On the action of β_1 in the stable homotopy of spheres at the prime 3

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Abstract. The element β_1 is the generator of the stable homotopy group $\pi_{10}(S^0)$. Here S^0 denotes the 3-localized sphere spectrum. Toda showed that $\beta_1^5 \neq 0$ and $\beta_1^6 = 0$. Here we generalize it to $\beta_1^4\beta_{9t+1} \neq 0$ and $\beta_1^5\beta_{9t+1} = 0$ for $\beta_{9t+1} \in \pi_{144t+10}(S^0)$ with $t \geq 0$. In particular, $\beta_1^4\beta_{10} \neq 0$ and $\beta_1^5\beta_{10} = 0$ for β_{10} shown to exist by Oka. This is proved by determining subgroups of $\pi_*(L_2S^0)$, where L_2 denotes the Bousfield localization functor with respect to $v_2^{-1}BP$.

1. Introduction

Let $E_r^*(X)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(X)$. Then Miller, Ravenel and Wilson [1] introduced β-elements $\beta_{s/j,i+1}$ in $E_2^2(S^0)$ for $(s, j, i+1) \in \boldsymbol{B}^+$, where p denotes a prime number, S^0 is the p-local sphere, and

$$\begin{aligned}\boldsymbol{B}^+ \;=\; \{(s, j, i+1) \in \boldsymbol{Z}^3 \mid\ & s = mp^n, n \geq 0, p \not| \, m \geq 1, j \geq 1, i \geq 0, \text{subject to} \\ & \text{i) } j \leq p^n \text{ if } m = 1, \text{ ii) } p^i | j \leq a_{n-i}, \text{ and iii) } a_{n-i-1} < j \text{ if } p^{i+1} | j\}\end{aligned}$$

for integers a_k defined by $a_0 = 1$ and $a_k = p^k + p^{k-1} - 1$. Here we use the abbreviation $\beta_{s/j,1} = \beta_{s/j}$ and $\beta_{s/1,1} = \beta_s$.

Let $V(1)$ denote the Toda-Smith spectrum, which is a cofiber of the Adams map $\alpha : \Sigma^{2p-2}V(0) \to V(0)$, where $V(0)$ is the mod p Moore spectrum. Since there exists a map $\beta : \Sigma^{2p^2-2}V(1) \to V(1)$ which induces v_2 on BP-homology at a prime > 3 by [9], we have homotopy elements $\beta_t \in \pi_{2t(p^2-1)-2p}(S^0)$ with $t > 0$. On the other hand, there is no such self map at the prime 3. But there are homotopy elements β_i for $i = 1, 2, 3, 5, 6, 10$

1991 *Mathematics Subject Classification.* 55Q45, 55Q52.

Key words and phrases. Homotopy groups of spheres, L_2-localization, β-elements, Adams-Novikov spectral sequence.

in this case due to Toda and Oka (*cf.* [2]). Besides, if we assume the existence of the self map $B : \Sigma^{144}V(1) \to V(1)$ that induces v_2^9 on BP-homology, then we see that there exists a family $\{\beta_{9t+i} \mid i = 0, 1, 2, 5, 6,\ t \geq 0\}$ in $\pi_*(S^0)$. The existence of B is claimed by Pemmaraju in his thesis. Furthermore, the existence of $\beta_{6/3} \in \pi_{82}(S^0)$ is shown by Ravenel [4]. Here we obtain a relation among β_{9t+1}, β_2 and $\beta_{6/3}$ as follows:

Theorem A. *Let t, i, j and k be non-negative integers. Then in the homotopy groups $\pi_*(S^0)$ of sphere spectrum localized away from 3,*

$$\beta_{9t+1}\beta_1^i \neq 0 \in \pi_*(S^0) \text{ if and only if } i < 5,$$
$$\beta_{9t+1}\beta_2\beta_1^j \neq 0 \in \pi_*(S^0) \text{ if and only if } j < 2, \text{ and}$$
$$\beta_{9t+1}\beta_{6/3}\beta_1^k \neq 0 \in \pi_*(S^0) \text{ if and only if } k < 4.$$

As is seen in [3, p.624], we have a relation

$$uv\beta_s\beta_t = st\beta_u\beta_v \quad \text{for } s + t = u + v$$

in the E_2-term $E_2^4(S^0)$. This implies

Corollary B. *In the homotopy groups $\pi_*(S^0)$ localized away from 3,*

$$\prod_{i=1}^k \beta_{9t_k+1} \neq 0 \text{ if and only if } k < 6, \text{and}$$
$$\left(\prod_{i=1}^k \beta_{9t_k+1}\right)\beta_{9t+2} \neq 0 \text{ if and only if } k < 3,$$

for integers $t, t_k \geq 0$. In particular, $\beta_{9t+1}^k \neq 0$ if and only if $k < 6$.

Remark. If the self map B does not exist, the above theorems are valid only for the homotopy elements such as β_1 and β_{10}.

We prove this by determining subgroups of $\pi_*(L_2S^0)$, where $L_2 : \mathcal{S}_{(3)} \to \mathcal{S}_{(3)}$ denotes the Bousfield localization functor on the category $\mathcal{S}_{(3)}$ of spectra localized away from 3 with respect to the Johnson-Wilson spectrum $E(2)$. In $\pi_*(L_2S^0)$, we have generalized β-elements $\beta_{s/j,i+1} \in E_2^2(L_2S^0)$ for $(s, j, i+1) \in \boldsymbol{B}$, where

$$\begin{aligned} \boldsymbol{B} \ = \ & \{(s, j, i+1) \in \boldsymbol{Z}^3 \mid s = mp^n, n \geq 0, 3 \not| \, m \in \boldsymbol{Z}, j \geq 1, i \geq 0, \\ & \text{such that } 3^i | j \leq a_{n-i} \text{ and either } 3^{i+1} \not| \, j \text{ or } a_{n-i-1} < j\}. \end{aligned}$$

Consider the sets

$$\begin{aligned}
\widehat{G} &= \sum_{t\in \boldsymbol{Z}} (B_5\{\beta_{9t+1}\} \bigoplus B_4\{\beta_{9t+1}\beta_{6/3}\} \\
&\qquad \bigoplus B_3\{\overline{\beta_{9t+7}\alpha_1}\} \\
&\qquad \bigoplus B_2\{\beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta'_1], [\beta_{9t+5}\beta'_1]\}), \\
\widehat{G}^* &= \sum_{t\in \boldsymbol{Z}} (B_5\{g_{16(9t+7)+15}\} \bigoplus B_4\{g_{16(9t+3)+7}\} \\
&\qquad \bigoplus B_2\{g_{144t}, g_{16(9t+5)+2}, g_{16(9t+8)+2}\} \\
&\qquad \bigoplus \sum_{n\geq 1} (B_3\{g_{16(3^{n+2}t+9u+3)} \mid u \in \boldsymbol{Z} - I(n)\} \\
&\qquad\quad \bigoplus B_2\{g_{16(3^{n+2}t+9u+3)} \mid u \in I(n)\})).
\end{aligned}$$

Here $B_k = \boldsymbol{Z}/3[\beta_1]/(\beta_1^k)$,

$$I(n) = \{x \in \boldsymbol{Z} \mid x = (3^{n-1}-1)/2 \text{ or } x = 5\cdot 3^{n-2} + (3^{n-2}-1)/2\},$$

$\overline{x}$ denotes a homotopy element detected by x in the E_2-term, $[x]$ is an element of $\pi_*(L_2S^0)$ such that $i_*([x]) = x \in \pi_*(L_2V(0))$ for the inclusion $i : S^0 \to V(0) = S^0 \bigcup_3 e^1$, and $g_i \in \pi_i(L_2S^0)$ is the generator. Then the subgroups $\widehat{G}\bigoplus\widehat{G}^*$ are generated by

$$\begin{aligned}
S = \{&\beta_{9t+1}, \beta_{9t+1}\beta_{6/3}, \overline{\beta_{9t+7}\alpha_1}, \beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta'_1], [\beta_{9t+5}\beta'_1], \\
&g_{16(9t+7)+15}, g_{16(9t+3)+7}, g_{144t}, g_{16(9t+5)+2}, \\
&g_{16(9t+8)+2}, g_{16(9t+3)} \mid t \in \boldsymbol{Z}\}
\end{aligned}$$

as a $\boldsymbol{Z}/3[\beta_1]$-module. In this paper, our key lemma is the following:

THEOREM C. *The homotopy groups $\pi_*(L_2S^0)$ contain subgroups $\widehat{G}\bigoplus\widehat{G}^*$.*

Consider the localization map $\iota : S^0 \to L_2S^0$. Then we see immediately the following:

