THE TOPOLOGY OF SPACES OF KNOTS

DEV P. SINHA

Abstract. We present two models for the space of knots which have endpoints at fixed boundary points in a manifold with boundary, one model defined as an inverse limit of mapping spaces and another which is cosimplicial. These models are homotopy equivalent to the corresponding knot spaces when the dimension of the ambient manifold is greater than three, and there are spectral sequences with identifiable E^1 terms which converge to their cohomology and homotopy groups. The combinatorics of the spectral sequences is comparable to combinatorics which arises in finite-type invariant theory.

Contents

1. Introduction

In this paper, we study the topology of spaces of embeddings of one-dimensional manifolds, $\mathrm{Emb}(I, M)$, where M is a smooth manifold with boundary and the embeddings must send the end-points of I to fixed boundary points of M with specified tangent vectors at those points. These spaces have gained considerable attention recently, especially in the work of Vassiliev [35] and Kontsevich [22, 23]. Also, Hatcher has been studying the topology of various components of the space of knots in $\mathbb{R}^3$ [20].

We build on the approach of Goodwillie and his collaborators [18, 17, 16, 36], which is known as the calculus of embeddings (or more generally of isotopy functors). Their approach produces a "Taylor Tower" of "polynomial approximations" to embedding spaces. From the spaces in this tower we produce two new models. The first, which we call a mapping space model, is useful in that it is simple, makes clear some

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relevant geometry (that of the evaluation map), and is closely related to point of view of Bott and Taubes [4] which produces knot invariants in dimension three and de Rham cohomology classes of knot spaces more generally [7]. The second model, which is cosimplicial, makes clear analogies with loop spaces and is especially convenient for calculations of homotopy and cohomology groups. Both of these models make use of compactifications of configuration spaces, essentially due Fulton and MacPherson [13] and done in the category of manifolds by Axelrod and Singer [2], but with some changes needed for the cosimplicial model [31].

Applying the cohomology and homotopy spectral sequences for the cosimplicial model we obtain spectral sequences for both cohomology and homotopy groups of $\mathrm{Emb}(I, M)$ which converge when the dimension of the ambient manifold is greater than three. When M is $\mathbb{R}^k \times I$, the cohomology spectral sequence is reminiscent of those of Vassiliev [35] and Kontsevich [22, 23]. The calculus of embeddings approach has a few advantages.

- There is a map directly from spaces of embeddings to their polynomial approximations.
- By work of Goodwillie and Klein [18] one may study embeddings of arbitrary manifolds when the domain has codimension three or greater, which is a much better range of dimensions than up to half the dimension, as Vassiliev and Kontsevich claim is possible with their techniques.
- Because one does not proceed through duality, it is possible to study homotopy groups as well as cohomology groups of embedding spaces.
- For knots our models are close to the geometry needed to define the Bott-Taubes and Kontsevich integrals.

The homotopy spectral sequence we produce in this paper gives the first computations of homotopy groups of embedding spaces. The rows in the E^1-term are similar to complexes defined by Kontsevich [23]. We study this spectral sequence in [27], giving explicit computations in low dimensions.

When the dimension of M is three it is not known whether our models are equivalent to knot spaces, but one can still pull back zero-dimensional cohomology classes (that is, knot invariants) from our models. These invariants seem to be connected in various ways with the theory of finite-type invariants. We indicate such connections as well as give directions for further work in Section 9.

1.1. **Basic definitions, notation and conventions.** We choose a variant of $\mathrm{Emb}(I, M)$ so that a knot depends only on its image. Fix a Riemannian metric on M. Note that many of our constructions will depend on this metric, but changing the metric will always result in changing these constructions by a homeomorphism, so we make little mention of the metric in general.

Definition 1.1. *Define* $\mathrm{Emb}(I, M)$ *to be the space of injective* C^∞ *maps of constant speed from* I *to* M*, whose values and unit tangent vectors at* 0 *and* 1 *are specified by fixed unit tangent vectors in* ∂M*.*

By convention, throughout this paper an embedding of any interval containing 0 or 1 must send that (those) point(s) to the designated points on the boundary of M, with the designated unit tangent vectors at that (those) point(s).

We use spaces of ordered configurations, and modifications thereof, extensively. There are two lines of notation for these spaces in the literature, namely $C_n(M)$ and $F(M, n)$. Persuaded by Bott, we choose to use the $C_n(M)$ notation. Note, however, that $C_n(M)$ in this paper is $C_n^0(M)$ in [4] and that $C_n[M]$ in this paper is $C_n(M)$ in [4]. Indeed, we warn the reader to pay close attention to the parentheses in our notation, because for the sake of brevity of notation they account for all of the distinction between various configuration spaces. We have attempted to make our choice of parentheses somewhat intuitive: $C_n(M)$ is the open configuration space; $C_n[M]$ is the Fulton-MacPherson compactification, the most natural closure; $C_n\langle[M]\rangle$ is a quotient of $C_n[M]$. In previous versions of this paper we had chosen to follow the $F(M, n)$ line of notation, and in homage to Fulton and MacPherson we named this compactification of $F(M, n)$ by $FM(M, n)$.

We work with stratifications of spaces, in a fairly naive sense. A stratification of a space X is a collection of subspaces $\{X_c\}$ such that the intersection of the closures of any two strata is the closure of some stratum. We associate a poset to a stratification by saying that $X_c \leq X_d$ if X_c is in the closure of X_d. If two spaces X and Y are stratified with isomorphic associated posets, then a stratum-preserving map is one in which the closure of X_c maps to the closure of Y_c for every c.

Through section 5 we try to assume minimal knowledge of homotopy limits and other constructions from algebraic topology, but in section 6 we do assume such familiarity.

There are many spaces, categories and functors involved in defining our constructions, so it may be helpful that we have tried to consistently follow some conventions. For the most part we use capitalized letters in the standard Roman font to denote spaces and categories. Functors are in the caligraphic font.

1.2. **Acknowledgments.** The author is deeply indebted to Tom Goodwillie. As mentioned in [17] (and also just briefly in [4]), Goodwillie has known for some time that one should be able to use Theorem 6.5 to show that the polynomial approximations to spaces of knots are given by partial totalizations of a cosimplicial space made from spaces homotopy equivalent to configuration spaces.

2. Heuristic understanding of the mapping space and cosimplicial models

Given a knot $\theta\colon I \to M$ and n distinct times, one may produce n distinct points in the target manifold by evaluating the knot at those points. Because a knot has nowhere vanishing derivative (which we may think of as having no infinitesimal self-intersections), one may in fact produce a collection of n unit tangent vectors at these distinct points. Let $Int(\Delta^n)$ be the open n-simplex and $C_n(M)$ be the configuration space of n distinct ordered points in M, defined as a subspace of M^n, and let $C'_n(M)$ be defined by the pull-back square

$$\begin{array}{ccc} C'_n(M) & \longrightarrow & (STM)^n \\ \downarrow & & \downarrow \\ C_n(M) & \longrightarrow & M^n, \end{array}$$

where STM is the unit tangent bundle of M.

We now define the evaluation map as follows.

Definition 2.1. *Given a knot θ let $ev_n(\theta)\colon Int(\Delta^n) \to C'_n(M)$ be defined by*

$$(ev_n(\theta))(t_1, \cdots t_n) = (u(\theta'(t_1)), \cdots, u(\theta'(t_n))),$$

where $u(v)$ is the unit tangent vector in the direction of the tangent vector v. Let $ev_n\colon \mathrm{Emb}(I, M) \to Maps(Int(\Delta^n), C'_n(M))$ be the map which sends θ to ev_θ.

The evaluation map is sometimes called a Gauss map, as it generalizes the map used to define the linking number.

For any n, ev_n maps the embedding space injectively to $\mathrm{Maps}(Int(\Delta)^n, C'_n(M)$. We cannot expect to use ev_n to study the embedding space as it stands, needing to add a boundary to the open n-simplex which we may then fix in order to define a mapping space with more topology. The appropriate boundary turns out to not be the standard boundary of an n-simplex but the one given by the Fulton-MacPherson compactifications, which we use to compactify $C'_n(M)$ as well. These compactifications are remarkable in that they have the same homotopy type as the open configuration spaces and they are functorial for embeddings. The Fulton-MacPherson compactifications are manifolds with corners, and for any θ the evaluation map $ev_n(\theta)$ respects the stratification. Our mapping space models are essentially the spaces of stratum-preserving maps (with one important additional technical condition). The maps from $\mathrm{Emb}(I, M)$ to these models are extensions of the ev_n.

Our cosimplicial models are closely related to these mapping space models. Recall the cosimplicial model for the based loop space, ΩM (see for example [25]). The nth entry of this cosimplicial model is given by the Cartesian product M^{n-1}. The coface maps are diagonal (or "doubling") maps, and the codegeneracy maps are projections (or "forgetting" maps). The map from the loop space to the nth total space of this cosimplicial space is the adjoint of the evaluation map from the simplex to M^{n-1}, which is the appropriate target of this evaluation map. The map from the loop space to the total space is a homeomorphism if $n \geq 2$. Correspondingly, the homotopy spectral sequence is trivial. On the other hand, the filtration which arises on homology gives rise to the Eilenberg-Moore spectral sequence [25].

To make a cosimplicial model of an embedding space it is natural to try to replace M^n by the configuration space $C'_n(M)$ as the nth cosimplicial entry, as we have seen above that $C'_n(M)$ is a natural target for the evaluation map for embeddings. The codegeneracy maps can be defined as for the loop space, by "forgetting" a point in a configuration. The coface maps are problematic, as one cannot "double" a point in a configuration to get a new configuration. One is tempted to add a point close to the point which needs to be doubled, but in order for the composition of doubling and forgetting to be the identity, one would need to add a point which is "infinitesimally close." The appropriate technical tool needed to overcome this difficulty is once again the Fulton-MacPherson compactification of configuration spaces. We will find that, while one can define diagonal maps for Fulton-MacPherson compactifications, these maps do not satisfy the cosimplicial axioms. We proceed to use a technical variant of these compactifications developed in [31] so that the cosimplicial axioms are satisfied.

3. Goodwillie's cutting method

The idea of one version of Goodwillie's cutting method is to approximate the space of embeddings of a manifold M in N with by using spaces of embeddings of $M - A$ in N for codimension zero submanifolds A of M. Fix a collection $\{J_i\}$, for $i = 1$ to n of disjoint sub-intervals of I.

Definition 3.1. *Define $E_S(M)$ for $S \subseteq \{1, \cdots, n\}$ to be the space of embeddings of $I - \bigcup_{s\in S} J_s$ in M whose speed is constant on each component of $I - \bigcup_{s\in S} J_s$.*

If $S \subseteq S'$ there is a restriction map from $E_S(M)$ to $E_{S'}(M)$. These restriction maps commute with one another, so if for example $S' = S \cup \{i, j\}$ there is a commutative square

$$\begin{array}{ccc} E_S(M) & \longrightarrow & E_{S\cup i}(M) \\ \downarrow & & \downarrow \\ E_{S\cup j}(M) & \longrightarrow & E_{S'}(M). \end{array} \tag{1}$$

Hence, given a knot one can produce a family of compatible elements of E_S for every non-empty S. Conversely, such a compatible family (if $n > 2$) determines a knot. Now consider instead of a family which is compatible on the nose, one which is compatible up to homotopy. So, for every square as in (1) above we consider $\alpha \in E_{S\cup i}$ and $\beta \in E_{S\cup j}$ with an isotopy between their restrictions in $E_{S'}$. Moreover, for every cube one gets by considering all subsets T such that $S \subseteq T \subseteq S \cup \{i, j, k\}$, one has an isotopy of these isotopies, namely a map of Δ^2 into $E_{S\cup\{i,j,k\}}$ whose restriction to the boundary is the three isotopies produced for the last three faces of the cube (the last three faces being the ones with $E_{S\cup\{i,j,k\}}$ as the terminal space). Such a system is more complicated than a compatible family, but is in fact more manageable from an algebraic topologist's perspective, and Goodwillie shows that spaces of families of punctured knots which are compatible up to homotopy are in fact good substitutes for spaces of knots (see Theorem 3.7).

The way in which one formalizes these compatible families up to homotopy is through the language of homotopy limits. We pause to state some basic properties of homotopy limits. A reader unfamiliar with them is encouraged to look at [5] and [15] for a more extensive treatment. A reader who has seen homotopy limits may want to skip the next subsection.

3.1. **Homotopy limits.** Limits in the category of spaces can change dramatically when the maps involved are changed by a homotopy, which means that when a space is defined as such a limit it is difficult, for example, to recover its cohomology and homotopy groups from those of the constituent spaces and the induced maps between them. For example, consider the diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y. \tag{2}$$

To see an example of a limit which is not homotopy invariant, take X and Y to both be points. This limit can either be a point or it can be empty, depending on whether the images of X and Y coincide, a condition which (if Z itself is path-connected and is not a single point) can change if f or g is changed by a homotopy. On the other hand, there are familiar cases in which this limit is homotopy invariant, if say $Y \xrightarrow{g} Z$ defines a fiber bundle, in which case the limit is the pull-back bundle. Indeed, if either f or g is a fibration, this limit is homotopy-invariant. One may think of the homotopy limit in general as the limit of the diagram one obtains by replacing spaces and maps in the original diagram by fibrations (using path-space constructions).

