Hurewicz images in $BP$ and related homology theories

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Abstract

In this paper $BP$-theory is used to give a proof that there exists a stable homotopy element in $\pi_{2n+1-2}^S(\mathbb{R}P^{\infty})$ with non-zero Hurewicz image in $ju$-theory if and only if there exists an element of $\pi_{2n+1-2}^S(S^0)$ which is represented by a framed manifold of Arf invariant one.

1 Introduction

1.1 Deciding whether or not framed manifolds with non-trivial Arf-Kervaire invariant exist is one of the longstanding problems in homotopy theory. In terms of stable homotopy theory this is concerned with the existence of certain two-primary classes in the stable homotopy group $\pi_m^S(S^0)$. It was shown in [?] that the problem only remains when $m = 2^n+1 - 2$. In fact calculations similar to those we are about to do give the result of [?] very easily.

At the moment it is known only that such framed manifolds exist when $n = 1, 2, 3, 4, 5$.

It is convenient to study an equivalent problem. There is a split surjection, called the Kahn-Priddy map, of the form

$$\pi_m^S(\mathbb{R}P^{\infty}) \longrightarrow \pi_m^S(S^0) \otimes \mathbb{Z}_2$$

and when $m = 2^n+1 - 2$ the condition that $\theta \in \pi_{2n+1-2}^S(\mathbb{R}P^{\infty})$ maps to a stable homotopy element represented by a framed manifold of Arf invariant one modulo 2 is equivalent to the Steenrod operation, $Sq^{2^n} : H^{2^n-1}(C(\theta); \mathbb{Z}/2) \longrightarrow H^{2^n+1-1}(C(\theta); \mathbb{Z}/2)$, being non-trivial on the mod 2 cohomology of the mapping cone, $C(\theta)$, of $\theta$.

The main result (Theorem ??) of this paper is to show that this happens if and only if the $ju$-theory Hurewicz homomorphism, $H_{ju} : \pi_{2n+1-2}^S(\mathbb{R}P^{\infty}) \longrightarrow ju_{2n+1-2}^S(\mathbb{R}P^{\infty})$ is non-trivial on $[\theta]$. This result was first conjectured a long while ago by Barratt and Mahowald, appearing in print in [?]. The proof given by Klippenstein and me in [?] unfortunately contains a gap. Namely the group in ([?] Lemma 3.1) is slightly larger than claimed, allowing some ruinous indeterminacy into the argument. In 1990 Knapp pointed out the problem and Klippenstein tried to repair his mistake, failing to do so before he left the academic profession.
A few years later Knapp produced a correct proof [?] based on work of Miller-Ravenel-Wilson [?]. In view of its history one cannot expect a non-technical proof of this conjecture but I believe that the proof given here is ‘elementary’ in the sense that it proceeds via a series of generalised homology calculations which are rather standard. Firstly Theorem ?? is reduced to a result in $J$ (Theorem ??), which is a generalised homology theory constructed from $BP$-theory. Then the proof of Theorem ?? requires only a basic familiarity with the rather awkward formula for the canonical anti-automorphism in $BP$-theory, discovered by Quillen and described in [?] or [?]. After that the proof rests on some easily obtained formulae involving binomial coefficients modulo 2 and a two-step induction argument.

In the past I was inclined to give $BP$ a wide berth. Therefore I am particularly grateful to Huajian Yang for introducing me to Quillen’s formulae in a very user-friendly manner. Huajian was my Britton post-doc at McMaster University before he, too, left the academic profession.

Here is a sketch of the contents of the paper. The reader will find that I have indulged in far too many tedious combinatorial details in §7 and §8. This is because this write-up of the proof is not intended for publication as it is. If published it will presumably require trimming to an extent which will make the tedious details difficult to resuturect. In §2 and §3 I recapitulate the facts and formulae which are needed about $BP$ and $\mathbb{R}P^\infty$ and introduce the homology theories, $J_*$ and $J'_*$, which are to $BP$ and $BP \wedge BP$ what $ju$ is to $bu$. The crux of the paper is to restrict the possibilities for Hurewicz images in these theories by analysing the canonical anti-involution induced by switching the factors in $BP \wedge BP$. In Theorem ?? we practice for the induction proof of Theorem ?? by proving a weaker result using $bu \wedge BP$ rather than $BP \wedge BP$. In §4 the deduction of Theorem ?? from Theorem ?? is explained. In §5 an induction argument, imitating the proof of Theorem ??, is given as a first step towards proving Theorem ???. In order to get around the point at which the argument of §5 falters we derive in §6 some complicated combinatorial identities modulo 2 which must be satisfied by the coefficients in our hypothetical Hurewicz image. The coefficients in question are either 0 or 1. In §7 and §8 it is shown how by studying the combinatorial identities of §6 in low degrees one can complete the induction argument for the proof of Theorem ???. Truth to tell, I have only included the complete details of this step (after all – enough is enough!) for the case when the integers $n$ is odd in the dimension, $2^{n+1} - 2$.

2 $BP$-theories and $\mathbb{R}P^\infty$

2.1 Let $BP$ denote the 2-adic Brown-Peterson spectrum ([?] pp.109-116; [?]) whose homotopy, $\pi_*(BP) = BP_*$, is isomorphic to $\mathbb{Z}_2[v_1, v_2, v_3, \ldots]$ where $\mathbb{Z}_2$ denotes the 2-adic integers and $\deg(v_i) = 2(2^i - 1)$. Then we have $BP^*(\mathbb{C}P^\infty) \cong BP^*[x]$ where $BP^* = BP_{-}$. The series $[2]x \in BP^*[x]$ is defined by $[2]x = f^*(x)$ where $f : \mathbb{C}P^\infty = BS^1 \to \mathbb{C}P^\infty$ is induced by the squaring map on the
circle, $S^1$. From ([?] Lemma 3.17 p.20) we have

$$[2]x \equiv 2x + \sum_{i \geq 1} v_i x^{2i} \pmod{<2, v_1, v_2, v_3, \ldots>^2 BP^*[x]}.$$

Now consider $BP^*(R^{2t}) \cong BP^* \oplus \tilde{BP}^*(R^{2t})$. The composition of $f$ with the canonical map, $i : R^{2t} \to CP^{2t} \to CP^{\infty}$, is trivial. Also $x^{t+1}$ is zero in $BP^*(CP^{2t})$ and there is an induced isomorphism of the form

$$i^* : BP^*[x]/<x^{t+1}, [2]x> \xrightarrow{\cong} BP^*(R^{2t}).$$

This isomorphism is proved together with the assertion that every element of $\tilde{BP}^{2m}(R^{2t})$ may be written uniquely as (the image under $i^*$ of) $\sum_{I,j} \epsilon_I v^I x^j$ where the sum is taken over all sequences of non-negative integers, $I = (i_1, \ldots, i_r)$, $v^I = v_1^{i_1} \cdots v_r^{i_r}$, $2j - \sum_{s=1}^r i_s 2(2s - 1) = 2m$ and $1 \leq j \leq t$ with each $\epsilon_I = 0$ or 1. To prove both assertions one observes that the Atiyah-Hirzebruch spectral sequence for the reduced group, $\tilde{BP}^*(R^{2t})$, collapses because it is concentrated in even total degree. The $E_2$-term is generated by $BP^*$ and $x$ so that $i^*$ is surjective. Also $E_2^{p,q}$ is zero unless $0 < p \leq 2t$ is even and $q = 2n$ in which case it is isomorphic to $BP^{2n} \otimes \mathbb{Z}/2$. Therefore the order of $\tilde{BP}^{2m}(R^{2t})$ is $2^n$ where $a = a_1 + \cdots + a_t$ and $a_j$ is equal to the number of sequences, $I$, such that $2j - \sum_{s=1}^r i_s 2(2s - 1) = 2m$. This is also the number of expressions of the form $\sum_{I,j} \epsilon_I v^I x^j$ in dimension $2m$. On the other hand, the form of $[2]x$ shows that every element of $\tilde{BP}^{2m}(R^{2t})$ may be written in at least one way in the desired form. Hence, by counting group orders, this expression must be unique and $i^*$ must be an isomorphism.

The S-dual of $R^{2t}$ is homotopy equivalent to $\Sigma^{1-2t} R^{2i-2} \otimes R^{2i-2t-2}$ for $i$ sufficiently large, by ([?] pp.205-208), and the previous discussion yields short exact sequences of the form

$$0 \to BP^{2i-2h}(R^{2i-2} \otimes R^{2i-2t-2}) \to BP^{2i-2h}(R^{2i-2}) \to BP^{2i-2h}(R^{2i-2t-2}) \to 0.$$ 

In addition, we have S-duals and isomorphisms of the form

$$BP^{2i-2h}(R^{2i-2} \otimes R^{2i-2t-2}) \cong BP^{1-2h}(\Sigma^{1-2t} R^{2i-2} \otimes R^{2i-2t-2})$$

$$\cong BP_{2h-1}(R^{2t}).$$

For $1 \leq h \leq t$ the element, $x^{2i-1-h}$ in $BP^{2i-2h}(R^{2i-2})$ maps to zero in $BP^{2i-2h}(R^{2i-2t-2})$ and we may define $x_{2h-1} \in BP_{2h-1}(R^{2t})$ to be equal to the image of $x^{2i-1-h}$ in $BP^{2i-2h}(R^{2i-2} \otimes R^{2i-2t-2})$ under the S-duality isomorphism.

Every element of $\tilde{BP}^{2i-2h}(R^{2i-2} \otimes R^{2i-2t-2})$ is uniquely writeable in the form $\sum_{I,j} \epsilon_I v^I x^j$ with $2i-1 - t \leq j \leq 2i-1 - 1$ and each $\epsilon_I \in \{0, 1\}$, Hence every
element of $BP_{2s-1}(RP^{2^t}) = BP_{2s-1}(RP^{2^t})$ is uniquely writeable in the form
\[ \sum_{l,k} \epsilon_l v^l x_{2k+1} \] with $1 \leq 2k + 1 \leq 2t - 1$ and each $\epsilon_l \in \{0, 1\}$. The relation that $x^{2^t-1-h} \cdot [2] x = 0$ translates into a congruence of the form
\[ 2x_{2h-1} + \sum_{j \geq 1} v_j x_{2h-2j+i+1} \equiv 0 \) (modulo $< 2, v_1, v_2, v_3, \ldots >^2 BP_*(RP^{2^t})$).

Recall (\cite{89}) p.89 that if $X$ is a commutative ring spectrum with unit, $\iota : S^0 \to X$, there are two maps, $\eta_L = 1 \wedge \iota$ and $\eta_R = \iota \wedge 1$, from $X$ to $X \wedge X$ which give $\pi_*(X \wedge X) = (X \wedge X)_*$ the structure of a left or right $\pi_*(X)$-module, respectively. When $X = BP$ there exist canonical elements, $t_i \in (BP \wedge BP)_2(2^{t-1})$, (\cite{89} Theorem 16.1 p.112; \cite{11} Theorem 3.11 p.17) such that $(BP \wedge BP)_* \cong BP_*[t_1, t_2, t_3, \ldots]$ as a left $BP_*$-module. From the collapsed Atiyah-Hirzebruch spectral sequence for $(BP \wedge BP)_*(RP^{2^t})$ there is an isomorphism of left $BP_*$-modules of the form
\[ (BP \wedge BP)_*(RP^{2^t}) \cong (BP \wedge BP)_* \otimes_{BP_*} BP_*(RP^{2^t}) \cong BP_*(RP^{2^t})_*[t_1, t_2, t_3, \ldots]. \]

Therefore every element of $(BP \wedge BP)_{2s-1}(RP^{2^t})$ is uniquely writeable in the form
\[ \sum_{l', k} \epsilon_{l'} v^{l'} t^i \cdot x_{2k+1} \] with $1 \leq 2k + 1 \leq 2t - 1$ and each $\epsilon_{l'} \in \{0, 1\}$. Here $t^{l'} = t^{(i_1, \ldots, i_e)}$ denotes $t^{i_1} \ldots t^{i_e}$.

3 $J_*$ and $J'*$

3.1 Let $\psi^3 : BP \to BP$ denote the Adams operation in $BP$-theory (\cite{89} Part II; \cite{10}; \cite{10} pp.59/60). Hence $\psi^3$ is equal to multiplication by $3^k$ on $BP_{2k}$ and by $3^{j+1}$ on $BP_{2^{j+1}}(RP^{2^t})$. The last fact follows easily from the formula $\psi^3(x) = 3^{-1} x$ (\cite{89} Corollary 4.3 p.60), that the S-duality isomorphism is given by slant product with the $BP$-Thom class of the tangent bundle of $RP^{2^t}$ (\cite{89} p.264) and that $\psi^3$ commutes with slant products. It also follows that $\psi^3 \wedge \psi^3 : BP \wedge BP \to BP \wedge BP$ is given by multiplication by $3^k$ on $(BP \wedge BP)_{2k}$ and by $3^{j+1}$ on $(BP \wedge BP)_{2^{j+1}}(RP^{2^t})$.

Define spectra, $J$ and $J'$, by the following cofibration sequences
\[ J \xrightarrow{\pi} BP \xrightarrow{\psi^3} BP \xrightarrow{\pi_1} \Sigma J \]
and
\[ J' \xrightarrow{\pi'} BP \wedge BP \xrightarrow{\psi^3 \wedge \psi^3} BP \wedge BP \xrightarrow{\pi'_1} \Sigma J'. \]

Since $\psi^3 \cdot \iota = \iota : S^0 \to BP$, $\eta_L$ and $\eta_R$ induce maps $\bar{\eta}_L, \bar{\eta}_R : J \to J'$, respectively. Also $\iota$ induces a (unique) map, $\bar{\iota} : S^0 \to J$, such that $\pi \cdot \bar{\iota} = \iota$.

Let $n \geq 1$ be an integer. Since $\psi^3 - 1$ is injective on $BP_{2n+1-2}(RP^{2n+1}) \cong BP_{2n+1-2}$, there is an isomorphism of the form
\[ (\pi_1)_* : BP_{2n+1-1}(RP^{2n+1}) \otimes \mathbb{Z}/2^{n+2} \cong J_{2n+1-2}(RP^{2n+1}), \]

\[ (\pi'_1)_* : BP_{2n+1-1}(RP^{2n+1}) \otimes \mathbb{Z}/2^{n+2} \cong J_{2n+1-2}(RP^{2n+1}), \]
since $3^{2n} - 1 = 2^{n+2}(2s + 1)$. Similarly there is an isomorphism of the form

$$(\pi'_1)_*: (BP \wedge BP)_{2n+1-1}(RP^{2n^2+1}) \otimes \mathbb{Z}/2^{n+2} \xrightarrow{\approx} J_{2n^2+1-2}(RP^{2n^2+1}).$$

By means of the isomorphisms, $(\pi'_1)_*$ and $(\pi'_1)_*$, we may represent elements of $J_{2n^2+1-2}(RP^{2n^2+1})$ and $J_{2n^2+1-2}(RP^{2n^2+1})$ by sums in degree $2^{n+1} - 1$ of the form $\sum_{I,k} e_I v_I t_{x_{2k+1}}$ and $\sum_{I',k} e_{I'} v_{I'} t_{x_{2k+1}}$, respectively, as in §??.

Now let $T: BP \wedge BP \xrightarrow{} BP \wedge BP$ be the map which interchanges the factors. Then $T \circ c$, the conjugation, on $(BP \wedge BP)_*(X)$. Let $bu$ denote the 2-adic connected, unitary $K$-theory spectra and let $\lambda: BP \xrightarrow{} bu$ be the canonical map corresponding to the $bu$-Thom class of the universal complex vector bundle ([?] Part II). Hence $\lambda_*: BP_* \xrightarrow{} bu_* \cong \mathbb{Z}_2[v_1]$ is given by $\lambda_*(v_i) = v_1$ and $\lambda_*(v_i) = 0$ for $i \geq 2$. By (?? pp.52-62)

$$(bu \wedge BP)_*(RP^{2n^2+1}) \cong (bu)_*(RP^{2n^2+1})[t_1, t_2, t_3, \ldots].$$

Recall also that $bu_{2k+1}(RP^{2n^2+1}) \cong \mathbb{Z}/2^{k+1}$ when $2k + 1 \leq 2^{n+1}$, generated by $\lambda_*(x_{2k+1})$, which we shall abbreviate to $x_{2k+1}$, and that $v_1 x_{2k+1} = 2(1+2s)x_{2k+3} \in bu_{2k+3}(RP^{2n^2+1})$ for some $s \in \mathbb{Z}$.

We are now going to study those elements, $u \in J_{2n^2+1-2}(RP^{2n^2+1})$, which can possibly satisfy the equation $(\tilde{\eta}_R)_*(u) \equiv c((\tilde{\eta}_L)_*(u)) = (\tilde{\eta}_L)_*(u)$. We shall do this by studying

$$(\lambda \wedge 1)_*(\pi'_1)_*(c((\tilde{\eta}_L)_*(u)) - (\tilde{\eta}_L)_*(u)) \in bu_*(RP^{2n^2+1})[t_1, t_2, t_3, \ldots] \otimes \mathbb{Z}/2^{n+2}.$$

For this purpose we shall need some formulae.

Recall that $BP_*$ embeds, via the Hurewicz homomorphism, into $H_*(BP; \mathbb{Z}_2) \cong \mathbb{Z}_2[m_1, m_2, \ldots]$ where $deg(m_i) = deg(v_i)$ and $v_i = 2m_i - \sum_{j=1}^{i-1} m_j v_i^{2j-1}$. Hence, by induction, $\lambda_*(m_k) = v_i^{2^{k-1} - 1}/2^k \in H_*(bu; \mathbb{Q}_2) \cong \mathbb{Q}_2[v_1]$.

**Lemma 3.2**

In $(bu \wedge BP)_* \cong \mathbb{Z}_2[v_1, t_1, t_2, t_3, \ldots]$

$$(\lambda \wedge 1)_*(c((\eta_L)_*(v_k))) = (\lambda \wedge 1)_*((\eta_R)_*(v_k)) \equiv 2t_k + v_1 t_{k-1}^2$$

modulo $< 2, v_1 >^2 \cdot (bu \wedge BP)_*$.

