote $BP$-theory mod 2 with coefficients $BP^*(2) = \mathbb{Z}/2[v_1,v_2,\ldots]$. We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BD(n);\mathbb{Z}/2) \otimes BP^* \Rightarrow BP^*(BD(n);\mathbb{Z}/2).$$

**Lemma 4.1.** The elements $x_i^2$ and $w_2^n$ are permanent cycles in the spectral sequence (4.1).

**Proof.** These elements are the top Chern classes of the representations $e_1 c$ and $\Delta c$, respectively. \hfill \Box

It is well-known that some of the differentials of (4.1) are given by

$$d_{2i+1-1}(x) = v_i \otimes Q_i x \mod (v_1,\ldots,v_{i-1}).$$

Since $Q_n w_2^n \neq 0$ by Corollary 2.3, we know that $w_2^n$ cannot be a permanent cycle, which implies $w_2^n \not\in \text{Im}(\rho_{\mathbb{Z}/2}: BP^*(BD(n)) \to H^*(BD(n);\mathbb{Z}/2))$. Thus the integral lift $\bar{w}_2^n$ of $w_2^n$ does not lie in the image of $\rho: BP^*(BD(n)) \to H^*(BD(n))$, either.

Let as above $\bar{W}$ denote a maximal elementary abelian subgroup of $D(n)$, and $w(\Delta)$ the total Stiefel-Whitney class of $\Delta$. Then

$$\text{Res}_{\bar{W}}^{D(n)}(w(\Delta)) = \prod (1 + x + z) = (1 + z)^{2^n} + D_1 (1 + z)^{2^{n-1}} + \cdots + D_n (1 + z)$$

$$= 1 + D_1 + \cdots + D_n + \text{Res}_{\bar{W}}^{D(n)}(w_2^n);$$

in particular,

$$\text{Res}_{\bar{W}}^{D(n)}(w_{2^n-1}(\Delta)) = D_1.$$ 

Hence, by (2.2), we can choose polynomials $\tilde{D}_i \in \mathbb{Z}/2[x_1,\ldots,x_{2n}] \cong H^*(BV;\mathbb{Z}/2)$ with $w_{2^n-1} = \tilde{D}_1$.

**Theorem 4.2.** There is a $BP^*$-module generator

$$x = c_{2^{n-1}}(\Delta c) - \tilde{D}_1(c_1(e_1 c),\ldots,c_1(e_{2n} c)) \in BP^*(BD(n))$$

such that

1. $\rho(x) = 2\bar{w}_2^n \mod H^*(BV)$,
2. $\rho_{\mathbb{Z}/2}(x) = 0$ in $H^*(BD(n);\mathbb{Z}/2)$. 

Proof. Since $x$ is defined via Chern classes, it is an element of $BP^*(BD(n))$. Assertion (2) is immediate from (4.3). Since $\bar{w}_2 \not\in \text{Im}(\rho)$, it suffices to prove (1) to show that $x$ is a $BP^*$-module generator. Let $F = \langle a_1a_2 \rangle \subset D(n)$; this is cyclic of order 4. By the double coset formula,

$$\text{Res}_F^{D(n)} \text{Ind}_{V_{odd}}^D(e_C) = \bigoplus_{FgV_{odd}} \text{Ind}_{F \cap g V_{odd}}^F \text{Res}_{F \cap g V_{odd}}^{gV_{odd}} (g^*e_C)$$

since the elements $g = a_2^i a_2^{2n}$, $i = 0$ or 1, form a complete set of double coset representatives. Notice that $\text{Ind}_N^H(e_C)$ decomposes as $e_F \oplus -e_F$ where $e_F$ is a faithful 1-dimensional complex representation of $\mathbb{Z}/4$. Thus the total Chern class of $\Delta_C$ restricts to $F$ as

$$\text{Res}_F(c(\Delta_C)) = ((1 + u)(1 - u))^{2n-1} = (1 - u^2)^{2n-1} \quad \text{with } H^*(BF) \cong \mathbb{Z}[u]/(4u).$$

Consequently, we have $\text{Res}_F(c_{2n-1}(\Delta_C)) = 2u^{2n-1}$ in $H^*(F)$. Since $\text{Res}_F(c_1(e_C)) = 2\lambda_i u$ for some $\lambda_i \in \mathbb{Z}/4$, we immediately obtain $\text{Res}_F(D) = 0$ and therefore (1). \hfill $\Box$

Now recall the following lemma of Totaro.