COROLLARY D. *For any element $x \in \pi_*(S^0)$ such that $\iota_*(x) \in S$, we have $x\beta_1 \neq 0 \in \pi_*(S^0)$.*

In [7], we showed that the β-elements $\beta_{s/j,i+1}$ for $(s, j, i+1) \in \boldsymbol{B}^c$ do not exist in $\pi_*(L_2S^0)$, where

$$\boldsymbol{B}^c = \{(9t+4,1,1), (9t+7,1,1), (9t+8,1,1), (9t+3,3,1), (9s,3,2), (3^is,3^i,1) \mid t \in \boldsymbol{Z}, s \in \boldsymbol{Z} - 3\boldsymbol{Z}, i > 1\}.$$

Moreover β-elements $\beta_{s/j,i+1}$ for $(s,j,i+1) \in \boldsymbol{B}^c$ do not exist in $\pi_*(S^0)$ if $t \geq 1$, $3 \not| \, s \geq 1$ and $i > 1$. We further showed in [7] the existence of β-elements β_{9t}, β_{9t+1} and β_{9t+5} in $\pi_*(L_2S^0)$ for $t \in \boldsymbol{Z}$. Here we show more β-elements in $\pi_*(L_2S^0)$:

THEOREM E. *$\beta_{s/j,i+1}$ for $(s,j,i+1) \in \boldsymbol{B} - \boldsymbol{B}^c$ survives to a homotopy element of $\pi_*(L_2S^0)$.*

Note that $\beta_{s/j,i+1}$ are homotopy elements for all $(s,j,i+1) \in \boldsymbol{B}$ at a prime > 5 ([8]).

The following sections 2 to 6 are devoted to show Theorem C and we give subgroups of $\pi_*(M^2)$ for an L_2-local spectrum M^2 with $E(2)_*(M^2) = E(2)_*/(3^\infty, v_1^\infty)$. Theorems A and E are actually corollaries of Theorem C, and proved in §7.

2. Basic properties of $H^*M_0^2$

Let $E(2)$ be the Johnson-Wilson spectrum with coefficient ring $E(2)_* = \boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}]$. Then $E(2)_*E(2)$ is a Hopf algebroid over $E(2)_*$ with $E(2)_*E(2) = E(2)_*[t_1, t_2, \ldots]/(\eta_R(v_i) : i > 2)$. For an $E(2)_*E(2)$-comodule M, $\mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$ is the cohomology of the cobar complex $\Omega^*M = \Omega^*_{E(2)_*E(2)}M$, and we will denote it by H^*M.

Recall the chromatic comodules N_j^i and M_j^i defined inductively by $N_0^0 = E(2)_*$, $N_1^0 = E(2)_*/(3)$, $N_2^0 = E(2)_*/(3, v_1)$, $M_j^i = v_{i+j}^{-1}N_j^i$ and the short exact sequence $0 \to N_j^i \to M_j^i \to N_j^{i+1} \to 0$ for $i+j+1 \leq 2$ [1]. Note that $N_j^i = M_j^i$ if $i+j = 2$. These have $E(2)_*E(2)$-comodule structure induced from the right unit $\eta_R : E(2)_* \to E(2)_*E(2)$. Consider the long exact sequences associated to these short ones $0 \to N_0^0 \to M_0^0 \to N_0^1 \to 0$ and $0 \to N_0^1 \to M_0^1 \xrightarrow{f} M_0^2 \to 0$, and denote $\delta' : H^sN_0^1 \to H^{s+1}E(2)_*$ and $\delta : H^sM_0^2 \to H^sN_0^1$ for the connecting homomorphisms. Then we see that $\delta'\delta : H^sM_0^2 \to H^{s+2}E(2)_*$ is an epimorphism if $s \geq 1$, and an isomorphism if $s > 1$, since $H^sM_0^0 = 0$ for $s \geq 1$ and $H^sM_0^1 = 0$ for $s > 1$ by [1]. In particular,

LEMMA 2.1. *$H^sE(2)_*$ for $s \geq 3$ consists of torsion elements.*

We have a short exact sequence $0 \to M_1^1 \xrightarrow{i} M_0^2 \xrightarrow{3} M_0^2 \to 0$ $(i(x) = x/3)$ which induces a long exact sequence

$$\cdots \longrightarrow H^{s-1}M_0^2 \xrightarrow{\delta} H^sM_1^1 \xrightarrow{i_*} H^sM_0^2 \xrightarrow{3} H^sM_0^2 \xrightarrow{\delta} \cdots.$$

An easy diagram chasing shows the following:

Lemma 2.2. ([1, Remark 3.11]) *Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
B^{s-1} & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & B^s & \xrightarrow{3} & B^s & \xrightarrow{\delta} & H^{s+1}M_1^1 \\
\downarrow{\scriptstyle f^{s-1}} & & \downarrow{\scriptstyle g^s} & & \downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle g^{s+1}} \\
H^{t-1}M_0^2 & \xrightarrow{\delta} & H^tM_1^1 & \xrightarrow{i_*} & H^tM_0^2 & \xrightarrow{3} & H^tM_0^2 & \xrightarrow{\delta} & H^{t+1}M_1^1
\end{array}$$

of modules with horizontal exact sequences and a 3 *torsion module* B^s. *If* g^s *and* g^{s+1} *are isomorphisms, then* f^s *is an epimorphism. Moreover, if* f^{s-1} *is an epimorphism, then* f^s *is an isomorphism.*

Let b_{10} denote the element of $H^2E(2)_*$ represented by the cocycle $-t_1 \otimes t_1^2 - t_1^2 \otimes t_1$. Then b_{10} acts on H^*M for any comodule M. In [7], we show the following:

Proposition 2.3. *The multiplication by* b_{10} *yields an isomorphism* $H^sM_1^1 \to H^{s+2}M_1^1$ *if* $s > 3$ *and an epimorphism if* $s = 3$.

This together with Lemma 2.2 implies

Corollary 2.4. *The multiplication by* b_{10} *yields an isomorphism* $H^sM_0^2 \to H^{s+2}M_0^2$ *if* $s > 3$ *and an epimorphism if* $s = 3$.

Corollary 2.5. $H^*M_0^2 \cong (H^4M_0^2 \bigoplus H^5M_0^2) \bigotimes \boldsymbol{Z}/3[b_{10}]$ *for* $* > 3$.

3. Some formulae in $\Omega^*E(2)_*$

The Adams-Novikov E_2-term for computing $\pi_*(L_2X)$ is the cohomology $H^*E(2)_*(X)$ of the cobar complex $\Omega^*E(2)_*(X)$, and in particular, $H^*E(2)_* = H^*N_0^0$ is the E_2-term for $\pi_*(L_2S^0)$.

For $X = V(1)$, the Toda-Smith spectrum, $E(2)_*(V(1)) = K(2)_* = \boldsymbol{Z}/3[v_2^{\pm 1}]$ and $H^*K(2)_* = K(2)_*[b_{10}]\otimes F\otimes\Lambda(\zeta_2)$ for $F = \boldsymbol{Z}/3\{1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\}$ (*cf.* [6]). Among these generators, we have relations (*cf.* [6, Prop. 5.9]):

$$
\begin{gathered}
v_2^2h_{10}b_{10} = h_{11}b_{11}, \quad v_2h_{11}b_{10} = -h_{10}b_{11} \\
b_{11}\xi = v_2h_{10}\psi_1 = v_2h_{11}\psi_0, \quad b_{10}\xi = -h_{10}\psi_0 = v_2^{-1}h_{11}\psi_1, \\
v_2^3b_{10}^2 = -b_{11}^2, \quad b_{10}\psi_1 = -v_2^{-1}b_{11}\psi_0, \quad b_{10}\psi_0 = v_2^{-2}b_{11}\psi_1
\end{gathered}
\tag{3.1}
$$

For $X = V(0)$, the mod 3 Moore spectrum, the E_2-term $H^*E(2)_*/(3)$ is computed in [7], in particular, we see the following:

LEMMA 3.2.

$$
\begin{aligned}
H^{2,0}E(2)_*/(3) &= 0 \\
H^{3,0}E(2)_*/(3) &= \{v_2^{-1}\psi_0, v_2^{-1}h_{10}b_{10}\} \\
H^{4,0}E(2)_*/(3) &= \{v_2^{-3}b_{10}b_{11}, v_2^{-1}\psi_0\zeta_2, v_2^{-1}h_{10}b_{10}\zeta_2\} \\
H^{5,0}E(2)_*/(3) &= \{v_2^{-3}b_{10}b_{11}\zeta_2\}.
\end{aligned}
$$

Consider the long exact sequence

$$
\cdots \longrightarrow H^{*-1}E(2)_*/(3) \xrightarrow{\delta} H^*E(2)_* \xrightarrow{3} H^*E(2)_* \xrightarrow{j_*} H^*E(2)_*/(3) \xrightarrow{\delta} \cdots
$$

associated to the short exact sequence $0 \to E(2)_* \xrightarrow{3} E(2)_* \xrightarrow{j} E(2)_*/(3) \to 0$.