**Proof**

We use the formulae of ([?] Theorem 16.1 p.112; [?] Theorem 3.11 p.17) from which we see that $(\eta_R)_*(v_1) = 2t_1 + v_1$ and now, by induction, we consider the case $k \geq 2$. In $(BP \wedge BP)_* \otimes \mathbb{Q}_2$ we have $(\eta_R)_*(m_k) = \sum_{j=0}^{k} m_j t_{k-j}$. Applying $(\lambda \wedge 1)_*$ we obtain the equation

$$(\lambda \wedge 1)_*((\eta_R)_*(v_k)) = 2 \sum_{j=0}^{k} v_{2j+1} - 2 - j t_{k-j}^{j} - \sum_{j=1}^{k-1} \sum_{s=0}^{j-1} (v_{2s-1} - 2^{s-j} t_{k-j} - 2^{s-1} - 2^{s-2} - 2^{s-3} - \cdots - 2^{s-j} t_{k-j}^{j})^2.$$

Since this expression lies in $\mathbb{Z}_2[v_1, t_1, t_2, t_3, \ldots]$ and $(\lambda \wedge 1)_*((\eta_R)_*(v_{2k-j}))$ lies in the ideal $< 2, v_1 >^2$, we see that $(\lambda \wedge 1)_*((\eta_R)_*(v_k)) \equiv 2 \sum_{j=0}^{k} v_{2j+1} - 2 - j t_{k-j}^j$ modulo the ideal $< 2, v_1 >^2$, as required. □
3.3 Let $E$ be a commutative ring spectrum and let $(\eta)_* : BP^*(CP^\infty) \to (E \wedge BP)^*(CP^\infty)$ denote the map induced by $\eta$, the unit of $E$. When $E = BP$, $(BP \wedge BP)^*(CP^\infty) \cong (BP \wedge BP)_-[[x]]$ where $x = (\eta_L)_*(x)$ in dimension two. On the other hand, $\eta = \eta_R$ so that the formula of ([?] Lemma 6.3 p.60; [?] Lemma 1.51 p.9) becomes

$$c(x) = \sum_{v \geq 0} b_v BP x^{v+1} \in (BP \wedge BP)^2(CP^\infty).$$

This formula also holds in $(BP \wedge BP)^2(RP^\infty)$. Since $(\eta_L)_*(x_{2k+1}) \in (BP \wedge BP)_*(RP^{2n+1})$ corresponds under S-duality to $x^{2i-1-k-1}$ in

$$(BP \wedge BP)^*(RP^{2i-2}/RP^{2i-2n+1-2}) \cong \left(\frac{x^{2i-2}}{x^{2n-1}Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}\right)/(2x)$$

(where this isomorphism follows from the canonical form for elements which was discussed in §??) then

$$c((\eta_L)_*(x_{2k+1})) = (\eta_R)_*(x_{2k+1}) = \sum_{w=0}^k b_{k,w} (\eta_L)_*(x_{2w+1})$$

where

$$\sum_{w=0}^k b_{k,w} x^{2i-1-w-1} = (\sum_{v \geq 0} b_v BP x^{v+1})^{2i-1-k-1}$$

in $(BP \wedge BP)^*(RP^{2i-2}/RP^{2i-2n+1-2})$. The coefficients, $b_v BP \in (BP \wedge BP)_* \cong Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$ satisfy ([?] Theorem 3.11(proof) p.17 and Theorem 1.48(c) p.8)

$$\sum_{i \geq 0} (\eta_L)_*(m_i) = \sum_{s \geq 0} (\eta_R)_*(m_s) (\sum_{v \geq 0} b_v BP)^{2s}.$$

This equation holds in $Q_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$ but setting each $v_i$ to zero we obtain the equation

$$0 = \sum_{i \geq 0} t_i \left(\sum_{v \geq 0} b_v BP\right)^{2i}$$

and this equation holds in $Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots]/<v_1, v_2, \ldots>$. Hence we find that

$$0 = \sum_{i \geq 0} t_i \sum_{v \geq 0} (b_v BP)^{2i} \in Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots]/<2, v_1, v_2, \ldots>$$

Since $t_0 = 1 = b_0$ one sees by induction that $b_v BP \in <2, v_1, v_2, \ldots> Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$ except when $v = 2^m - 1$ for some $m$ and for each $v \geq 1$

$$0 \equiv \sum_{j=0}^w t_j (b_{2v-j-1}^{BP})^{2j} \pmod{2, v_1, v_2, \ldots} Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots].$$
Lemma 3.4

In the notation of §3.4 and 3.5, the element \( x_{2n+1-1} \in BP_{2n+1-1}(RP^{2n+1}) \) satisfies

\[
(\lambda \wedge 1)_*(c((\eta_L)_*(x_{2n+1-1})) - (\eta_L)_*(x_{2n+1-1})) \ll 2, v_1 > bu_* (RP^{2n+1})[t_1, t_2, \ldots]
\]

in \((bu \wedge BP)_* (RP^{2n+1})\).

Proof

By §3.4, \( c((\eta_L)_*(x_{2n+1-1})) \) corresponds under S-duality to \((\sum_{l \geq 0} t_{b_0} P_{x^n+1})^{2l+1 - (2^n-1)}\) which is congruent modulo \( < 2, x^{2n-1} > \) to \( x^{2n-1} + 2^n \), since \( b_0 = 1 \), and this corresponds to \( x_{2n+1-1} \), as required. \( \square \)

Theorem 3.5

In the notation of §3.4 and 3.5, let \( u \in J_{2n+1-2}(RP^{2n+1}) \) be represented as

\[
u = \sum_l \epsilon_l v^l x_{2n+1-\deg(u')-1}
\]

If

\[
(\tilde{\eta}_R)_*(u) = (\tilde{\eta}_L)_*(u) \in J_{2n+1-2}(RP^{2n+1})
\]

then, either for some \( 0 \leq d \leq n+1 \) and \( \epsilon = 1 \) or for some \( d \geq n+2 \) and \( \epsilon = 0 \) we have

\[
u = \epsilon 2^d x_{2n+1-1} + \sum_{l(l') \geq d+1} \epsilon_{l'} v^{l'} x_{2n+1-\deg(u')-1}
\]

Here the length of \( I = (i_1, \ldots, i_t) \) is defined to be equal to \( l(I) = i_1 + \ldots + i_t \).

Proof

By the discussion of §3.4 and 3.5, there exists an integer, \( 0 \leq d \), such that \( u = \sum_{l(I) \geq d} \epsilon_l v^l x_{2n+1-\deg(u')-1} \). If \( d \geq n+2 \) we have nothing to prove, since \( 2^{n+2} \) annihilates \( J_{2n+1-2}(RP^{2n+1}) \). Hence we may suppose that \( 0 \leq d \leq n+1 \). Also, since \( x_{2n+1-1} \) is the only possible term with \( l(I) = 0 \) there is nothing to prove if \( d = 0 \). Hence we assume that \( 1 \leq d \leq n+1 \). Then, by §3.4, \( 2^d x_{2n+1-1} \) may be written as \( \sum_{l(l')} \tilde{\epsilon}_{l'} v^{l'} x_{2n+1-\deg(u')-1} \) with \( \tilde{\epsilon}_{(d,0,0,\ldots)} = 1 \). Accordingly let \( z \in J_{2n+1-2}(RP^{2n+1}) \) denote whichever of \( u \) or \( u - 2^d x_{2n+1-1} \) has the form \( z = \sum_{l(I) \geq d} \epsilon_l v^l x_{2n+1-\deg(u')-1} \) with \( \epsilon_{(d,0,0,\ldots)} = 0 \).

We shall now show that \( z \) contains no terms with \( l(I) = d \), which will establish the result by induction on \( d \).

Let \( J'' \) be defined by the cofibre sequence

\[
J'' \longrightarrow bu \wedge BP (\psi^3/\psi^3)^{1-1} bu \wedge BP \longrightarrow \Sigma J''
\]

so that \( J''_{2n+1-2}(RP^{2n+1}) \cong (bu \wedge BP)_{2n+1-1}(RP^{2n+1}) \otimes \mathbb{Z}/2^{n+2} \). Also \( \lambda \wedge 1 \) induces a map, \( \lambda \wedge 1 : J' \longrightarrow J'' \), such that, by Lemma 3.5, \( 0 = (\lambda \wedge 1)_*(\tilde{\eta}_R)_*(z) - (\tilde{\eta}_L)_*(z) \) in \( J''_{2n+1-2}(RP^{2n+1}) \otimes \mathbb{Z}/2^{d+1} \cong (bu \wedge BP)_{2n+1-1}(RP^{2n+1}) \otimes \mathbb{Z}/2^{d+1} \).

On the other hand, since \( \epsilon_{(d,0,0,0,\ldots)} = 0 \) in the canonical form for \( z \) we have \( 0 = (\lambda \wedge 1)_*(\tilde{\eta}_L)_*(z) \) (modulo \( 2^{d+1} \)) because \( \lambda \wedge 1 \) annihilates \( v_i \) if \( i \geq 2 \).
Now let us examine \((\lambda \wedge 1)_*((\tilde{\eta}_R)_* (v_1^i \ldots v_r^i x_{2k+1}))\) (modulo \(2^{d+1}\)) where \(l(I) = \sum_{j=1}^{r} i_j = d\) and \(k = 2^n - 1 - \deg (v^l)/2\). From Lemma ?? and §??, in \((bu \wedge BP)_* (R P^{2^n-1} \otimes Z/2^{d+1})\),

\[ (\lambda \wedge 1)_*((\tilde{\eta}_R)_* (v_1^i \ldots v_r^i x_{2k+1})) = \Pi_{j=1}^{l} (2t_j + v_1 t_{j-1}^2)^{i_j} (\sum_{w=0}^{k} b_{k,w} (\tilde{\eta}_L)_* (x_{2w+1})) \]

\[ \equiv 2^d \sum_{0 \leq i_j \leq i_1} t_1^{i_1} \ldots t_{r-1}^{2(i_r-1)} \left( \begin{array}{c} i_1 \\ l_1 \\ \vdots \\ i_r \\ l_r \end{array} \right) (\sum_{w=0}^{k} b_{k,w} (\tilde{\eta}_L)_* (x_{2w+2e(\mathcal{L}-1)})) \]

where, for \(\mathcal{L} = (l_1, \ldots, l_r), \epsilon(\mathcal{L}) = d - \sum_{j=1}^{r} l_j\), since \(2x_{2s+3} \equiv v_1 x_{2s+1} \pmod{4}\).

It is more convenient to study the S-dual of this expression, which lies in the \(Z/2\)-vector space

\[ 2^d((bu \wedge BP)_* (R P^{2^n-1} \otimes Z/2^{d+1}) \equiv 2^d((\frac{x_{2r-1}}{x_{2^n-1} Z_2[t_1, \ldots] [[x]]} \otimes Z/2^{d+1}) \]

Since \(x_{2w+1}\) corresponds under S-duality to \(x_{2^n-1-w-1}, (\lambda \wedge 1)_*((\tilde{\eta}_R)_* (v_1^i \ldots v_r^i x_{2k+1}))\) corresponds to

\[ \Pi_{j=1}^{l} (2t_j + v_1 t_{j-1}^2)^{i_j} (\sum_{w=0}^{k} b_{k,w} x_{2^n-1-w-1}) \]

\[ \equiv 2^d \sum_{0 \leq i_j \leq i_1} t_1^{i_1} \ldots t_{r-1}^{2(i_r-1)} \left( \begin{array}{c} i_1 \\ l_1 \\ \vdots \\ i_r \\ l_r \end{array} \right) (\sum_{w=0}^{k} b_{k,w} x_{2^n-1-w-\mathcal{L}-1}) \]

\[ \equiv 2^d \Pi_{j=1}^{l} (t_j + x^{-1} t_{j-1}^2)^{i_j} (\sum_{w=0}^{k} b_{k,w} x_{2^n-1-w-1}) \]

\[ \equiv 2^d \Pi_{j=1}^{l} (t_j + x^{-1} t_{j-1}^2)^{i_j} x_{2^n-1-k-d-1} (\sum_{w=0}^{k} b_{k,w} x_{k-w}). \]

From the discussion of §?? we know that \(b_{k,k} = 1\) so that the term of lowest degree in \(x\) in the above expression is \(2^d t_1^{2i_1} t_2^{2i_2} \ldots t_{r-1}^{2i_{r-1}} x_{2^n-1-2^n+\deg (v^l)/2-d} \pmod{2^{d+1}}\).

Suppose now that \(z = \sum_{l(I) \geq d} \epsilon_I v^l x_{2n+1-1-\deg (v^l)}\) and let \(2m = \min \{ \deg (v^l) \mid l(I) = d, \epsilon_I = 1 \}\). Only those terms with \(l(I) = d\) can contribute non-trivially to \((\lambda \wedge 1)_*((\tilde{\eta}_R)_* (z)) \pmod{2^{d+1}}\). Thus the terms in this expression with the lowest degree in \(x\) contribute

\[ 2^d \sum_{\deg (v^l) = 2m, \epsilon_I = 1, l(I) = d} t_1^{2i_1} \ldots t_{r-1}^{2i_{r-1}} x_{2^n-1-2^n-m-d} \pmod{2^{d+1}}. \]

Therefore this expression has to be zero and, because \(2^n - m = k+1 \geq 1, d \geq 1\), this sum can only be zero if

\[ 2^d \sum_{\deg (v^l) = 2m, \epsilon_I = 1, l(I) = d} t_1^{2i_1} \ldots t_{r-1}^{2i_{r-1}} \]

\[ \]
vanishes identically modulo $2^{d+1}$. However, $t_1^{j_1} \ldots t_r^{j_r} x_1^{2^{j_1} - 2^{j_2} + \ldots - 2^{j_r} + m - d}$ can only come from $v^j \ldots v^r x_{2n+1-1-2m}^j$ if $j_2 + \ldots + j_r = d$ and from $v^j \ldots v^r x_{2n+1-1-2m}^j$ otherwise. Therefore no terms $\epsilon_{l'}v^j x_{2n+1-1-2m}$ with $l(I) = d$ and $\epsilon_l = 1$ appear in the expression for $z$. The proof is now easily completed by induction on $m$ and $d$. □

In §6 we shall show that the only possible value of $d$ in Theorem ?? is $d = n + 1$. In §?? we shall improve Theorem ?? to the following more difficult result.

**Theorem 3.6**

In the notation of §6?? and ??, let $u \in J_{2n+1-2}(R^{2n+1})$ be represented as

$$u = \epsilon 2^{n+1} x_{2n+1-1} + \sum_{l(I') \geq n+2} \epsilon_{l'} v^j x_{2n+1-1-\deg(v')} - 1$$

satisfy

$$\tilde{\eta}_{l'}(u) = (\tilde{\eta}_{l})(u) \in J_{2n+1-2}(R^{2n+1})$$

where $\epsilon, \epsilon_{l'} \in \{0, 1\}$. Then $\epsilon_{l'} = 0$ if $l(I') = n + 2$.

## 4 Im(J)-theory and the Kervaire invariant

### 4.1 Suppose that $\theta : S^{2n+1-2} \rightarrow R^{2n+1} \rightarrow \mathbb{R}$ is an $S$-map whose mapping cone is denoted by $C(\theta)$. In dimension $2n+1 - 2$ the Kahn-Priddy map $[?]$ gives a split surjection of stable homotopy groups

$$\pi_{2n+1-2}^{S}(R^{2n+1}) \rightarrow \pi_{2n+1-2}^{S}(S^{0}) \otimes \mathbb{Z}_2$$

onto the 2-Sylow subgroup of the stable homotopy groups of spheres. The Kervaire invariant ([?][?][?]) of a framed manifold yields a homomorphism from $\pi_{2n+1-2}^{S}(S^{0}) \otimes \mathbb{Z}_2$ to the group of order two. Furthermore, it is well-known that the image of $[\theta] \in \pi_{2n+1-2}^{S}(R^{2n+1})$ has non-trivial Kervaire invariant if and only if the Steenrod operation ([?][?])

$$Sq^{2n} : H^{2n-1}(C(\theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \rightarrow H^{2n+1-1}(C(\theta); \mathbb{Z}/2)$$

is non-trivial.

Now let $bu$ denote 2-adic connective K-theory and define $ju$-theory by means of the fibration $ju \rightarrow bu \pi_{2n+1-2}^{S} \rightarrow bu$. Hence $ju*$ is a generalized homology theory for which $ju_{2n+1-2}(R^{2n+1}) \cong \mathbb{Z}/2^{n+2}$. Recall that, if $l \in ju_{2n+1-2}(S^{2n+1-2}) \cong \mathbb{Z}_2$ is a choice of generator, the associated $ju$-theory Hurewicz homomorphism

$$H_{ju} : \pi_{2n+1-2}^{S}(R^{2n+1}) \rightarrow ju_{2n+1-2}(R^{2n+1}) \cong \mathbb{Z}/2^{n+2}$$

is defined by $H_{ju}([\theta]) = \theta_{*}(u)$.

We are now ready to state the main result of this section.
Theorem 4.2
For $n \geq 1$ the image of $[\theta] \in \pi_{2n+1-2}^S(\mathbb{R}P^\infty)$ under the ju-theory Hurewicz homomorphism

$$H_{ju}([\theta]) \in ju_{2n+1-2}(\mathbb{R}P^\infty) \cong \mathbb{Z}/2^{n+2}$$

is non-trivial if and only if $Sq^{2^n}$ is non-trivial on $H^{2n-1}(C(\theta); \mathbb{Z}/2)$.

In any case, $2H_{ju}([\theta]) = 0$.

Proof
Consider the following commutative diagram.

$$
\begin{array}{ccc}
J_{2n+1-2}(S^{2n+1-2}) & \longrightarrow & J_{2n+1-2}(\mathbb{R}P^\infty) \\
\downarrow & & \downarrow \\
BP_{2n+1-1}(C(\theta)) & \longrightarrow & BP_{2n+1-2}(S^{2n+1-2}) \\
\downarrow \psi^3 - 1 & & \downarrow \psi^3 - 1 \\
BP_{2n+1-1}(\mathbb{R}P^\infty) & \longrightarrow & BP_{2n+1-1}(C(\theta))
\end{array}
$$

Let $\tilde{\imath} \in J_{2n+1-2}(S^{2n+1-2})$ be the class given by the J-theory unit as in §??.

The J-theory Hurewicz image is given by $\theta_*(\tilde{\imath}) \in J_{2n+1-2}(\mathbb{R}P^\infty) \cong J_{2n+1-2}(\mathbb{R}P^{2n+1})$.

It is an element satisfying the conditions of Theorems ?? and ??.

The image of $\tilde{\imath}$ in $BP_{2n+1-2}(S^{2n+1-2})$ is $\imath$ of §??, which lifts to $\imath'' \in BP_{2n+1-1}(C(\theta))$. The
\[(\psi^3 - 1)(\iota'')\) lifts to \(\alpha \in BP_{2n+1-1}(R^{P\infty})\) and, by ([?]) Proposition 2 pp.241-2, the image of \(\alpha\) in \(J_{2n+1-2}(R^{P2n+1})\) is equal to \(\theta_s(\tilde{\iota})\). Therefore, by Theorem ??, we obtain an equation of the form
\[\psi^3(\iota'') = \iota'' + \epsilon 2^d x_{2n+1-1} + \sum_{l(l') \geq d+1} \epsilon_{l'} v' x_{2n+1-deg(v')-1} + 2^{n+2} \beta \in BP_{2n+1-1}(C(\theta))\]
for some \(0 \leq d \leq n+1, \epsilon = 0, 1\) and \(\beta \in BP_{2n+1-1}(R^{P\infty})\).