We formalize as follows. A diagram of spaces is simply a functor from a small category C, which we sometimes refer to as an indexing category, to the category of spaces. Recall that if C is a small category, the realization of the nerve of C, denoted $|C|$, is a simplicial complex with a vertex for every object in the category, an edge for every morphism, a two simplex for every composition of two morphisms, and so forth. Also, recall that if c is an object of C, the category $C \downarrow c$ of objects over c has objects which are maps with target c and morphisms given by morphisms in C which commute with these structure maps. Note that the nerve of $C \downarrow c$ is contractible since it has a final object, namely c mapping to itself by the identity morphism. A morphism g from c to d induces a map $|g|$ from $C \downarrow c$ to $C \downarrow d$.

Definition 3.2. *The homotopy limit of a functor $\mathcal{F}$ from a small category C to the category of spaces is the subspace of*

$$\prod_{c \in C} Maps(|C \downarrow c|, \mathcal{F}(c))$$

of collections of maps $\{f_c\}$ such that for all morphisms $c \to d$ in C the following square commutes

$$\begin{array}{ccc} |C \downarrow c| & \xrightarrow{f_c} & \mathcal{F}(c) \\ {\scriptstyle |g|}\big\downarrow & & {\scriptstyle X(g)}\big\downarrow \\ |C \downarrow d| & \xrightarrow{f_d} & \mathcal{F}(d). \end{array}$$

The following special case of the definition is useful. Let C be a poset with a unique maximal element m, and let $\mathcal{F}$ be a functor from C to based spaces in which all morphisms are inclusions of NDR pairs in which the subspace in each pair is closed. Note that for any object c of C the category $C \downarrow c$ is naturally a sub-category of $C \downarrow m$, and that $C \downarrow m$ is itself isomorphic to C. Hence, the $|C \downarrow c|$ define a stratification of $|C \downarrow m|$, and the poset associated to this stratification is C. Similarly, if all morphisms of $\mathcal{F}$ are inclusions of closed subspaces then $\mathcal{F}(m)$ is stratified by the $\mathcal{F}(c)$ with associated poset isomorphic to C once again. The following proposition follows immediately from unraveling the definitions.

Proposition 3.3. *Let C be a poset with a unique maximal element m, and let $\mathcal{F}$ be a functor from C to the category of spaces in which all morphisms are inclusions of NDR pairs in which the subspace in each pair is closed.. Then the homotopy limit of $\mathcal{F}$ is the space of all stratum-preserving maps from $|C \downarrow m| = |C|$ to $\mathcal{F}(m)$.*

Some basic properties of homotopy limit are the following (see [15] and [5]).

- Homotopy limit is a contravariant functor from the category of diagrams of spaces (that is, functors from some small category to spaces) to the category of spaces.

- The homotopy limit is homotopy invariant in the sense that if there is a map of diagrams indexed by the same category $\mathcal{F} \to \mathcal{G}$ in which each $\mathcal{F}(c) \to \mathcal{G}(c)$ is a (weak) homotopy equivalence, then the induced map on homotopy limits is a (weak) homotopy equivalence.
- There is a canonical map from the limit to the homotopy limit, as the subspace in which each f_d is constant, which is a homotopy equivalence if "enough of the maps in the diagram are fibrations."
- If a functor $\mathcal{F}$ from D to C is "left cofinal" then composition with $\mathcal{F}$ induces a weak equivalence on homotopy limits. Let $\mathcal{F} \downarrow c$ be the category whose objects are pairs (d, f) where d is an object of D and f is a morphism from $\mathcal{F}(d)$ to c, and morphisms are given by morphisms g in D such that $\mathcal{F}(g)$ commutes with the structure maps. A functor $\mathcal{F}$ is left cofinal if $|\mathcal{F} \downarrow c|$ is contractible for every object $c \in C$.

3.2. **Polynomial approximations to spaces of knots.** With homotopy limits in hand, we may define Goodwillie's models. In the calculus of isotopy functors [36, 16], a suitable functor from the category of open subsets of a manifold to the category of spaces is polynomial roughly if it is determined by its value on open subsets which are diffeomorphic to the disjoint union of n or fewer Euclidean balls. One must keep track of restrictions between these distinguished open sets, so the main diagrams which appear are cubical diagrams.

Definition 3.4. *Let $[n]$ be the category of all subsets of $\{0, \cdots, n\}$ where morphisms are defined by inclusion. Let $[n]_0$ be the full subcategory of non-empty subsets. A cubical diagram is a functor from $[n]$ to the category of based spaces.*

Note that the nerve of $[n]$ is an $n+1$-dimensional triangulated cube whereas the nerve of $[n]_0$ consists of n faces of that cube and is in fact isomorphic to the barycentric subdivision of the n-simplex.

We now formalize the notion of families of punctured knots which are compatible up to homotopy.

Definition 3.5. *Let $\{J_i\}$, for $i = 1$ to n be a collection of sub-intervals of I. Let $\overline{\mathcal{E}_n}(M)$ be the cubical diagram which sends $S \in [n]$ to the space of embeddings $E_S(M)$, which by definition is $Emb(I - \bigcup_{s \in S} J_s, M)$ and sends the inclusion of $S \subset S'$ to the restriction map. Let $\mathcal{E}_n(M)$ be the functor one obtains by precomposing $\mathcal{E}_n(M)$ with the inclusion of $[n]_0$ in $[n]$.*

A cubical diagram such as $\overline{\mathcal{E}_n}(M)$ determines a map from the initial space in the cube, in this case $\mathrm{Emb}(I, M)$, to the limit of the rest of the cube because by definition the initial space maps to all others compatibly. For $n \geq 3$ this map is a homeomorphism, since as mentioned above a family of compatible embeddings of $I - J_i$ for $i = 1$ to n uniquely determine an embedding of I and vice-versa. Such a cube also determines a map from the initial space in the cube to the homotopy limit of the rest of the cube.

Definition 3.6. *Let $P_n\mathrm{Emb}(I, M)$ be the homotopy limit of $\mathcal{E}_n(M)$, and let α_n be the canonical map from $\mathrm{Emb}(I, M)$, which is the initial space in the cube above, to $P_n\mathrm{Emb}(I, M)$.*

$P_n\mathrm{Emb}(I, M)$ is a degree n polynomial approximation to the space of knots in the sense of the calculus of isotopy functors [36]. The following result states that these polynomial approximations to spaces of knots converge when the dimension of the ambient manifold is greater than three. We take this the starting point of our work, which focuses on understanding these models better. We recommend consulting [18, 16, 36] to understand the beautiful underlying theory. A good starting point is to understand weaker statements of the following theorem (which apply in our case to when the range manifold has dimension five or greater), which essentially use only dimension counting arguments and the Blakers-Massey theorem. These simple arguments are to appear in notes from expository lectures by Goodwillie [19].

Theorem 3.7 ([18]). *If the dimension of M is greater than three, then the map α_n from $\mathrm{Emb}(I, M)$ to $P_n\mathrm{Emb}(I, M)$ induces isomorphisms on homology and homotopy groups through dimension $(n-1)(\dim M - 3)$.*

Note that $[n]_0$ is a sub-category of $[n+1]_0$ (using the standard injection of $\{1, \cdots, n\}$ into $\{1, \cdots, n+1\}$ for definiteness), and we may choose the intervals S_j for the homotopy limit defining $P_n\mathrm{Emb}(I, M)$ to be a subset of those defining $P_{n+1}\mathrm{Emb}(I, M)$. Thus $P_{n+1}\mathrm{Emb}(I, M)$ maps to $P_n\mathrm{Emb}(I, M)$ through a restriction map r_n such that $r_n \circ \alpha_{n+1} = \alpha_n$. By the above theorem, the maps α_n are inducing isomorphism on homology and homotopy groups through a range which always increases, so we deduce the following.

Corollary 3.8. *If the dimension of the ambient manifold is greater than three, the map from* $\mathrm{Emb}(I, M)$ *to the inverse limit*

$$P_0\mathrm{Emb}(I, M) \leftarrow P_1\mathrm{Emb}(I, M) \leftarrow P_2\mathrm{Emb}(I, M) \leftarrow \cdots$$

given rise to by the α_n*'s is a weak equivalence.*

A homotopy limit of a diagram essentially only depends on the spaces and maps in the diagram up to homotopy. Because the spaces $E_S(M)$ are homotopy equivalent to $C'_n(M)$ (see the proof of Proposition 5.14), we are lead to search for models equivalent to $P_n\mathrm{Emb}(I, M)$ which involve configuration spaces (with tangential data). Appropriate compactifications of these configuration spaces are essential to our construction of such models.

4. Fulton-MacPherson compactifications and categories of trees

We use two different versions of Fulton-MacPherson compactifications of configuration spaces [13] in the setting of manifolds as first defined by [2]. In this section we recall some of their properties, all of which are proved in [31]. In outline, we define compactifications of configuration spaces, with the important property that the inclusion of the open configuration space into each of these compactifications is a homotopy equivalence. These compactifications are naturally stratified, and the in the case of configurations in the interval the compactification is isomorphic to the realization of the nerve of the category of strata. When we include tangent vectors at the points in configurations there are natural diagonal (or coface) maps as well as projections between these spaces.

We start with the Fulton-MacPherson compactification of the space of ordered configurations of n points in a manifold M.

4.1. $C_n[M]$**, the basic compactification and** Ψ_n**, the basic category of trees.** The definition of these compactifications is not entirely enlightening on its own. The structure is better understood through analysis of the stratification of these spaces, which we will give below.

In our approach, the key case to define is the case of Euclidean space. Let $x = (x_1, \ldots, x_n) \in C_n(\mathbb{R}^k)$. Define $\alpha_{ij} \colon C_n(\mathbb{R}^k) \to S^{k-1}$ as sending x to the unit vector in the direction of $x_i - x_j$. Define $\beta_{ijk} \colon C_n(\mathbb{R}^k) \to I$ as sending x to $\arctan(||x_i - x_j||/||x_i - x_k||)$.

Definition 4.1. *Let* $C_n[\mathbb{R}^k]$ *to be the closure of* $C_n(\mathbb{R}^k)$ *included in* $(\mathbb{R}^k)^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$*, where the map from* $C_n(\mathbb{R}^k)$ *to this product is the standard inclusion on the first factor, the map* $\prod_{i<j} \alpha_{ij}$ *on the second factor, and the map* $\prod_{i<j<k} \beta_{ijk}$ *on the third factor. For a general compact manifold, one may embed the manifold in some* $\mathbb{R}^k$ *and thus restrict the maps* α_{ij} *and* β_{ijk}*. One may then define the compactification as the closure of* $C_n(M)$ *in* $M^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$

We show that this definition for general compact manifolds is, up to diffeomorphism, independent of embedding in [31].

Remark. *The standard definition of these compactifications, developed in the setting of differential topology by Axelrod and Singer, takes the closure inside a product of blow-ups of* M^i*. The definition above, with which we work in* [31]*, is more elementary.*

We now list some important properties of these compactifications.

- $C_n[M]$ is compact if M is compact.

- $C_n[M]$ is a manifold with corners and is thus naturally stratified.
- There is a canonical inclusion $C_n(M) \to C_n[M]$ which realizes $C_n(M)$ as the interior of $C_n[M]$. Thus, these spaces are homotopy equivalent, since a topological manifold with boundary is homotopy equivalent to its interior.
- There is a map from $C_n[M]$ onto M^n which extends the inclusion of $C_n(M)$ in M^n, namely the restriction to $C_n[M]$ of the projection of $M^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$ onto M^n. By projecting further onto the factors of M^n we may thus refer to the "ith point in a (degenerate) configuration in $C_n[M]$." If a point in $C_n[M]$ has image $x_1, \cdots x_n$ in M^n in which two (or more) of the x_i coincide we say that such a point has a double (or triple, etc.) point.
- If in the projection of $x \in C_n[M]$ onto M^n some $x_i = x_j$, then there is a well-defined $v_{ij} \in STM$, which we call the relative vector, which gives their "infinitesimal relative position". When M is $\mathbb{R}^k$, this v_{ij} is given by the restriction of the projection of $(\mathbb{R}^k)^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$ onto the ij factor of S^{k-1}.
- Given an embedding $f : M \hookrightarrow N$, there is an associated inclusion of $C_n[M]$ of $C_n[N]$ which extends the inclusion of $C_n(M)$ in $C_n(N)$, whose existence follows from using $M \hookrightarrow N \hookrightarrow \mathbb{R}^k$ to define $C_n[M]$. By abuse, we also call the map on $C_n[M]$ the evaluation map and denote it $ev_n(f)$. This map is an inclusion of manifolds with corners.

Better understanding of these compactifications comes from studying the stratification of $C_n[M]$ defined by its structure as a manifold with corners. The spaces in this stratification are built from spaces of "infinitesimal configurations."

Definition 4.2. *Define $IC_i(M)$ to be the space of i distinct points in TM all lying in one fiber, modulo translation and scaling in that fiber. For example, $IC_2(M)$ is diffeomorphic to STM. Let p be the projection of $IC_i(M)$ onto M.*

Let $\pi \colon C_n(M) \to M$ be the projection onto the last coordinate.

Theorem 4.3. *The codimension one strata of $C_n[M]$ are diffeomorphic to spaces Z_i given as pullbacks*

$$\begin{array}{ccc} Z_i & \longrightarrow & IC_i(M) \\ \downarrow & & \downarrow p \\ C_{n-i+1}(M) & \xrightarrow{\pi} & M. \end{array}$$

Informally, a sequence of configurations in $C_n(M)$ - the interior of $C_n[M]$ - may degenerate by having i points in M approach one another in a generic fashion. If this sequence converges, the limit of these i points will define an element of $IC_i(M)$ as an "infinitesimal configuration". The limit point in M of these i approaching points along with the $n-i$ points in M which are not approaching one another together define a configuration of $n-i+1$ points.