LEMMA 3.3. *The map* $d_1 = j_*\delta : H^*E(2)_*/(3) \to H^{*+1}E(2)_*/(3)$ *sends* $v_2^{-1}h_{10}b_{10}$ *(resp.* $v_2^{-1}h_{10}b_{10}\zeta_2$*) to* $v_2^{-3}b_{10}b_{11}$ *(resp.* $v_2^{-3}b_{10}b_{11}\zeta_2$*).*

PROOF. Note that $v_2^{-1}h_{10}b_{10}$ is represented by a cochain whose leading term is $v_2^{-3}t_1^3 \otimes b_{11}$. Since $d(t_1^{3^{i+1}}) = 3b_{1i}$ by definition, we compute

$$
d(v_2^{-3}t_1^3 \otimes b_{11}) = 3(v_2^{-3}b_{10} \otimes b_{11} + \cdots),
\tag{3.4}
$$

which shows $\delta(v_2^{-1}h_{10}b_{10}) = v_2^{-3}b_{10}b_{11} + \cdots$. For $v_2^{-1}h_{10}b_{10}\zeta_2$, it follows immediately from (3.4) and Proposition 4.2 in the next section. q.e.d.

Let x denote a cochain that represents ξ. Then it is shown in [7, Lemma 4.4] that $d(x) \equiv v_1^2 f_0 \bmod (3)$ for f_0 that represents $-v_2^{-1}\psi_0 \bmod (3, v_1)$ (In [7], x is denoted by $X(0)$). So we have a cochain A such that

(3.5) $$d(x) \equiv v_1^2 f_0 \mod (3) \quad \text{and} \quad d(f_0) = 3A$$

in the cobar complex $\Omega^* E(2)_*$. Then A is a cocycle of $H^{4,0}E(2)_*$. Furthermore, we have

Lemma 3.6. $d(f_0) = \pm 3 f_0 \otimes z \bmod (9)$ *in the cobar complex* $\Omega^4 E(2)_*$. *Here* z *denotes a cocycle that represents the generator* ζ_2.

Proof. The projection $E(2)_* \to E(2)_*/(3)$ sends A in (3.5) to a cocycle which is also denoted by A. By virtue of Lemmas 3.2 and 3.3, we put

$$[A] = k_1 v_2^{-1}h_{10}b_{10}\zeta_2 + k_2 v_2^{-1}\psi_0\zeta_2,$$

where $[A]$ denotes a cohomology class represented by $A \in \Omega^* E(2)_*/(3)$. In fact, f_0 may be replaced by $f_0 + k v_2^{-3} t_1^3 \otimes b_{11}$ for some $k \in \boldsymbol{Z}/3$ if necessary. Since $d_1([A]) = 0$ and $d_1(v_2^{-1}\psi_0) = 3[A]$ by definition,

$$0 = k_1 v_2^{-3} b_{10}b_{11}\zeta_2 + k_2(k_1 v_2^{-1}h_{10}b_{10}\zeta_2 + k_2 v_2^{-1}\psi_0\zeta_2)\zeta_2,$$

by Lemma 3.3. Noticing that $\zeta_2^2 = 0$ and $v_2^{-3}b_{10}b_{11}\zeta_2 \neq 0$, we see that $k_1 = 0$.

On the other hand, if A represents 0, then we have an element $v_2^{-1}\psi_0$ in $H^{3,0}E(2)_*$. Since $H^{2,0}E(2)_*/(3) = 0$ by Lemma 3.2, $v_2^{-1}\psi_0$ generates a $\boldsymbol{Z}_{(3)}$-free submodule in $H^{3,0}E(2)_*$, which contradicts to Lemma 2.1. Therefore A represents a non-zero element. This means that $k_2 = \pm 1$. q.e.d.

Lemma 3.7. *Put* $\widetilde{v_2t_1} = v_2t_1 + v_1\tau$ *and* $\widetilde{v_2^2t_1^3} = v_2^2t_1^3 + v_1v_2t_1^6$. *Then* $d(\widetilde{v_2t_1}) = v_1^2 b_{10}$ *and* $d(\widetilde{v_2^2t_1^3}) = -v_1^2 b_{11}$. *Furthermore, we have a cochain* $u \in \Omega^2 E(2)_*$ *such that*

$$d(u) \equiv \widetilde{v_2t_1} \otimes b_{11} + \widetilde{v_2^2t_1^3} \otimes b_{10} \mod (3, v_1^2).$$

Proof. The first statement is checked by a routine computation.

For the second, we have an element $u' \in \Omega^2 E(2)_*$ such that $d(u') \equiv v_2t_1 \otimes b_{11} + b_{10} \otimes v_2^2t_1^3 \bmod (3, v_1)$, since we have a relation $v_2h_{11}b_{10} = -h_{10}b_{11}$ in $H^3E(2)_*/(3, v_1)$ by (3.1). Put $d(u') \equiv \widetilde{v_2t_1} \otimes b_{11} + b_{10} \otimes \widetilde{v_2^2t_1^3} + v_1w \bmod (3, v_1^3)$ for some cochain w. Send this by d, and we have $0 \equiv v_1^2b_{10} \otimes b_{11} - b_{10} \otimes v_1^2b_{11} + v_1d(w) \bmod (3, v_1^3)$. Then $w \in H^{3,52}E(2)_*/(3, v_1^2)$, which is 0 since $H^{3,52}E(2)_*/(3, v_1) = \{v_2b_{11}\zeta_2\}$ by [6, Th. 5.8] and $d(v_2b_{11}\zeta_2) \not\equiv 0 \bmod (3, v_1^2)$ by [7, Lemma 3.3]. Therefore, we see that there is a cochain $\overline{w}$ such that $d(\overline{w}) \equiv w$, and we have $d(u'') \equiv \widetilde{v_2t_1} \otimes b_{11} + b_{10} \otimes \widetilde{v_2^2t_1^3} \bmod (3, v_1^3)$ if we put $u'' = u' - v_1\overline{w}$. There is also a cochain a such that $d(a) \equiv \widetilde{v_2^2t_1^3} \otimes b_{10} - b_{10} \otimes \widetilde{v_2^2t_1^3} \bmod (3, v_1^2)$, and so we have the lemma by putting $u = u'' + a$. q.e.d.

Lemma 3.8. *There exists a cochain w such that*

$$d(w) \equiv \widetilde{v_2t_1} \otimes f_0 - x' \otimes b_{10} \mod (3, v_1^3).$$

Here x' denotes a cocycle that represents $\xi \bmod (3, v_1)$ and $t_1 \otimes x'$ is homologous to $t_1 \otimes x \bmod (3, v_1^3)$.

Proof. This is shown in the same way as the above lemma. By the equation $h_{10}\psi_0 = -\xi b_{10}$ in (3.1), we have a cochain w' such that $d(w') \equiv v_2t_1 \otimes f_0 - b_{10} \otimes x \bmod (3, v_1)$. Put $d(w') \equiv \widetilde{v_2t_1} \otimes f_0 - b_{10} \otimes x + v_1a$ for a cochain a. Send this by d, and we see that a is a cocycle of $\Omega^{4,16}E(2)_*/(3, v_1^2)$. Since we see that $H^{4,16}E(2)_*/(3, v_1^2) = \{h_{10}b_{10}\zeta_2\}$ by [7], $a = kt_1 \otimes z \otimes b_{10}$ for some $k \in \boldsymbol{Z}/3$. Furthermore, $b_{10} \otimes x$ is homologous to $x \otimes b_{10}$, and so we have cochain w such that $d(w) \equiv \widetilde{v_2t_1} \otimes f_0 - (x - kv_1t_1 \otimes z) \otimes b_{10}$. Now put $x' = x - kv_1t_1 \otimes z$, and we have the lemma. q.e.d.

Lemma 3.9. *In the cobar complex $\Omega^*E(2)_*$, there exists a cochain y such that*

$$d(y) \equiv t_1 \otimes x - v_1v_2^{-1}f_1 - v_1z \otimes x - kv_1v_2^{-2}t_1 \otimes b_{11} \mod (3, v_1^2)$$

for some $k \in \boldsymbol{Z}/3$. Here f_1 denotes a cocycle that represents ψ_1.