Define \(ju\) by the 2-local cofibring, \(ju \rightarrow bu \xrightarrow{\psi^3-1} bu\), so that \(ju_{2n+1-2}(R^{P\infty}) \cong \mathbb{Z}/2^{n+2}\), generated by \(\lambda_*(x_{2n+1-1})\) where \(\lambda_* : J_*(X) \rightarrow ju_*(X)\) is the \(ju\)-theory analogue of the map used in the proof of Theorem ???. Therefore the \(ju\)-theory Hurewicz image of \(\theta\) is \(\epsilon 2^d \lambda_*(x_{2n+1-1})\).

First we must show that \(d \geq n+1\) which will imply that the \(ju\)-theory Hurewicz image of \(\theta\) is trivial unless \(d = n+1\) and in that case is non-trivial if and only if \(\epsilon = 1\). If \(\epsilon = 1\) and \(d < n+1\), we replace \(\iota\) by \(2^{n+1-d} \iota\) then the argument which is to follow shows that \(2^{n+1-d} \theta\) is detected by \(Sq^{n}\) on the mod 2 cohomology of its mapping cone. This is impossible easily seen to be impossible, by comparing the mapping cone sequences for \(2^{n+1-d} \theta\) and \(2^n \theta\).

The fact that \(d \geq n+1\) implies that \(2H_{ju}(\theta) = 0\).

Now write \(s_n : BP_*(X) \rightarrow BP_{*-2n}(X)\) for the Landweber-Novikov operation in BP-homology corresponding to \(s_{(n,0,0,\ldots)}\) in ([?]) p.12). We are going to study the consequences of the relations, \(3^n \psi^3 s_m = s_m v^3\). This relation is established by observing that the sum of the left and right sides over \(m\) correspond to two ring operations in BP-cohomology and therefore are equal if and only if these cohomology operations agree on \(x \in BP^2(CP^{\infty})\), which is easily verified. In addition, if \(T : BP \rightarrow H\mathbb{Z}/2\) corresponds to the Thom class then a similar argument shows that \((Sq^{2n}), T_n = T_m : BP_*(X) \rightarrow H_{*-2m}(X; \mathbb{Z}/2)\).

Also, if \(0 \leq m \leq t\), using the formulae of ([?]) Part I §5 and §8.1) it is not difficult to show (we shall only need this formula modulo 2) that
\[s_m(x_{2t+1}) = (-1)^m \left( \begin{array}{c} m + t \\ m \end{array} \right) x_{2t-2m+1} \in BP_{2t-2m+1}(R^{P\infty}).\]

Bearing in mind the previous discussion about what to do if \(d < n+1\), we may suppose that \(d = n+1\) and write
\[\psi^3(\iota'') = \iota'' + \epsilon 2^{n+1} x_{2n+1-1} + 2^{n+2} \beta + \gamma \in BP_{2n+1-1}(C(\theta)).\]
Here \(\beta \in BP_{2n+1-1}(R^{P\infty}) \subset BP_{2n+1-1}(C(\theta))\) and, by Theorem ??,
\[\gamma = \sum_{l(l') \geq n+2} \epsilon_{l'} v' x_{2n+1-deg(v')-1}.\]

Applying the relation with \(m = 2^n-1\) we obtain the following equation in \(BP_{2n-1}(C(\theta)) \cong BP_{2n-1}(R^{P\infty})\):
\[s_{2n-1}(\iota'') + \epsilon 2^{n+1} \left( \begin{array}{c} 2^{n-1} + 2^n - 1 \\ 2^{n-1} \end{array} \right) x_{2n-1} + 2^{n+2} s_{2n-1}(\beta) + s_{2n-1}(\gamma) = 3^{2n-1} 3^{2^n-1} s_{2n-1}(\iota'').\]
because $\psi^3$ acts like multiplication by $3^{2^n-1}$ on $BP_{2^n-1}(R P^\infty)$.

We are going to apply

$$
\lambda_* : BP_{2^n+1-1}(R P^\infty) \to bu_{2^n+1-1}(R P^\infty) \pmod{2^{n+3}}
$$

to the above equation, bearing in mind that $\lambda_*(v_k) = 0$ for $k \geq 2$ and that

$$0 = v_1 x_{2j-1} + 2x_{2j+1} \in bu_{2j+1}(R P^\infty).$$

In $bu_*(R P^\infty)$ consider $\lambda_*(s_m(v_k))x_{2j+1}$. If $m \neq 2^k-1, 2^k-2$ then $\lambda_*(s_m(v_k))$ is a multiple of $v_{1}^{2^k+e}$ for some $e \geq 0$ and therefore $\lambda_*(s_m(v_k))x_{2j+1} \in 4bu_{2j+2^{k+1}-1-2m}(R P^\infty)$. Similarly one sees that $\lambda_*(s_{2^k-2}(v_k))x_{2j+1} \in 2bu_{2j+3}(R P^\infty)$. Also, since $s_{2^k-1}$ cannot decrease Adams filtration, $\lambda_*(s_{2^k-1}(v_k)) \in 2bu_0(S^0)$ and $\lambda_*(s_{2^k-1}(v_k))x_{2j+1} \in 2bu_{2j+1}(R P^\infty)$.

Now consider $v_{i_1} v_{i_2} \ldots v_{i_t} x_{2j+1} = BP_{2^n+1-1}(R P^\infty)$ and

$$
\lambda_* \left( s_{2^n-1}(v_{i_1} v_{i_2} \ldots v_{i_t} x_{2j+1}) \right) = \sum_{a_1 + \ldots + a_t = 2^n-1} \lambda_*(s_{a_1}(v_{i_t})) \ldots \lambda_*(s_{a_t}(v_{i_1})) \lambda_*(s_{a_{t+1}}(x_{2j+1})).
$$

The above discussion shows that this lies in $2^t bu_{2n-1}(R P^\infty)$ unless $t = 0$. Also $\lambda_*(s_{2^n-1}(x_{2n+1-1})) \in 2bu_{2n-1}(R P^\infty)$ since

$$
\begin{pmatrix}
2^{n-1} + 2^n - 1 \\
2^{n-1}
\end{pmatrix} = 2(2s + 1)
$$

for some $s$. Hence both $2^{n+2} \lambda_*(s_{2^n-1}(\beta))$ and $\lambda_*(s_{2^n-1}(\gamma))$ lie in $2^{n+3} bu_{2n-1}(R P^\infty)$.

From this discussion, in the previous notation, our equation implies the following congruence in $bu_{2n-1}(R P^\infty) \cong \mathbb{Z}/2^{n-1}$:

$$
(3^{2^n} - 1) \lambda_*(s_{2^n-1}(t'')) \equiv 2^{n+2} \epsilon \pmod{2^{n+3} bu_{2n-1}(R P^\infty)}.
$$

However, for $n \geq 1$, $(3^{2^n} - 1) = 2^{n+2}(2w + 1)$ for some $w$ so that $\epsilon = 1$ if and only if $\lambda_*(s_{2^n-1}(t''))$ is a generator of $bu_{2n-1}(R P^\infty)$. The factorisation, $T : BP \xrightarrow{\lambda} bu \to H\mathbb{Z}/2$, implies that $\epsilon = 1$ if and only if the dual Steenrod operation, $Sq^{2^n}$, is non-trivial on $H_{2n+1}(C(\theta); \mathbb{Z}/2)$, which is equivalent to $Sq^{2^n}$ being non-trivial on $H^{2n-1}(C(\theta); \mathbb{Z}/2)$. This completes the proof. □

5 Theorem ?? – the induction step

5.1 In this section we begin the proof of Theorem ??, establishing the main part of an induction argument.

Consider once more the isomorphism

$$(BP \wedge BP)^*(R P^{2i-2}/R P^{2i-2n+1-2}) \cong \left( \frac{x^{2i-1-2^n}Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][x]}{x^{2i-1}Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][x]} \right)/(2|x)$$
established in §?? by means of the canonical form for elements, discussed in §??.
Recall that there are isomorphisms of the form
\[(BP \wedge BP)^{2i-2n+1}(R P^{2i-2}/R P^{2i-2n+1-2}) \otimes Z/2^{n+2}\]
\[\cong (BP \wedge BP)_{2n+1-1}(R P^{2n+1}) \otimes Z/2^{n+2}\]
\[\cong J_{2n+1-2}(R P^{2n+1}).\]

Suppose that \(u \in J_{2n+1-2}(R P^{2n+1})\) is represented by
\[u = \epsilon^{2n+1} x_{2n+1-1} + \sum_{l(I') \geq n+2} \epsilon_I u^{I'} x_{2n+1-\deg(u^{I'})-1}\]
with \(\epsilon, \epsilon_I \in \{0, 1\}\). If \(u\) satisfies \((\tilde{\eta}_R)_*(u) = (\tilde{\eta}_L)_*(u) \in J_{2n+1-2}(R P^{2n+1})\) we wish to show that \(\epsilon_I\) is zero for all \(I'\) with \(l(I') = n + 2\). For this purpose we shall compute in \((BP \wedge BP)^{2i-2n+1}(R P^{2i-2}/R P^{2i-2n+1-2}) \otimes Z/2^{n+2}\) or rather in a (graded) quotient, denoted by \(E^*\) for brevity.

Let \(I = \prec 2, v_1, v_2, \ldots > \prec BP^* = Z_2[v_1, v_2, \ldots]\) be the ideal generated by 2, \(v_1, v_2, \ldots\). According to §??, since \([2] x = 0\), we have, for all \(j \geq 0\), \(2x^{j+1} \equiv \sum_{i \geq 1} v_i x^{2i+j}\) (modulo \([2][x]\)) in \(BP_*/[x]/([2]x)\). By induction on \(d \geq 0\) we have, for all \(j \geq 0\),
\[2^{d+1} x^{d+1+j} \equiv \sum_{i_1, \ldots, i_{d+1} \geq 1} v_{i_1} \ldots v_{i_{d+1}} x^{2i_1+2i_2+\ldots+2i_{d+1}+j} \mod (I^{d+2}[x]).\]

Similarly, since \(i\) is much larger than \(n\), we have, in \((BP \wedge BP)^*(R P^{2i-2}/R P^{2i-2n+1-2})\),
\[2^n v^I t^I' x^{2i-1-k-1} = (\sum_{i_1, \ldots, i_{n+2} \geq 1} v_{i_1} \ldots v_{i_{n+2}} x^{2i_1+2i_2+\ldots+2i_{n+2}+n-2}) v^I t^I' x^{2i-1-k-1}\]
\[+ \sum_{l(I'') \geq n+3} \epsilon_I u^{I''} t^{I''} x^j.\]

Therefore, if we set
\[u_n = \sum_{i_1, \ldots, i_{n+2} \geq 1} v_{i_1} \ldots v_{i_{n+2}} x^{2i_1+2i_2+\ldots+2i_{n+2}+n-2}\]
then
\[E^* = \left(\frac{x^{2i-1-2n} Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}{x^{2i-1} Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}\right) \approx \frac{Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}{([2] x) \otimes Z/2^{n+2}} \rightarrow E^*\]
where \(\approx\) denotes the ideal generated by elements of the forms \([2][x], v^I t^I' x^{2i-1-2n}, l(I) \geq n + 3\) and \(u_n t^{I''} x^{2i-1-2n-(n+2)}\). Then \(E^*\) is a quotient of the required form. Let
\[\rho : \left(\frac{x^{2i-1-2n} Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}{x^{2i-1} Z_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}\right) / ([2] x) \otimes Z/2^{n+2} \rightarrow E^*\]
denote the canonical quotient map.
Consider the (graded) subgroup, $D^* \subseteq E^*$, generated by the elements of the form $\rho(v^I t^{\prime} x^{2i-1-k-1})$ with $l(I) = n + 2$. By induction, using the canonical form of §?? for elements of $(BP \wedge BP)^*(\mathbb{R}P^{2i-2}/\mathbb{R}P^{2i-2n+1-2})$ we see that $D^*$ is a $\mathbb{Z}/2$-vector space on generators of the form $v^I t^{\prime} x^j$ with $l(I) = n + 2$ and $2i-1 - 2^n \leq j \leq 2i-1 - 1$ modulo the relations $0 = u_n x^j$ for $j \geq 2i-1 - 2^n - n - 2$. In particular, a homogeneous element of $D^*$ represented by an element of the form $\sum_{l(I) = n + 2} \epsilon_{l,k} v^I t^{l} x^{2i-1-k-1}$ with $\epsilon_{l,k} \in \{0, 1\}$ and $\epsilon_{(n+2,0,0,...),k} = 0$ for all $k$ can be zero if and only if each $\epsilon_{l,k}$ is zero.

We are now ready for the induction argument, which will put severe restrictions on the canonical form for elements, $u \in J_{2^n+1-2}(\mathbb{R}P^{2n+1})$ in Theorem ??.

The proof imitates that of Theorem ??.

5.2 **Analysis of the leading terms**

We begin by observing that $u = 2^{n+1} x^{2^n+1-1}$ satisfies the condition of the theorem. This is because $2^{n+1}(\eta_R)_*(x_{2^n+1-1})$ and $2^{n+1}(\eta_L)_*(x_{2^n+1-1})$ are S-dual to $2^{n+1}c(x)^{2i-1-2^n}$ and $2^{n+1}c(x)^{2i-1-2^n}$, respectively. However,

$$2^{n+1}c(x)^{2i-1-2^n} = 2^{n+1} \left( \sum_{v \geq 0} b_v^{BP} x^{v+1} \right)^{2^{i-1}-2^n},$$

which is congruent to $2^{n+1} x^{2i-1-2^n}$ modulo $< 2^{n+2}, x^{2i-1}>$, as required.

Therefore we may modify $u$ so that $\epsilon = 0$ and we may write

$$u = \sum_{l(I) \geq n+2,k} \epsilon_{l,k} v^I t^{l} x_{2k+1}$$

with $2^{n+1} - \deg(v^I) - 2 = 2k$, whose S-dual is

$$D(u) = \sum_{l(I) \geq n+2,k} \epsilon_{l,k} v^I t^{l} x^{2i-1-k-1}.$$ 

Note that this is also the expression for the S-dual of $(\eta_L)_*(u)$.

Next we shall compare the images in $E^{2i-1-2^n}$ of the S-duals of $(\eta_L)_*(u) = \sum_{l(I') \geq n+2,k} \epsilon_{l',k} v^{I'} t^{l'_{2k+1}}$ and $(\eta_R)_*(u)$. To calculate the second image we need to sharpen Lemma ??.

The proof of Lemma ?? easily yields, for $k \geq 1$,

$$(\eta_R)_*(v_k) = 2t_k + \sum_{j \geq 1} v_j t^{2j}_{k-j} \pmod{I^2[t_1, t_2, \ldots]}.$$ 

Therefore, in $E^{2i-1-2^n}$, the image of the S-dual of $(\eta_R)_*(u)$ is given by

$$\sum_{I', k} \epsilon_{I',k} \prod_{j=1}^r \left( \sum_{a \geq 1} v_a (t_j x^{2a-1} + t_{j-a}^{2a}) \right)^j x^{2i-1-k-1} \left( \sum_{q \geq 0} b_q^{BP} x^q \right)^{2^{i-1-k-1}}$$

where $I' = (i_1, \ldots, i_r)$ and $l(I') = n + 2$. In this expression the multiple of $v_1^{n+2}$ is equal to

$$\sum_{I', k} \epsilon_{I',k} v_1^{n+2} \prod_{j=1}^r \left( t_j x + t_{j-1}^2 \right)^{i_j} x^{2i-1-k-1} \left( \sum_{q \geq 0} b_q^{BP} x^q \right)^{2^{i-1-k-1}}.$$
In $E^*$ the expression under consideration is unchanged by adding the following linear combination of multiples of $u_n$,

$$\sum_{I',k} \epsilon_{I',k} u_n \prod_{j=1}^r (t_j x + t_{j-1}^2)^{i_j} x^{2i-1-k-n-3} (\sum_{q \geq 0} b_q^{BP} x^q)^{2i-1-k-1}.$$ 

Hence, in $E^{2i-2n+1}$, $u$ is also represented by the expression

$$\sum_{I',k} \epsilon_{I',k} \prod_{j=1}^r (\sum_{a \geq 1} v_a (t_j x^{2a-1} + t_{j-a}^{2a} \cdot x^{2i-1-k-1}) (\sum_{q \geq 0} b_q^{BP} x^q)^{2i-1-k-1}$$

$$- \sum_{I',k} \epsilon_{I',k} u_n \prod_{j=1}^r (t_j x + t_{j-1}^2)^{i_j} x^{2i-1-k-n-3} (\sum_{q \geq 0} b_q^{BP} x^q)^{2i-1-k-1}.$$ 

Bearing in mind that $t_0 = 1 = b_0^{BP}$, expanding this expression and collecting all the terms which do not involve any $t_j$’s should give zero. Also this expression contains no monomials of the form $v_i^{n+2} \cdot x^q$ so that we may apply the criterion of §2 to decide whether or not this element is zero in $D^*$. 

Now consider the non-zero terms of smallest degree in $x$. That is, consider the maximal $k$ such that there exist $\epsilon_{I',k}$’s which are non-zero. Since every term except $v_1^{n+2} x^{n+2}$ in $u_n$ has degree at least $n + 3$, the terms of degree $2i-1-k-1$ in $x$ in the modified representative for $u$ are given by

$$\sum_{I'} \epsilon_{I',k} \prod_{j=1}^r (v_j + v_{j-1} t_1^{2i-1} + \ldots + v_{1} t_{j-1}^2)^{i_j} x^{2i-1-k-1}$$

$$- \sum_{I'} \epsilon_{I',k} v_1^{n+2} \prod_{j=1}^r (t_j^2)^{i_j} x^{2i-1-k-1}.$$ 

Consider the subsum for the same maximal $k$, over $I' = (i_1, \ldots, i_r)$ such that $i_2, \ldots, i_{r-1}$ are not all zero and, of course, $i_r$ is non-zero. This means that $r \geq 3$. This subsum contributes

$$\sum_{(i_2, \ldots, i_{r-1}) \neq 0} \epsilon_{I',k} v_1^{i_1+i_r} v_2^{i_2} \ldots v_{r-1}^{i_{r-1}} t_{r-1}^{2i_r} x^{2i-1-k-1} \in D^*.$$ 

By induction on the maximal value of $r$ for which $\epsilon_{I',k}$ is non-zero in this subsum, we see that $\epsilon_{I',k} = 0$ if $(i_2, \ldots, i_{r-1}) \neq 0$. This is because, for such terms, $v_1^{i_1+i_r} v_2^{i_2} \ldots v_{r-1}^{i_{r-1}} t_{r-1}^{2i_r} x^{2i-1-k-1}$ can only originate from $\epsilon_{I',k} v_1^{i_1} v_2^{i_2} \ldots v_{r-1}^{i_{r-1}} v_{i_r} x^{2i-1-k-1}$.