Higher codimension strata arise from two phenomena related to this kind of degeneration of a configuration. Disjoint subsets within a configuration may degenerate. Or, a subset of i points may degenerate but not have a well defined limit in $IC_i(M)$ because a subset of k points in this subset "degenerates faster". Indeed, suppose that for some compact M there were a sequence $\{(x_1, \cdots, x_n)_i\}$ of points in $C_n(M)$ in which $\{x_1\}_i$, $\{x_2\}_i$ and $\{x_3\}_i$ were approaching the same point x in M. If these three points were not approaching generically in the sense that $\lim(d(x_1, x_2)/d(x_1, x_3))$ were zero, then one would not be able to define a point in $IC_3(M)$ from their limit in $C_n[M]$. Instead, one may define two points in $IC_2(M)$, one coming from the limit of x_1 and x_2 and another coming from the limit of x_1 (or x_2) and x_3.

In order to account for different nested subsets in a configuration which may be degenerating, we introduce a category of trees.

Definition 4.4. *Define an f-tree to be a rooted, connected tree, with labelled leaves and with no bivalent vertices. Thus, an f-tree is a connected acyclic graph with a specified vertex called the root and denoted*

v_r. The root may have any valence, but other vertices may not be bivalent. The univalent vertices (other than perhaps the root) are called leaves, and each leaf is labelled with a distinct integer between one and the number of leaves, inclusively. Let $\#v$ be equal to one if v is a leaf vertex or the valence minus one for internal vertices. Note that there is a unique path from any vertex to the root vertex, which we call the root path for that vertex. For any edge, the vertex which is closer to the root is called its initial vertex, and the other vertex is called the terminal vertex.

We will be interested in the set of f-trees as a set of objects in a category in which morphisms are defined by contracting edges.

Definition 4.5. *Given a tree T and a set of non-root, non-leaf edges E the contraction of T along E is the tree T' obtained as follows. For each edge $e \in E$, we identify its initial vertex v_i with its terminal vertex v_t and remove e from the set of edges. Each edge whose initial vertex is v_t is modified so as to have initial vertex v_i.*

Definition 4.6. *Define Ψ_n to be the category whose objects are f-trees with n leaves. There is a morphism in Ψ_n from T to T' if T' is isomorphic to a contraction of T along some set of edges.*

We use f-trees to label strata roughly by associating to each vertex the space of infinitesimal configurations of k-points, where k is the number of edges emanating from that vertex. Let $E(T)$ be the set of root edges of T and let $r(T)$ be its cardinality, namely the valence of the root vertex. For a root edge e let $V(e)$, the set of vertices over e, be the set of vertices for whom the root path goes through e.

Definition 4.7. *Let T be an f-tree. Given a root edge e of T define $IC_e(M)$ to be subspace of the product $\prod_{v\in V(e)} IC_{\#v}(M)$ of tuples of infinitesimal configurations all sitting over the same point in M, and let p_e be the projection from $Z_e(M)$ onto M. Define the space $C_T(M)$ as a pullback as follows*

$$\begin{array}{ccc} C_T(M) & \longrightarrow & C_{r(T)}(M) \\ \big\downarrow & & \big\downarrow \\ \prod_{e\in E(T)} IC_e(M) & \xrightarrow{\prod p_e} & M^{r(T)}. \end{array}$$

The following theorem, which determines $C_n[M]$ as a point-set, is proven in [31].

Theorem 4.8. *Let M have dimension greater than one. The category of strata of $C_n[M]$ as a manifold with corners is isomorphic to Ψ_n. The stratum corresponding to a tree T is diffeomorphic to $C_T(M)$.*

It is a worthwhile exercise to check that the codimension of the stratum corresponding to T is the number of internal (that is non-root, non-leaf) vertices of T. Note as well that since the only closed manifolds in the definition of these strata are $IC_2(M)$, the only strata which are closed are those labelled by trees whose root vertex is univalent, and all of whose internal vertices are trivalent, which are the strata with the largest codimension.

To capture the way in which the strata of $C_n[M]$ are topologized as a whole, we will indicate the behavior of sequences in $C_n(M)$ which converge to the stratum $C_T(M)$. We start with the following.

Definition 4.9. *Define an exclusion relation R on a set A to be a subset of A^3 such that the following properties hold*

1. *$(x,x), y \in R$ for all $x \neq y$ in A.*
2. *If $(x,y), z \in R$ then $(y,x), z \in R$.*
3. *If $(x,y), z \in R$ and $(w,x), y \in R$ then $(w,x), z \in R$.*

Next note that once two root paths from v_1 and v_2 meet at some vertex v, they must coincide. We call this path from v to the root the coincident path of the two vertices and denote it $Co(v_1, v_2)$.

Definition 4.10. *Given an f-tree T, let $R(T)$ be the exclusion relation on the set of labels of leaves in which $(i,j),k \in R(T)$ if for leaves l_i, l_j and l_k $Co(l_i,l_j)$ contains $Co(l_i,l_k)$ and $Co(l_j,l_k)$.*

Note that a tree is not quite determined by its associated exclusion relation. If the root vertex of T is univalent and T' is obtained from T by contracting the root edge, then $R(T) = R(T')$.

From the following theorem, one can determine how the strata named in Theorem 4.8 are topologized.

Theorem 4.11. *Let T be an f-tree. Let $s = \{(x_1, \cdots, x_n)_i\}$ be a sequence of of points in $C_n(M)$ which converges in $C_n[M]$. If the root valence of T is greater than one, then s converges to a point in the closure of $C_T(M)$ if and only if the limit of $d(x_i,x_j)/d(x_i,x_k)$ approaches zero for every $(i,j),k \in R(T)$. If the root valence of T equals one, s converges to a point in the closure of $C_T(M)$ if and only if the previous condition is satisfied and all of the x_i approach the same point in M.*

4.2. $C_n[M,\partial]$ **for manifolds with boundary, associahedra, and planar trees.**

Definition 4.12. *If M is a manifold with boundary with two distinguished points y_0 and y_1 in its boundary (as in the definition of $\mathrm{Emb}(I,M)$), define $C_n[M,\partial]$ to be the subspace of $C_{n+2}[M]$ of points whose image under the projection to M^{n+2} is of the form $(y_0, x_1, \cdots x_n, y_1)$.*

Note that the category of strata of $C_n[M,\partial]$ is isomorphic to the subcategory of Ψ_{n+2} of trees such that the coincident path of the first and last leaf in the tree is simply the root vertex, which implies that the valence of the root vertex of such a tree is at least two.

An important case of such a manifold is that of the interval with its two boundary points. Recall that $C_n(I)$ is a disjoint union of $n!$ copies of the subspace of a closed simplex consisting of the interior and two faces. We consider a single component, say the component in which the points in the interval are in the standard order. Recall that the Stasheff polytope, or associahedron, which we denote A_n has vertices labelled by ways in which one can parenthesize $n+2$ letters and edges given by applications of the associative law. We will give a more full description below.

Proposition 4.13 (See [31]). *Each component of $C_n[I,\partial]$ is isomorphic as a manifold with corners to the nth Stasheff polytope, A_n.*

For example, $A_2 = C_2[I,\partial]$ is a pentagon. Three of its faces correspond to faces of the simplex, where if $0 \le t_1 < t_2 \le 1$ is a point in $C_2(I)$, these faces are the "$t_1 = 0$", "$t_2 = 1$" and "$t_1 = t_2$" faces. The two other faces project onto the "$t_1 = t_2 = 0$" and "$t_1 = t_2 = 1$" vertices of the simplex. The limit point of a sequence of points in the interior of the two-simplex in which t_1 and t_2 approach zero is given by $\lim \frac{t_1}{t_2}$. See Figure 4.2.

As we have mentioned, one of the main objects of study in this paper is the evaluation map for a knot, which we may now see extends to a map from A_n to $C_n[M,\partial]$. The evaluation map preserves strata, but note that since the interval is one-dimensional, the category of non-empty strata of A_n is a proper subcategory of Ψ_n.

Definition 4.14. *Define Φ_n to be subcategory of Ψ_{n+2} consisting of trees which satisfy two conditions:*

- *The coincident path of the first and last leaf in the tree is simply the root vertex.*
- *There is a planar embedding of the tree such that the ordering on the leaves is compatible with the orientation on the plane.*

The category Φ_n must essentially be due to Stasheff, though he claims no credit for it. Further information about this category is given by [26].

Not only is Φ_n isomorphic to the category of strata of A_n, but there is a stronger relationship between the two. First note that Φ_n is a poset and has a terminal object, namely the tree which has no interval vertices. Observe that the minimal elements in Φ_n, namely trees all of whose vertices are trivalent, correspond to ways to parenthesize a word with $n+2$ letters. These observations lead to the following proposition, which is illustrated in Figure 4.2.

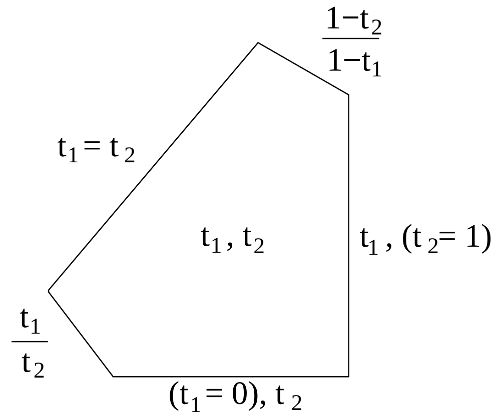

FIGURE 1. Natural coordinates on $C_2[I, \partial]$, illustrating Proposition 4.13

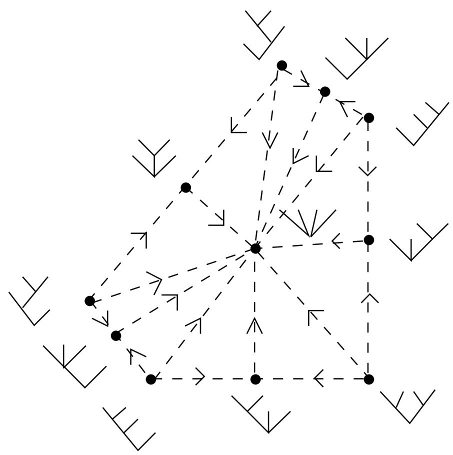

FIGURE 2. Illustration of Proposition 4.15 for $n = 2$. The dots represent objects of Φ_2 and so are labeled by trees. The dashed arrows represent morphisms.

Proposition 4.15. *The nerve of Φ_n is isomorphic as a polyhedron to the barycentric subdivision of the nth Stasheff polyhedron A_n.*

We need to refine this correspondence for further applications. Let T be a tree with $n+2$ leaves. We let A_T denote the face of A_n which corresponds to T by Proposition 4.15, so that the nerve of $\Phi_n \downarrow t$ is isomorphic as a polyhedron to the barycentric subdivision of A_T. Explicitly, A_T is isomorphic to $A_{|v_r|-2} \times \prod_{v \neq v_r} A_{|v|-3}$, where v runs over all non-root vertices of T. If there is a morphism between T and T', let $i_{T,T'}$ denote the corresponding inclusion of A_T as a face of $A_{T'}$. Let A_T^{nr} denote $\prod_{v \neq v_r} A_{|v|-3}$. Note that $i_{T,T'}$ descends to a map, which we will by abuse give the same name, from A_T^{nr} to $A_{T'}^{nr}$.

The poset Φ_n is an analogue of the poset $[n]_0$ of non-empty subsets of the set $\{0, \cdots, n\}$. Indeed, the realization of the nerve of $[n]_0$ is isomorphic to the barycentric subdivision of the n-simplex Δ^n, of which A_n is a blow-up. Corresponding to the canonical projection from A_n to Δ^n is a functor from Φ_n to $[n]_0$ which we now define.

We say two leaves are root-joined if their coincident path consists only of the root vertex. Note that for trees in Φ_n, the ordering on leaves of a tree gives an ordering on adjacent pairs of leaves. If there is a morphism from T to T' in Φ_n, then if the ith adjacent pair of leaves is root-joined in T it is root-joined in t' as well. Hence, a morphism in Φ_n gives rise to an inclusion of the subsets of $\{0, \cdots, n\}$ which index pairs of adjacent leaves which are root-joined.

Definition 4.16. *Let $\mathcal{F}_n$ be the functor from Φ_n to $[n]_0$ which sends a tree to the set of its root-joined adjacent pairs of vertices.*

In the next section, we will be taking homotopy limits over the categories Φ_n and $[n]_0$. Recall from subsection 3.1 the notion of cofinality of a functor (between indexing categories).

Proposition 4.17. *The functors $\mathcal{F}_n \colon \Phi_n \to [n]_0$ are cofinal.*

Proof. For any object S of $[n]_0$, the category $F_n \downarrow S$ is a poset with a terminal object, namely the unique tree of smallest level which maps to S. □

4.3. $C_n\langle[M]\rangle$ **the "cosimplicial" variant, tangential data, and diagonal maps.** We now come to our final variant of these compactifications, a new variant which is of technical importance because its "diagonal maps" satisfy cosimplicial identities. As in the definition of $C_n[M]$, we embed M in some $\mathbb{R}^k$ and restrict the maps $\alpha_{ij} \colon C_n(\mathbb{R}^k) \to S^{k-1}$ to $C_n(M)$.

Definition 4.18. *Define $C_n\langle[M]\rangle$ to be the closure of $C_n(\mathbb{R}^k)$ in $M^n \times (S^{k-1})^{\binom{n}{2}}$.*

First note that there is a map from $C_n[M]$ to $C_n\langle[M]\rangle$ since a point in the closure of $C_n(M)$ in $M^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$ will project onto a point in its closure in $M^n \times (S^{k-1})^{\binom{n}{2}}$. In [31] we show that this map is a quotient map. We first exhibit $C_n\langle[M]\rangle$ as a union of manifolds and then sketch a description of this quotient map.