Proof. It is shown in [7, Lemma 6.4] that there exists a cochain Y_0 such that $d(Y_0) \equiv t_1 \otimes X + v_1v_2^{-3}\tau^3 \otimes X + v_1^2v_2^{-1}t_1^3 \otimes X \bmod (3, v_1^3)$. It is also shown that $x \equiv X + v_1v_2^{-1}Y_1 + kv_1v_2^{-2}b_{11}$ for some $k \in \boldsymbol{Z}/3$ in [7, Proof of Lemma 4.4.]. Take now y to be Y_0. Then mod $(3, v_1^2)$,

$$d(y) \equiv t_1 \otimes (x - v_1v_2^{-1}Y_1 - kv_1v_2^{-2}b_{11}) - v_1z \otimes X + v_1v_2^{-1}t_2 \otimes X.$$

Since $f_1 = t_1 \otimes Y_1 - t_2 \otimes X$ by the proof of [7, Lemma 4.4], we have the result.
q.e.d.

4. The E_2-terms $H^sM_0^2$ for $s > 3$.

In [7], $H^*M_1^1$ is given as the direct sum of three $E(2,1)_* = \boldsymbol{Z}/3[v_1, v_2^{\pm 3}]$-modules A_i:

$$H^*M_1^1 = A_0 \bigoplus A_1 \bigoplus A_2.$$

In order to describe the modules A_i, we use notations:

$$\begin{aligned}
k(1)_* &= \boldsymbol{Z}/3[v_1] \\
K(1)_* &= \boldsymbol{Z}/3[v_1^{\pm 1}] \\
PE &= \boldsymbol{Z}/3[b_{10}] \bigotimes \Lambda(\zeta_2) \\
E(2,n)_* &= \boldsymbol{Z}/3[v_1, v_2^{\pm 3^n}] \\
F_{(h)} &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2/v_1, v_2h_{10}/v_1, v_2^2h_{11}/v_1, v_2b_{11}/v_1\} \\
F_{(t)} &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2^{-1}/v_1, v_2h_{10}/v_1^2, v_2^2h_{11}/v_1^2, v_2^{-1}b_{11}/v_1\} \\
F_{(h)}^* &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1, \psi_0/v_1, v_2\psi_1/v_1, b_{11}\xi/v_1\} \\
F_{(t)}^* &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1^2, v_2^{-2}\psi_0/v_1, v_2^{-1}\psi_1/v_1, b_{11}\xi/v_1^2\} \\
F_n &= E(2, n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n - 1}, v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n + 1}, \\
&\qquad v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n + 1}, v_2^{3^n(5\pm 3) + (3^n - 1)/2}\xi/v_1^{4\cdot 3^n}\}.
\end{aligned}$$

Then the modules A_i are given as follows:

$$\begin{aligned}
A_0 &= (K(1)_*/k(1)_*) \bigotimes \Lambda(h_{10}, \zeta_2) \\
A_1 &= \sum_{n \geq 0} F_n \bigotimes \Lambda(\zeta_2) \\
A_2 &= (F_{(h)} \bigoplus F_{(t)} \bigoplus F_{(h)}^* \bigoplus F_{(t)}^*) \bigotimes PE,
\end{aligned}$$

Consider the exact sequence $H^{1,0}M_1^1 \xrightarrow{\delta} H^{2,0}E(2)_*/(3, v_1^{3^i}) \to H^{2,-4\cdot 3^i}M_1^1 \xrightarrow{v_1^{3^i}}$ associated to the short exact sequence $0 \to E(2)_*/(3, v_1^{3^i}) \xrightarrow{1/v_1^{3^i}} M_1^1 \xrightarrow{v_1^{3^i}} M_1^1 \to 0$. Then the structure of $H^*M_1^1$ shows immediately the following:

Lemma 4.1. *For each $i > 0$, each element of $H^{2,0}E(2)_*/(3, v_1^{3^i})$ is divisible by $v_1^{3^i - 3^{i-1}}$.*

In the same way as [5, Lemma 2.6] with the above lemma, we obtain

Proposition 4.2. *For each integer $i > 0$, there exists a cocycle z_i of $\Omega^{1,0}E(2)_*/(3^{i+1}, v_1^{3^i})$ such that $z_i = z \in \Omega^1 K(2)_*$.*

By virtue of this proposition, we abuse the notation z for a cocycle that represents ζ_2 as we did in the previous papers for a prime > 3.

Consider the connecting homomorphism $\delta : H^s M_0^2 \to H^{s+1} M_1^1$ associated to the short exact sequence $0 \to M_1^1 \xrightarrow{i} M_0^2 \xrightarrow{3} M_0^2 \to 0$.

Lemma 4.3. $\qquad \delta(v_2^2\xi/3v_1^3) = v_2\psi_0/v_1.$

Proof. In [7, Lemma 4.4], it is also shown that there exists a cochain $X(2)$ such that $d(X(2)) \equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} f_1^3 \mod (3, v_1^5)$. Since $H^{3,20}M_2^0 = 0$, the congruence holds mod $(3, v_1^6)$ if we replace $X(2)$ by a suitable cochain x''. Put now $d(x'') \equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} f_1^3 + 3A \mod (9, v_1^6)$. Then $0 \equiv 3v_1^3 t_1 \otimes z^3 \otimes X^3 - 3v_1^3 t_1 \otimes v_2^{-3} f_1^3 + 3d(A) \mod (9, v_1^4)$, and we see that $A \in H^{3,40}E(2)_*/(3, v_1^3)$, which is seen to be $\{v_1^2 v_2 \psi_0\}$ by [7]. Note that $3v_1^3 t_1 \otimes z^3 \otimes X^3$ and $v_1^3 t_1 \otimes v_2^{-3} f_1^3$ are homologous to zero and $v_1^3 b_{11} \otimes X \mod (9, v_1^4)$ by Lemma 3.9 and (3.1), respectively, and that $d(-v_1^2 v_2 (v_2 f_0)) \equiv v_1^3 v_2 t_1^3 \otimes f_0 + v_1^3 b_{11} \otimes X \mod (3, v_1^4)$ by [7, Lemma 4.3] which is homologous to $-v_1^2 b_{11} \otimes X$ by (3.1). Thus we see that A is homologous to $-v_1^2 v_2^2 f_0$, and $d(x'') \equiv -3v_1^2 v_2^2 f_0 \mod (9, v_1^3)$. Since x'' and f_0 represent $v_2^2\xi$ and $-v_2^{-1}\psi_0$, respectively, we have the lemma. q.e.d.

Put

$$G^s = (i_*(F_{(h)} \bigoplus F_{(h)}^*) \bigotimes E(2,1)_*[b_{10}] \bigotimes \Lambda(\zeta_2))^s \subset H^s M_0^2$$

for $i_* : H^* M_1^1 \to H^* M_0^2$ given by $i_*(x) = x/3$.

Lemma 4.4. *For the connecting homomorphism $\delta : H^s M_0^2 \to H^{s+1} M_1^1$, $\delta(x)$ for a generator x of G^s is obtained by the following equations*:

$$\delta(v_2/3v_1) \quad = \quad -v_2 h_{10}/v_1^2,$$

$$\begin{aligned}
\delta(v_2h_{10}/3v_1) &= v_2^{-1}b_{11}/v_1 + v_2h_{10}\zeta_2/v_1,\\
\delta(v_2^2h_{11}/3v_1) &= v_2^2b_{10}/v_1 + v_2^2h_{11}\zeta_2/v_1,\\
\delta(v_2b_{11}/3v_1) &= v_2^2h_{11}b_{10}/v_1^2;\\
\delta(v_2(v_2^{-1}\psi_0)/3v_1) &= \xi b_{10}/v_1^2 \pm v_2(v_2^{-1}\psi_0)\zeta_2/v_1,\\
\delta(\xi/3v_1) &= v_2^{-1}\psi_1/v_1 + (1\pm 1)\xi\zeta_2/v_1 + kv_2^{-1}h_{11}b_{10}/v_1\\
\delta(b_{11}\xi/3v_1) &= v_2^2(v_2^{-1}\psi_0)b_{10}/v_1 + (1\pm 1)b_{11}\xi\zeta_2/v_1 + kv_2h_{10}b_{10}^2/v_1,\\
\delta(v_2\psi_1/3v_1) &= b_{11}\xi/v_1^2 \pm v_2\psi_1\zeta_2/v_1.
\end{aligned}$$