So far then, we have shown that the terms of degree $2i-1-k-1$ in $x$ in the modified representative for $(\eta_R)_*(u)$ are given by

$$\sum_{I'= (i_1, 0, 0, \ldots, 0, i_r)} \epsilon_{I',k} (v_r + v_{r-1} t_1^{2i-1} + \ldots + v_1 t_{r-1}^2)^{i_r} - v_1^{n+2} t_{r-1}^{2i_r} x^{2i-1-k-1} \in D^*$$

where $i_1 + i_r = n + 2$ in this sum.
We write this representative for \((\eta_R)_*(u)\) as a sum of three parts in the form
\[
\sum_{I'= (i_1, 0, \ldots, 0, i_r), \ r \geq 4} \epsilon_{I', k}(v_1^{i_1}(v_r + v_{r-1} t_1^{2^r-1} + \ldots + v_1 t_1^{2^r-1}))^{i_r} - v_1^{n+2} t_1^{2i_r} x^{2^i_r - 1 - k - 1}
\]
\[
+ \sum_{I'=(i_1, 0, i_3)} \epsilon_{I', k}(v_1^{i_1}(v_3 + v_2 t_1^4 + v_1 t_2^{i_3}) - v_1^{n+2} t_1^{2i_3}) x^{2^{i_3} - 1 - k - 1}
\]
\[
+ \sum_{I'=(i_1, i_2)} \epsilon_{I', k}(v_1^{i_1}(v_2 + v_1 t_1^{i_2}) - v_1^{n+2} t_1^{2i_2}) x^{2^{i_2} - 1 - k - 1}.
\]

Next, in this simplified expression, consider the subsum
\[
\sum_{I'=(i_1, 0, \ldots, 0, i_r), \ r \geq 4} \epsilon_{I', k} v_1^{i_1} (v_r + v_{r-1} t_1^{2^r-1} + \ldots + v_1 t_1^{2^r-1})^{i_r} - v_1^{n+2} t_1^{2i_r} x^{2^i_r - 1 - k - 1} \in D^*.
\]

Expanding out the first subsum and considering the terms
\[
\sum_{I'=(i_1, 0, \ldots, 0, i_r), \ r \geq 4} \epsilon_{I', k} v_1^{i_1} (v_r - t_1^{2^r-1})^{i_r} x^{2^i_r - 1 - k - 1} \in D^*
\]
shows that \(\epsilon_{I', k} = 0\) for each \(I' = (i_1, 0, 0, \ldots, 0, i_r)\) with \(r \geq 4\). This is because, in the simplified expression, \(\epsilon_{(i_1, 0, 0, \ldots, 0, i_r), k} v_1^{i_1} (v_r - t_1^{2^r-1})^{i_r} x^{2^i_r - 1 - k - 1}\) can only originate from \(\epsilon_{(i_1, 0, 0, \ldots, 0, i_r), k} v_1^{i_1} v_r^{i_r} x^{2^{i_r} - 1 - k - 1}\), if \(r \geq 4\).

Thus we have shown that the terms of degree \(2^{i_r} - 1 - k - 1\) in \(x\) in the modified representative for \((\eta_R)_*(u)\) are given by
\[
\sum_{i_1} \epsilon_{i_1, 0, n+2-i_1, k} (v_1^{i_1}(v_3 + v_2 t_1^4 + v_1 t_2^{n+2-i_1}) - v_1^{n+2} t_1^{2(n+2-i_1)}) x^{2^{i_1} - 1 - k - 1}
\]
\[
+ \sum_{i_1} \epsilon_{i_1, n+2-i_1, k} (v_1^{i_1}(v_2 + v_1 t_1^{n+2-i_1}) - v_1^{n+2} t_1^{2(n+2-i_1)}) x^{2^{n+2-i_1} - 1 - k - 1}
\]

By expanding this expression and considering the terms involving some \(v_3\)'s and \(t_j\)'s, we see that this expression can equal
\[
\sum_{i_1} \epsilon_{i_1, 0, n+2-i_1} v_1^{i_1} v_3^{n+2-i_1} x^{2^{i_1} - 1 - k - 1} + \sum_{i_1} \epsilon_{i_1, n+2-i_1} v_1^{i_1} v_2^{n+2-i_1} x^{2^{n+2-i_1} - 1 - k - 1} \in D^*,
\]
the terms in the S-dual of \((\eta_L)_*(u)\) of degree \(2^{i-1} - k - 1\) in \(x\), if and only if somewhere in the first sum \(n + 2 - i_1 = 2^\alpha\) and the remaining \(\epsilon_{(i_1, 0, n+2-i_1), k}\) must vanish.

For the maximal \(k\) under consideration there can be only one remaining such term, namely when \(k = 2^n - i_1 - 7i_3 - 1 = 2^n - n - 3 - 2^{\alpha+1}\). Therefore we have whittled down the terms of degree \(2^{i-1} - k - 1\) in our representative for \((\eta_R)_*(u)\) to the form

\[
\epsilon_{(n+2-2^\alpha, 0, 2^\alpha), k} v_1^{n+2-2^\alpha} \left(v_3^{2^\alpha} + v_2^{2^\alpha} t_1^{2^\alpha+2}\right) x^{2^{i-1} - k - 1}
\]

\[
+ \sum_{i_1} \epsilon_{(i_1, n+2-i_1), k} \sum_{a=1}^{n+2-i_1} \left(\begin{array}{c} n + 2 - i_1 \\ a \end{array}\right) v_1^{n+2-a} v_2^a t_1^{2(n+2-i_1-a)} x^{2^{i-1} - k - 1}.
\]

To cancel the terms involving \(t_j\)'s we must have either \(\epsilon_{(n+2-2^\alpha, 0, 2^\alpha), k} = 0\) or there is a non-zero term with \(a = 2^\alpha\) and \(2(n+2-i_1-a) = 2^{\alpha+2}\). However, this implies that \(n + 2 - i_1 = 2^{\alpha+1}\) with the result that the binomial coefficient

\[
\left(\begin{array}{c} n + 2 - i_1 \\ a \end{array}\right) = \left(\begin{array}{c} 2^{\alpha+1} \\ 2^\alpha \end{array}\right)
\]

is even.

Therefore the terms of degree \(2^{i-1} - k - 1\) in our representative for \((\eta_R)_*(u)\) to the form

\[
\sum_{i_1} \epsilon_{(i_1, n+2-i_1), k} \sum_{a=1}^{n+2-i_1} \left(\begin{array}{c} n + 2 - i_1 \\ a \end{array}\right) v_1^{n+2-a} v_2^a t_1^{2(n+2-i_1-a)} x^{2^{i-1} - k - 1}.
\]

Similarly, by expanding this expression and considering the terms involving some \(v_2\)'s and \(t_j\)'s, we see that all the \(\epsilon_{(i_1, n+2-i_1), k}\) must vanish except for possibly one for which \(n + 2 - i_1 = 2^\beta\) and \(k = 2^n - i_1 - 3i_2 - 1 = 2^n - n - 3 - 2^{\beta+1}\). Therefore the terms of degree \(2^{i-1} - k - 1\) in our representative for \((\eta_R)_*(u)\) by now have the form

\[
\epsilon_{(n+2-2^\beta, 2^\beta), k} v_1^{n+2-2^\beta} v_2^{2^\beta} x^{2^{i-1} - k - 1},
\]

which does equal the terms of degree \(2^{i-1} - k - 1\) in the S-dual of \((\eta_L)_*(\epsilon_{(n+2-2^\beta, 2^\beta), k} v_1^{n+2-2^\beta} v_2^{2^\beta} x_{2k+1})\).

The following result recapitulates the progress of the induction argument so far.

**Proposition 5.3**
In Theorem ??, the element \( u \in J_{2^{n+1}-2}(\mathbb{R}^\infty) \) may be assumed to have the form
\[
\begin{align*}
u = 2^n x_{2^n+1} + \epsilon(n+2-3\beta) v_1 x_{2^{n+1}-2n-2+2-5} + \\
+ \sum_{\text{deg}(v') > 2n+2+4} \epsilon' v' x_{2^n+1-\text{deg}(v')-1} + \\
+ \sum_{l(I') \geq n+3} \epsilon' v' x_{2^n+1-\text{deg}(v')-1}
\end{align*}
\]
with \( \epsilon, \epsilon(n+2-3\beta), \epsilon' \in \{0, 1\} \).

6 Some important combinatorial identities

6.1 We are going to study \((\eta_R)_*(v_1^{i_1} \ldots v_r^{i_r} x^{2^{i_r}-k-1})\) with \( i_1 + \ldots + i_r = n + 2, 2^{i_1-1} - k - 1 \geq 2^{i_r-1} - 2^n + n + 2 + 2^{j+1} \) in the quotient of \( E^* \) of §?? given by setting \( 0 = t_1 = t_2 = \ldots = t_{j-1} = t_{j+1} = t_{j+2} = \ldots \). In fact, we shall therefore be in \( D^* = < t_1, t_2, \ldots, t_{j-1}, t_{j+1}, t_{j+2}, \ldots > \). This is the vector space over \( \mathbb{F}_2 \) on a basis given by \( v_1 t_1 x^\delta \) with \( 2^{i_1-1} - 2^n \leq \delta \leq 2^{i_r-1} - 1, l(I) = n + 2 \) and \( I \neq (n+2, 0, 0, \ldots) \).

In order to compute these elements we shall need the following result.

Lemma 6.2

For \( j \geq 1 \), in
\( \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]/ < 2, v_1, v_2, \ldots, t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots > \)
we have
\[
b_v^{BP} \equiv \begin{cases} 
  t_j^{(2^m-1)/(2^j-1)} & \text{if } v = 2^m - 1, \\
  0 & \text{otherwise.}
\end{cases}
\]

Proof

We know already that \( b_v^{BP} \equiv 0 \) unless \( v = 2^s - 1 \). From the congruence
\[
0 \equiv \sum_{j=0}^{v} t_j (b_v^{BP})_{2^m-j-1} 2^j
\]
we find, in the quotient of the statement of the lemma,
\[
0 \equiv t_j (b_v^{BP})_{2^m-j-1} 2^j + t_0 b_v^{BP}
\]
so that, by induction,
\[
b_v^{BP} \equiv t_j^{1+2^j(2^{m-1}-j-1)/(2^j-1)} = t_k^{(2^m-1)/(2^j-1)},
\]
as required. It is easy to see that \( b_v^{BP} \equiv 0 \) in the other cases. \( \Box \)
6.3 The case $j = 1$

In canonical form in $D^*/<t_2, t_3, t_4, \ldots>$, if $i_1 + i_2 < n + 2$, Lemma ?? implies that

$$(\eta_R)_*(v_{i_1}^{i_1} \cdots v_{i_r}^{i_r} x^j) = (v_1 + \sum_{k=1}^{\infty} v_k t_1 x^{2^k-1})^{i_1} (v_2 + v_1 t_1^2)^{i_2} (v_3 + v_2 t_1^4)^{i_2}$$

$$\cdots (v_r + v_{r-1} t_1^{2^{r-1}})^{i_r} x^j (\sum_{v=0}^{\infty} t_1^{2^v-1} x^{2^v-1})^j.$$ 

If $i_1 + i_2 = n + 2$ we have

$$(\eta_R)_*(v_1^{n+2-i_2} v_2^{i_2} x^j) = ((v_1 + \sum_{k=1}^{\infty} v_k t_1 x^{2^k-1})^{n+2-i_2} (v_2 + v_1 t_1^2)^{i_2}$$

$$- (\sum_{k=1}^{\infty} v_k x^{2^k-2})^{n+2} (1 + x t_1)^{n+2-i_2} t_1^{2i_2}) x^j (\sum_{v=0}^{\infty} t_1^{2^v-1} x^{2^v-1})^j.$$ 

Now suppose that, in (cohomological) dimension $2^i - 2^{n+1}$, we have an element as in Proposition ?? but with $\epsilon = 0$ in the leading term. Such an element has an S-dual of the form

$$D(u) = \sum_I \epsilon_I v^I x^{2^i-1-k-1}$$

with $\epsilon_I \in \{0, 1\}$ and where the lowest degree terms in $x$ consist solely of the term $v_1^{n+2-2^\beta} v_2^{2^\beta} x^{2^i-1-2^{n+2}+2^\beta+1}$.

Now consider all those terms whose $(\eta_R)_*$-image contains some monomials of the form $v_1^{i_1} v_2^{i_2} x^s t_1^I$. These can only come from monomials with $I = (i_1, i_2, i_3)$.

Let us calculate the monomials of the form $v_1^{n+2-2^\beta} v_2^{2^\beta} x^{n+2+2i_2+6i_3}$ occurring in

$$(\eta_R)_*(v_1^{i_1} v_2^{i_2} v_3^{i_3} x^{2^i-1-2^{n+2+2i_2+6i_3}})$$

where $n + 2 = i_1 + i_2 + i_3$.

If $i_3$ is non-zero we obtain these terms from

$$(v_1 + v_1 t_1 x + v_2 t_1 x^3)^{i_1} (v_2 + v_1 t_1^2)^{i_2} (v_2 t_1^4)^{i_3} (x \sum_{v=0}^{\infty} t_1^{2^v-1} x^{2^v-1})^{2^i-1-2^{n+2+2i_2+6i_3}}$$
and they are equal to

$$\sum_b \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b \\
\end{array} \right) (v_1(1 + t_1 x))^{n+2-i_2-i_3} (v_2 t_1 x^3)^b$$

$$\left( \frac{i_2}{i_2 + i_3 + b - 2^\beta} \right) (v_1 t_1^2 (2^i + 0)^{i_2} (v_2 t_1^4)^i_3 (x \sum_{b=0}^{\infty} t_1^{2b-1} x^{2b-1})^{2i-1-2n+n+2+2i_2+6i_3}$$

$$= \sum_b v_1^{n+2-2^\beta} v_2^{2^\beta} \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b \\
\end{array} \right) (1 + t_1 x)^{n+2-i_2-i_3} (t_1 x^3)^b$$

$$\left( \frac{i_2}{i_2 + i_3 + b - 2^\beta} \right) t_1^{2i_2+2i_3+2b-2^\beta+1} t_1^{4i_3} (x \sum_{b=0}^{\infty} t_1^{2b-1} x^{2b-1})^{2i-1-2n+n+2+2i_2+6i_3}$$

$$= \sum_b v_1^{n+2-2^\beta} v_2^{2^\beta} \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b \\
\end{array} \right) (1 + t_1 x)^{n+2-i_2-i_3}$$

$$\left( \frac{i_2}{i_2 + i_3 + b - 2^\beta} \right) (x t_1)^{2i_2+6i_3+3b-2^\beta+1} x^{2i-1-2n+n+2+2^\beta+1} (x \sum_{b=0}^{\infty} t_1^{2b-1} x^{2b-1})^{2i-1-2n+n+2+2i_2+6i_3}$$

Setting $w = xt_1$ and $L(w) = \sum_{b=0}^{\infty} t_1^{2b-1} x^{2b-1} = \sum_{b=0}^{\infty} w^{2b-1}$ the previous expression becomes

$$v_1^{n+2-2^\beta} v_2^{2^\beta} (x L(w))^{2i-1-2n+n+2+2^\beta+1} \sum_b \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b \\
\end{array} \right) \left( \begin{array}{c}
  i_2 \\
  i_2 + i_3 + b - 2^\beta \\
\end{array} \right)$$

$$(1 + w)^{n+2-i_2-i_3} w^{2i_2+6i_3+3b-2^\beta+1} L(w)^{2i_2+6i_3-2^\beta+1}$$

If $i_3 = 0$ then the terms in question coming from $(\eta_R)_*(v_1 v_2^{i_2} x^{2i-1-2n+n+2+2i_2})$ are the same as come from

$$(v_1 + v_1 x t_1 + v_2 t_1 x^3)^{n+2-i_2} (v_2 + v_1 t_1^{i_2})$$

$$-(v_1 + v_2 x^2)^{n+2} (1 + x t_1)^{n+2-i_2} t_1^{2i_2} (x \sum_{b=0}^{\infty} t_1^{2b-1} x^{2b-1})^{2i-1-2n+n+2+2i_2}.$$
These terms are

\[ v_1^{n+2-2\beta} v_2^{2\beta} (x L(w))^{2i_1-2n+2+2\beta+1} \left( \sum_b \binom{n + 2 - i_2}{b} \left( \binom{i_2}{i_2 + b - 2\beta} \right) \right) \]

\[ (1 + w)^{n+2-i_2-2i_2+3b-2\beta+1} L(w)^{2i_2-2\beta+1} \]

\[ - \binom{n + 2}{2\beta} v_1^{n+2-2\beta} v_2^{2\beta} x^{2i_1} (1 + w)^{n+2-i_2+2i_1} (x L(w))^{2i_1-2n+2+2i_1} \]

\[ = v_1^{n+2-2\beta} v_2^{2\beta} (x L(w))^{2i_1-2n+2+2\beta+1} \left( \sum_b \binom{n + 2 - i_2}{b} \left( \binom{i_2}{i_2 + b - 2\beta} \right) \right) \]

\[ (1 + w)^{n+2-i_2-2i_2+3b-2\beta+1} L(w)^{2i_2-2\beta+1} \]

\[ - \binom{n + 2}{2\beta} (1 + w)^{n+2-i_2} w^{2i_2} L(w)^{2i_2-2\beta+1} \]

When \( i_2 = 2\beta \) and \( i_3 = 0 \) this expression becomes

\[ v_1^{n+2-2\beta} v_2^{2\beta} (x L(w))^{2i_1-2n+2+2\beta+1} \]

\[ + \binom{n + 2 - 2\beta}{2\beta} (1 + w)^{n+2-2\beta+1} w^{2i_1} \]

\[ + \binom{n + 2}{2\beta} (1 + w)^{n+2-2\beta+1} w^{2i_1} \]

More generally, let us calculate the monomials of the form \( v_1^{n+2-j} v_2^j x^{a} t_1^b \) occurring in

\[ (\eta_R)_s (v_1^{i_1} v_2^{i_2} v_3^{i_3} x^{2i_1-2n+2+2i_2+6i_3}) \]

where \( n + 2 = i_1 + i_2 + i_3 \).