First, to label the strata of $C_n\langle[M]\rangle$ we must enlarge the category Ψ_n.

Definition 4.19. *Define Γ_n to be the category of f-trees whose vertices are each colored, say red or blue, with both the root vertex and any trivalent vertex being colored blue. We define a contraction of a set of edges of colored f-trees as we did for f-trees with the additional requirement that the initial vertex of each contracted edge must be relabeled blue. There is a morphism in Γ_n from T to T' if T' is obtained from T by contracting a collection of edges and recoloring some red vertices blue.*

The red vertices will correspond to spaces of collinear infinitesimal configurations, and the blue vertices will correspond to the non-collinear infinitesimal configurations.

Definition 4.20. *For $n > 2$ define $L_n(M)$ to be the space whose points are lines through the origin in the tangent bundle of M along with n labelled points points on each line up to orientation-preserving diffeomorphism in the line.*

Thus, $L_n(M)$ is diffeomorphic to $n!/2$ copies of STM.

Definition 4.21. *Let $IC_n^o(M)$ be the subspace of $IC_n(M)$ of n points in a tangent space of M which are not all collinear up to translation and scaling.*

Definition 4.22. *Let T be a colored f-tree. Given a root edge e of T define $IC_e(M)$ to be subspace of the product*

$$\prod_{v \in V(e), v \text{ blue}} IC^o_{\#v}(M) \times \prod_{v \in V(e), v \text{ red}} L_{\#v}(M)$$

of tuples of infinitesimal configurations and lines in the tangent bundle all sitting over the same point in M, and moreover such that if v and w are red vertices which are connected by an edge, then the lines *corresponding to v and w must be distinct. Let p_e be the projection from $IC_e(M)$ onto M. Define the space $C_T\langle[M]\rangle$ as a pullback as follows*

$$\begin{tikzcd} C_T\langle[M]\rangle \arrow[r] \arrow[d] & C_{r(T)}(M) \arrow[d] \\ \prod_{e\in E(T)} IC_e(M) \arrow[r, "\prod p_e"] & M^{r(T)}. \end{tikzcd}$$

Theorem 4.23. *If M has dimension greater than one, $C_n\langle[M]\rangle$ is a union of the manifolds $C_T\langle[M]\rangle$ for $T \in \Gamma_n$. Moreover, these manifolds define a stratification whose associated category is isomorphic to Γ_n.*

It is not immediate from the structure given by this stratification to realize $C_n\langle[M]\rangle$ as a quotient of $C_n[M]$. We sketch briefly how to define this quotient map explicitly. First one must refine the stratification of $C_n[M]$, by considering $IC_n(M)$ for $n > 2$ as the union of $IC_n^o(M)$ and its complement, which is diffeomorphic to $L_n(M) \times \mathrm{Int}(\Delta^{n-2})$. One then notes that the closure of each $L_n(M) \times \mathrm{Int}(\Delta^{n-2})$ in $C_n[M]$ is diffeomorphic to $L_n(M) \times A_{n-2}$. The quotient map from $C_n[M]$ to $C_n\langle[M]\rangle$ takes each copy of $L_n(M) \times A_{n-2}$ and projects it onto a copy of $L_n(M)$.

We go further in [31] to show the following.

Theorem 4.24. *The projection from $C_n[M]$ to $C_n\langle[M]\rangle$ is a homotopy equivalence. Moreover, the restriction of this projection to any stratum of $C_n[M]$ is a homotopy equivalence.*

As mentioned in section 2, we need versions of our configuration spaces which include tangent vectors at all of the points in a configuration.

Definition 4.25. *Let $C'_n\langle[M]\rangle$ be the closure of $C'_n(M)$ in $(STM)^n \times (S^{k-1})^{\binom{n}{2}}$ where the maps onto S^{k-1} are defined as the composite of the projection of $C'_n(M)$ onto $C_n(M)$ and the maps α_{ij}. Similarly, let $C'_n[M]$ be the closure of $C'_n(M)$ in $(STM)^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$.*

It is easy to show that $C'_n[M]$ fits in a pull-back square similar to that which defines $C'_n(M)$. We choose stratifications of these spaces so that the associated categories are still Ψ_n and Γ_n. Namely, to a tree T in Ψ_n we associate the pullback $C'_T[M]$ defined as

$$\begin{tikzcd} C'_T[M] \arrow[r] \arrow[d] & (STM)^{r(T)} \arrow[d] \\ C_T[M] \arrow[r, "f"] & M^{r(T)}, \end{tikzcd}$$

where $r(T)$ is the valence of the root vertex and f is the canonical projection. There is a similar construction for $C'_R\langle[M]\rangle$. Note that the union of the subspaces in each of these stratifications is no longer the entire space as it was for $C_n[M]$ and $C_n\langle[M]\rangle$.

Define $C_n\langle[M,\partial]\rangle$ as before, and for a manifold with two distinguished tangent vectors on its boundary, as in the definition of $\mathrm{Emb}(I, M)$, define $C'_n\langle[M,\partial]\rangle$, respectively $C'_n[M,\partial]$, to be the subspace of $C'_{n+2}\langle[M,\partial]\rangle$, respectively $C'_{n+2}[M,\partial]$, which sits over points in $(STM)^{n+2}$ whose first and last coordinates are given by the distinguished tangent vectors.

Perhaps the most commonly used maps between products M^n are diagonal maps and projections, used for example to define the cosimplicial model of the loop space of M. Unlike the open configuration spaces, these compactifications have canonical diagonal maps as well as projections. Moreover, for the $C_n\langle[M]\rangle$ these maps satisfy cosimplicial identities.

First consider the square

$$\begin{array}{ccc} C'_n(M) & \longrightarrow & (STM)^n \times \prod_{1\leq i<j\leq n} S^{k-1} \times \prod_{1\leq i<j<k\leq n} I \\ \downarrow & & \downarrow{\scriptstyle s^l} \\ C'_{n-1}(M) & \longrightarrow & (STM)^{n-1} \times \prod_{1\leq i<j\leq n-1} S^{k-1} \times \prod_{1\leq i<j<k\leq n-1} I, \end{array}$$

where the left vertical arrow forgets the lth point in a configuration and s^l projects by forgetting all factors in which the label l appears (and then relabels). Because this square commutes and s^l is an open map, we may deduce the following.

Proposition 4.26. *There are projection maps s^l from $C'_n[M]$ to $C'_{n-1}[M]$, or by abuse from $C'_n\langle[M]\rangle$ to $C'_{n-1}\langle[M]\rangle$, which extend the lth projection from $C'_n(M)$ to $C'_{n-1}(M)$ and lift the lth projection map from $(STM)^n$ to $(STM)^{n-1}$.*

The diagonal or coface maps are slightly more difficult to define since there are no corresponding maps on the open configuration spaces which one can extend. For purposes of reindexing, let $\sigma_l(i)$ equal i if $i \leq l$ or $i-1$ if $i > l$. Let F be our embedding of M in $\mathbb{R}^k$ and let sF' be the composite $STM \xrightarrow{dF} T^{\times}\mathbb{R}^k \xrightarrow{u} S^{k-1}$, where $T^{\times}\mathbb{R}^k$ is the bundle of non-zero tangent vectors of $\mathbb{R}^k$ and u sends a non-zero vector to its corresponding unit vector.

Definition 4.27. *Let*

$$\delta^l\colon (STM)^n \times \prod_{1\leq i<j\leq n} S^{k-1} \to (STM)^{n+1} \times \prod_{1\leq i<j\leq n+1} S^{k-1}$$

send $((x_1,\ldots,x_n),(a_{12},\ldots,a_{n-1,n}))$ to $((x'_1,\ldots,x'_{n+1}),(a'_{12},\ldots,a'_{n,n+1}))$ where $x'_i = x_{\sigma_l(i)}$ and

$$a'_{ij} = \begin{cases} a_{\sigma_l(i)\sigma_l(j)} & \text{if } i,j \neq l, l+1 \\ sF'(x_i) & \text{if } i,j = l, l+1. \end{cases}$$

We define our diagonal, or doubling, maps on $C'_n\langle[M]\rangle$ by restricting δ^l. We prove the following in [31].

Theorem 4.28. *The map δ^l restricted to $C'_n\langle[M]\rangle$ maps to $C'_{n+1}\langle[M]\rangle$. It lifts the lth diagonal map from $(STM)^n$ to $(STM)^{n+1}$ and extends the map from $C'_n(M)$ to $IC_2(M) \times_M C'_n(M) \subset C'_{n+1}\langle[M]\rangle$ which sends $v_1,\ldots,v_n$ to $v_i,(v_1,\ldots,v_n)$.*

Note that there is a version of δ^l on $C_n\langle[M]\rangle$ itself once one has a non-zero section of the tangent bundle with which one may define $a'_{i,i+1}$, the relative vector at the doubled point.

Both δ^l and s^l restrict to maps on $C'_n\langle[M,\partial]\rangle$, where for δ^l we have $0 \leq l \leq n+1$ and for s^l we require $0 < l < n+1$. Informally, δ^0 and δ^{n+1} add a point to a configuration at one of the chosen boundary points.

It is easy to check that on the ambient spaces $(STM)^n \times \prod_{1\leq i<j\leq n} S^{k-1}$ the maps s^l and δ^l satisfy the cosimplicial axioms. We deduce the following important result.

Theorem 4.29. *Between the $C'_n\langle[M,\partial]\rangle$, the maps δ^i and s^i satisfy the cosimplicial identities.*

We now define versions of diagonal maps for $C'_n[M]$. Let $\delta^l(m)\colon (STM)^n \times (S^{k-1})^{\binom{n}{2}} \to (STM)^{n+m} \times (S^{k-1})^{\binom{n+m}{2}}$ denote the composite of δ^l with itself m times. Next, note that there is a map from the $m-1$st Stasheff polytope A_{m-1} to $I^{\binom{m+1}{3}}$, which we call ι given by the product of the maps β_{ijk} in the identification of A_{m-1} as $C_{m-1}[I,\partial]$. Also, let σ_l^m be defined as

$$\sigma_l^m(i) = \begin{cases} i & \text{if } i < l \\ l & \text{if } l \leq i \leq l+m \\ i-m & \text{if } i > l+m. \end{cases}$$

Finally, let $\tau_l^m \colon I^{\binom{n}{3}} \to I^{\binom{n+m}{3}-\binom{m}{3}}$ send (b_{ijk}) for $1 \leq i < j < k \leq n$ to (b'_{ijk}) for $1 \leq i < j < k \leq m+n$ but not $l+1 \leq i < j < k \leq l+m$ where $b'_{ijk} = b_{\sigma_l^m(i),\sigma_l^m(j),\sigma_l^m(k)}$.

Theorem 4.30. *The maps*

$$\delta^l(m) \times \tau_m \times \iota \colon \left((STM)^n \times (S^{k-1})^{\binom{n}{2}}\right) \times I^{\binom{n}{3}} \times A_{m-1} \to \left((STM)^{n+m} \times (S^{k-1})^{\binom{n+m}{2}}\right) \times I^{\binom{n+m}{3}-\binom{m+1}{3}} \times I^{\binom{m+1}{3}} \tag{3}$$

restrict to maps $\delta^l(m) \colon C'_n[M] \times A_{m-1} \to C'_{n+m}[M]$ *which lift the diagonal maps* $(STM)^n \to (STM)^m$.

Clearly, the maps δ^i and s^i commute with the quotient maps from $C'_n[M]$ to $C'_n\langle[M]\rangle$.

Note that the restriction of $\delta^l(m)$ to $C'_n[M] \times v$, where v is a vertex of A_{m-1} is a composite of $\delta^i(1)$'s. Thus, if we think of $\delta^l(1)$ as trying to be a coface map, the map $\delta^l(m)$ provides canonical homotopies, parameterized by A_{m-1}, between composites of such coface maps which would agree in a cosimplicial setting.

5. The mapping space model

As we have mentioned, the evaluation map of a knot θ, namely $ev_n(\theta)$ from A_n (one component of $C_n[I,\partial]$) to $C'_n[M,\partial]$ is stratum preserving. But note as well that for a point on the boundary of A_n which has a triple point (or greater) the image of that triple point in $C'_n[M,\partial]$ will be an infinitesimal triangle (or n-gon) which is degenerate in the sense that all points are aligned along a single vector, namely the tangent vector to the knot.

Definition 5.1. *A point $x \in C_n[M]$ is aligned if, naming its image in M^n by $x_1, \cdots, x_n$, for each collection $x_i = x_j = x_k = \cdots$, the relative vectors v_{ij} with $i < j$ are all equal. A point in $C'_n[M]$ is aligned if its projection onto $C_n[M]$ is aligned and, in the notation above, the relative vector v_{ij} serves as the tangent vector at x_i and x_j.*

The subspace of a stratum consisting of points which are aligned is called the aligned sub-stratum of that stratum.

Definition 5.2. *A stratum-preserving map from A_n to $C'_n[M,\partial]$ is aligned if points in its image are aligned. Let $AM_n(M)$ denote the space of aligned stratum-preserving maps from A_n to $C'_n[M,\partial]$.*

$AM_n(M)$ maps to $AM_{n-1}(M)$ by restricting an aligned map to a principal face. Let $AM_\infty(M)$ denote the homotopy inverse limit of the $AM_n(M)$. As noted above, the evaluation map ev_n maps $\mathrm{Emb}(I,M)$ to $AM_n(M)$. The ev_n give rise to a map ev_∞ from $\mathrm{Emb}(I,M)$ to $AM_\infty(M)$. The following are the main theorems of this section.