Proof. Note that v_2h_{10} is represented by a cochain $\widetilde{v_2t_1} = \eta_R(v_2)t_1 - v_1t_2$ of Lemma 3.7. In the cobar complex $\Omega^*E(2)_*/(9, v_1^3)$, we compute

$$\begin{aligned}
d(v_1^2v_2) &= 6v_1t_1\eta_R(v_2) + 3v_1^2t_2 = 6v_1\widetilde{v_2t_1};\\
d(v_1^2v_2t_1) &= 6v_1t_1\otimes v_2t_1 + 3v_1^2t_2\otimes t_1\\
&= \underline{6v_1v_2t_1\otimes t_1}_1 + \underline{6v_1^2\tau\otimes t_1}_2\\
d(3v_1v_2t_1^2) &= \underline{3v_1^2t_1^3\otimes t_1^2}_3 - \underline{6v_1v_2t_1\otimes t_1}_1\\
d(3v_1^2t_1\tau) &= -3v_1^2(\underline{t_1^4\otimes t_1}_2 + \underline{t_1^3\otimes t_1^2}_3 + t_1\otimes\tau + \underline{\tau\otimes t_1}_2)\\
d(-3v_1^2v_2^{-2}t_3) &= 3v_1^2v_2^{-2}(t_1\otimes t_2^3 + \underline{t_2\otimes t_1^9}_2 + v_2b_{11});\\
d(v_1^2v_2^2t_1^3) &= 6v_1t_1\otimes v_2^2t_1^3 + 6v_1^2v_2t_2\otimes t_1^3 + 3v_1^2v_2^2b_{10}\\
&= \underline{6v_1v_2^2t_1\otimes t_1^3}_4 + \underline{12v_1^2v_2t_1^4\otimes t_1^3}_7\\
&\quad + \underline{6v_1^2v_2t_2\otimes t_1^3}_7 + \underline{3v_1^2v_2^2b_{10}}_6\\
d(6v_1v_2^2t_2) &= \underline{12v_1^2v_2t_1^3\otimes t_2}_5 - \underline{6v_1v_2^2t_1\otimes t_1^3}_4 - \underline{6v_1^2v_2^2b_{10}}_6\\
d(3v_1^2v_2^{-7}t_3^3) &= -3v_1^2v_2^{-7}(t_1^3\otimes t_2^9 + t_2^3\otimes t_1^{27} + v_2^3b_{11}^3)\\
&= -3v_1^2v_2^{-7}(\underline{v_2^8t_1^3\otimes t_2}_5 + \underline{v_2^{-6}t_2^3\otimes t_1^3}_7 + \underline{v_2^9b_{10})}_6.
\end{aligned}\tag{4.5}$$

Underlined terms with the same number sum up to zero except for the terms numbered 6 and 7. The terms numbered 6 and 7 sum up to $3v_1^2v_2^2b_{10}$ and $6v_1^2v_2^2\zeta_2\otimes t_1^3$, respectively. These imply the first three equations. In fact, $\delta([a/3v_1]) = [(i_*)^{-1}d(v_1^2a)/9v_1^3]$, where $[a]$ denotes a homology class represented by a. Since $\delta(b_{11}a) = b_{11}\delta(a)$, the first equation gives $\delta(v_2b_{11}/3v_1) = -v_2h_{10}b_{11}/v_1^2$, which equals to $v_2^2h_{11}b_{10}/v_1^2$ by Lemma 3.7. Thus we have the fourth equation.

By Lemma 3.6 and the first equation in (4.5), we compute

$$d(v_1^2v_2f_0) \equiv 6v_1\widetilde{v_2t_1}\otimes f_0 \pm 3v_1^2v_2f_0\otimes z \mod (9, v_1^3). \tag{4.6}$$

Now by Lemma 3.8, we have the fifth equation. Multiply h_{10} to the fifth equation, and we have $\delta(h_{10}\psi_0/3v_1) = h_{10}\xi b_{10}/v_1^2 + h_{10}\psi_0\zeta_2/v_1$. Lemma 3.9

says that $h_{10}\xi = v_1v_2^{-1}\psi_1 + v_1\xi\zeta_2 + kv_1v_2^{-2}h_{10}b_{11}$ for some $k \in \boldsymbol{Z}/3$. Since $h_{10}\psi_0 = -\xi b_{10}$, we have the sixth equation. The seventh follows immediately from a product of b_{11} and the sixth equation. A multiplication of b_{10} and fifth equation gives the last one by using relations in (3.1). Here note that $b_{10}^2 = -v_2^{-3}b_{11}^2$ holds in $H^4E(2)_*/(3, v_1^2)$. q.e.d.

Proposition 4.7. *$H^sM_0^2 = G^s$ for $s = 4, 5$. In other words, $H^4M_0^2$ and $H^5M_0^2$ are $\boldsymbol{Z}/3[v_2^{\pm 3}]$-modules generated by*

$$G^4 : \begin{array}{c} v_2b_{10}^2/3v_1, v_2h_{10}b_{10}\zeta_2/3v_1, v_2^2h_{11}b_{10}\zeta_2/3v_1, v_2b_{11}b_{10}/3v_1; \\ \psi_0\zeta_2/3v_1, \xi b_{10}/3v_1, b_{11}\xi/3v_1, v_2\psi_1\zeta_2/3v_1, \end{array}$$

and

$$G^5 : \begin{array}{c} v_2b_{10}^2\zeta_1/3v_1, v_2h_{10}b_{10}^2/3v_1, v_2^2h_{11}b_{10}^2/3v_1, v_2b_{11}b_{10}\zeta_2/3v_1; \\ \psi_0b_{10}/3v_1, \xi b_{10}\zeta_2/3v_1, b_{11}\xi\zeta_2/3v_1, v_2\psi_1b_{10}/3v_1, \end{array}$$

respectively.

Proof. Put $B^s = G^s$. Then there is a canonical map $f^s : B^s \to H^sM_0^2$ sitting in the commutative diagram

$$\begin{array}{ccccccccc} H^{s-1}M_0^2 & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & B^s & \xrightarrow{3} & B^s & \xrightarrow{\delta} & H^{s+1}M_1^1 \\ \| & & \| & & \downarrow f^s & & \downarrow f^s & & \| \\ H^{s-1}M_0^2 & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & H^sM_0^2 & \xrightarrow{3} & H^sM_0^2 & \xrightarrow{\delta} & H^{s+1}M_1^1. \end{array}$$

Lemma 4.4 implies that the δ-images of the generators of B^s are linearly independent. Therefore we see that the above sequence is exact, and Lemma 2.2 shows that f^s is an isomorphism. q.e.d.

5. On the E_2-terms $H^sM_0^2$ for $s \leq 3$

The submodule $A_2^s \subset H^sM_1^1$ is:

$$\begin{aligned} A_2^0 &= \boldsymbol{Z}/3\{v_2/v_1, v_2^{-1}/v_1\} \\ A_2^1 &= \boldsymbol{Z}/3\{v_2h_{10}/v_1^2, v_2h_{10}/v_1, v_2^2h_{11}/v_1^2, v_2^2h_{11}/v_1, \end{aligned}$$

$$
\begin{aligned}
& v_2\zeta_2/v_1, v_2^{-1}\zeta_2/v_1\} \\
A_2^2 &= \boldsymbol{Z}/3\{v_2b_{11}/v_1, v_2^{-1}b_{11}/v_1, v_2h_{10}\zeta_2/v_1^2, v_2h_{10}\zeta_2/v_1, \\
& v_2^2h_{11}\zeta_2/v_1^2, v_2^2h_{11}\zeta_2/v_1, v_2b_{10}/v_1, v_2^{-1}b_{10}/v_1, \\
& \xi/v_1^2, \xi/v_1\} \\
A_2^3 &= \boldsymbol{Z}/3\{v_2b_{11}\zeta_2/v_1, v_2^{-1}b_{11}\zeta_2/v_1, v_2h_{10}b_{10}/v_1^2, v_2h_{10}b_{10}/v_1, \\
& v_2^2h_{11}b_{10}/v_1^2, v_2^2h_{11}b_{10}/v_1, v_2b_{10}\zeta_2/v_1, v_2^{-1}b_{10}\zeta_2/v_1, \\
& \psi_0/v_1, v_2\psi_0/v_1, v_2\psi_1/v_1, v_2^{-1}\psi_1/v_1, \\
& \xi\zeta_2/v_1^2, \xi\zeta_2/v_1\}.
\end{aligned}
$$

Now consider the map $d_1 = \delta i_* : H^sM_1^1 \to H^{s+1}M_1^1$. Then [1, Prop. 6.9] shows

$$
d_1(v_2^3/v_1^3) = v_2^2h_{11}/v_1^2. \tag{5.1}
$$

Here we compute:

LEMMA 5.2. *The Bockstein differential $d_1 = \delta i_*$ acts up to sign as follows:*

$$
\begin{aligned}
d_1(v_2/v_1) &= v_2h_{10}/v_1^2, \\
d_1(v_2^{-1}/v_1) &= v_2^{-1}\zeta_2/v_1; \\
d_1(v_2h_{10}/v_1) &= v_2^{-1}b_{11}/v_1 + v_2h_{10}\zeta_2/v_1, \\
d_1(v_2^2h_{11}/v_1) &= v_2^{-1}b_{10}/v_1 + v_2^{-1}h_{11}\zeta_2/v_1, \\
d_1(v_2\zeta_2/v_1) &= v_2h_{10}\zeta_2/v_1^2; \\
d_1(v_2b_{11}/v_1) &= v_2^2h_{11}b_{10}/v_1^2, \\
d_1(v_2h_{10}\zeta_2/v_1) &= v_2^{-1}b_{11}\zeta_2/v_1^2, \\
d_1(v_2^2h_{11}\zeta_2/v_1) &= v_2^2b_{10}\zeta_2/v_1, \\
d_1(v_2b_{10}/v_1) &= v_2h_{10}b_{10}/v_1^2, \\
d_1(\xi/v_1^2) &= \xi\zeta_2/v_1^2, \\
d_1(\xi/v_1) &= v_2^{-1}\psi_1/v_1 + \xi\zeta_2/v_1; \\
d_1(v_2b_{11}\zeta_2/v_1) &= v_2^2h_{11}b_{10}\zeta_2/v_1^2, \\
d_1(v_2h_{10}b_{10}/v_1) &= v_2^{-1}b_{11}b_{10}/v_1^2 + v_2h_{10}b_{10}\zeta_2/v_1, \\
d_1(v_2^2h_{11}b_{10}/v_1) &= v_2^{-1}b_{10}^2/v_1 + v_2^{-1}h_{11}b_{10}\zeta_2/v_1, \\
d_1(v_2b_{10}\zeta_2/v_1) &= v_2h_{10}b_{10}\zeta_2/v_1^2; \\
d_1(\xi\zeta_2/v_1) &= v_2^{-1}\psi_1\zeta_2/v_1, \\
d_1(\psi_0/v_1) &= \xi b_{10}/v_1^2 + \psi_0\zeta_2/v_1, \\
d_1(v_2\psi_1/v_1) &= b_{11}\xi/v_1^2.
\end{aligned}
$$

The other elements of A_2 that do not appear in the left hand side are in the image of d_1.

PROOF. Lemma 4.3 and (5.1) show that $v_2^{-2}\psi_0/v_1$ and $v_2^2h_{11}/v_1^2$ are in the image of d_1. The other parts follow from Lemma 4.4, except for d_1 on v_2^{-1}/v_1 and ξ/v_1^2.

For the exceptional cases, consider the diagram

$$\begin{array}{ccccccccc}
H^{s-1}M_0^2 & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & H^sM_0^2 & \xrightarrow{3} & H^sM_0^2 & \xrightarrow{\delta} & H^{s+1}M_1^1 \\
\downarrow{\scriptstyle b_{10}} & & \downarrow{\scriptstyle b_{10}} & & \downarrow{\scriptstyle b_{10}} & & \downarrow{\scriptstyle b_{10}} & & \downarrow{\scriptstyle b_{10}} \\
H^{s+1}M_0^2 & \xrightarrow{\delta} & H^{s+2}M_1^1 & \xrightarrow{i_*} & H^{s+2}M_0^2 & \xrightarrow{3} & H^{s+2}M_0^2 & \xrightarrow{\delta} & H^{s+3}M_1^1.
\end{array}$$

If we have a relation $\delta(\alpha/3) = \beta b_{10} + \alpha\zeta_2$ in Lemma 4.4, then we see that $\beta/3 = -\alpha\zeta_2/3$ in $H^*M_0^2$, since $i_*(x) = x/3$. Therefore, we compute

$$b_{10}\delta(\beta/3) = \delta(\beta b_{10}/3) = -\delta(\alpha\zeta_2/3) = -\delta(\alpha/3)\zeta_2 = -\beta b_{10}\zeta_2,$$

and so we obtain

$$\delta(\beta/3) = -\beta\zeta_2$$

up to Ker b_{10}. Note that b_{10} acts monomorphically on A_2. Now take β to be the exceptional cases, and we have all d_1. q.e.d.

Hence, we have

PROPOSITION 5.3. *$H^sM_0^2$ contains $E(2,1)_*/(3,v_1)$-module as follows*:

$$\begin{array}{lll}
H^0M_0^2 & \supset & E(2,1)_*\{v_2^{\pm1}/3v_1\} \\
H^1M_0^2 & \supset & E(2,1)_*\{v_2h_{10}/3v_1, v_2^2h_{11}/3v_1, v_2\zeta_2/3v_1\} \\
H^2M_0^2 & \supset & E(2,1)_*\{v_2b_{11}/3v_1, v_2h_{10}\zeta_2/3v_1, v_2^2h_{11}\zeta_2/3v_1, v_2b_{10}/3v_1 \\
& & \quad \xi/3v_1^2, \xi/3v_1\} \\
H^3M_0^2 & \supset & E(2,1)_*\{v_2b_{11}\zeta_2/3v_1, v_2h_{10}b_{10}/3v_1, v_2^2h_{11}b_{10}/3v_1, v_2b_{10}\zeta_2/3v_1 \\
& & \quad \xi\zeta_2/3v_1, \psi_0/3v_1, v_2\psi_1/3v_1\}.
\end{array}$$

6. The Adams-Novikov differentials

Now consider spectra defined by cofiber sequences:

$$(6.1)\quad S^0 \to p^{-1}S^0 \to N^1, \quad N^1 \to L_1N^1 \to N^2, \quad V(0) \to v_1^{-1}V(0) \to W,$$

and $M^2 = L_2N^2$. The Adams-Novikov differentials on $\pi_*(L_2W)$ is determined in [7]. Let $\iota : L_2W \to M^2$ denote the canonical map that induces $i : M_1^1 \to M_0^2$. Suppose that $d_r(x) = y$ in the E_r-term for L_2W. Then $d_r(x/3) = d_r(i_*x) = i_*y = y/3$. In this way, we have the following except for $d_9(v_2^{-1}/3v_1)$ and $d_5(v_2^{3t}\xi/3v_1^2)$:

Lemma 6.2. *The Adams-Novikov differential d_r is given (up to sign) by:*

$$\begin{gathered}
d_r(v_2/3v_1) = 0, \quad d_5(v_2^4/3v_1) = v_2^2h_{11}b_{10}^2/3v_1, \quad d_5(v_2^7/3v_1) = v_2^5h_{11}b_{10}^2/3v_1, \\
d_r(v_2^2/3v_1) = 0, \quad d_r(v_2^5/3v_1) = 0, \quad d_9(v_2^{-1}/3v_1) = v_2^{-5}b_{11}b_{10}^3\zeta_2/3v_1, \\
d_r(v_2h_{10}/3v_1) = 0, \quad d_9(v_2^4h_{10}/3v_1) = v_2b_{10}^5/3v_1, \quad d_r(v_2^7h_{10}/3v_1) = 0, \\
d_r(v_2^2h_{11}/3v_1) = 0, \quad d_r(v_2^5h_{11}/3v_1) = 0, \quad d_9(v_2^8h_{11}/3v_1) = v_2^4b_{11}b_{10}^4/3v_1, \\
d_5(v_2b_{11}/3v_1) = v_2h_{10}b_{10}^3/3v_1, \quad d_r(v_2^4b_{11}/3v_1) = 0, \quad d_5(v_2^7b_{11}/3v_1) = v_2^7h_{10}b_{10}^3/3v_1, \\
d_5(\xi/3v_1^2) = v_2^{-3}b_{11}\xi b_{10}\zeta_2/3v_1, \quad d_r(v_2^3\xi/3v_1^2) = 0, \quad d_5(v_2^6\xi/3v_1^2) = v_2^3b_{11}\xi b_{10}\zeta_2/3v_1, \\
d_r(\xi/3v_1) = 0, \quad d_r(v_2^3\xi/3v_1) = 0, \quad d_9(v_2^6\xi/3v_1) = v_2^3\psi_0b_{10}^4/3v_1, \\
d_5(\psi_0/3v_1) = v_2^{-3}b_{11}\xi b_{10}^2/3v_1, \quad d_r(v_2^3\psi_0) = 0, \quad d_5(v_2^6\psi_0/3v_1) = v_2^3b_{11}\xi b_{10}^2/3v_1, \\
d_5(v_2\psi_1/3v_1) = \xi b_{10}^3/3v_1, \quad d_5(v_2^4\psi_1/3v_1) = v_2^3\xi b_{10}^3/3v_1, \quad d_r(v_2^7\psi_1/3v_1) = 0. \\
d_9(b_{11}\xi/3v_1) = v_2^{-2}\psi_1b_{10}^5/3v_1, \quad d_r(v_2^3b_{11}\xi/3v_1) = 0, \quad d_r(v_2^6b_{11}\xi/3v_1) = 0.
\end{gathered}$$