If \( i_3 \) is non-zero we obtain these terms from

\[ (v_1 + v_1 t_1 x + v_2 t_1^3)^{i_1} (v_2 + v_1 t_1^{i_2})^{i_2} (v_2 t_1^{i_3})^{i_3} (x \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1})^{2i_1-2n+2+2i_2+6i_3} \]
and they are equal to

\[ \sum_b \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b
\end{array} \right) (v_1(1 + t_1 x)^{n+2-i_2-i_3-b}(v_2 t_1 x^3)^b \]

\[ \left( \begin{array}{c}
  i_2 \\
  i_2 + i_3 + b - j
\end{array} \right) (v_1 t_1^2)^{i_2} v_2^{i_3-i_2-i_3-b} (v_2 t_1^4)^{i_3} (x \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1}) 2^{i_1-1} - 2^n + n + 2 + 2i_2 + 6i_3 \]

\[ = \sum_b v_1^{n+2-j} v_2^j \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b
\end{array} \right) (1 + t_1 x)^{n+2-i_2-i_3-b} (t_1 x^3)^b \]

\[ \left( \begin{array}{c}
  i_2 \\
  i_2 + i_3 + b - j
\end{array} \right) t_1^{2i_2+2i_3+2b-2j} t_1^{4i_3} (x \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1}) 2^{i_1-1} - 2^n + n + 2 + 2i_2 + 6i_3 \]

\[ = \sum_b v_1^{n+2-j} v_2^j \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b
\end{array} \right) (1 + t_1 x)^{n+2-i_2-i_3-b} \]

\[ \left( \begin{array}{c}
  i_2 \\
  i_2 + i_3 + b - j
\end{array} \right) (x t_1)^{2i_2+6i_3+3b-2j} x^{2i_1-1} - 2^n + n + 2 + 2j (x \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1}) 2^{i_1-1} - 2^n + n + 2 + 2i_2 + 6i_3 \]

As before, setting \( w = x t_1 \) and \( L(w) = \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1} = \sum_{v=0}^{\infty} w^{2v-1} \) the previous expression becomes

\[ v_1^{n+2-j} v_2^j (x L(w))^{2^{i_1-1} - 2^n + n + 2 + 2j} \sum_b \left( \begin{array}{c}
  n + 2 - i_2 - i_3 \\
  b
\end{array} \right) \left( \begin{array}{c}
  i_2 \\
  i_2 + i_3 + b - j
\end{array} \right) \]

\[ (1 + w)^{n+2-i_2-i_3-b} w^{2i_2+6i_3+3b-2j} L(w)^{2i_2+6i_3-2j} \]

If \( i_3 = 0 \) then the terms in question coming from \((\eta_R)_*(v_1 v_2^2 x^{2i_1-1} - 2^n + n+2+2i_2)\) are the same as come from

\[ (v_1 + v_1 x t_1 + v_2 t_1 x^3)^{n+2-i_2} (v_2 + v_1 t_1^2)^{i_2} \]

\[ -(v_1 + v_2 x^2)^{n+2}(1 + x t_1)^{n+2-i_2} t_1^{2i_2} (x \sum_{v=0}^{\infty} t_1^{2v-1} x^{2v-1}) 2^{i_1-1} - 2^n + n + 2 + 2i_2. \]
These terms are
\[ v_1^{n+2-j}v_2^j(xL(w))^{2i-1-2n+n+2+2j}(\sum_b \binom{n+2-i_2-i_3}{b} \binom{i_2}{2} \binom{i_2+b-j}{i_2+b-j}) \]
\[ (1 + w)^{n+2-i_2-b} w^{2i_2+3b-2j} L(w)^{2i_2-2j} \]
\[ - \left( \begin{array}{c} n+2 \\ j \end{array} \right) v_1^{n+2-j} v_2^j x^{2j} (1 + w)^{n+2-iz} (xL(w))^{2i-1-2n+n+2+2i_2} \]
\[ = v_1^{n+2-j} v_2^j (xL(w))^{2i-1-2n+n+2+2j}(\sum_b \binom{n+2-i_2}{b} \binom{i_2}{2} \binom{i_2+b-j}{i_2+b-j}) \]
\[ (1 + w)^{n+2-i_2-b} w^{2i_2+3b-2j} L(w)^{2i_2-2j} \]
\[ - \left( \begin{array}{c} n+2 \\ j \end{array} \right) (1 + w)^{n+2-iz} w^{2i_2} L(w)^{2i_2-2j} \]

The relation \((\eta_\Re)_*(u) = u\) gives us, in the subring
\[ \mathbb{Z}/2[w]/(w^{2n-n-2-2}) \subseteq \mathbb{Z}/2[x, t_1]/(x^{2n-n-2-2}), \]
the following equation for each \(j\):
\[ \epsilon_{(n+2-j, j)} \]
\[ = \sum_{(i_1, i_2, i_3), i_1+i_2+i_3=n+2} L(w)^{2i-1-2n+n+2+2j}(\sum_b \binom{n+2-i_2-i_3}{b} \binom{i_2}{2} \binom{i_2+i_3+b-j}{i_2+i_3+b-j}) \]
\[ \epsilon_{(i_1, i_2, i_3)} (1 + w)^{n+2-i_2-i_3-b} w^{2i_2+6i_3+3b-2j} L(w)^{2i_2+6i_3-2j} \]
\[ - \sum_{(i_1, i_2), i_1+i_2=n+2} \epsilon_{(i_1, i_2)} \left( \begin{array}{c} n+2 \\ j \end{array} \right) (1 + w)^{n+2-iz} w^{2i_2} L(w)^{2i_2-2j} \]

Since \(j \leq n+2\) we may view all these equations as being in the same ring, for example \(\mathbb{Z}/2[w]/(w^{2n-3n-6})\) and work in some suitable range in which \(2^n\) is large compared with \(n+6\). Therefore, in this ring we may re-write the equation so as to yield the following result.
Proposition 6.4

In the notation of Proposition ??, if \( j \leq n + 2 \),

\[ \epsilon(n + 2 - j, j) \]

\[ = \sum_{i_1, i_2, i_3, i_1 + i_2 + i_3 = n + 2} b L(w)^{2i_1 - 2n + n + 2 + 2i_2 + 6i_3} \left( \begin{array}{c} n + 2 - i_2 - i_3 \\ b \end{array} \right) \left( \begin{array}{c} i_2 \\ i_2 + i_3 + b - j \end{array} \right) \]

\[ \epsilon(i_1, i_2, i_3)(1 + w)^{n + 2 - i_2 - i_3} \left( 1 + \frac{w}{1 + w} \right)^{i_1} (1 + w^2)^{i_2} w^{4i_3} \]

\[ - \sum_{i_1, i_2} L(w)^{2i_1 - 2n + n + 2 + 2i_2} \epsilon(i_1, i_2) \left( \begin{array}{c} n + 2 \\ j \end{array} \right) (1 + w)^{n + 2 - i_2} w^{2i_2}. \]

In \( \mathbb{Z}/2[w]/(w^{2n - 3n - 6}) \).

Remark 6.5 Adding over \( j \) and noticing that

\[ \sum_j \left( \begin{array}{c} n + 2 \\ j \end{array} \right) = 2^{n + 2} \equiv 0 \pmod{2} \]

we obtain

\[ \sum_j \epsilon(n + 2 - j, j) \]

\[ = \sum_{i_1, i_2, i_3, i_1 + i_2 + i_3 = n + 2} L(w)^{2i_1 - 2n + n + 2 + 2i_2 + 6i_3} \]

\[ \epsilon(i_1, i_2, i_3)(1 + w)^{n + 2 - i_2 - i_3} (1 + \frac{w}{1 + w})^{i_1} (1 + w^2)^{i_2} w^{4i_3} \]

\[ = \sum_{i_1, i_2, i_3} L(w)^{2i_1 - 2n + n + 2} \epsilon(i_1, i_2, i_3)(1 + w)^{i_1} (1/(1 + w))^{i_2} L(w)^2 (1 + w^2)^{i_2} (L(w)^6 w^4)^{i_3} \]

\[ = \sum_{i_1, i_2, i_3} L(w)^{2i_1 - 2n + n + 2} \epsilon(i_1, i_2, i_3)(L(w)^2 (1 + w^2))^{i_2} (L(w)^6 w^4)^{i_3}. \]

6.6 The case \( j = 2 \)

Next we calculate the monomials of the form \( v_1^{n + 2 - j} v_2^j a_1 t_2^b \) occurring in

\[ (\eta_R)_* (v_1^{i_1} v_2^{i_2} \ldots v_r^{i_r} x^s) \in D^{2i - 2n + 1} / < t_1, t_3, t_4, \ldots >. \]

Since

\[ (\eta_R)_* (v_s^{i_s}) \equiv \left( \sum_{a \geq 1} v_a (t_a x^{2a - 1} + t_{2a-s-a}^2) \right)^{i_s} \]

the term under consideration can only come from monomials with \( I = (i_1, i_2, i_3, i_4). \)
If \( M = \sum_{m \geq 0} t_2^{2m-1}/(2^m-1) x^{2m-1} \) then \( c(x) = xM \). Therefore, if \( i_4 \) is non-zero,

\[
v_1^{i_1} v_2^{i_2} v_3^{i_3} v_4^{i_4} x^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}
\]

contributes

\[
v_1^{i_1} (v_1 t_2 x + v_2 t_2 x^3 + v_2)^{i_2} (v_1 t_2^4)^{i_3} (v_2 t_2^4)^{i_4} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}
\]

and, when \( i_4 = 0 \), it contributes

\[
-(v_1 + v_2 x^2)^{n+2} (t_2 x)^{i_2} (t_2^4)^{i_3} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}.
\]

The term of the form \( v_1^{n+2-j} v_2^j x t_2^4 \) occurring in

\[
(\eta R)_* (v_1^{i_1} v_2^{i_2} v_3^{i_3} v_4^{i_4} x^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4})
\]
is equal, when \( i_4 \) is non-zero, to

\[
v_1^{i_1} \left( \begin{array}{c} i_2 \\ j-i_4 \end{array} \right) (v_1 t_2 x)^{i_2+i_4-j} v_2^{j-i_4} (1 + t_2 x^3)^{j-i_4} (v_1 t_2^4)^{i_3} (v_2 t_2^4)^{i_4} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}
\]

\[
= \left( \begin{array}{c} i_2 \\ j-i_4 \end{array} \right) v_1^{n+2-j} v_2^j (1 + t_2 x^3)^{j-i_4} t_2^{i_2+2i_3+5i_4-j} x^{i_2+i_4-j} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}
\]

and, when \( i_4 = 0 \), it is equal to

\[
- \left( \begin{array}{c} n+2 \\ j \end{array} \right) v_1^{n+2-j} v_2^j x^{i_2+i_4} t_2^{i_2+2i_3+5i_4} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3}.
\]

Next we observe that

\[
\left( \begin{array}{c} i_2 \\ j-i_4 \end{array} \right) v_1^{n+2-j} v_2^j (1 + t_2 x^3)^{j-i_4} t_2^{i_2+2i_3+5i_4-j} x^{i_2+i_4-j} (xM)^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}
\]
is equal to \( v_1^{n+2-j} v_2^j x^{2i_1-1-2^n+n+2+2j} \) times

\[
\left( \begin{array}{c} i_2 \\ j-i_4 \end{array} \right) (1 + t_2 x^3)^{j-i_4} (t_2 x^3)^{i_2+2i_3+5i_4-j} x^{i_2+i_4-j} M^{2i_1-1-2^n+n+2+2i_2+6i_3+14i_4}.
\]
Hence, if we write $y = t_2 y^3$, this term is the product of $v_1^{n+2-j} v_2^j x^{i_1-i_3-2^n+n+2+2j}$ with

$$
\left( \begin{array}{c}
i_2 \\
j - i_4
\end{array} \right) (1 + y)^{j-i_4} y^{i_2+2i_3+5i_4-j} M(y)^{2i_1-2^n+n+2+2i_2+6i_3+14i_4}
$$

where $M(y) = \sum_{m \geq 0} y^{(2^m-1)/(2^2-1)}$.

Also

$$
\left( \begin{array}{c}
n + 2 \\
j
\end{array} \right) v_1^{n+2-j} v_2^j x^{i_2+2j}\ M(x)^{2i_1-2^n+n+2+2i_2+6i_3}
$$

is equal to the product of $v_1^{n+2-j} v_2^j x^{i_1-i_3-2^n+n+2+2j}$ with

$$
\left( \begin{array}{c}
n + 2 \\
j
\end{array} \right) y^{i_2+2i_3} M(y)^{2i_1-2^n+n+2+2i_2+6i_3}.
$$

Therefore, equating coefficients of $v_1^{n+2-j} v_2^j x^{i_1-i_3-2^n+n+2+2j}$ we obtain the following identity in $\mathbb{Z}/2[y]/(y^{(2^n-3n-6)}/3)$.

**Proposition 6.7**

In the notation of Proposition ??, if $j \leq n + 2$,

$$
\epsilon_{n+2-j,j}
$$

$$
= \sum_{(i_1,i_2,i_3,i_4),i_1+i_2+i_3+i_4=n+2} \epsilon_{(i_1,i_2,i_3,i_4)} \left( \begin{array}{c}
i_2 \\
j - i_4
\end{array} \right) (1 + y)^{j-i_4} y^{i_2+2i_3+5i_4-j} M(y)^{2i_1-2^n+n+2+2i_2+6i_3+14i_4}
$$

$$
- \sum_{(i_1,i_2,i_3),i_1+i_2+i_3=n+2} \epsilon_{(i_1,i_2,i_3)} \left( \begin{array}{c}
n + 2 \\
j
\end{array} \right) y^{i_2+2i_3} M(y)^{2i_1-2^n+n+2+2i_2+6i_3}
$$

in $\mathbb{Z}/2[y]/(y^{(2^n-3n-6)}/3)$.

**Remark 6.8** Adding over $j$ we obtain
\[
\sum_j \epsilon_{n+2-j,j} = \sum_{(i_1,i_2,i_3,i_4), i_1+i_2+i_3+i_4=n+2} \sum_j \epsilon_{(i_1,i_2,i_3,i_4)} \binom{j}{i_2} (j-i_4)
\]

\[
(1+y)^{-i_4}y^{i_2+i_3+i_4} M(y)^{2i_1-1-2n+n+2+2i_2+6i_3+14i_4}
\]

\[
- \sum_{(i_1,i_2,i_3), i_1+i_2+i_3=n+2} \sum_j \epsilon_{(i_1,i_2,i_3)} \binom{n+2}{j} y^{i_2+2i_3} M(y)^{2i_1-1-2n+n+2+2i_2+6i_3}
\]

\[
= \sum_{(i_1,i_2,i_3,i_4), i_1+i_2+i_3+i_4=n+2} \sum_j \epsilon_{(i_1,i_2,i_3,i_4)} (1+y)^{i_2} y^{2i_3+4i_4} M(y)^{2i_1-1-2n+n+2+2i_2+6i_3+14i_4}
\]

\[
\equiv \sum_{(i_1,i_2,i_3,i_4), i_1+i_2+i_3+i_4=n+2} \epsilon_{(i_1,i_2,i_3,i_4)} y^{2i_3+4i_4} M(y)^{2i_1-1-2n+n+2+2i_2+6i_3+14i_4}
\]

in the ring \(\mathbb{Z}/2[y]/(y^{2n-3n-6}/3)\).

**6.9 The case \(j \geq 3\)**

Now, rather than \(t_1\) or \(t_2\), let us try this process with \(t_k\) for \(k \geq 3\). Hence we calculate

\[
(\eta_R)_*(v_1^{i_1} v_2^{i_2} \cdots v_r^{i_r} x^s) \in D^{2n-1} / < t_1, t_2, \ldots, t_{k-1}, t_{k+1}, \ldots >
\]

collecting the terms of the form \(v_1^{n+2-j} v_2^j x^a t_k^b\).

Since

\[
(\eta_R)_*(v_s^{i_s}) \equiv (\sum_{a \geq 1} v_a(t_3 x^{2a-1} + t_{s-a}^{2n-1}))^{i_s}
\]

the term under consideration can only come from monomials with \(I = (i_1, i_2, 0, \ldots, 0, i_k, i_{k+1}, i_{k+2})\) and

\[
(\eta_R)_*(v_1^{i_1} v_2^{i_2} v_k^{i_k} v_{k+1}^{i_{k+1}} v_{k+2}^{i_{k+2}} x^{2i_1-1-2n+n+2+2i_2+(2k-2)i_k+(2k+1-2)i_{k+1}+(2k+2-2)i_{k+2}})
\]

\[
\equiv v_1^{i_1} v_2^{i_2} (v_1 t_{k} x + v_2 x^3 t_k)^{i_k} (v_1 t_{k}^2)^{i_{k+1}} (v_2 t_{k}^4)^{i_{k+2}} (x M_k)^{2i_1-1-2n+n+2+2i_2+(2k-2)i_k+(2k+1-2)i_{k+1}+(2k+2-2)i_{k+2}}
\]

where \(M_k = \sum_{m \geq 0} t_k^{(2m-1)/2} x^{2m-1}/(2k-1)\).

Consider the term from the binomial expansion of \((v_1 t_{k} x + v_2 x^3 t_k)^{i_k}\)

\[
\binom{i_k}{a} (v_1 t_{k} x)^a (v_2 x^3 t_k)^{i_k-a}
\]

to contribute \(v_1^{n+2-j} v_2^j\) we must have

\[
a + i_1 + i_{k+1} = i_1 + i_2 + i_k + i_{k+1} + i_{k+2} - j
\]
Adding over Remark 6.11

\[ a = i_2 + i_k + i_{k+2} - j \]

This means that we obtain a contribution of the form

\[ \left( \begin{array}{c} i_k \\ j - i_2 - i_{k+2} \end{array} \right) \left( v_1^j v_2^j (v_1 t_k x)^{i_2 + i_k + i_{k+2} - j} (v_2 t_k x^3)^{j - i_2 - i_{k+2}} (v_1 t_k^2)_{i_{k+1}} \right) \]

\[ = \left( \begin{array}{c} i_k \\ j - i_2 - i_{k+2} \end{array} \right) v_1^{n+2-j} v_2^j (t_k x^{2^k-1})^{i_k + 2i_{k+1} + 4i_{k+2}} \]

\[ M_k^{2^i-1-2^n+n+2+2i_2 + (2^k-2)i_k + (2^k+1-2)i_{k+1} + (2^k+2-2)i_{k+2}} \]

When \( i_2 = 0 = i_{k+2} \) we have to subtract the usual correction term.

If we set \( y_k = t_k x^{2^k-1} \) this gives the following result.