Theorem 5.3. *Let the dimension of M be greater than three. The map $ev_\infty \colon \mathrm{Emb}(I,M) \to AM_\infty(M)$ is a weak homotopy equivalence.*

We will relate these mapping space models to models produced by the calculus of embeddings. Recall that $P_n\mathrm{Emb}(I,M)$ denotes the nth polynomial approximation to the space of knots in M.

Theorem 5.4. *$AM_n(M)$ is weakly homotopy equivalent to $P_n\mathrm{Emb}(I,M)$ for all n including $n = \infty$. Moreover, the evaluation map ev_n coincides with the map α_n in the homotopy category.*

Theorem 5.3 follows from combining Theorem 5.4 and Corollary 3.8, so the rest of this section is devoted to proving Theorem Theorem 5.4. We will first identify the mapping space model as a homotopy limit. Then, we will find a sequence of equivalences between this homotopy limit and the homotopy limit defining $P_n\mathrm{Emb}(I,M)$. This sequence first involves changing the shape of the homotopy limit involved and then interpolating between configuration spaces and embedding spaces by defining a space which incorporates both configurations and embeddings.

For reference, we list the sequence of equivalences and the lemmas in which they are proved now, even though the intermediate homotopy limits have not been defined. In Lemma 5.6 we show that $AM_n(M)$ is homeomorphic to the homotopy limit of a diagram $\mathcal{D}_n[M]$. We then relate the diagram $\mathcal{D}_n[M]$ to others which interpolate between it and $\mathcal{E}_n(M)$, whose homotopy limit is $P_n\mathrm{Emb}(I,M)$, as outlined here.

$$\mathcal{D}_n[M] \underset{5.8}{\to} \widetilde{\mathcal{D}_n}\langle[M]\rangle \underset{5.11}{\leftarrow} \mathcal{D}_n\langle[M]\rangle \underset{5.18}{\to} \mathcal{E}_n\langle[M]\rangle \underset{5.18}{\leftarrow} \mathcal{E}_n(M). \tag{4}$$

We proceed with our sequence of lemmas. First note that the closure of the aligned substrata of $C'_n[M,\partial]$ are homeomorphic to $C'_{n-m}[M,\partial] \times \prod A_{k_i}$, for some k_i with $\sum k_i = m-2$. In fact, we see that these strata are precisely in the image of the maps $\delta^l(m)$.

Given a root edge e of a tree T, let v_e be the terminal vertex of e and let $\#e$ be the valence of v_e minus three. Recall that A^{nr}_T is isomorphic to $\prod_{v\neq v_r} A_{|v|-3}$, where v runs over all non-root vertices of T.

Definition 5.5. *Let e be an edge of T and let T' be the tree resulting from the contraction of T, so $A_T = A_{\#e} \times A_{T'}$. Let $\mathcal{D}_n[M]$ be the functor from Φ_n to spaces which sends a tree T to $C'_{|v_r|-2}[M,\partial]\times A^{nr}_T$, which sends the contraction of a root edge e of T to*

$$\delta^i(\#e+1)\times id\colon \left(C'_{|v_r|-2}[M,\partial]\times A_{\#e}\right)\times A_{T'} \to C'_{|v_r|+\#e-1}[M,\partial]\times A_{T'},$$

and which sends the contraction of a non-root edge to $id \times i_{T,T'}$.

The following is now straightforward, following from Proposition 3.3, Propositions 4.13 and 4.15 and the identification of the aligned strata of $C'_n[M,\partial]$ as the images of the maps $\delta^i(m)$.

Lemma 5.6. *$AM_n(M)$ is homeomorphic to the homotopy limit of $\mathcal{D}_n[M]$.*

Now that we have identified $AM_n(M)$ as the homotopy limit of $\mathcal{D}_n[M]$, we interpolate between $\mathcal{D}_n[M]$ and $\mathcal{E}_n(M)$. The spaces $C'_n\langle[M]\rangle$ play an important role in this interpolation. An aligned stratum-preserving map from A_n to $C'_n[M]$ can be composed with the quotient map from $C'_n[M]$ to $C'_n\langle[M]\rangle$. Because as stated in Theorem 4.24 the restriction of the quotient map from $C'_n[M]$ to $C'_n\langle[M]\rangle$ to any stratum is a homotopy equivalence, the corresponding mapping spaces are homotopy equivalent. We phrase this equivalence in terms of homotopy limits as follows.

Note that the aligned strata for $C'_n\langle[M,\partial]\rangle$ are simply of the form $C'_{n-k+1}\langle[M,\partial]\rangle$.

Definition 5.7. *Let $\widetilde{\mathcal{D}}_n\langle[M]\rangle$ be the functor from Φ_n to spaces which sends a tree T to $C'_{|v_r|-2}\langle[M,\partial]\rangle$, which sends the contraction of the ith root edge to $\delta^i(\#e)$, and which sends the contraction of a non-root edge to the identity map.*

There is a map of diagrams between $\mathcal{D}_n[M]$ and $\widetilde{\mathcal{D}}_n\langle[M]\rangle$ defined by quotient maps. Because these quotient maps are homotopy equivalences, we deduce the following.

Lemma 5.8. *The map from the homotopy limit of $\mathcal{D}_n[M]$ to the homotopy limit of $\widetilde{\mathcal{D}}_n\langle[M]\rangle$ defined by the quotient maps is a homotopy equivalence.*

For the next step in our series of equivalences, note that since many of the maps in the definition of $\widetilde{\mathcal{D}}_n\langle[M]\rangle$ are the identity they may be eliminated. This elimination corresponds to a replacement of the category Φ_n by $[n]_0$.

Definition 5.9. *Let $\mathcal{D}_n\langle[M]\rangle$ be the functor from $[n]_0$ to spaces which sends S to $C'_{\#S}\langle[M,\partial]\rangle$ and sends the inclusion $S\subseteq S'$ where $S' = S\cup i$ to $\delta^i(k)$ where k is the number of elements of S less than i.*

By construction, we have the following.

Lemma 5.10. *$\widetilde{\mathcal{D}}_n\langle[M]\rangle$ is the composite of $\mathcal{D}_n\langle[M]\rangle$ with the functor $\mathcal{F}_n$ of Definition 4.16.*

The following lemma, which gives the next link in our chain of equivalences, now follows immediately from Proposition 4.17 and the fact that cofinal functors induce equivalences on homotopy limits.

Lemma 5.11. *The homotopy limit of $\widetilde{\mathcal{D}}_n\langle[M]\rangle$ is weakly homotopy equivalent to the homotopy limit of $\mathcal{D}_n\langle[M]\rangle$.*

To interpolate between $\mathcal{D}_n\langle[M]\rangle$ and $\mathcal{E}_n(M)$, we incorporate both embeddings and configurations in one space. As in Definition 3.5, let $\{J_i\}$, for $i = 1$ to n be a collection of sub-intervals of I. Let $\{I_\alpha\}$ be the set of connected components of $I - \bigcup_{s\in S} J_s$, and given an embedding f of $I - \bigcup_{s\in S} J_s$ let f_α be the restriction of f to I_α. Let ev_S be the evaluation map from $E_S(M) \times \prod I_\alpha$ to $C'_{\#S+1}(M)$.

Definition 5.12. *Given a metric space X define $\mathcal{H}(X)$ to be the space whose points are compact subspaces of X and with a metric defined as follows. Let A and B be compact subspaces of X and let x be a point in X. Define $d(x, A)$ to be $\liminf_{a\in A} d(x, a)$. We define $d(A, B)$ to be the greater of $\limsup_{b\in B} d(b, A)$ and $\limsup_{a\in A} d(a, B)$, which is sometimes called the Hausdorff metric.*

Note that since STM is metrizable, as is of course S^k, then so is $C'_n\langle[M]\rangle$.

Definition 5.13. *Let $S \in [n]_0$. Define $E_S\langle[M]\rangle$ to be the union of $E_S(M)$ and $C'_{\#S-1}\langle[M,\partial]\rangle$ as a subspace of $\mathcal{H}(C'_{\#S+1}\langle[M]\rangle)$, where $C'_{\#S-1}\langle[M,\partial]\rangle \subset C'_{\#S+1}\langle[M]\rangle$ is a subspace of one-point subsets and the image of $f \in E_S(M)$ in $\mathcal{H}(C'_{\#S+1}\langle[M]\rangle)$ is given by the image of $ev_S(f \times \prod I_\alpha)$.*

It is helpful to think of $E_S\langle[M]\rangle$ as a space of embeddings which may be degenerate by having all of the embeddings of components "shrink" until they are tangent vectors, which we think of as infinitesimal embeddings.

Proposition 5.14. *The inclusions of $C'_{\#S-1}\langle[M,\partial]\rangle$ and $E_S(M)$ into $E_S\langle[M]\rangle$ are weak homotopy equivalences.*

In fact we first define a map from $E_S\langle[M]\rangle$ onto $C'_{\#S-1}\langle[M,\partial]\rangle$ which is a deformation retraction.

Definition 5.15. *Let $\epsilon\colon E_S\langle[M]\rangle \to C'_{\#S-1}\langle[M,\partial]\rangle$ be the map which is the identity on $C'_{\#S}\langle[M,\partial]\rangle$ and on $E_S(M)$ is defined by sending f to $ev_S(f, m)$, where $m \in \prod I_\alpha$ has first coordinate $m_0 = 0$, last coordinate $m_1 = 1$, and other coordinates given by defining m_α to be the mid-point of I_α.*

Proof of Proposition 5.14. We define a homotopy between ϵ and the identity map on $E_S\langle[M]\rangle$ which is the identity on $C'_{\#S-1}\langle[M,\partial]\rangle$. On $E_S(M)$, letting s be the homotopy variable we set $f_\alpha(t)_s = f((1-s)t+sm_\alpha)$ for $s < 1$.

Next, note that the restriction of $\epsilon\colon E_S(M) \to C'_{\#S+1}(M,\partial)$ is a fibration, essentially by the isotopy extension theorem. We show that the fiber of this map, namely the space of embeddings of $I - \bigcup_{s\in S} J_s$ with a given tangent vectors at the m_α, is weakly contractible. Suppose we have a family of such embeddings parameterized by a compact space. First, we may apply a reparametrizing homotopy (as above, of the form $f(t,s) = f((1-s)t + sm_\alpha)$ for $s \in [0, a]$ for some a) until the image of each component lies in a fixed Euclidean chart in M about the image of m_α. By compactness there is an a which works for the entire family.

Noting that the unit tangent vector at m_α is by definition fixed for all points in a fiber of ϵ so we may a priori choose coordinates in each chart around these points so that each $f_\alpha(t) = (f_{\alpha,1}(t), f_{\alpha,2}(t), \dots)$ with each $f_{\alpha,j}(m_\alpha) = 0$ and $f'_\alpha(m_\alpha) = (a, 0, 0, \dots)$ for some $a > 0$. Next, for each component, consider the "projection" homotopy defined in coordinates by

$$f_\alpha(t)_s = (f_{\alpha,1}(t), sf_{\alpha,2}(t), sf_{\alpha,3}(t), \dots).$$

This homotopy is not necessarily an isotopy, but it will always be on some neighborhood of m_α since the derivative there is bounded away from zero throughout the homotopy. By compactness of the parameter space, there is some non-zero b such that this homotopy is an isotopy on a neighborhood N of m_α of length b for all points in the parameter space. The composite of the first reparametrizing homotopy, a second reparametrizing homotopy which changes the image of each interval so as to be the image of N, and the projection homotopy defines a homotopy between the given family of embeddings and one which is essentially a constant family (up to scaling on the first coordinates of the fixed charts). $\square$

Hence, the spaces $E_S\langle[M]\rangle$ are a suitable interpolation between $E_S(M)$ and $C'_{\#S-1}\langle[M,\partial]\rangle$. Of course, to define a suitable diagram interpolating between $\mathcal{D}_n\langle[M]\rangle$ and $\mathcal{E}_n(M)$ we need maps as well as spaces.

Proposition 5.16. *Let $S \subset S'$ in $[n]_0$ where $S' = S \cup i$. There is a map $\rho_{S,S'} \colon E_S\langle[M]\rangle \to E_{S'}\langle[M]\rangle$ whose restriction to $E_S(M)$ is the map to $E_{S'}(M)$ defined by restriction of embeddings and whose restriction to $C'_{\#S-1}\langle[M,\partial]\rangle$ is the map δ^i.*

Proof. We simply need to check that the function so defined is continuous. It is continuous when restricted to either $E_S(M)$ or $C'_{\#S-1}\langle[M,\partial]\rangle$. Because $E_S(M)$ is open and dense in $E_S\langle[M]\rangle$, it suffices to check continuity on $C'_{\#S-1}\langle[M,\partial]\rangle$. Since δ^i is continuous, it suffices to show that for every ϵ there is a δ such that if distance between $\theta \in E_S(M)$ and $x \in C'_{\#S-1}\langle[M,\partial]\rangle$ is less than δ then the distance between $\rho_{S,S'}(\theta)$ and $\delta^i(x)$ is less than ϵ.