**Lemma 4.3.** ([T1]) Let $p$ be a prime and $X$ any space. If $\rho_{\mathbb{Z}/p} : BP^*(X) \otimes_{BP^*} \mathbb{Z}_p \rightarrow H^*(X; \mathbb{Z}/p)$ is not injective, then $\rho : BP^{*+2}(X \times \mathbb{Z}/p) \otimes_{BP^*} \mathbb{Z}_p \rightarrow H^{*+2}(X \times \mathbb{Z}/p)$ is also not injective. \hfill $\Box$

Let $\rho' : CH^*(-) \rightarrow H^*(-)$ denote the cycle map respectively $\rho'_{\mathbb{Z}/2}$ the cycle map followed by reduction modulo 2. Since Chow rings have Chern classes, we easily deduce

**Corollary 4.4.** There is a non-zero element $x'$ in $CH^{2n}(BD(n))$ satisfying

1. $\rho'(x') = 2\bar{w}_{2n} \mod H^*(BV)$;
2. $\rho'_{\mathbb{Z}/2}(x') = 0$.

Hence $\rho' : CH^{2n+2}(B(D(n) \times \mathbb{Z}/2)) \rightarrow H^{2n+2}(B(D(n) \times \mathbb{Z}/2))$ is not injective. \hfill $\Box$

**Remark.** First note that the above argument does not hold for $n = 1$. Indeed, in that case $H^*(BD_8) \subset \text{Im}(\rho)$ modulo $H^*(BV)$. Similar facts hold for 2-groups $G$ which have a cyclic maximal normal subgroup $[S]$, i.e. dihedral, semidihedral, quasidihedral, and generalized quaternion groups of order a power of 2. Moreover $BP^*(BG)$ is generated by Chern classes for these groups. The extraspecial 2-groups of order $2^{2n+1}$ are of two types. Quillen calls them the real and the quaternionic type, where the real type corresonds to the groups $D(n)$.
considered above, and the quaternionic group of order $2^{n+1}$ is the central product of $D(n-1)$ with the quaternion group $Q_8$ of order 8. Consider now this second case, and denote this group by $D'(n)$; it also has center $\mathbb{Z}/2$ with quotient $V \cong (\mathbb{Z}/2)^{2n}$. In Quillen’s notation $[Q]$, this corresponds to $h = n + 1$ and $r = 2$. The quadratic form (extension class) is

$$f = x_1^2 + x_1x_2 + x_2^2 + \sum_{i=2}^{n} x_{2i-1}x_{2i},$$

and the cohomology is given by

$$H^*(BD'(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2n+1}] \otimes \mathbb{Z}/2[x_1, \ldots, x_{2n}]/(f, Q_0f, \ldots, Q_{n-1}f).$$

Here the $x_i$ are as before the generators of $H^*(BD(n); \mathbb{Z})$ inflated to $D'(n)$, and $w_{2n+1}$ is the Euler class of the $2^{n+1}$-dimensional irreducible representation $\Delta$. The cohomology of $D'(n)$ is also detected on subgroups $\tilde{W} \cong Q_8 \times W$ in one-to-one correspondence with maximal isotropic subspaces, i.e. there is an injection

$$H^*(BD'(n); \mathbb{Z}/2) \twoheadrightarrow \prod_W H^*(B(Q_8 \times W); \mathbb{Z}/2)$$

where $W$ ranges over the maximal isotropic subspaces of $V$ (which have dimension $n-1$). The Stiefel-Whitney classes $w_j(\Delta)$ are zero except for the following values of $j$ ([Q], (5.6)):

$$\text{Res}_{Q_8 \times W}(w_j(\Delta)) = \begin{cases} (D'_i)^4 & \text{for } j = 2^h - 2^{h-i}, 1 \leq i \leq n-1 \\ \sum_{i=0}^{n-2} e^{2i}(D'_{n-i-1})^4 & \text{for } j = 2^{n+1} \end{cases}$$