Proof. Here we show the exceptional cases. Lemma 4.4 shows

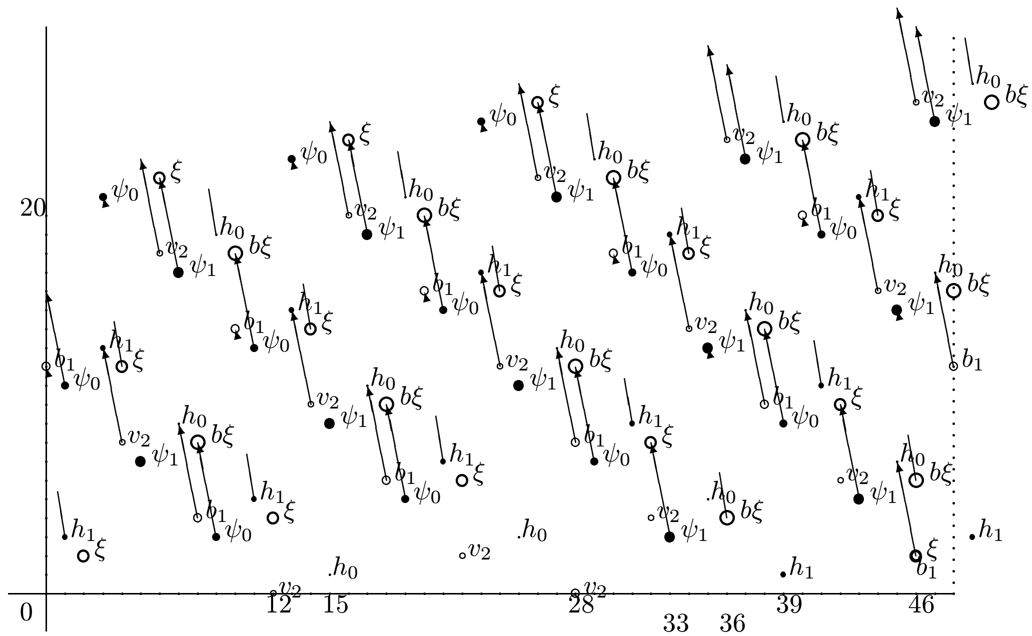
$$v_2^{-1}b_{10}/3v_1 = -v_2^{-1}h_{11}\zeta_2/3v_1 \quad \text{and} \quad v_2^{3t}\xi b_{10}/3v_1^2 = \pm v_2^{3t}\psi_0\zeta_2/3v_1.$$

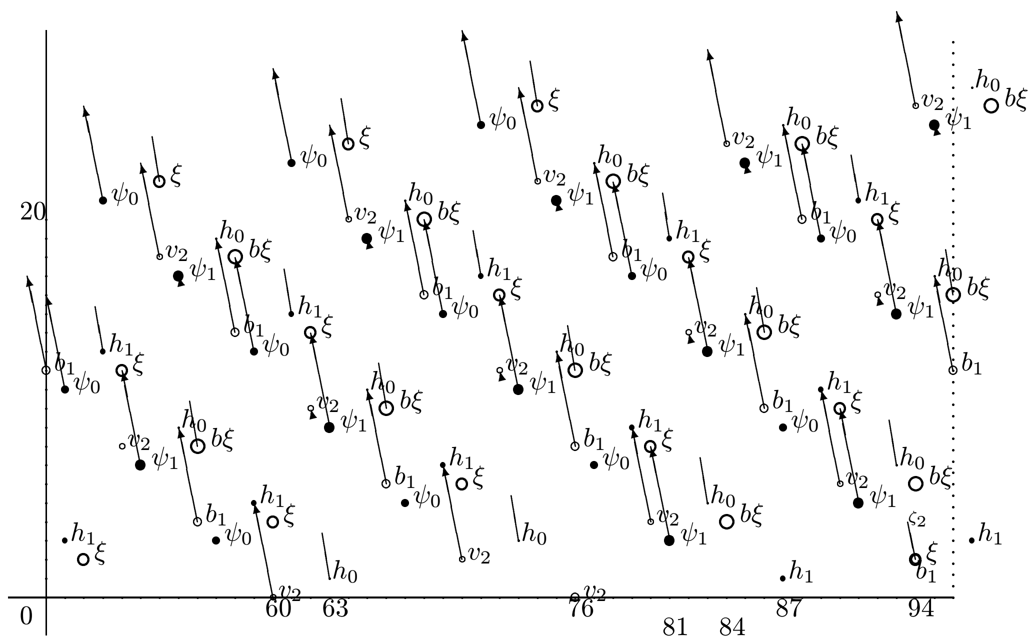
Now we compute

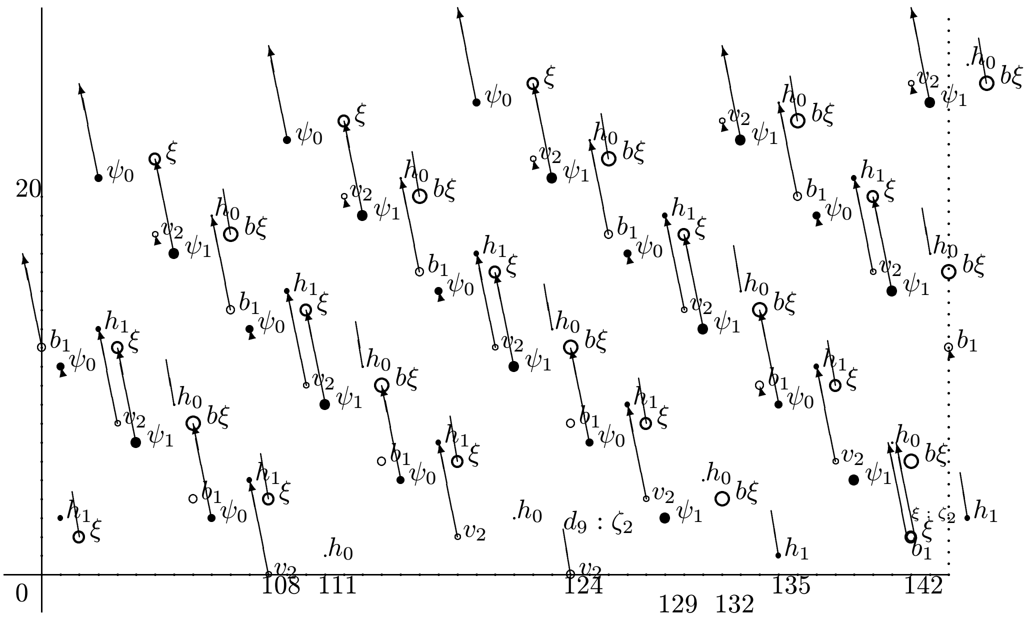
$$b_{10}d_9(v_2^{-1}/3v_1) = d_9(v_2^{-1}b_{10}/3v_1) = -d_9(v_2^{-1}h_{11}\zeta_2/3v_1) = v_2^{-5}b_{11}b_{10}^4\zeta_2/3v_1,$$

and we have $d_9(v_2^{-1}/3v_1) = v_2^{-5}b_{11}b_{10}^3/3v_1$ as desired. In the same way, we have the other case. q.e.d.

Now we display the chart of the Adams-Novikov spectral sequence:


20
0
12
15
28
33
36
39
46


20
0
60
63
76
81
84
87
94



Furthermore, since the elements of the target are shown not to be killed by the differentials in Lemma 6.2, we have other differentials derived from [7]:

(6.3)

$$\begin{aligned}
d_5(v_2^{9t+3}/3v_1^3) &= v_2^{9t+1}h_{10}b_{10}^2/3v_1,\\
d_5(v_2^{9t-1}h_{11}/9v_1^2) &= v_2^{9t-2}h_{10}b_{10}^2\zeta_2/3v_1,\\
d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/3v_1^{2\cdot 3^{n+1}+1}) &= \pm v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/3v_1 \ (n \geq 0),\\
d_5(v_2^{3^{n+2}t+8\cdot 3^n}h_{10}/3v_1^{10\cdot 3^n+1}) &= -v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/3v_1 \ (n > 1).
\end{aligned}$$

This shows that,

Theorem 6.4. *The E_∞-term of $\pi_*(M^2)$ contains the module $\widetilde{G}\bigoplus\widetilde{G}^*\bigoplus\widetilde{GZ}\bigoplus\widetilde{GZ}^*$.*