We consider $C'_{\#S-1}\langle[M,\partial]\rangle$ as it is defined, as a subspace of $(STM)^n \times (S^{k-1})^{\binom{n}{2}}$ where $n = \#S + 1$, and use the notation of Definition 4.27 so that $x = ((x_1, \ldots, x_n), (a_{12}, \ldots, a_{n-1,n}))$. Also, let F be the embedding of M in $\mathbb{R}^k$ used to define the maps to S^{k-1}. A bound on the distance between x and θ is equivalent to a bound for each α between x_α and $\theta'(t)$ for all $t \in I_\alpha$, as well as a bound on the distance between a_{ij} and the unit vector in the direction of $F \circ \theta(t) - f \circ \theta(s)$ for $t \in I_i$ and $s \in I_j$. Such bounds clearly give rise to the same bounds on the distance between $x'_\beta = x'_{\sigma_i(\beta)}$ and $\theta'(t)$ for all $t \in I_\beta$, as well as better bounds on all factors of S^{k-1} other than the factor labelled by $i, i+1$. In this last case, note that we may choose our bound on the distance between x_i and $\theta'(t)$ for all $t \in I_i$ so that the distance between $sF'(x_i)$ and $s(F \circ \theta(t_i) - F \circ \theta(t_{i+1}))$ for all $t_i \in I_i, t_{i+1} \in I_{i+1}$ is arbitrarily small. $\square$

Definition 5.17. *Let $\mathcal{E}_n\langle[M]\rangle$ be the functor from $[n]_0$ to spaces which sends S to $E_S\langle[M]\rangle$ and sends the inclusion $S \subseteq S'$ where $S' = S \cup i$ to $\rho_{S,S'}$.*

We have constructed the maps $\rho_{S,S'}$ so that both $\mathcal{D}_n\langle[M]\rangle$ and $\mathcal{E}_n(M)$ map to $\mathcal{E}_n\langle[M]\rangle$ through the inclusions entry-wise. By Proposition 5.14 these inclusions are weak equivalences, so we may deduce the following.

Lemma 5.18. *The homotopy limits of $\mathcal{D}_n\langle[M]\rangle$ and $\mathcal{E}_n(M)$ are weakly equivalent to the homotopy limit of $\mathcal{E}_n\langle[M]\rangle$.*

We may now piece together the proof of the main theorem of this section.

Proof of Theorem 5.4. The fact that $AM_n(M)$ is weakly equivalent to $P_n\mathrm{Emb}(I, M)$ follows from the string of equivalences given by Lemmas 5.6, 5.8, 5.11, and 5.18.

It remains to show that the evaluation map ev_n coincides in the homotopy category with the map α_n. Clearly ev_n coincides with other evaluation maps (which by abuse we also call ev_n) in the equivalences of Lemmas 5.8 and 5.11. Thus we focus on the equivalences of Lemma 5.18 and show that the composite of ev_n and the inclusion of $\mathrm{holim}\mathcal{D}_n\langle[M]\rangle$ in $\mathrm{holim}\mathcal{E}_n\langle[M]\rangle$ is homotopic to the composite of α_n and the inclusion of $\mathrm{holim}\mathcal{E}_n(M)$ in $\mathrm{holim}\mathcal{E}_n\langle[M]\rangle$.

Recall that $\mathrm{holim}\mathcal{E}_n\langle[M]\rangle$ is the subspace of the product $\prod_{S \in [n]_0} \mathrm{Maps}(\Delta^{\#S-1}, E_S\langle[M]\rangle)$ of maps $\{f_S\}$ which are compatible in that if $S \subset S'$ then the restriction of $f_{S'}$ to the face of $\Delta^{\#S'-1}$ identified with $\Delta^{\#S-1}$ is the composite of f_S and the restriction map from $E_S\langle[M]\rangle$ to $E_{S'}\langle[M]\rangle$. The map α_n sends $\mathrm{Emb}(I, M)$ to this space as the subspace in which each f_S is constant as a function on $\Delta^{\#S-1}$, with image given by the restriction from $\mathrm{Emb}(I, M)$ to $E_S(M)$. We will homotop these f_S by "shrinking towards the evaluation points".

Let $\rho(J, t, s)$ be the interval which linearly interpolates, with parameter s, between the interval J and the degenerate interval $[t, t]$. Explicitly, if $J = [a, b]$ then $\rho(J, t, s)$ is the interval $[(1-s)a + st, (1-s)b + st]$.

Let $x = 0 \leq t_1 \leq \ldots \leq t_{\#S-1}$ be a point in $\Delta^{\#S-1}$ and by convention let $t_0 = 0$ and $t_{\#S} = 1$, and let $\theta \in \mathrm{Emb}(I, M)$. We define our homotopy of f_S by defining $h_S(\theta, x, s)$ for $s < 1$ to be the embedding of $\bigcup_\alpha I_\alpha$ which is the composite of the linear isomorphism between $\bigcup_\alpha I_\alpha$ and $\bigcup_\alpha \rho(I_\alpha, t_\alpha, s)$ and the restriction of θ to $\bigcup_\alpha \rho(I_\alpha, t_\alpha, s)$. We define $h_S(\theta, x, 1)$ to be $(ev_n(\theta))(x)$. It is straightforward to check that h_S is well-defined and continuous, and that the various h_S for differing S are compatible. □

Note that Goodwillie's Theorem 3.7, upon which we build, can be proved for knots in manifolds of dimension five or greater by dimension-counting arguments (sharper versions of this theorem require surgery theory and the results of Goodwillie's thesis [14]). We wonder if one can prove that the inclusion of the embedding space into the space of aligned maps through the evaluation map can be shown to be highly connected by more direct arguments. Is there a dimension-counting argument to determine the greatest number of parameters with which one can homotop an aligned map into an evaluation map of some knot? This could be an important technical question, especially in the application of these ideas to classical knots, for which the analogue of Theorem 3.7 is not known.

6. The cosimplicial model

We now produce cosimplicial models of spaces of knots. We take as our starting point the model defined by the homotopy limit of $\mathcal{D}_n\langle [M] \rangle$. We show that $\mathcal{D}_n\langle [M] \rangle$ is a special kind of diagram indexed by $[n]_0$, namely it is pulled back from a cosimplicial diagram.

To set notation, we recall some standard constructions. Let Δ be the cosimplicial category, whose objects are the sets $\underline{n} = \{0, \cdots, n\}$ and where a morphism from $\underline{m}$ to $\underline{n}$ is an order preserving map. Special order preserving maps generate this category, namely s^i which is the surjection of $\underline{n}$ onto $\underline{n-1}$ which sends both i and $i+1 \in \underline{n}$ to $i \in \underline{n-1}$ and d^i which is the inclusion of $\underline{n}$ into $\underline{n+1}$ for which i is not in the image. Let Δ_n be the full subcategory of Δ whose objects are the sets $\underline{i}$ for $0 \leq i \leq n$.

A cosimplicial space is a functor from Δ to the category of spaces. There is a canonical example, $\Delta^\bullet$, whose nth entry is Δ^n and whose coface and codegeneracy maps are inclusions of faces and projections between simplices. Given a cosimplicial space $\mathcal{X}^\bullet$ let $i_n\mathcal{X}^\bullet$ be the restriction of $\mathcal{X}^\bullet$ to Δ_n.

Recall Theorem 4.29, that the diagonal and projection maps between $C'_n\langle [M, \partial] \rangle$ satisfy cosimplicial axioms.

Definition 6.1. *Let $\mathcal{C}^\bullet\langle [M] \rangle$ be the cosimplicial space whose nth entry is $C'_n\langle [M, \partial] \rangle$, whose coface maps maps are given by the δ^i, and whose codegeneracy maps are given by projections.*

The following theorem is the main theorem of this section. Recall that the totalization of a cosimplicial space $\mathcal{X}^\bullet$ is the space of maps from $\Delta^\bullet$ to $\mathcal{X}^\bullet$.

Theorem 6.2. *The space $\mathrm{Emb}(I, M)$ is weakly equivalent to the totalization of a fibrant replacement of the cosimplicial space $\mathcal{C}^\bullet\langle [M] \rangle$.*

This theorem will follow from Proposition 6.4, which relates $i_n\mathcal{C}^\bullet\langle [M] \rangle$ to our models $\mathcal{D}_n\langle [M] \rangle$, and Theorem 6.5, which is a general fact about homotopy limits over $[n]_0$ which are pulled back from Δ_n.

The fact that the nerve of $[n]_0$ is isomorphic to the barycentric subdivision of an n-simplex is related to the existence of a canonical functor from $[n]_0$ and Δ_n.

Definition 6.3. *Let $\mathcal{G}_n \colon [n]_0 \to \Delta_n$ be the functor which sends a subset S to the object in Δ_n with the same cardinality, and which sends an inclusion $S \subseteq S'$ to the composite $\underline{i} \cong S \subset S' \cong \underline{j}$, where $\underline{i}$ and $\underline{j}$ are isomorphic to S and S' respectively as ordered sets. By abuse let $\mathcal{G}_n$ also denote the composite of $\mathcal{G}_n$ with the inclusion of Δ_n in Δ.*

The first step in proving Theorem 6.2 is to relate $\mathcal{D}_n\langle[M]\rangle$ to $\mathcal{C}^\bullet\langle[M]\rangle$. The following proposition is immediate from unraveling Definitions 5.9, 6.1 and 6.3.

Proposition 6.4. *$\mathcal{D}_n\langle[M]\rangle$ is the composite of $i_n\mathcal{C}^\bullet\langle[M]\rangle$ and $\mathcal{G}_n$.*

The next step in proving Theorem 6.2 is the immediate application of a general theorem.

Theorem 6.5. *Let $\mathcal{X}^\bullet$ be a cosimplicial space. The homotopy limit of $\mathcal{G}_n \circ \mathcal{X}^\bullet$ is weakly equivalent to the nth totalization of a fibrant replacement of $\mathcal{X}^\bullet$.*

Before proving this theorem in general, it is enlightening to establish its first case. Consider the homotopy limit

$$H = \mathrm{holim}\ X_0 \xrightarrow{s_0} X_1 \xleftarrow{s_1} X_0,$$

where X_0 and X_1 are entries of a fibrant cosimplicial space $\mathcal{X}^\bullet$ with structure maps s_0, s_1 and π. By definition then, s_0 s_1 are sections of π, which is a fibration. We claim that this homotopy limit is weakly equivalent to Tot^1 of the given cosimplicial space. The homotopy limit H naturally fibers over ${X_0}^2$ with fiber ΩX_1, the based loop space of X_1. On the other hand, the first total space fibers over X_0 with fiber equal to $\Omega(\text{fiber }\pi)$, so the equivalence is not a triviality.

Considering the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s_0"] \arrow[d, "id"] & X_1 \arrow[d, "\pi"'] & X_0 \arrow[l, "s_1"'] \arrow[d, "id"'] \\
X_0 \arrow[r, "id"] & X_0 & X_0 \arrow[l, "id"']
\end{tikzcd}$$

we see H also fibers over H_0, the homotopy limit of $X_0 \xrightarrow{id} X_0 \xleftarrow{id} X_0$, which is simply the space of paths in X_0, which is homotopy equivalent to X_0 through a deformation retraction onto the constant paths. The fiber of this map over a constant path is homotopy equivalent to $\Omega(\text{fiber }\pi)$. In fact if we lift the homotopy equivalence of H_0 with X_0 defined by shrinking a path to a constant path, we get a homotopy equivalence of H with a subspace of H which is homeomorphic to $Tot^1\mathcal{X}^\bullet$. Note that the standard filtration of the totalization of $\mathcal{X}^\bullet$ is more efficient than our first filtration of H.

We break the proof of Theorem 6.5 into two theorems.

Theorem 6.6. *The functor $\mathcal{G}_n$ is left cofinal.*

The second theorem is straight from Bousfield and Kan [5].

Theorem 6.7. *The homotopy limit of $i_n\mathcal{X}^\bullet$ is weakly equivalent to the nth totalization of a fibrant replacement of $\mathcal{X}^\bullet$.*

These two theorems along with Proposition 6.4 give a chain of equivalences

$$\mathrm{holim}\mathcal{D}_n\langle[M]\rangle \cong \mathrm{holim}\mathcal{G}_n \circ i_n\mathcal{C}^\bullet\langle[M]\rangle \xleftarrow{\simeq} \mathrm{holim} i_n\mathcal{C}^\bullet\langle[M]\rangle \simeq \mathrm{Tot}^n(\mathcal{C}^\bullet\langle[M]\rangle),$$

which establishes Theorem 6.2. We now proceed to prove Theorem 6.6.

Lemma 6.8. *The simplicial complex $|\mathcal{G}_n \downarrow \underline{d}|$ is isomorphic to the barycentric subdivision of the complex whose i simplices correspond to pairs (S, f) where $S \subseteq \underline{n}$ is of cardinality $i+1$ and f is an order preserving map from S to $\underline{d}$, and whose face structure is defined by restriction of these maps.*

Proof. Note that $\mathcal{G}_n \downarrow \underline{d}$ is a poset since $[n]_0$ is. Since for any S, $S \cong \mathcal{G}_n(S)$ as ordered sets, we may consider elements of $\mathcal{G}_n \downarrow \underline{d}$ to be pairs (S, f) as in the statement of the lemma. If $S' \subseteq S$, then (S', f') maps to (S, f) if and only of f restricts to f'. Therefore, the subcategory of objects in $\mathcal{G}_n \downarrow \underline{d}$ which map to a given (S, f) is isomorphic to $[\#S - 1]_0$, whose realization is an $\#S - 1$-simplex. Moreover, restating from above, the realization of the objects under (S', f') is a face of the objects under (S, f) if and only if $S' \subseteq S$ and f' is the restriction of f. $\square$

We also use the following straightforward result.