where $e \in H^4(Q_8; \mathbb{Z}/2)$ is the Euler class of the obvious 4-dimensional irreducible representation of $Q_8$ and $D'_i$ is the degree $(2^{n-1} - 2^{n-1-i})$ Dickson invariant for rank $n-1$. Thus almost all arguments for $D(n)$ work in this case, too, except for $Q_m w_j(\Delta) = 0$. For example, we can define $x = c_{2^n}(e \xi) - (\tilde{D}_1)^4$ in $BP^*(BD'(n))$; this class satisfies $\rho(x) = 2\tilde{w}_{2n+1}$ and $\rho_{\mathbb{Z}/2}(x) = 0$. However, it seems that we can not prove that $x$ is a $BP^*$-module generator of $BP^*(BD'(n))$ because $\text{Res}_N(c_{2^n}(\text{Ind}_{\mathbb{Z}/2})^4(e_F))) = u^{2^n}$ and $w_{2n+1}(\Delta) \in \text{Im}(\rho)$ mod $(H^*(BV))$.

5. Permanent cycles

This section is concerned with the following statement:

**Assumption 5.1.** In the Atiyah-Hirzebruch spectral sequence converging to $BP^*(BD(n))$, every element in $(2, v_1, \ldots, v_{n-1}) \otimes \tilde{w}_{2^n}$ is a permanent cycle.
Unfortunately, we can not prove this in full generality, only a rather weak version (covering the cases \( n = 3, 4 \)), which nevertheless seems to justify why we expect such classes to be permanent.

Let \( BP(\mathbb{Z}, n-1) \) denote the cohomology theory with coefficients \( BP(\mathbb{Z}, n-1) = \mathbb{Z}/(p[v_{n-2}, v_{n-1}] \). Then there are natural transformations

\[
BP^*(-) \longrightarrow BP(\mathbb{Z}, n-1)^*(-) \longrightarrow H^*(-; \mathbb{Z}/2) .
\]

**Proposition 5.2.** In the Atiyah-Hirzebruch spectral sequence converging to \( BP(\mathbb{Z}, n-1)^*(BD(n)) \), every element in \( (v_{n-2}) \otimes w_{2n} \) is a permanent cycle.

**Proof.** First note that \( w_{2n} \) is not in the image of \( Q_{n-2} \), which is easily seen by restricting to \( N \): \( Res_N(w_{2n}) = z^{2^n} = Q_{n-2} 2^{2^{n-1}+1} \), whereas the image of \( Res_{D(n)}^N \) is generated by \( z^{2^n} \).

This means that no element in \( (v_{n-2}) \otimes w_{2n} \) lies in the image of the first potentially non-zero differential \( d_2n-1 \). The next non-zero differential is \( d_{2n-3} = v_{n-2}^2 \otimes - \). Dimensional reasons easily imply that \( (v_{n-2}) \otimes w_{2n} \notin \text{Im}(d_r) \) for any \( r \).

Suppose first that \( d_r w_{2n} = 0 \) for all \( r < 2^n - 1 \). Then \( d_{2n-1} w_{2n} = v_{n-1} \otimes Q_{n-1} w_{2n} \neq 0 \). Thus by naturality of the spectral sequence and Corollary 2.2, we see that

\[
\text{Res}_{W}(d_{2n-1} w_{2n}) = v_{n-1} \otimes Q_{n-1} w' = v_{n-1} \otimes D_n w' .
\]

Since there are classes \( \tilde{D}_i \in H^*(BD(n); \mathbb{Z}/2) \) which restrict to \( D_i \) on each \( \tilde{W} \), Quillen’s detection result (2.3) shows

\[
d_{2n-1} w_{2n} = v_{n-1} \otimes \tilde{D}_n w_{2n} . \tag{5.1}
\]

Furthermore, \( \tilde{D}_n w_{2n} \) is \( v_{n-2} \)-torsion, since \( d_{2n-1} \tilde{D}_i = v_{n-2} \otimes \tilde{D}_n \), whence \( v_{n-2} \otimes w_{2n} \) persists to \( E_{2n+1}^* \). So assume now that \( d_r w_{2n} \neq 0 \) for some \( r < 2^n - 1 \). For dimensional reasons again, the only possibility for such a differential is

\[
d_{2n-3} w_{2n} = v_{n-2}^2 \otimes (aw_{2n} + b) \quad \text{with} \quad a, b \in H^*(BV; \mathbb{Z}/2) . \tag{5.2}
\]