Proposition 6.10

In the notation of Proposition ??, if \( j \leq n + 2 \) and \( k \geq 3 \),

\[ \epsilon(n+2-j,j) \]

\[ = (\sum_{i_1,i_2,\ldots,i_k,i_{k+1},i_{k+2})i_1+i_2+i_k+i_{k+1}+i_{k+2}=n+2 \epsilon(i_1,\ldots) \left( \begin{array}{c} i_k \\ j - i_2 - i_{k+2} \end{array} \right) \left( v_1^{i_k+2i_{k+1}+4i_{k+2}} \right) y_k^{i_k+2i_{k+1}+4i_{k+2}} \]

\[ - \sum_{i_1,0,\ldots,i_k,i_{k+1})i_1+i_k+i_{k+1}=n+2 \epsilon(i_1,\ldots) \left( \begin{array}{c} n + 2 \\ j \end{array} \right) y_k^{i_k+2i_{k+1}} \]

\[ M_k^{2^i-1-2^n+n+2+2i_2 + (2^k-2)i_k + (2^k+1-2)i_{k+1} + (2^k+2-2)i_{k+2}} \]

in \( \mathbb{Z}/2[y_k]/(y_k^{2^{k-1}}) \).

Remark 6.11 Adding over \( j \) and noting that

\[ \sum_j \left( \begin{array}{c} i_k \\ j - i_2 - i_{k+2} \end{array} \right) \equiv 0 \ (\text{modulo } 2) \]

except when \( i_k = 0 \), when it is equal to one, we obtain

\[ \sum_j \epsilon(n+2-j,j) \]

\[ = (\sum_{i_1,i_2,0,\ldots,0,i_{k+1},i_{k+2})i_1+i_2+i_{k+1}+i_{k+2}=n+2 \epsilon(i_1,i_2,0,\ldots,0,i_{k+1},i_{k+2})y_k^{2i_{k+1}+4i_{k+2}} \]

\[ M_k(y_k)^{2^i-1-2^n+n+2+2i_2 + (2^k-2)i_k + (2^k+1-2)i_{k+1} + (2^k+2-2)i_{k+2}}. \]
For all expressions to be in \( \mathbb{Z}/2[w]/(w^{2n} - 3n - 6) \) we should put \( y_k = w^{2k-1} \) to give

\[
\sum_j \epsilon(n+2-j,j) = (\sum_{j=0}^{n+2} \epsilon(n+2,j)_{i_1,i_2,i_3})(1 + w)^{n+2-j} L(w) \left( \begin{array}{c} n + 2 - i_2 - i_3 \nonumber \\
2^j \end{array} \right) \left( \begin{array}{c} i_2 \\
B@ \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
B@ \end{array} \right)
\]

\[
\epsilon(i_1,i_2,i_3)(1 + w)^{n+2-J} \left( \begin{array}{c} n + 2 - i_2 - i_3 \\
B@ \end{array} \right) L(w) \left( \begin{array}{c} n + 2 - i_2 - i_3 \\
J \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
J \end{array} \right) w^{4i_3}
\]

\[-\sum_{j=0}^{n+2} \epsilon(i_1,i_2)(1 + w)^{n+2-j} L(w) \left( \begin{array}{c} n + 2 - i_2 - i_3 \\
J \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
J \end{array} \right) w^{2i_2}.\]

In \( \mathbb{Z}/2[w]/(w^{2n} - 3n - 6) \) and we shall consider the terms of degree at most seven in \( w \).

In this case we shall be working modulo \( < w^8, w^9, \ldots > \) so that \( L(w) \equiv 1 + w + w^3 + w^7 \). The only contributing terms in the first sum must have \( i_3 = 0, 1 \). If \( i_3 = 1 \) and \( i_2 + 1 + b - j = 0, 1 \) and \( 0 \leq b \leq 3 \).

Working modulo \( < w^8, w^9, \ldots > \) will ensure that the relevant terms from the final sum of \( (i_1, i_2) \)-terms have \( i_2 \leq 3 \). In addition, if \( n \geq 5 \) then \( 2^n - 3n - 6 \geq 8 \).

### 7 Theorem ?? – the second step

#### 7.1

In the remaining sections we investigate the low degree terms of the identities of Propositions ??, ?? and ?? to show that the coefficients must all vanish. This section gets us as far as \( \beta \geq 3 \), which starts off our induction on \( \beta \).

In the circumstances of Theorem ?? we may suppose, by Proposition ??, that \( \epsilon = 0 \) and that \( \epsilon(n+2-j',2j') = 0 \) for \( 0 \leq j \leq \beta - 1 \). Then the first step of the induction shows that the lowest \( x \)-degree in the \( \tilde{S} \)-dual, \( D(u) \), is \( 2^j - 1 - 2^n + n + 2 + 2^{j+1} \), which means that \( \epsilon(i_1,i_2,i_3) = 0 \) for all \( i_1 + i_2 + i_3 = n + 2 \) and \( 2i_2 + 6i_3 \leq 2^{\beta+1} \) – except possibly \( \epsilon(n+2-j',2j') \), which we are trying to show is zero in order to complete the last step of the main induction.

Next, we are going to take the equation from Proposition ??

\[
\epsilon(n+2-j,j)
\]

\[
= \sum_{(i_1,i_2,i_3), \ i_1+i_2+i_3=n+2, \ b} L(w) \left( \begin{array}{c} n + 2 - i_2 - i_3 \\
J \end{array} \right) \left( \begin{array}{c} i_3 \\
J \end{array} \right) \left( \begin{array}{c} i_2 \\
J \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
J \end{array} \right) \left( \begin{array}{c} n + 2 \\
J \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
J \end{array} \right) w^{4i_3}
\]

\[-\sum_{(i_1,i_2), \ i_1+i_2=n+2} L(w) \left( \begin{array}{c} n + 2 - i_2 \\
J \end{array} \right) \left( \begin{array}{c} i_3 + b - j \\
J \end{array} \right) w^{2i_2}.\]
and $2^{i_{1}-1} - 2^i$ is divisible by 8 so that the relations reduce to

$$\epsilon_{(n+2-j,i)}$$

$$= \sum_{(i_{1}, i_{2}), \ i_{1} + i_{2} = n + 2, \ b} (1 + w + w^3 + w^7)^{n + 2 + 2i_{2}} \binom{n + 2 - i_{2}}{b} \binom{i_{2}}{i_{2} + b - j}$$

$$\epsilon_{(i_{1}, i_{2})}(1 + w)^{n + 2 - i_{2} - b} w^{2i_{2} + 3b - 2j}$$

$$+ \sum_{(i_{1}, i_{2}, 1), \ i_{1} + i_{2} = n + 1, \ b} (1 + w + w^3 + w^7)^{n + 2 + 2i_{2}} \binom{n + 1 - i_{2}}{b} \binom{i_{2}}{i_{2} + 1 + b - j}$$

$$\epsilon_{(i_{1}, i_{2}, 1)}(1 + w)^{n + 1 - i_{2} - b} w^{2i_{2} + 6 + 3b - 2j}$$

$$- \sum_{(i_{1}, i_{2}), \ i_{1} + i_{2} = n + 2} (1 + w + w^3 + w^7)^{n + 2 + 2i_{2}} \epsilon_{(i_{1}, i_{2})} \binom{n + 2}{j} (1 + w)^{n + 2 - i_{2} - 2i_{2}}$$

in $\mathbb{Z}/2[w]/(w^8)$.

Now we shall tabulate the values of the parameters which can contribute non-trivially.

If $i_3$ is non-zero then $i_3 = 1, i_2 + 1 + b - j = 0, 1, 0 \leq b \leq 3$ and $2i_2 + 6 + 3b - 2j \leq 7$. Hence, when $i_3 = 1$, we have the following table of possibilities:

<table>
<thead>
<tr>
<th>$i_3$</th>
<th>$i_2 + 1 + b - j$</th>
<th>$b$</th>
<th>$i_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$j - 1$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$j - 2$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>$j - 3$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>$j - 4$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$j$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$j - 1$</td>
</tr>
</tbody>
</table>

When $i_3 = 0$ in the first sum the only contributing terms must satisfy $0 \leq i_2 + b - j \leq 3, 0 \leq b \leq 7$ and $2i_2 + 3b - 2j \leq 7$. The table of possibilities is as
follows:

<table>
<thead>
<tr>
<th>$i_3$</th>
<th>$i_2 + b - j$</th>
<th>$b$</th>
<th>$i_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
<td>$j + 3$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>1</td>
<td>$j + 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$j + 2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$j + 1$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>$j$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>$j - 1$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$j + 1$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$j$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$j - 1$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
<td>$j - 2$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
<td>$j - 3$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>5</td>
<td>$j - 4$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$j$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$j - 1$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>$j - 2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>$j - 3$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>$j - 4$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5</td>
<td>$j - 5$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>6</td>
<td>$j - 6$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>7</td>
<td>$j - 7$</td>
</tr>
</tbody>
</table>

In the third sum we must have $0 < i_2 \leq 3$, since $\epsilon_{(n+2,0)} = 0$.

**7.2 The first identity modulo $w^8$**

Some rather mechanical algebra leads to the following identity in $\mathbb{Z}/2[w]/(w^8)$:
\begin{align*}
\epsilon_{(n+2-j, j)} &= \binom{j+3}{3} \epsilon_{(n-j-1, j+3)} (1 + w)^{2n+j-1} w^6 \\
&\quad + (n - j) \binom{j+2}{3} \epsilon_{(n-j, j+2)} w^7 \\
&\quad + (1 + w + w^3)^{n+2j+6} \binom{j+2}{2} \epsilon_{(n-j, j+2)} (1 + w)^{n-j} w^4 \\
&\quad + (n + 1 - j) \binom{j+1}{2} \epsilon_{(n-j+1, j+1)} (1 + w)^{2n+j+4} w^5 \\
&\quad + \binom{n+2-j}{2} \binom{j}{2} \epsilon_{(n-j+2, j)} (1 + w)^{2n+j+2} w^6 \\
&\quad + \binom{n-j+3}{3} \binom{j-1}{2} \epsilon_{(n-j+3, j-1)} w^7 \\
&\quad + (1 + w + w^3)^{n+4+2j} (j+1) \epsilon_{(n-j+1, j+1)} (1 + w)^{n+1-j} w^2 \\
&\quad + (1 + w + w^3)^{n+2+2j} (n-j) j \epsilon_{(n-j+2, j)} (1 + w)^{n+1-j} w^3 \\
&\quad + (1 + w + w^3)^{n+2j} \binom{n+3-j}{2} (j-1) \epsilon_{(n-j+3, j-1)} (1 + w)^{n-j+1} w^4 \\
&\quad + \binom{n+4-j}{3} j \epsilon_{(n-j+4, j-2)} (1 + w)^{2n+j-1} w^5
\end{align*}
\[
+ \binom{n + 5 - j}{4} (j - 3) \varepsilon_{n-j+5,j-3} (1 + w)^{2n+j-3} w^6 \\
+ \binom{n - j + 6}{5} j \varepsilon_{n-j+6,j-4} w^7 \\
+ (1 + w + w^3 + w^7)^{n+2} 2^j \varepsilon_{n-j+2,j} (1 + w)^{n+2-j} \\
+ (1 + w + w^3)^{n+2j} (n + 3 - j) \varepsilon_{n-j+3,j-1} (1 + w)^{n+2-j} w \\
+ (1 + w + w^3)^{n+2j-2} \left( \binom{n - j + 4}{2} \varepsilon_{n-j+4,j-2} (1 + w)^{n+2-j} w^2 \\
+ (1 + w + w^3)^{n+2j-4} \left( \binom{n - j + 5}{3} \varepsilon_{n-j+5,j-3} (1 + w)^{n+2-j} w^3 \\
+ (1 + w + w^3)^{n+2j-6} \left( \binom{n - j + 6}{4} \varepsilon_{n-j+6,j-4} (1 + w)^{n+2-j} w^4 \\
+ (1 + w)^{2n+j-6} \left( \binom{n - j + 7}{5} \varepsilon_{n-j+7,j-5} w^5 \\
+ \binom{n - j + 8}{6} \varepsilon_{n-j+8,j-6} (1 + w)^{2n+j-8} w^6 \\
+ \binom{n - j + 9}{7} \varepsilon_{n-j+9,j-7} w^7 \\
+ (1 + w + w^3)^{n+2j-2} \varepsilon_{n-j+2,j-1,1} (1 + w)^{n-j+2} w^4
\]
Lemma 7.3
For all $n \geq 5$, in the notation of §??, $\epsilon(n_{-1,1}) = 0$.

Proof
In §?? set $j = 1$ and consider the coefficient of $w$. This yields the congruence $0 \equiv \epsilon(n_{+1,1}) \pmod{2}$. □

Lemma 7.4
For all $n \geq 5$, in the notation of §??, $\epsilon(n_{+2}) = \epsilon(n_{-1,3}) = 0$.

Proof
If we set $j = 2$ in §?? then, using Lemma ?? and considering the coefficients of $w^2$ and $w^3$ yields the equations $0 \equiv \epsilon(n_{-1,3}) + (n + 3)\epsilon(n_{+2}) \pmod{2}$ and $0 \equiv \epsilon(n_{-1,3}) + n\epsilon(n_{+2}) \pmod{2}$, respectively. □

7.5 Now we set $j = 1, 2, 3, 4, 5, 6$ in §?? then, using Lemma ?? and Lemma ?? recalling from §?? that they imply $\epsilon(i_{1, j_2, 1}) = 0$ for $i_2 = 0, 1$ and gathering results as we go. This yields the following relations in $\mathbb{Z}/2[w]/(w^8)$:

Here we try $j = 1$ using the same results:
\[0 \equiv n \epsilon_{(n-4,6)}.\]

Here we try \(j = 2\) using the same results:
\[
0 = \binom{n + 2}{2} \epsilon_{(n-4,6)}
\]

Here we try \(j = 3\) using the same results:
\[
0 \equiv (1 + w + w^3)^{n+4} \epsilon_{(n-1,2,1)} (1 + w)^{n-1} w^4
\]
\[
+ (1 + w)^{2n+4} \epsilon_{(n-2,3,1)} w^6
\]
\[
+ (1 + w)^{2n+7} \epsilon_{(n-4,6)} \binom{n + 2}{3} w^6
\]
from which (degrees 4, 6 and 7 resp.) \(\epsilon_{(n-1,2,1)} = \epsilon_{(n-2,3,1)} = 0\) and
\[
0 \equiv \binom{n + 2}{3} \epsilon_{(n-4,6)}.
\]

Here we try \(j = 4\) using the same results plus \(j = 3\) results:
\[
\epsilon_{(n-2,4)} = \epsilon_{(n-5,7)} (w^6 + w^7)
\]
\[
+ (1 + w + w^3)^{n+10} \epsilon_{(n-4,6)} (1 + w)^{n-4} w^4
\]
\[
+ (1 + w + w^3)^{n+12} \epsilon_{(n-3,5)} (1 + w)^{n-3} w^2
\]
\[
+ (1 + w + w^3 + w^7)^{n+10} \epsilon_{(n-2,4)} (1 + w)^{n-2}
\]
\[
- \epsilon_{(n-4,6)} \binom{n + 2}{4} (w^6 + w^7).
\]

The coefficient of \(w^2\) yields \(0 \equiv \epsilon_{(n-3,5)} + n \epsilon_{(n-2,4)} \pmod{2}\) while that of \(w^3\) yields \(0 \equiv \epsilon_{(n-3,5)} + n \epsilon_{(n-2,4)} \pmod{2}\), too.
Here we try $j = 5$ using the same results plus $j = 3$ results:

$$\begin{align*}
\epsilon_{(n-3,5)} &= (n-1)\epsilon_{(n-5,7)}w^7 \\
&\quad + (1 + w + w^3)^n\epsilon_{(n-5,7)}(1 + w)^{n-5}w^4 \\
&\quad + n\epsilon_{(n-4,6)}(1 + w)^{2n+9}w^5 \\
&\quad + (1 + w + w^3)^{n+12}(n-1)\epsilon_{(n-3,5)}(1 + w)^{n-4}w^3 \\
&\quad + (1 + w + w^3 + w^7)^{n+12}\epsilon_{(n-3,5)}(1 + w)^{n-3}w \\
&\quad + (1 + w + w^3)^{n+10}n\epsilon_{(n-2,4)}(1 + w)^{n-3}w \\
&\quad + (1 + w + w^3)^n\epsilon_{(n-3,4,1)}(1 + w)^{n-3}w^4 \\
&\quad + (1 + w)^{2n+6}\epsilon_{(n-4,5,1)}w^6 \\
&\quad -(1 + w)^{2n+7}\epsilon_{(n-4,6)}\left(\frac{n+2}{5}\right)w^6.
\end{align*}$$

Here we try $j = 6$ using the same results plus $j = 3$ results:
\[ \epsilon_{(n-4,6)} \]
\[ = (n + 1)\epsilon_{(n-5,7)}(1 + w)^{2n+10}w^5 \]
\[ + \binom{n-4}{2}\epsilon_{(n-4,6)}(1 + w)^{2n}w^6 \]
\[ + (1 + w + w^3)^n\epsilon_{(n-5,7)}(1 + w)^{n-5}w^2 \]
\[ + (1 + w + w^3)^{n+4}\binom{n-3}{2}\epsilon_{(n-2,4)}(1 + w)^{n-5}w^4 \]
\[ + (1 + w + w^3 + w^7)^{n+6}\epsilon_{(n-4,6)}(1 + w)^{n-4} \]
\[ + (1 + w + w^3)^{n+4}(n + 1)\epsilon_{(n-3,5)}(1 + w)^{n-4}w \]
\[ + (1 + w + w^3)^{n+2}\binom{n-2}{2}\epsilon_{(n-2,4)}(1 + w)^{n-4}w^2 \]
\[ + (1 + w + w^3)^{n+2}\epsilon_{(n-4,5,1)}(1 + w)^{n-4}w^4 \]
\[ + (1 + w)^{2n+4}(n - 1)\epsilon_{(n-3,4,1)}w^5 \]
\[ + n\epsilon_{(n-4,5,1)}w^7 \]
\[ - (1 + w)^{2n+7}\epsilon_{(n-4,6)}\binom{n+2}{6}w^6. \]

### 7.6 The second identity modulo \( w^8 \)

Now consider the equation of Proposition 7, setting \( y = w^3 \) and \( M(y) = 1 + w^3 + w^{15} + \ldots \). We examine the terms of degree at most six in \( w \). These terms can only come from \((i_1, i_2, i_3, i_4)\)-terms with \( i_4 = 0 \) and \( i_3 = 0, 1 \) and from \((i_1, i_2, i_3)\)-terms with \((i_2, i_3) = (0, 1), (1, 0), (2, 0)\). By the above remarks these terms of degree at most six in \( w \) are the same as in the following equation in \( \mathbb{Z}/2[w]/(w^8) \) for all \( n \geq 5 \):
\[
\epsilon_{n+2-j,j} \\
\equiv \epsilon_{(n+1-j,j,1)} w^6 + \epsilon_{(n+2-j,j)} (1 + w^3)^{n+2+3j} \\
+ \epsilon_{(n+1-j,j+1)} (j + 1) w^3 (1 + w^3)^{n+4+3j} + \epsilon_{(n-j,j+2)} \binom{j + 2}{j} w^6
\]

From the terms in \( w^3 \) and \( w^6 \) we obtain the following congruences modulo 2:

\[
0 \equiv (n + j) \epsilon_{(n+2-j,j)} + (j + 1) \epsilon_{(n+1-j,j+1)}
\]

and

\[
0 \equiv \binom{n + 2 + 3j}{2} \epsilon_{(n+2-j,j)} + (j + 1)n \epsilon_{(n+1-j,j+1)}
+ \binom{j + 2}{j} \epsilon_{(n-j,j+2)} + \epsilon_{(n+1-j,j,1)}.
\]

**Proposition 7.7**

For all \( n \geq 5 \), in the notation of \( §?? \), \( \epsilon_{(n,4)} = 0 \).