Lemma 6.9. *Let Y be a simplicial complex which is a union of simplices which are indexed by a partially-ordered set A which has a minimal element. Suppose that each simplex σ_a of Y, except for the minimal, shares at least one face with some $\sigma_b < \sigma_a$, and that each such σ_a has a face which is not shared with any such σ_b. Then Y is contractible.*

Proof. We build Y inductively by adjoining simplices in an order which does not violate the partial ordering of A. By assumption, at every step we adjoin a simplex along a set of faces which is non-empty and not the set of all faces, so by induction we are adjoining a contractible complex along a contractible complex to a contractible complex at every step. □

We are now ready to prove Theorem 6.6.

Proof of Theorem 6.6. By definition to show $\mathcal{G}_n$ is left cofinal is to show $|\mathcal{G}_n \downarrow \underline{d}|$ is contractible. By Lemma 6.9, $|\mathcal{G}_n \downarrow \underline{d}|$ is homeomorphic to $Y_n(d)$, which is the n-dimensional simplicial complex whose i simplices are labeled by pairs S, f where $S \subseteq \underline{n}$ is of cardinality $i+1$ and f is an order preserving map from S to $\underline{d}$ and whose face structure is defined by restriction of these maps.

Because any such (S, f) admits a map to some $(\underline{n}, \bar{f})$, say by letting $\bar{f}(i) = f(s(i))$ where $s(i)$ is the greatest element of S which is less than or equal to i, $Y_n(d)$ is a union of its n-simplices, which are indexed by the set A of order-preserving maps from $\underline{n}$ to $\underline{d}$. We put a partial order on such maps where $f \geq g$ if $f(i) \geq g(i)$ for all i, so that A has unique minimal and maximal elements given by $f(i) = 0$ for all i and $f(i) = d$ for all i respectively. In all cases except for the maximal element, an f in A shares a faces with a greater g, defined by increasing a single value of f. Similarly, in all cases except for the minimal element, an f shares a face with a lesser g. In the case of the maximal element, the face defined by $\{0, \cdots, n-1\} \mapsto \underline{d}$ is not shared by any other simplex. Hence we may apply Lemma 6.9 to $Y_n(d)$ to deduce it is contractible and finish our proof. □

Because the spaces $C'_n[M, \partial]$ are more familiar than $C'_n\langle[M, \partial]\rangle$ and are manifolds with corners, one would want to use them to define a cosimplicial model, just as as we used them for the mapping space model. Indeed, we have defined maps δ^i on $C'_n[M, \partial]$ which one could use in conjunction with projections to try to define a cosimplicial space. Unfortunately, for these compactifications $\delta^i\delta^i$ is not equal to $\delta^{i+1}\delta^i$. These two maps are related by a canonical homotopy, and in fact we claim that one can define a sort of A_∞ cosimplicial space with the $C'_n[M, \partial]$ as entries. Alternatively, one can try to change the category Δ so as to capture the identities which are satisfied. One would like to define a category C whose relationship to Δ is much like the relationship between Φ_n and $[n]_0$, and then find an appropriate C-space model. We took such an approach in an earlier version of this paper. Such a category C is touched upon in [26] as well. Because the algebra of our spectral sequences is ultimately cosimplicial, we opted for our current approach as the most direct. Nonetheless, defining a model based on the combinatorics of the associahedra might be useful in further applications of the theory, in particular in connecting with Bott-Taubes integrals and in connecting with Kontsevich's models based on operads.

We end this section by solidifying the analogy between our cosimplicial model for a knot space and the cosimplicial model of a loop space. The machinery which has culminated in this cosimplicial model for embeddings can be applied for immersions of an interval in M, namely $Imm(I, M)$, as well. Because immersions may self-intersect globally we have that the nth degree approximation from embedding calculus is a homotopy limit over $[n]_0$ of spaces $Imm(I - \bigcup_{s \in S} I_s, M) \cong (STM)^{\#S-1}$. Following the arguments in this paper, we get from these polynomial approximations a cosimplicial model which has nth entry $(STM)^n$. This cosimplicial space is precisely the cosimplicial model for $\Omega(STM)$, which is homotopy equivalent to the space of immersions by theorems of Hirsch and Smale [32]. The spectral sequence in cohomology for this cosimplicial model is the Eilenberg-Moore spectral sequence, which is thus analogous to the spectral sequences we develop in Section 8.

7. The cohomology of ordered configurations in Euclidean space

The computations which we review now are standard (see [9, 10, 1]). To compute the cohomology ring $H^*(C_n(\mathbb{R}^{k+1}); A)$ for any ring A and $k > 1$ we appeal to the Leray-Serre spectral sequence for the fibering $\bigvee_n S^k \to C_n(\mathbb{R}^{k+1}) \xrightarrow{f} C_{n-1}(\mathbb{R}^{k+1})$, where f forgets the last point in a configuration. Assuming that $k > 1$ the base of this fibration is simply connected, so the coefficient system of the spectral sequence is trivial (which is also true for $k = 1$ [9]). We find inductively that the E_2-term of this spectral sequence is comprised of free modules concentrated in bi-degrees p, q such that both p and q are divisible by k, implying that the spectral sequence collapses at E_2. Let $p_n(t)$ denote the Poincare series of $C_n(\mathbb{R}^{k+1})$. Inductively we deduce that $p_n(t) = (1 + (n-1)t^k)p_{n-1}(t)$, so that $p_n(t) = \prod_{i=1}^{n-1}(1 + it^k)$.

Determining the ring structure requires more detailed analysis. Recall that map $\alpha_{ij} \colon C_n(\mathbb{R}^{k+1}) \to S^k$ as sends $(x_1, \cdots, x_n)$ to $\frac{x_i - x_j}{|x_i - x_j|}$. Orient S^k and let $\iota \in H^k(S^k)$ denote the dual to the fundamental class. Let $a_{ij} = \alpha_{ij}^*(\iota)$. We have the following relations, the first two pulled back from $H^*(S^k)$ and the last simply restating graded-commutativity.

$$a_{ij}{}^2 = 0 \tag{5}$$

$$a_{ij} = (-1)^{k+1} a_{ji} \tag{6}$$

$$a_{ij}a_{lm} = (-1)^k a_{lm}a_{ij} \tag{7}$$

Now consider the special case of $C_3(\mathbb{R}^{k+1})$. The Leray-Serre spectral sequence has

$$E_\infty^{p,q} = \begin{cases} A & p,q = 0,0 \text{ or } k,0 \\ A^2, & p,q = k,0 \text{ or } k,k \\ 0 & \text{otherwise} \end{cases}$$

The product map from $E_\infty^{0,k} \otimes E_\infty^{k,0} \to E_\infty^{k,k}$ is an isomorphism. As the Leray-Serre spectral sequence is a spectral sequence of algebras, we deduce that the products $a_{12}a_{23}$, $a_{12}a_{13}$ and $a_{13}a_{23}$ span $H = H^{2k}(C_3(\mathbb{R}^{k+1}))$. Because $H \cong A^2$, there must be a relation of the form

$$c_1\ a_{12}a_{23} + c_2\ a_{23}a_{31} + c_3\ a_{31}a_{13} = 0, \tag{8}$$

where at least one of the c_i is non-zero. The cyclic group of order three acts on $C_3(\mathbb{R}^{k+1})$ by cyclically permuting the points in a configuration, which gives rise to an action on the cohomology group H. Under this action, the classes $a_{12}a_{23}$, $a_{23}a_{31}$ and $a_{31}a_{13}$ get cyclically permuted, so that the relation above also holds after cyclic permutation of the coefficients. Because $a_{12}a_{23}$, $a_{23}a_{31}$ and $a_{31}a_{13}$ span H which is of rank two, the permutations of this relation must give redundant relations so that $c_1 = c_2 = c_3 = 1$.

For $C_n(\mathbb{R}^{k+1})$ in general, we consider the map to $C_3(\mathbb{R}^{k+1})$ defined by forgetting all but the ith, jth and kth points. Equation 8 pulls back to

$$a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0, \tag{9}$$

which we call the Jacobi identity.

Let R_n denote the A-algebra generated by classes x_{ij} in degree k for $1 \leq i \neq j \leq n$ with relations given by 5 through 7 and the Jacobi identity.

Theorem 7.1. *As rings, $H^*(C_n(\mathbb{R}^{k+1})) \cong R_n$.*

Proof. Given what we have shown to this point, it suffices to show that the surjection $R_n \to H^*(C_n(\mathbb{R}^{k+1}))$ which sends x_{ij} to a_{ij} is an isomorphism by comparing ranks degree-wise. We may compute the Poincare series of R_n, which we call $q_n(t)$, inductively by noting that R_n is generated as an R_{n-1}-module by the unit and the classes x_{in}. Terms involving products of more than one x_{in} can be reduced to this basis since the $x_{in}x_{jn} = \pm x_{ij}x_{in} \pm x_{ij}x_{jn}$ through the Jacobi identity. We deduce that $q_n(t) = (1 + (n-1)t^k)q_{n-1}(t)$, which is the same inductive relation obeyed by $p_n(t)$, the Poincare series of $H^*(C_n(\mathbb{R}^{k+1}))$. Because $q_1(t) = p_1(t) = 1$, we have $q_n(t) = p_n(t)$ in general, which is what we wanted to show. □

For our applications, we will need the cohomology ring of $C'_n(\mathbb{R}^{k+1})$ which is homotopy equivalent to $C_n(\mathbb{R}^{k+1}) \times (S^k)^n$. By the Künneth theorem, this cohomology ring is isomorphic to that of $C_n(\mathbb{R}^{k+1})$ tensored with an exterior algebra on n generators, which we call b_1 through b_n.

To connect with the combinatorics of [4, 7, 35], we give a description of the cohomology groups of $C'_n(\mathbb{R}^{k+1})$ in terms of chord diagrams. Consider the free module generated by linear chord diagrams with vertices labeled $1, \ldots, n$ and m edges. Note that a vertex need not be attached to any edge. The edges in the diagram are labeled (if k is odd) or oriented (if k is even). Let $\widetilde{W}_{n,m}$ denote the quotient of this module by the following relations:

- If C has two edges connecting the same vertices, then $C = 0$.
- If C and D are diagrams which differ by a change of orientation of an edge (even case) or by the transposition of labels of two edge (odd case), then $C = -D$.
- Fix vertices i, j and k. Let T be a diagram with $m+1$ edges, three of which are α connecting i to j, β connecting j to k and γ connecting k to i. Let $T_{\hat{\alpha}}$ (respectively, β or γ) denote the diagram with m chords which include all chords in T, with the same labeling or orientation, except for α. The final relation in $\widetilde{C}_{n,m}$ is that $T_{\hat{\alpha}} + T_{\hat{\beta}} + T_{\hat{\gamma}} = 0$.

By construction, we have the following.

Proposition 7.2. $\widetilde{W}_{n,m} \cong H^{mk}(C_n(\mathbb{R}^{k+1}))$.

This isomorphism is realized by sending $a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_m j_m}$ to the diagram with a chord between every i_l and j_l, ordered from 1 to m as given or oriented so that i_l is the positive end of the chord.

We may similarly give a more combinatorial description of the cohomology groups of $C'_n(\mathbb{R}^{k+1})$. Using the Künneth theorem, we may realize $H^{nk}(C'_n(\mathbb{R}^{k+1}))$ as a module generated by chord diagrams with n vertices, i of which are marked, and $m - i$ edges, with i ranging from 0 to m and relations as in the definition of $\widetilde{W}_{n,m}$. We call this module $W_{n,m}$.

8. The spectral sequences

In this section we give spectral sequences which converge to the homotopy and cohomology groups of $\mathrm{Emb}(I, M)$, when M has dimension at least four. We focus particular attention on the case of $M = \mathbb{R}^k$.

Recall that the spectral sequence for the homotopy groups of a cosimplicial space $\mathcal{X}$ is straightforward to construct [5]. It is simply the spectral sequence for the tower of fibrations

$$\mathrm{Tot}^0 \mathcal{X} \leftarrow \mathrm{Tot}^1 \mathcal{X} \leftarrow \cdots .$$

Theorem 8.1. *Let M have dimension four or greater. There is a second quadrant spectral sequence converging to $\pi_*(\mathrm{Emb}(I, M))$ whose E_1 term is given by*

$$E_1^{-p,q} = \bigcap \, ker \, {s^k}_* \subseteq \pi_q(C'_p\langle[M]\rangle)).$$

The d_1 differential is the restriction to this kernel of the map

$$\Sigma_{i=0}^{p+1} (-1)^i {\delta^i}_* \colon \pi_q(C'_{p-1}\langle[M]\rangle) \to \pi_q(C'_p\langle[M]\rangle).$$

We give more explicit computations of the homotopy spectral sequence when M is $\mathbb{R}^k \times I$ in [27]. The rows of this spectral sequence are reminiscent of complexes defined by Kontsevich in [23].

We now discuss the cohomology spectral sequence. We proceed by taking the homology spectral sequence first studied in [25] and dualizing through the universal coefficient theorem. The convergence of the homology spectral sequence is more delicate than that of the homotopy spectral sequence, arguments often starting with the convergence of the standard Eilenberg-Moore spectral sequence as input, and has been studied in [3, 6, 29].

For a cosimplicial space $\mathcal{X}^\bullet$ let $\overline{H}_*(\mathcal{X}^n)$, the normalized homology of $\mathcal{X}^n$, be the intersection of the kernels of the codegeneracy maps $s^i \colon H_*(\mathcal{X}^n) \to H_*(\mathcal{X}^n - 1)$. Theorem 3.2 of [6] states that the mod-p homology spectral sequence of a cosimplicial space $\mathcal{X}^\bullet$ converges (strongly) when three conditions are met, namely that $\mathcal{X}^n$ is simply connected, $\overline{H}_m(\mathcal{X}^n) = 0$ for $m \leq n$, and for any given k only finitely many $\overline{H}_m(\mathcal{X}^n)$ with $m - n = k$ are non-zero. The last two conditions are satisfied if $\overline{H}_*(\mathcal{X}^n)$ vanishes through degree cn for some $c > 1$, which we call the vanishing condition. We check the vanishing condition for $\mathcal{C}^\bullet\langle[M]\rangle$ where M is simply connected.