Note that the subgroup of the automorphisms of \( G \) stabilizing the center \( N \) is the orthogonal group \( O(V) \) of \( V \) associated to the quadratic form \( q \) ([B-C], p. 216). Since \( \Delta \) is the unique irreducible representation which acts non-trivially on \( N \), the element \( w_{2n} \) is invariant under the orthogonal group ([Q], Remark 4.7). Let as before \( V^{odd} = \mathbb{Z}/2 \{ a_{2i-1} \mid 1 \leq i \leq n \} \), and \( V^{ev} = \mathbb{Z}/2 \{ a_{2i} \mid 1 \leq i \leq n \} \). Both are maximal isotropic subspaces, and \( V = V^{odd} \oplus V^{ev} \).
Then

\[(5.3) \quad O(V) = \{g \oplus t^g^{-1} \mid g \in \text{GL}_n(\mathbb{Z}/2)\}\]

(where \(t\) is transposition).

Any maximal \(q\)-isotropic subspace \(W\) has dimension \(n\). Interchanging \(a_{2i-1}\) with \(a_{2i}\) if necessary, we can turn the projection

\[pr: W \subset V = V^{\text{odd}} \oplus V^{\text{ev}} \longrightarrow V^{\text{odd}}\]

into an isomorphism (see \([Q]\), p. 201). Consider the commutative diagram

\[(5.4) \quad \begin{array}{ccc}
V^{\text{odd}} & \longrightarrow & W \\
\downarrow g & & \downarrow g \\
V^{\text{odd}} & \longrightarrow & gW \\
& & \downarrow \\
& & D(n)
\end{array}\]

Let \(a\) be as in (5.2) and suppose \(\text{Res}_W(a) \neq 0\). Since \(pr^*: H^*(BV^{\text{odd}}; \mathbb{Z}/2) \cong H^*(BW; \mathbb{Z}/2)\), we have \((pr^*)^{-1}(a) \neq 0\) in \(H^*(BV^{\text{odd}}; \mathbb{Z}/2)\). The image of \(a\) in \(H^*(BD(n); \mathbb{Z}/2)\) is invariant under the action of \(O(V)\). Hence \((pr^*)^{-1}(a)\) is invariant under the GL\(_n\)-action, by (5.3).

But \(|a| = 2^n - 3\), and there is no invariant of that degree, whence \(a = 0\). Using similar reasoning, we can prove \((pr^*)^{-1}(b) = D_{n-1}D_n\). But in the spectral sequence converging to \(BP(n-2, n-1)^*(BD(n))\), there is the differential \(d_{2^n-1}(\bar{D}_{n-1}) = v_{n-2} \otimes \bar{D}_{n-1}\).

Hence we can remove \(v_{n-1} \otimes b\) from (5.2).

Finally, we shall prove \(d_r(v_{n-1} \otimes w_{2^n}) = 0\) for all \(r \geq 2^n\). Suppose

\[d_r(v_{n-1} \otimes w_{2^n}) = \sum v_{n-2}^kv_{n-1}^la_{kl} \quad \text{for some } a_{kl} \in H^*(BD(n); \mathbb{Z}/2).\]

Then each \(a_{kl}\) is invariant under the action of \(O(V)\), and moreover the degrees of the \(a_{kl}\) are odd. Arguing as above, we see that

\[(5.5) \quad \text{Res}_{\tilde{V}^{\text{odd}}}(a_{kl}) = \sum D_I \quad \text{with } D_I \in (D_n)\]

where \(D_I = D_{i_1} \cdots D_{i_m}\), \(1 \leq i_j \leq n + 1\), identifying \(D_{n+1}\) with \(w_{2^n}\). From \(Q_{n-1}D_i = D_iD_n\) we have \(Q_{n-1}D_I = mD_ID_n\). Hence if \(Q_{n-1}D_I \neq 0\), the number of factors must be odd.

Thus we assume the number of \(D_i\)'s in (5.5) is even. But for each such \(D_I\) there is a corresponding \(\tilde{D}_I \in H^*(BD(n); \mathbb{Z}/2)\), and all those \(\tilde{D}_I\) are \((v_{n-2}, v_{n-1})\)-torsion. Thus \(d_r(v_{n-1} \otimes w_{2^n}) = 0\) for dimensional reasons. \(\square\)
6. Transfers of Chern classes

To study Chern classes, we consider the restriction to the center $N \cong \mathbb{Z}/2$ of $D(n)$. Let $I$ denote the ideal $(2, v_1, v_2, \ldots)$ in $BP^*$. Then

$$\rho_{\mathbb{Z}/2} : BP^* BN / I \cong \mathbb{Z}/2[z^2] \subset H^*(BN; \mathbb{Z}/2).$$