**Proof**

When \( n \) is even and \( j \) is odd the equation of \( §?? \) derived from the coefficient of \( w^3 \) yields \( 0 \equiv \epsilon_{(n+2-j,j)} \). Hence, if \( n \) is even, considering the coefficient of \( w^2 \) in the congruence of \( §?? \) for \( j = 6 \) yields

\[
0 \equiv \binom{2n + 2}{2} \epsilon_{(n-4,6)} + \binom{n - 2}{2} \epsilon_{(n-2,4)} \equiv \epsilon_{(n-4,6)} + \binom{n - 2}{2} \epsilon_{(n-2,4)}.
\]

When \( n \equiv 0 \) (modulo 4) the congruence of \( §?? \) when \( j = 2 \) shows that \( 0 = \epsilon_{(n-4,6)} \) and hence \( 0 = \epsilon_{(n-2,4)} \), as required.

When \( n \) is odd the congruence of \( §?? \) when \( j = 1 \) shows that \( \epsilon_{(n-4,6)} = 0 \). Also, for odd \( n \) and even \( j \), the equation of \( §?? \) derived from the coefficient of \( w^3 \) yields \( \epsilon_{(n+2-j,j)} = \epsilon_{(n+1-j,j+1)} \) so that \( \epsilon_{(n-2,4)} = \epsilon_{(n-3,5)} \) and \( \epsilon_{(n-5,7)} = 0 \). Under these circumstances the congruence of \( §?? \) when \( j = 4 \) takes the following form,
in $\mathbb{Z}/2[w]/(w^7)$:

\[
\epsilon_{(n-2,4)} \equiv (1 + w + w^3)^{n+12} \epsilon_{(n-2,4)} (1 + w)^{n-3} w^2 \\
+ (1 + w + w^3 + w^7)^{n+10} \epsilon_{(n-2,4)} (1 + w)^{n-2} \\
\equiv \epsilon_{(n-2,4)} (1 + w + w^3)^{n+10} (1 + w)^{n-3} ((1 + w + w^3)^2 w^2 + (1 + w)) \\
\equiv \epsilon_{(n-2,4)} (1 + w + w^3)^{n+10} (1 + w)^{n-3} (1 + w^2 + w^4) \\
\equiv \epsilon_{(n-2,4)} (1 + w + w^2 + w^4)((1 + w)^{2n+7} + (n + 10)w^3(1 + w)^{2n+6} \\
+ \left( \begin{array}{c}
    n + 10 \\
    2 \\
\end{array} \right) w^6 (1 + w)^{2n+5} \\
\equiv \epsilon_{(n-2,4)} (1 + w + w^2 + w^4)(1 + w + w^4(2n + 6)/4 + w^5(2n + 6)/4 + w^3 \\
+ \left( \begin{array}{c}
    n + 10 \\
    2 \\
\end{array} \right) w^6 \\
\equiv \epsilon_{(n-2,4)} (1 + w + w^2 + w^4 + w + w^2 \\
+ w^3 + w^5 + w^4(2n + 6)/4 + w^5(2n + 6)/4 \\
+ w^6(2n + 6)/4 + w^5(2n + 6)/4 + w^6(2n + 6)/4 + w^3 + w^4 + w^5 \\
+ \left( \begin{array}{c}
    n + 10 \\
    2 \\
\end{array} \right) w^6 \\
\equiv \epsilon_{(n-2,4)} (1 + w^4(2n + 6)/4 + \left( \begin{array}{c}
    n + 10 \\
    2 \\
\end{array} \right) w^6)
\]

If $n \equiv 3$ (modulo 4) then $(2n + 6)/4$ is odd while $\left( \begin{array}{c}
    n + 10 \\
    2 \\
\end{array} \right)$ is odd if $n \equiv 1$ (modulo 4) so that $\epsilon_{(n-2,4)} = 0$ for all odd $n$.

There remains the case when $n \equiv 2$ (modulo 4). In this case $\left( \begin{array}{c}
    n - 2 \\
    2 \\
\end{array} \right)$ is even and so the congruence at the start of the proof shows that $\epsilon_{(n-4,6)} = 0$. Also $0 = \epsilon_{(n+2-j,j)}$ for all odd $j$. Hence, putting $j = 4, 5$ in the last congruence of §??
shows that \( \varepsilon_{(n-3,4,1)} = 0 = \varepsilon_{(n-4,5,1)} \). Therefore the congruence of \( \S ?? \) when \( j = 6 \) becomes, in \( \mathbb{Z}/2[w]/(w^8) \):

\[
0 \equiv (1 + w + w^3)^{n+4} \binom{n-3}{2} \varepsilon_{(n-2,4)}(1 + w)^{n-5}w^4 + (1 + w + w^3)^{n+2} \binom{n-2}{2} \varepsilon_{(n-2,4)}(1 + w)^{n-4}w^2 \\
\equiv (1 + w + w^3)^{n+4} \varepsilon_{(n-2,4)}(1 + w)^{n-5}w^4,
\]

which shows that \( \varepsilon_{(n-2,4)} = 0 \) in this case, too. \( \Box \)

8 Theorem ?? – the final step

8.1 In this section we press on with the combinatorics, proving that Theorem ?? is true when \( n \equiv 1 \) (modulo \( 4 \)) in Proposition ??, when \( n \equiv 3 \) (modulo \( 8 \)) in Proposition ?? and when \( n \equiv 7 \) (modulo \( 8 \)) in \( \S ?? \).

We shall suppose, by induction, that \( \varepsilon_{(n+2-2^j,2j)} = 0 \) for \( 0 \leq j \leq \beta - 1 \), where we rely on Proposition ?? to ensure that this is true for \( \beta = 3 \). Then the main induction shows, if \( u \in J_{2^n+1-2}(RP^\infty) \) is as in Proposition ??, then the lowest \( x \)-degree in the S-dual, \( D(u) \), is \( 2^{j-1} - 2^n + n + 2 + 2^{\beta+1} \). This means that \( \varepsilon_{(i_1,i_2,i_3)} = 0 \) for all \( i_1 + i_2 + i_3 = n + 2 \) and \( 2i_1 + 6i_3 \leq 2^{\beta+1} \) – except possibly \( \varepsilon_{(n_2-2^\beta,2^\beta)} \), which we are trying to show is zero in order to complete the last step of the main induction of \( \S ?? \) and \( \S ?? \).

We are going to show that \( \varepsilon_{(n+2-2^\beta,2^\beta)} = 0 \) by imitating the argument used to prove Proposition ??.

Now we set \( j = 2^\beta - 1, 2^\beta, 2^\beta + 1, 2^\beta + 2 \) in \( \S ?? \) using the facts that

\[
\varepsilon_{(n+2-2j,2j)} = 0 \ for \ 0 \leq j \leq \beta - 1
\]

and

\[
\varepsilon_{(n+1-j,j,1)} = 0 \ for \ 0 \leq j \leq \beta - 3
\]

and gathering results as we go.

This yields the following relations in \( \mathbb{Z}/2[w]/(w^8) \).

Here we try \( j = 2^\beta - 1 \) using the above results to obtain:

\[
0 \equiv (1 + w + w^3)^{n+2^\beta+1-4} \varepsilon_{(n-2^\beta,3\cdot2^\beta-2,1)}(1 + w)^{n-2^\beta+3}w^4 \\
+ (1 + w)^{2n+2^\beta} \varepsilon_{(n-2^\beta+2\cdot2^\beta-1,1)}w^6
\]

which proves the first part of the following result.
Lemma 8.2
For all $n \geq 5$, in the notation of §7, $\epsilon_{(n-2^3,2^3-2,1)} = 0 = \epsilon_{(n-2^3+2,2^3-1,1)}$.
Also, when $n \equiv 2 \pmod{4}$,

\[ \epsilon_{(n-2^3,2^3+2)} = \epsilon_{(n-2^3+1,2^3,1)}. \]

Proof
From the preceding discussion it remains only to prove the last statement, which follows by setting $j = 2^\beta$ in the second congruence of §7 and observing from the first congruence of §7 that $\epsilon_{(n+2-j,j)}$ is zero when $j$ is odd and $n$ is even. □

8.3 Here we try $j = 2^\beta$ using the above results, including Lemma 7.7, to obtain:

\[ \epsilon_{(n+2-2^\beta,2^\beta)} \]
\[ \equiv \begin{pmatrix} 2^\beta + 3 \\ 3 \end{pmatrix} \epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{2n+2^\beta-1}w^6 + (1 + w + w^3)^{n+2^\beta+1+w^6}(2^\beta + 2) \epsilon_{(n-2^\beta,2^\beta+2)}(1 + w)^{n-2^\beta}w^4 \]
\[ + (1 + w + w^3)^{n+4+2^\beta+1}(2^\beta + 1)\epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^{n+1-2^\beta}w^2 \]
\[ + (1 + w + w^3 + w^7)^{n+2+2^\beta+1} \epsilon_{(n-2^\beta+2,2^\beta)}(1 + w)^{n+2-2^\beta} \]
\[ \equiv \epsilon_{(n-2^\beta-1,2^\beta+3)}(w^6 + w^7) + (1 + w + w^3)^{n+6} \epsilon_{(n-2^\beta,2^\beta+2)}(1 + w)^nw^4 \]
\[ + (1 + w + w^3)^{n+4} \epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^{n+1}w^2 \]
\[ + (1 + w + w^3 + w^7)^{n+2} \epsilon_{(n-2^\beta+2,2^\beta)}(1 + w)^{n+2} \]

Lemma 8.4
Suppose that $n \geq 5$.
(i) If $n \equiv 0,1 \pmod{4}$ then $\epsilon_{(n-2^\beta,2^\beta+2)} = \epsilon_{(n-2^\beta+2,2^\beta)}$,
(ii) If $n \equiv 2,3 \pmod{4}$ then $\epsilon_{(n-2^\beta,2^\beta+2)} = 0$. 
Proof

If \( n \) is even then \( \epsilon_{(n+2-j,j)} \) vanishes for odd \( j \), by \( \S \). In this case the congruence of \( \S \) yields the following identity in \( \mathbb{Z}/2[w]/(w^8) \):

\[
\epsilon_{(n+2-2^j,2^j)} \\
\equiv (1 + w + w^3)^{n+6} \epsilon_{(n-2^j,2^j+2)} (1 + w)^n w^4 \\
+ (1 + w^2 + w^3 + w^4 + w^7)^{n+2} \epsilon_{(n-2^j+2,2^j)}.
\]

If \( n \equiv 0 \) (modulo 4) this identity reduces to

\[
\epsilon_{(n+2-2^j,2^j)} \\
\equiv (1 + w^2) \epsilon_{(n-2^j,2^j+2)} w^4 \\
+ (1 + w^4 + w^6)^{(n+2)/2} \epsilon_{(n-2^j+2,2^j)} \\
\equiv (w^4 + w^6) \epsilon_{(n-2^j,2^j+2)} \\
+ (1 + w^4 + w^6) \epsilon_{(n-2^j+2,2^j)},
\]

as required.

If \( n \equiv 2 \) (modulo 4) the identity reduces to

\[
\epsilon_{(n+2-2^j,2^j)} \\
\equiv \epsilon_{(n-2^j,2^j+2)} (1 + w^2) w^4 \\
+ \epsilon_{(n-2^j+2,2^j)},
\]

as required.

By \( \S \), if \( n \) is odd then \( \epsilon_{(n+2-j,j)} = \epsilon_{(n-j+1,j+1)} \) when \( j \) is even. In this case
the congruence of \( \xi = \gamma \) yields the following identity in \( \mathbb{Z}/2[w]/(w^8) \):

\[
\epsilon_{(n+2-2^3,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^6 + w^7) \\
+(1 + w + w^3)n^6 \epsilon_{(n-2^3,2^3+2)}(1 + w)n w^4 \\
+(1 + w + w^3)n^4 \epsilon_{(n-2^3+2,2^3)}(1 + w)n^{1+1} w^2 \\
+(1 + w^2 + w^3 + w^4 + w^7)n^2 \epsilon_{(n-2^3+2,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^6 + w^7) \\
+(1 + w^3)w^4 \epsilon_{(n-2^3,2^3+2)} \\
+((1 + w)^{2n+5} + w^3(1 + w)^{2n+4})w^2 \epsilon_{(n-2^3+2,2^3)} \\
+(1 + w^2 + w^3 + w^4 + w^7)n^2 \epsilon_{(n-2^3+2,2^3)}.
\]

If \( n \equiv 1 \) (modulo 4) we obtain

\[
\epsilon_{(n+2-2^3,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^4 + w^6) \\
+((1 + w)^7 + w^3(1 + w)^6)w^2 \epsilon_{(n-2^3+2,2^3)} \\
+(1 + w^2 + w^3 + w^4 + w^7)^3 \epsilon_{(n-2^3+2,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^4 + w^6) \\
+(w^2 + w^3 + w^4 + w^6) \epsilon_{(n-2^3+2,2^3)} \\
+(1 + w^2 + w^3 + w^4 + w^7 + w^4 + w^6 + (w^2 + w^3)^3) \epsilon_{(n-2^3+2,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^4 + w^6) + (1 + w^4 + w^6) \epsilon_{(n-2^3+2,2^3)}
\]

which completes the proof of part (i).

Similarly, if \( n \equiv 3 \) (modulo 4) we obtain

\[
\epsilon_{(n+2-2^3,2^3)} \\
\equiv \epsilon_{(n-2^3,2^3+2)}(w^4 + w^6) \\
+\epsilon_{(n-2^3+2,2^3)},
\]

which completes the proof of part (ii).
Corollary 8.5

For \( n \geq 5 \),

\[
\epsilon_{(n+1-2^3,2^3,1)} = \begin{cases} 
0 & \text{if } n \equiv 0, 2 \pmod{4}, \\
\epsilon_{(n+2-2^3,2^3)} & \text{if } n \equiv 1, 3 \pmod{4}.
\end{cases}
\]

Proof

Setting \( j = 2^3 \) in the second congruence of \( \S ?? \) yields the identity

\[
0 \equiv \begin{pmatrix} n+2 \\ 2 \end{pmatrix} \epsilon_{(n+2-2^3,2^3)} + n\epsilon_{(n+1-2^3,2^3+1)}
\]

modulo 2.

When \( n \equiv 0 \pmod{4} \) this yields

\[
0 \equiv \epsilon_{(n+2-2^3,2^3)} + \epsilon_{(n-2^3,2^3+2)} + \epsilon_{(n+1-2^3,2^3,1)}
\]

so that, by Lemma \( ?? \), \( \epsilon_{(n+1-2^3,2^3,1)} = 0 \).

When \( n \equiv 1 \pmod{4} \) we obtain

\[
0 \equiv \epsilon_{(n+2-2^3,2^3)} + \epsilon_{(n+2-2^3,2^3)} + \epsilon_{(n-2^3,2^3+2)} + \epsilon_{(n+1-2^3,2^3,1)}
\]

so that \( \epsilon_{(n-2^3,2^3+2)} = \epsilon_{(n+1-2^3,2^3,1)} \).

When \( n \equiv 2 \pmod{4} \) we obtain

\[
0 \equiv \epsilon_{(n-2^3,2^3+2)} + \epsilon_{(n+1-2^3,2^3,1)}.
\]

so that \( 0 = \epsilon_{(n-2^3,2^3+2)} = \epsilon_{(n+1-2^3,2^3,1)} \).

Finally, when \( n \equiv 3 \pmod{4} \) we obtain

\[
0 \equiv \epsilon_{(n+2-2^3,2^3)} + \epsilon_{(n-2^3,2^3+2)} + \epsilon_{(n+1-2^3,2^3,1)}.
\]

so that \( \epsilon_{(n+2-2^3,2^3)} = \epsilon_{(n+1-2^3,2^3,1)} \). \( \square \)

Corollary 8.6

For \( n \geq 5 \), \( \epsilon_{(n-2^3,2^3+1,1)} = 0 \).