We first concentrate on $C_n(M)$. Producing the same spectral sequence as Cohen and Taylor [11] obtained by different methods, Totaro [34] studies the Leray spectral sequence of the inclusion of $C_n(M)$ in M^n. The stalks of the sheaf representing the cohomology of the fiber are products of the cohomology of $C_i(\mathbb{R}^k)$. Next note that the inclusion of $C'_n(M)$ in $(STM)^n$ has the same fibers. Because the projections s^i are compatible for $C_n(M)$, $(STM)^n$ and $C_n(\mathbb{R}^k)$, we claim that the vanishing condition for $\mathcal{C}^\bullet\langle[M]\rangle$ follows from the vanishing condition for $\mathcal{C}^\bullet\langle[\mathbb{R}^k]\rangle$, which follows from Corollary 8.4 below, and the vanishing condition for the cosimplicial model of $\Omega(STM)$, which is standard (the normalized homology vanishes up to dimension $(k+1)n$ where k is the connectivity of STM).

We may deduce the following.

Theorem 8.2. *Let M be simply connected and have dimension four or greater. There is a second quadrant spectral sequence converging to $H^*(\mathrm{Emb}(I, M); \mathbb{Z}/p)$ whose E_1 term is given by*

$$E_1^{-p,q} = \textit{coker}\ \Sigma(s^i)^* \colon H^q(C'_{p-1}(M); \mathbb{Z}/p) \to H^q(C'_p(M); \mathbb{Z}/p).$$

The d_1 differential is the passage to this cokernel of the map

$$\Sigma(-1)^i(\delta^i)^* \colon H^q(C'_p(M); \mathbb{Z}/p) \to H^q(C'_{p-1}(M); \mathbb{Z}/p).$$

We now give a more combinatorial identification of this E^1 term of our spectral sequence when $M = \mathbb{R}^k \times I$. Because the cohomology of $C'(\mathbb{R}^k \times I)$ is torsion free, we may combine the spectral sequences above to work integrally. Recall from Section 7 that $H^{np}(C'_n(\mathbb{R}^{k+1})) \cong W_{n,p}$, where $W_{n,p}$ is a module of chord diagrams. Our first task is to identify the homomorphisms $(s^k)^*$ in terms of these modules.

Let $\tau_l(i)$ is equal to i if $i \leq l$ and $i+1$ if $i > l$. For $i, j \neq l$, the map $\alpha_{ij} \colon C_n(\mathbb{R}^{k+1}) \to S^k$ factors as $\alpha_{\tau_l(i)\tau_l(j)} \circ s^l$. Hence $a_{ij} \in H^k(C_{n-1}(\mathbb{R}^{k+1}))$ maps to $a_{\tau_k(i)\tau_k(j)} \in H^k(C_n(\mathbb{R}^{k+1}))$ under s_k^*. Translating through the isomorphism of Proposition 7.2, $(s^l)^*$ takes a chord diagram with $n-1$ vertices, relabels the vertices according to τ_l and adds a vertex, not attached to any edges nor marked, labeled l. Hence the sub-module generated by the images of $(s^l)^*$ is the sub-module $N_{n,p}$ of chord diagrams in which at least one vertex is not attached to an edge. The quotient map $W_{n,p} \to W_{n,p}/N_{n,p}$ is split, so that $W_{n,p}/N_{n,p}$ is isomorphic to the submodule of $W_{n,p}$ generated by chord diagrams in which every vertex is attached or marked, which we call $V_{n,p}$.

Next we identify the homomorphisms $(\delta^\ell)^*$. Recall that $\sigma_\ell(i)$ is equal to i if $i \leq \ell$ and $i-1$ if $i > \ell$. If $(i,j) \neq (\ell, \ell+1)$ then the composite $\alpha_{ij} \circ \delta_\ell$ coincides with $\alpha_{\tau_\ell(i),\tau_\ell(j)}$, which implies that a_{ij} maps to $a_{\tau_\ell(i),\tau_\ell(j)}$ under δ_ℓ^*. On the other hand, $\alpha_{\ell,\ell+1} \circ \delta_\ell$ is the projection of $C'_n(\mathbb{R}^{k+1})$ onto its ℓth factor of S^k, so that $\delta_\ell^*(a_{\ell,\ell+1}) = b_\ell$. We extend these computations to define δ_ℓ^* on all of $H^*(C'_n(\mathbb{R}^{k+1}))$ using the cup product.

These homomorphisms have a beautiful interpretation in terms of chord diagrams. Let c_ℓ be the map on $W_{n,p}$ defined simply by contracting the linear edge between the ℓth and $\ell+1$st vertices in a chord diagram. Moreover, we mark the ℓth vertex in the new diagram if there was an edge between these two in the original diagram, and if this vertex is already marked, we set the result to zero.

Corollary 8.3. *There is a spectral sequence converging to $H^*(E(I,\mathbb{R}^{k+1}))$ whose E_1 term is given by*

$$E_1^{-p,q} = \begin{cases} V_{n,p} \text{ for } q = kn \\ 0 \text{ otherwise.} \end{cases}$$

with $d^1 = \Sigma(-1)^\ell c_\ell$.

Note that up to regrading, the E_2-term of this spectral sequence depends only on the parity of k. At first glance this spectral sequence looks similar to Vassiliev's [35] as well as to complexes of Cattaneo, Cotta-Ramussino and Longoni [7]. We comment on potential relationships below.

We end by proving a vanishing result. Recall from Section 7 that the cohomology of $C_n(\mathbb{R}^{k+1})$ vanishes above degree $(n-1)k$, so that the cohomology of $C_n(\mathbb{R}^{k+1}) \times (S^k)^n$ vanishes above degree $(2n-1)k$. Note also that the module $V_{n,p}$ is zero of $n < p/2$, as each edge connects two vertices. These two observations establish the existence of vanishing lines in the spectral sequence above.

Corollary 8.4. *The E_1 term of the spectral sequence of Corollary 8.3 vanishes above the line $2p-1 = q/k$ and below the line $p/2 = q/k$.*

9. Further work and open questions

This paper is meant to serve as a foundation for further study of spaces of knots. We see three interrelated problems which may serve as guideposts for such further study.

- Gain explicit understanding of cohomology and homotopy groups of spaces of knots.
- Develop invariant descriptions of (E^2-approximations of) cohomology and homotopy groups of spaces of knots, perhaps by connecting with theory of operads.
- Determine the consequences of this study of spaces of knots for the problem of classification of knots in a three-manifold.

We end this paper by indicating what is known to us about each of these three problems.

In [22] Kontsevich outlined a program to compute cohomology of spaces of knots in manifolds of dimension four or greater which is standard from the point of view of algebraic topology, namely to use a spectral sequence as an upper bound and then produce enough classes to match that upper bound. There is a similar program for homotopy groups, which one would only expect to be manageable rationally. The spectral sequence in Theorem 8.1 is the first such spectral sequence computing homotopy groups of knot spaces. The spectral sequence of Theorem 8.2 is the first to address cohomology of spaces of knots in a general manifold. Even for Euclidean spaces, the spectral sequence in Corollary 8.3 differs at first glance from Vassiliev's [35], thus giving new progress on the first step of this program.

The next step in the program, namely producing cohomology classes, has been carried out for de Rham cohomology of spaces of knots in Euclidean spaces by Cattaneo, Cotta-Ramussino and Longoni [7]. Generalizing the techniques of Bott and Taubes [4], they use integrals in de Rham theory to produce complexes whose cohomology map to the de Rham cohomology of the space of knots with gradings compatible with the spectral sequence of Theorem 8.2. The situation is analogous to that of loop spaces, for which Chen's iterated integrals [8] are used in some cases to prove collapse of an Eilenberg-Moore spectral sequence. The complexes of [7] would provide a lower bound for this cohomology if their cohomology maps in injectively. It is shown in [7] that this map is injective in certain (bi-)degrees by pairing the classes with explicit homology classes. Hence, to complete the program of understanding rational cohomology of knots in Euclidean space it would suffice to show that the complexes of [7] are quasi-isomorphic to the rows of the spectral sequence of Corollary 8.3 and to find homology classes which pair non-trivial with all of the forms defined in [7]. Such explicit homology classes are desirable in any case in order to have a complete understanding of these computations.

Another computational area which is seemingly related is the homology of the loopspace of a configuration space, studied by F. Cohen and Gitler [12], in which combinatorics similar to that of Theorem 8.2 appears.

The computational program is not as far along for homotopy groups. The author along with Scannell made first computations in [27]. We proved vanishing results and made computations in low degrees, showing the d^1 differential to be of high rank but still giving plenty of non-zero classes, some of which must survive to E^∞ of this spectral sequence. The geometry of the spherical families of knots representing these classes presents an interesting open question.

Even if collapse of our spectral sequences were known in some cases and we had a handle on some relevant geometry, our knowledge of these cohomology and homotopy groups would not be complete. It would be as if we knew that the bar complex gave computations of the cohomology of the loop space in some cases but we had no global understanding of the functor Tor. Our second problem above is to gain such understanding, which could touch on fields far from topology.

There has been some progress on this problem. Kontsevich outlined an approach to spaces of knots and embeddings more generally based on the language of operads [23]. We can see operads occur in our approach since the spaces A_n and more generally $C_n[\mathbb{R}^k]$ form the entries of an operad equivalent to the little disks operad. So we expect to be able to translate our mapping space model and, if we "blow up" the cosimplicial category as indicated at the end of Section 6, a cosimplicial model into the language of operads. Such connections are already bearing fruit. By applying a theorem of McClure and Smith from their solution of Deligne's conjecture [24], a theorem saying that the totalization of a cosimplicial space with a compatible operad structure is a two-fold loop space, we claim that the space of knots in $\mathbb{R}^k \times I$ is a two-fold loop space. The existence of this two-fold loop structure implies that the homology of these knot spaces is a Gerstenhaber algebra, which should be useful for explicit computations. Note that Tourtchine [33] has announced a Gerstenhaber structure on Vassiliev's spectral sequence, which we conjecture agrees with ours.

Finally, we remark on what is known about potential consequences of our work in the case of knot spaces of the most interest, namely knots in dimension three. There seem to be many connections with the theory of finite-type invariants. At a foundational level, the calculus of embeddings approximates knot spaces through homotopy limits or cosimplicial spaces while Vassiliev studies the Spanier-Whitehead duals of these spaces through homotopy colimits or simplicial spaces (see [30]), so we are optimistic about the possibility of direct relationships between these approaches.

As mentioned in the introduction, because there is a map from the space of knots to our models, we can pull back invariants of the set of components of our models to define knot invariants. There are two routes which one can take to do so. One route is to first pass from the spaces defining our models ($E_J(M)$ or $C'_i\langle[M]\rangle$) to the free abelian groups on those spaces. For a cosimplicial space, one defines its homology spectral sequence by passing to the free abelian group of each of its entries, so invariants which one defines in this way are enumerated by the cohomology spectral sequence of $\mathcal{C}\langle[M]\rangle$. In his thesis under Goodwillie, Volic has shown the group $E^2_{-2p,2p}$ in this spectral sequence is isomorphic to the module of chord diagrams modulo the four-term relations from finite-type knot theory. Volic has also shown that the invariants one pulls back from these models are in fact of finite type p. The open question is as to whether one can pull back all finite-type invariants from these models.

One approach to this question of factoring finite type invariants through our models, at least with real coefficients, would be to extend the definition of the differential forms by configuration space integrals on $\mathrm{Emb}(I, M)$ of [4] to models closely related to ours. We suspect that an extension should exist, noting that general aligned stratum preserving maps share many properties with the evaluation map of a knot which are used in the proofs of [4]. But there are serious technical issues. One important piece which is missing from our mapping space model is that the restrictions of an aligned map to the various principal faces of A_n need not coincide. One expects better results with a cosimplicial model (or in particular a model based on a category which is a blow-up of the cosimplicial category), but there one runs into the difficulty that for example $\mathcal{C}^\bullet\langle[M]\rangle$ is not fibrant. The categorical fibrant replacement ruins the geometry needed to do de Rham theory, so one would need to find a geometric fibrant model.

Less is known about knot invariants one can define from our approach by not first passing to free abelian groups on configuration spaces. These invariants are the main topic of study in [28], which builds on [27] since these invariants are enumerated by the homotopy spectral sequence. Finite-type invariants are supposed to be analogs of linking number, which can be defined combinatorially, analytically, homologically or homotopically through the Gauss or evaluation map. An interpretation in homotopy theory or differential topology of finite-type invariants has been missing from the theory. The question has been as to the correct point of view on the evaluation map. Our mapping space model seems to be the right home for the evaluation map, so we are lead to search for invariants of components of that model. We have not fully established a connection with finite-type invariants, but there is good evidence for one. The number of these invariants agrees with the number of finite-type invariants in low degrees. The modules of these invariants are quotients of the modules $Lie(n)$ used to define the Lie operad, giving a potential connection with Lie algebras and Feynman diagrams. We have also found first constructions of these invariants through differential topology, which give a beautiful connection with the linking number.

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Department of Mathematics, University of Oregon, Eugene OR and Department of Mathematics, Brown University, Providence RI

E-mail address: dps@math.brown.edu