Since the image of the restriction $H^*(BD(n); \mathbb{Z}/2) \to H^*(BN; \mathbb{Z}/2)$ is generated by $w_{2^n} \notin \text{Im}(\rho_{\mathbb{Z}/2})$, we see that

$$\text{Im}[BP^*(BD(n)) \to BP^*(BN)/I] = \mathbb{Z}/2[u^{2^n}],$$

where $u$ denotes the obvious generator in degree 2. Let $\xi$ be a complex representation of $D(n)$; it restricts to $N$ as the sum of $m$ copies (say) of the nontrivial character $c_C$ plus some trivial representations. Then there is an element $u' \equiv u \mod I$ in $BP^*(BN)$ with

$$\text{Res}_N(c(\xi)) = (1 + u'^m)$$

where $c(\xi)$ denotes as usual the total Chern class of $\xi$. Then $u^m \in \text{Im}[BP^*(BD(n)) \to BP^*(BN)/I]$, whence $m$ has to be divisible by $2^n$.

**Proposition 6.1.** Suppose Assumption 5.1 holds and $n \geq 3$. Then the permanent cycles $[v_1 \bar{w}_{2^n}], \ldots, [v_{n-1} \bar{w}_{2^n}]$ are not represented by Chern classes.

**Proof.** Suppose $[v_i \bar{w}_{2^n}]$ is the Chern class of some representation $\xi$, which must satisfy (6.2) for some $m = 2^n m'$. Then

$$v_i u^{2^{n-1}} = \text{Res}_N(c_{2^{n-1}2^{k-1}}(\xi)) \mod I^2.$$

But the restriction of the total Chern class of $\xi$ is given by

$$\text{Res}_N(c(\xi)) = 1 + 2m'(u')^{2^{n-1}} \mod (I^2, u^{2^n})$$

$$= 1 + m' (v_1 u^{2^{n-1}+1} + \cdots + v_i u^{2^{n-1}+2^i-1} + \cdots) \mod (I^2, u^{2^n})$$

which does not contain the term $v_i u^{2^{n-1}}$, a contradiction. \qed

**Theorem 6.2.** Suppose Assumption 5.1 holds and $n \geq 3$. Then $[v_1 \bar{w}_{2^n}], \ldots, [v_{n-2} \bar{w}_{2^n}]$ are not represented by transfers of Chern classes.
Proof. Let $H$ be a subgroup of $D(n)$, and suppose $[v_j w_{2n}] = \text{Tr}_H^{D(n)}(x)$ for some $x \in BP^*(BH)$. By the double coset formula,

$$\text{Res}_N^{D(n)} \text{Tr}_H^{D(n)}(x) = \sum_{g \in N} \text{Tr}_{g^{-1} H g \cap N} \text{Res}_{g^{-1} H g \cap N}(g^* x)$$

where the sum ranges over double coset representatives $g$ of $H \backslash G \cap N$. If $H$ intersects $N$ trivially, then so does any conjugate of $H$. Hence we need only consider subgroups $H$ containing the center, and the double coset formula evaluates to $|D(n)/H| \cdot \text{Res}_N(x)$. Since this element is represented by

$$\text{Res}_N [v_i w_{2n}] = v_i u^{2n-1} \neq 0 \mod I^2,$$

we get $|D(n)/H| = 2$ and thus $H \cong D(n-1) \times \mathbb{Z}/2$.

The total Chern class $c(\zeta)$ of any representation $\zeta$ of $D(n-1)$ restricts as

$$\text{Res}_N (c(\zeta)) = (1 + u')^{2^{n-1}}m = 1 + mu^{2n-1} \mod (I, u^2).$$

Hence we have

$$\text{Res}_N (2c(\zeta)) = 2 + 2mu^{2n-1}$$

$$= (v_1 u^2 + \cdots + v_i u^{2i} + \cdots) + m(v_1 u^{2n-1+1} + \cdots + v_i u^{2n-1+2^{i-1} + \cdots}) \mod (I^2, u^{2n}),$$

which does not contain $v_i u^{2n-1}$. This is a contradiction. \(\square\)

7. $BP^*(\text{BSpin}(7))$

The mod 2 cohomology of $\text{BSpin}(n)$ was computed by Quillen [Q]:

$$H^*(\text{BSpin}(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2k}(\Delta)] \otimes \mathbb{Z}/2[w_2, \ldots, w_n]/(w_2, Q_0 w_2, \ldots, Q_{h-1} w_2)$$

where $\Delta$ is a spin representation of $\text{Spin}(n)$ and $2^h$ the Radon-Hurwitz number (see [Q] §6).