Here $E(2,1)_*$-modules are given as follows:

$$\begin{aligned}
\widetilde{G} &= B_5(2,2)_*\{v_2/3v_1\}\bigoplus B_4(2,2)_*\{v_2^4b_{11}/3v_1\}\\
&\quad\bigoplus B_3(2,2)_*\{v_2^7h_{10}/3v_1\}\\
&\quad\bigoplus B_2(2,2)_*\{v_2h_{10}/3v_1, v_2^2h_{11}/3v_1, v_2^5h_{11}/3v_1\},\\
\widetilde{G}^* &= B_5(2,2)_*\{v_2^7\psi_1/3v_1\}\bigoplus B_4(2,2)_*\{v_2^3\psi_0/3v_1\}\\
&\quad\bigoplus B_2(2,2)_*\{\xi/3v_1, v_2^3b_{11}\xi/3v_1, v_2^6b_{11}\xi/3v_1\}\\
&\quad\bigoplus \textstyle\sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u\in \boldsymbol{Z}-I(n)\}\\
&\qquad\qquad\bigoplus B_2(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u\in I(n)\}),\\
\widetilde{GZ} &= B_5(2,2)_*\{v_2\zeta_2/3v_1\}\\
&\quad\bigoplus B_3(2,2)_*\{v_2^4b_{11}\zeta_2/3v_1\}\\
&\quad\bigoplus B_2(2,2)_*\{v_2h_{10}\zeta_2/3v_1, v_2^2h_{11}\zeta_2/3v_1, v_2^5h_{11}\zeta_2/3v_1, v_2^7h_{10}\zeta_2/3v_1\},\\
\widetilde{GZ}^* &= B_5(2,2)_*\{v_2^7\psi_1\zeta_2/3v_1\}\bigoplus B_4(2,2)_*\{v_2^3\psi_0\zeta_2/3v_1\}\\
&\quad\bigoplus B_2(2,2)_*\{\xi\zeta_2/3v_1\}\\
&\quad\bigoplus B_1(2,2)_*\{v_2^3b_{11}\xi\zeta_2/3v_1, v_2^6b_{11}\xi\zeta_2/3v_1\}\\
&\quad\bigoplus \textstyle\sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u\in \boldsymbol{Z}-I(n)\}\\
&\qquad\qquad\bigoplus B_2(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u\in I(n)\}),
\end{aligned}$$

for $B_k(2,n)_* = (\boldsymbol{Z}/3)[v_2^{\pm 3^n}, b_{10}]/(b_{10}^k)$ *and* $I(n)$ *given in the introduction.*

Proof. Suppose that $d_r(x) = y \neq 0$ in the Adams-Novikov spectral sequence for $\pi_*(M^2)$. Then y is in the image of $i_* : H^*M_1^1 \to H^*M_0^2$, since y has filtration ≥ 5. Lemma 5.2 shows that $\delta(y) \neq 0$ for the connecting homomorphism $\delta : H^*M_0^2 \to H^*M_1^1$, and so $\delta(x) \neq 0$ and we have $d_r(\delta(x)) = \delta(y)$ in the Adams-Novikov spectral sequence for $\pi_*(L_2W)$. Observing the differentials given in [7] with Lemma 4.4, we see that there is no more new differentials, and obtain the theorem. q.e.d.

7. Application to β-elements

In [1], $H^0M_0^2$ is determined and we see that

$$v_2^s/3^{i+1}v_1^j \in H^0M_0^2 \quad \text{if and only if} \quad (s,j,i+1) \in \boldsymbol{B}.$$

Consider the universal Greek letter map $\eta = \delta'\delta : H^0M_0^2 \to H^2E(2)_*$, where $\delta : H^0M_0^2 \to H^1N_0^1$ and $\delta' : H^1N_0^1 \to H^2E(2)_*$ are the connecting homomorphisms associated to the short exact sequences $0 \to N_0^1 \to M_0^1 \to M_0^2 \to 0$

and $0 \to E(2)_* \to M_0^0 \to N_0^1 \to 0$, respectively. Then the β-elements are defined by

$$\beta_{s/j,i+1} = \eta(v_2^s/3^{i+1}v_1^j).$$

We obtain the following immediately.

LEMMA 7.1. *Mod* $(3, v_1)$, $\beta_1 \equiv b_{10}$, $\beta_2 \equiv v_2 h_{11}\zeta_2$ *and* $\beta_{6/3} \equiv v_2^3 b_{11}$ *in the* E_2*-term* $E_2(S^0)$.

Furthermore, note that $\beta_1' = h_{11} \in E_2(V(0))$ and $\alpha_1 = h_{10} \in E_2(S^0)$. The generators of $\widetilde{G}$ then yields the following elements:

$$\begin{gathered}\eta(v_2^{9t+1}/3v_1) = \beta_{9t+1}, \quad \eta(v_2^{9t+4}b_{11}/3v_1) = \beta_{9t+1}\beta_{6/3},\\ \eta(v_2^{9t+7}h_{10}/3v_1) = \beta_{9t+7}\alpha_1, \quad \eta(v_2^{9t+1}h_{10}/3v_1) = \beta_{9t+1}\alpha_1,\\ \eta(v_2^{9t+2}h_{11}/3v_1) = [\beta_{9t+2}\beta_1'], \quad \eta(v_2^{9t+5}h_{11}/3v_1) = [\beta_{9t+5}\beta_1'].\end{gathered}$$

Now we prove the theorems in the introduction.

PROOF OF THEOREM C. Consider the long exact sequences

$$\cdots \to \pi_*(L_0S^0) \to \pi_*(L_2N^1) \to \pi_{*+1}(L_2S^0) \to \cdots$$

and

$$\cdots \to \pi_*(L_1N^1) \to \pi_*(M^2) \to \pi_{*+1}(L_2N^1) \to \cdots$$

associated to the cofiber sequences of (6.1). Note that $\pi_*(L_0S^0) = \boldsymbol{Q}$ and

$$\pi_*(L_1N^1) = \boldsymbol{Q}/\boldsymbol{Z}_{(3)} \bigotimes \Lambda(y) \bigoplus A$$

shown in [1], where A is the $\boldsymbol{Z}_{(3)}$-module generated by $v_1^{sp^i}/3^{i+1}$ for $i \geq 0$ and $3 \not| \, s \in \boldsymbol{Z}$. Therefore, the module $\widetilde{G} \bigoplus \widetilde{G}^* \bigoplus \widetilde{GZ} \bigoplus \widetilde{GZ}^*$ given in Theorem 6.4 is isomorphically sent to $\pi_*(L_2S^0)$. Theorem C now follows. q.e.d.

PROOF OF THEOREM A. Consider the localization map $\iota : S^0 \to L_2S^0$. Since the induced map $\iota_* : \pi_*(S^0) \to \pi_*(L_2S^0)$ sends a β-element to the same named one, the non-triviality of products of β-elements in $\pi_*(S^0)$ is deduced from the one in $\pi_*(L_2S^0)$. 'if' part now follows immediately from Theorem C except for β_2. For β_2, note that $\beta_{9t+1}\beta_2 = [\beta_{9t+2}\beta_1']\zeta_2 \in \eta(\widetilde{GZ})$ for the universal Greek letter map η. Thus 'if' part for β_2 is shown.

In Lemma 6.2, we have $d_9(v_2^{9t+4}h_{10}/3v_1) = v_2^{9t+1}b_{10}^5/3v_1$ and $d_9(v_2^{9t+8}h_{11}/3v_1) = v_2^{9t+4}b_{11}b_{10}^4/3v_1$, which yield

$$d_9(\beta_{9t+4}\alpha_1) = \beta_1^5\beta_{9t+1} \quad \text{and} \quad d_9(\beta_{9t+8}h_{11}) = \beta_{9t+1}\beta_{6/3}\beta_1^4 \tag{7.2}$$

in the E_9-term $E_9^*(L_2S^0)$ by considering the image of the universal Greek letter map. In the same manner, the equation $d_5(v_2^{9t+4}\zeta_2/3v_1) = v_2^{9t+2}h_{11}\zeta_2b_{10}^2/3v_1$ in Lemma 6.2 yields

$$d_5(\beta_{9t+4}\zeta_2) = \beta_{9t+1}\beta_2b_{10}^2 \tag{7.3}$$

in the E_9-term $E_5^*(L_2S^0)$. If $t \geq 0$, then the equations (7.2) and (7.3) also hold in the Adams-Novikov spectral sequence for $\pi_*(S^0)$, since the elements appeared in (7.2) and (7.3) are also defined in $E_2(S^0)$. q.e.d.

Proof of Theorem E. In the proof of Theorem 6.4, we read off that the elements on the 0th line hit nothing except for the β-elements given by $\boldsymbol{B}^c$. Therefore, we obtain Theorem E. q.e.d.

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