This is proved by calculating the Serre spectral sequence of the fibration

$$\mathbb{Z}/2 \longrightarrow \text{BSpin}(n) \longrightarrow BSO(n).$$

We consider the case $n = 7$. Then $h = 3$ and the mod 2 cohomology of $\text{BSpin}(n)$ is a polynomial algebra on the Stiefel-Whitney classes $w_4, w_6, w_7, w_8$ of a spin representation, i.e.

$$H^*(\text{BSpin}(7); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8].$$
Recall that Spin(7) has the exceptional Lie group $G_2$ as a subgroup. $G_2$ contains a rank three elementary abelian 2-subgroup, and its mod 2 cohomology is isomorphic to the rank three Dickson invariants, i.e. $H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3]$. Here we may identify the Dickson invariants with the Stiefel-Whitney classes of the restriction of the spin representation to $G_2$, namely $D_1 = w_4$, $D_2 = w_6$, and $D_3 = w_7$. In particular, we have $H^*(B\text{Spin}(7);\mathbb{Z}/2) \cong \mathbb{Z}/2[D_1, D_2, D_3] \otimes \mathbb{Z}/2[w_8]$.

The Brown-Peterson cohomology of $B\text{Spin}(7)$ is given in [K-Y]. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(B\text{Spin}(7);\mathbb{Z}) \otimes BP^* \Longrightarrow BP^*(B\text{Spin}(7)).$$

Since $Q_0w_6 = w_7$ and since there is no higher 2-torsion, the $E_2$-term is isomorphic to $BP^*[w_4, w_6^2, w_7, w_8]/(2w_7)$. It is shown in [K-Y] that all non-zero differentials are of the form $d_{2m-1} = v_{m-1} \otimes Q_{m-1}$. Indeed,

$$d_3w_4 = v_1w_7, \quad d_7w_7 = v_2w_7^2, \quad d_7w_8 = v_2w_7w_8, \quad d_{15}(w_7w_8) = v_3w_7^2w_8^2.$$

Thus

$$E_16^{*,*} \cong (BP^*[1, 2w_4, 2w_8, 2w_4w_8, v_1w_8]) \oplus BP^*/(2, v_1, v_2)[w_7^2\{w_8^2\}] \oplus BP^*/(2, v_1, v_2, v_3)[w_7^2\{w_8^2\}] \otimes \mathbb{Z}(2)[w_4^2, w_6^2, w_8^2].$$

This page is generated by even degree elements, hence $E_{16} = E_{\infty}$. Note that the reason for the permanency of $v_1w_8$ is that $d_7w_8 = v_2w_7w_8$ but $d_3(w_4w_8) = v_1w_7w_8$.

**Remark.** The terms $B/(2, v_1)[w_8]$ in the formulas (6.10) and (6.11) of [K-Y] are typing errors and should be replaced with $B(2, v_1)[w_8]$.

**Lemma 7.1.** The element $[v_1w_8]$ is not represented by a Chern class, i.e. there is no representation $\rho$ with $[v_1w_8] = c_2(\rho)$.

**Proof.** This follows from Proposition 6.1 by looking at the commutative diagram

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin}(7) \longrightarrow SO(7) \longrightarrow 1 \quad \text{or} \quad 0 \longrightarrow \mathbb{Z}/2 \longrightarrow D(3) \longrightarrow (\mathbb{Z}/2)^6 \longrightarrow 0$$

whose rows are central extensions. \qed

The same diagram gives the following theorem as a consequence of Theorem 6.2.
Theorem 7.2. The element $[v_1 \otimes w_8]$ is not represented by the transfer of a Chern class in $BP^*(BD(3))$. □

Similar arguments work for Spin(9) and $D(4)$; in this case the Radon-Hurwitz number is 16.

Theorem 7.3. In $BP^*(BD(4))$, the elements $[v_1 \otimes w_{16}]$ and $[v_2 \otimes w_{16}]$ are not transfers of Chern classes in $BP^*(BD(n))$, $n < 4$.

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