Proof

Setting \( j = 2^3 + 1 \) in the second congruence of \( \S ?? \) yields the identity

\[
0 \equiv \begin{pmatrix} n+5 \\ 2 \end{pmatrix} \epsilon_{(n+1-2^3,2^3+1)}
\]

\[+\epsilon_{(n-2^3-1,2^3+3)} + \epsilon_{(n-2^3,2^3+1,1)}.
\]
When \( n \) is even and \( j \) is odd the first congruence of \( \S \) implies that \( \epsilon_{(n+2-j,j)} = 0 \) and hence \( \epsilon_{(n-2^\beta,2^\beta+1,1)} = 0 \) when \( n \) is even. Applying the first congruence of \( \S \) we obtain the relation

\[
0 \equiv \binom{n+5}{2} \epsilon_{(n+2-2^\beta,2^\beta)} + \epsilon_{(n-2^\beta,2^\beta+2)} + \epsilon_{(n-2^\beta,2^\beta+1,1)}
\]

which yields \( \epsilon_{(n-2^\beta,2^\beta+1,1)} = 0 \), by Lemma \( \S \). \( \square \)

8.7 Here we try \( j = 2^\beta + 1 \) using the above results, including Lemma \( \S \), to obtain:
\[
\varepsilon_{(n+1-2^\beta,2^\beta+1)} \\
\equiv (n - 2^\beta - 1) \left(\frac{2^\beta + 3}{3}\right) \varepsilon_{(n-2^\beta-1,2^\beta+3)} w^7 \\
+ (1 + w + w^3)^{n+2^\beta+1} \left(\frac{2^\beta + 3}{2}\right) \varepsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{n-2^\beta-1} w^4 \\
+ (n - 2^\beta) \left(\frac{2^\beta + 2}{2}\right) \varepsilon_{(n-2^\beta,2^\beta+2)} (1 + w)^{2n+2^\beta+5} w^5 \\
+ (1 + w + w^3)^{n+4+2^\beta+1} (n - 2^\beta - 1)(2^\beta + 1) \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n-2^\beta} w^3 \\
+ (1 + w + w^3 + w^7)^{n+4+2^\beta+1} \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n-2^\beta+1} w \\
+ (1 + w + w^3)^{n+2^\beta+1} (n + 3 - j) \varepsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{n-2^\beta+1} w \\
+ (1 + w + w^3)^{n+2^\beta+1} \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n-2^\beta+1} w^4 \\
+ (1 + w)^{2n+2^\beta+2} (2^\beta + 1) \varepsilon_{(n-2^\beta,2^\beta+1,1)} w^6 \\
\equiv (n - 1) \varepsilon_{(n-2^\beta-1,2^\beta+3)} w^7 \\
+ (1 + w + w^3)^n \varepsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{n-1} w^4 \\
+ n \varepsilon_{(n-2^\beta,2^\beta+2)} (1 + w)^{2n+5} w^5 \\
+ (1 + w + w^3)^{n+4}(n - 1) \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^n w^3 \\
+ (1 + w + w^3 + w^7)^{n+4} \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n+1} \\
+ (1 + w + w^3)^{n+2} \varepsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{n+1} w \\
+ (1 + w + w^3)^n \varepsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n+1} w^4 \\
+ (1 + w)^{2n+2} \varepsilon_{(n-2^\beta,2^\beta+1,1)} w^6 \\
\]

Here we try \( j = 2^\beta + 2 \) using the above results, including Lemma ??, to
obtain:

\[ \epsilon_{(n-2^\beta,2^\beta+2)} \]

\[ \equiv (n - 2^\beta - 1) \binom{2^\beta + 3}{2} \epsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{2n+2^\beta+6} w^5 \]

\[ + \binom{n - 2^\beta}{2} \binom{2^\beta + 2}{2} \epsilon_{(n-2^\beta,2^\beta+2)} (1 + w)^{2n+2^\beta+4} w^6 \]

\[ + (1 + w + w^3)^{n+2^\beta+1+8(2^\beta + 3)\epsilon_{(n-2^\beta-1,2^\beta+3)}} (1 + w)^{n-2^\beta-1} w^2 \]

\[ + (1 + w + w^3)^{n+2^\beta+1+4} \binom{n + 1 - 2^\beta}{2} (2^\beta + 1) \epsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n-2^\beta-1} w^4 \]

\[ + (1 + w + w^3 + w^7)^{n+2^\beta+1+6\epsilon_{(n-2^\beta,2^\beta+2)}} (1 + w)^{n-2^\beta} \]

\[ + (1 + w + w^3)^{n+2^\beta+1+4(n + 1 - 2^\beta)\epsilon_{(n-2^\beta+1,2^\beta+1)}} (1 + w)^{n-2^\beta} w \]

\[ + (1 + w + w^3)^{n+2^\beta+1+2} \binom{n - 2^\beta + 2}{2} \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{n-2^\beta} w^2 \]

\[ + (1 + w + w^3)^{n+2^\beta+1+2\epsilon_{(n-2^\beta,2^\beta+1,1)}} (1 + w)^{n-2^\beta} w^4 \]
\[(1 + w)^{2n+2^\beta}(n - 2^\beta + 1)\epsilon_{(n-2^\beta+1,2^\beta,1)}w^5\]
\[+(n - 2^\beta)(2^\beta + 1)\epsilon_{(n-2^\beta,2^\beta+1,1)}w^7\]
\[\equiv (n - 1)\epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{2n+6}w^5\]
\[+ \binom{n}{2}\epsilon_{(n-2^\beta,2^\beta+2)}(1 + w)^{2n+4}w^6\]
\[+(1 + w + w^3)^n\epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{n-1}w^2\]
\[+(1 + w + w^3)^{n+4}\binom{n + 1}{2}\epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^{n-1}w^4\]
\[+(1 + w + w^3 + w^7)^{n+6}\epsilon_{(n-2^\beta,2^\beta+2)}(1 + w)^n\]
\[+(1 + w + w^3)^{n+4}(n + 1)\epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^nw\]
\[+(1 + w + w^3)^{n+2}\binom{n + 2}{2}\epsilon_{(n-2^\beta+2,2^\beta)}(1 + w)^nw^2\]
\[+(1 + w + w^3)^{n+2}\epsilon_{(n-2^\beta,2^\beta+1,1)}(1 + w)^nw^4\]
\[+(1 + w)^{2n}(n + 1)\epsilon_{(n-2^\beta+1,2^\beta,1)}w^5\]
\[+n\epsilon_{(n-2^\beta,2^\beta+1,1)}w^7\]

Here we try \(j = 2^\beta + 3\) using the above results, including Lemma ??, to
obtain:

\[\epsilon_{(n-2^\beta-1,2^\beta+3)}\]

\[\equiv \left( \begin{array}{c} n - 2^\beta - 1 \\ 2 \end{array} \right) \epsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{2n+2^\beta+5} w^6\]

\[+ \left( \begin{array}{c} n - 2^\beta \\ 3 \end{array} \right) \left( \begin{array}{c} 2^\beta + 2 \\ 2 \end{array} \right) \epsilon_{(n-2^\beta,2^\beta+2)} w^7\]

\[+(1 + w + w^3)^{n+2^\beta+1+8}(n - 1)\epsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{n-2^\beta} w^3\]

\[+ \left( \begin{array}{c} n + 1 - 2^\beta \\ 3 \end{array} \right) \epsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{2n+2^\beta+2} w^5\]

\[+ \left( \begin{array}{c} n - 2^\beta + 3 \\ 5 \end{array} \right) \epsilon_{(n-2^\beta+3,2^\beta-1)} w^7\]

\[+(1 + w + w^3 + w^7)^{n+2^\beta+1+8}\epsilon_{(n-2^\beta-1,2^\beta+3)} (1 + w)^{n-2^\beta-1}\]

\[+(1 + w + w^3)^{n+2^\beta+1+6}n\epsilon_{(n-2^\beta,2^\beta+2)} (1 + w)^{n-2^\beta-1} w\]

\[+(1 + w + w^3)^{n+2^\beta+1+4} \left( \begin{array}{c} n - 2^\beta + 1 \\ 2 \end{array} \right) \epsilon_{(n-2^\beta+1,2^\beta+1)} (1 + w)^{n-2^\beta-1} w^2\]

\[+(1 + w + w^3)^{n+2^\beta+1+2} \left( \begin{array}{c} n - 2^\beta + 2 \\ 3 \end{array} \right) \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{n-2^\beta-1} w^3\]

\[+(1 + w + w^3)^{n+2^\beta+1+4} \epsilon_{(n-2^\beta-1,2^\beta+2,1)} (1 + w)^{n-2^\beta-1} w^4\]

\[+(1 + w)^{2n+2^\beta+1} n\epsilon_{(n-2^\beta,2^\beta+1,1)} w^5\]

\[+(1 + w)^{2n+2^\beta} \left( \begin{array}{c} n - 2^\beta + 1 \\ 2 \end{array} \right) \epsilon_{(n-2^\beta+1,2^\beta,1)} w^6\]

\[+(1 + w)^{2n+2^\beta+4} \epsilon_{(n-2^\beta-2,2^\beta+3,1)} w^6\]
\[ \equiv \binom{n-1}{2} \epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{2n+1}w^6 + \binom{n}{3} \epsilon_{(n-2^\beta,2^\beta+2)}w^7 \\
+ (1 + w + w^3)^n(n - 1)\epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{n-2}w^3 + \binom{n+1}{3} \epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^{2n+2}w^5 \\
+ \binom{n+3}{5} \epsilon_{(n-2^\beta+3,2^\beta-1)}w^7 \\
+ (1 + w + w^3 + w^7)^n \epsilon_{(n-2^\beta-1,2^\beta+3)}(1 + w)^{n-1} \\
+ (1 + w + w^3)^{n+6}n\epsilon_{(n-2^\beta,2^\beta+2)}(1 + w)^{n-1}w \\
+ (1 + w + w^3)^{n+4} \left( \binom{n+1}{2} \epsilon_{(n-2^\beta+1,2^\beta+1)}(1 + w)^{n-1}w^2 \\
+ (1 + w + w^3)^{n+2} \left( \binom{n+2}{3} \epsilon_{(n-2^\beta+2,2^\beta)}(1 + w)^{n-1}w^3 \\
+ (1 + w + w^3)^{n+4} \epsilon_{(n-2^\beta-1,2^\beta+2,1)}(1 + w)^{n-1}w^4 \\
+ (1 + w)^{2n+1}n\epsilon_{(n-2^\beta,2^\beta+1,1)}w^5 \\
+ (1 + w)^{2n-1} \left( \binom{n+1}{2} \epsilon_{(n-2^\beta+1,2^\beta+1,1)}w^6 \\
+ (1 + w)^{2n} \epsilon_{(n-2^\beta-2,2^\beta+3,1))w^6} \right) \\
\] 

Here we try \( j = 2^\beta + 4 \) using the above results, including Lemma ??, to
obtain:

\[ \epsilon_{(n-2\beta,2\beta+4)} \]
\[ = \epsilon_{(n-2\beta-5,2\beta+7)}(1 + w)w^6 \]
\[ + (1 + w + w^3)n^2\epsilon_{(n-2\beta-4,2\beta+6)}(1 + w)^n w^4 \]
\[ + \left( \begin{array}{c} n-1 \\ 3 \end{array} \right) \epsilon_{(n-2\beta-1,2\beta+3)}w^7 \]
\[ + (1 + w + w^3)n^4\epsilon_{(n-2\beta-3,2\beta+5)}(1 + w)^{n-3}w^2 \]
\[ + (1 + w + w^3)n\left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \epsilon_{(n-2\beta-1,2\beta+3)}(1 + w)^{n-3}w^4 \]
\[ + \left( \begin{array}{c} n+1 \\ 4 \end{array} \right) \epsilon_{(n-2\beta+1,2\beta+1)}(1 + w)w^6 \]
\[ + (1 + w + w^3 + w^7)n^2\epsilon_{(n-2\beta-2,2\beta+4)}(1 + w)^{n-2} \]
\[ + (1 + w + w^3)n(n + 1)\epsilon_{(n-2\beta-1,2\beta+3)}(1 + w)^{n-2}w \]
\[ + (1 + w + w^3)n^6\left( \begin{array}{c} n \\ 2 \end{array} \right) \epsilon_{(n-2\beta,2\beta+2)}(1 + w)^{n-2}w^2 \]
\[ + (1 + w + w^3)n^4\left( \begin{array}{c} n+1 \\ 3 \end{array} \right) \epsilon_{(n-2\beta+1,2\beta+1)}(1 + w)^{n-2}w^3 \]
\[ + (1 + w + w^3)n^2\left( \begin{array}{c} n+2 \\ 4 \end{array} \right) \epsilon_{(n-2\beta+2,2\beta)}(1 + w)^{n-2}w^4 \]
\[ + (1 + w + w^3)n^2\epsilon_{(n-2\beta-2,2\beta+3,1)}(1 + w)^{n-2}w^4 \]
\[ + (1 + w)^2n+2(n + 1)\epsilon_{(n-2^{g-1},2^{g+2},1)}w^5 \]
\[ + \binom{n}{2} \epsilon_{(n-2^{g+2},1)}w^6 \]
\[ + \binom{n+1}{3} \epsilon_{(n-2^{g+2},1)}w^7 \]
\[ + n\epsilon_{(n-2^{g-2},2^{g+3},1)}w^7 \]

**Proposition 8.8**

If \( n \geq 5 \) then for all \( j \)

\[ \epsilon_{(n+1-j,1,j,1)} = \epsilon_{(n+2-j,1,1)} \]

if \( n \) is even and

\[ 0 \equiv \epsilon_{(n+1-j,1)} + \epsilon_{(n+2-j,1)} + \epsilon_{(n+2-j,1)} \pmod{2} \]

if \( n \) is odd.

**Proof**

In Proposition ?? when \( k = 3 \) we have \( M_3 = 1 + y_3 + y_3^5 + \ldots \) so that the terms in \( y_3^g \) with \( g = 0, 1 \) come only from terms in the congruence with \( i_4 = 0 = i_5 \). From these terms we obtain a relation in \( \mathbb{Z}/2[y_3]/(y_3^2) \) of the form

\[ \epsilon_{(n+2-j,1)} \equiv \epsilon_{(n+2-j,1)}(1 + y_3)^n + \epsilon_{(n+1-j,1)}y_3 + \epsilon_{(n+2-j,1)}y_3. \]

The result follows by considering the terms of degree one in \( y_3 \). \( \Box \)

**8.9** If \( n \) is even then Proposition ?? and induction on \( j \) (c.f. ??) shows that \( \epsilon_{(n+1-j,1)} = 0 \) for all \( j \). If \( n \equiv 2 \pmod{4} \) Lemma ?? implies that \( \epsilon_{(n-2^{g+3},2^{g+2}+2)} = 0 \).

However, from the second congruence of ?? we obtain

\[ 0 \equiv \binom{n+2+12t}{2} \epsilon_{(n+2-4t,4t)} + (4t + 1)n\epsilon_{(n+1-4t,4t+1)} \]
\[ + \binom{4t+2}{4t} \epsilon_{(n-4t,4t+2)} + \epsilon_{(n+4t+1,1)} \]

modulo 2, which implies for all \( t \) that \( \epsilon_{(n-4t,4t+2)} = 0 \) if \( n \equiv 2 \pmod{4} \) and \( \epsilon_{(n-4t,4t+2)} = \epsilon_{(n+2-4t,4t)} \) if \( n \equiv 0 \pmod{4} \).
Proposition 8.10

If \( n \equiv 0 \pmod{4} \) then

\[
0 \equiv \epsilon_{(n-2^\beta-2,2^\beta+4)} + \binom{n+2}{4} \epsilon_{(n-2^\beta+2,2^\beta)} \pmod{2}.
\]

Proof

If \( n \equiv 0 \pmod{4} \) then \( \epsilon_{(n+1-j,j,1)} = 0 \) for all \( j \) and \( \epsilon_{(n+2-j,j)} = 0 \) for all odd \( j \). Also \( \epsilon_{(n-2^\beta-2,2^\beta+4)} = \epsilon_{(n-2^\beta-4,2^\beta+6)} \). Substituting these identities into the congruence obtained when \( j = 2^\beta + 4 \) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):

\[
\epsilon_{(n-2^\beta-2,2^\beta+4)} = (1 + w^4)\epsilon_{(n-2^\beta-2,2^\beta+4)}
\]

\[
+ \binom{n+2}{4} \epsilon_{(n-2^\beta+2,2^\beta)} w^4
\]

so that

\[
0 \equiv \epsilon_{(n-2^\beta-2,2^\beta+4)} + \binom{n+2}{4} \epsilon_{(n-2^\beta+2,2^\beta)} \pmod{2}
\]

as required. \( \square \)

Corollary 8.11

If \( n \equiv 0 \pmod{4} \) then \( \epsilon_{(n-6,8)} = 0 \).

Proof

Put \( \beta = 2 \) in Proposition ???. \( \square \)

8.12 Suppose that \( n \equiv 0 \pmod{4} \). Then \( \epsilon_{(n+1-j,j,1)} = 0 \) for all \( j \) and \( \epsilon_{(n+2-j,j)} = 0 \) for all odd \( j \) and \( \epsilon_{(n-4t,4t+2)} = \epsilon_{(n+2-4t,4t)} \) for all \( t \). Substituting these identities into the congruence of ?? when \( j = 4t+4 \) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):
\[\begin{align*}
\epsilon_{(n-4t-2,4t+4)} & \\
\equiv & (1 + w + w^3)^{n+6} \epsilon_{(n-4t-2,4t+4)} (1 + w)^{n-4t-4}w^4 \\
+ & (1 + w + w^3 + w^7)^{n+2} \epsilon_{(n-4t-2,4t+4)} (1 + w)^{n+2-4t-4} w^4 \\
+ & (1 + w + w^3)^{n-6} \binom{n-4t+2}{4} \epsilon_{(n-4t+2,4t)} (1 + w)^{n+2-4t-4}w^4 \\
\equiv & \epsilon_{(n-4t-2,4t+4)} (w^4 + ((n/2) + 1)w^6) \\
+ & (1 + w^2 + w^3 + w^4 + w^7)^{n+2} \epsilon_{(n-4t-2,4t+4)} (1 + (t+1)w^4) \\
+ & (1 + w^2 + w^3 + w^4)^{n+2}w^4 \binom{n-4t+2}{4} \epsilon_{(n-4t+2,4t)} \\
\equiv & \epsilon_{(n-4t-2,4t+4)} (w^4 + w^6) \\
+ & (1 + tw^4 + w^6) \epsilon_{(n-4t-2,4t+4)} \\
+ & w^4 \binom{n-4t+2}{4} \epsilon_{(n-4t+2,4t)} \\
\equiv & \epsilon_{(n-4t-2,4t+4)} (1 + (t+1)w^4) \\
+ & w^4 \binom{n-4t+2}{4} \epsilon_{(n-4t+2,4t)}.
\end{align*}\]

Therefore

\[(t+1) \epsilon_{(n-4t-2,4t+4)} \equiv \binom{n-4t+2}{4} \epsilon_{(n-4t+2,4t)} \pmod{2}.\]

Setting \(t = 2k\) proves the following result:

**Lemma 8.13**

Suppose that \(n \equiv 0 \pmod{4}\).

Then, modulo 2,

\[\epsilon_{(n-8k-2,8k+4)} \equiv \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{8}, \\
\epsilon_{(n-8k+2,8k)} & \text{if } n \equiv 4 \pmod{8}.
\end{cases}\]
\textbf{8.14} Suppose that $n \equiv 2$ (modulo 4). Then $\epsilon_{(n+1-j,j,1)} = 0$ for all $j$ and $\epsilon_{(n+2-j,j)} = 0$ for all odd $j$ and $\epsilon_{(n-4t,4t+2)} = 0$ for all $t$. Substituting these identities into the congruence of §2 when $j = 4t$ we obtain the following relation in $\mathbb{Z}/2[w]/(w^8)$:

$$
\epsilon_{(n+2-4t,4t)}
\equiv (1 + w + w^3 + w^7) \epsilon_{(n+4t+2,4t)} (1 + w)^{n+2-4t}
$$

$$
+ (1 + w + w^3)^{n-6} \binom{n-4t+6}{4} \epsilon_{(n-4t+6,4t-4)} (1 + w)^{n+2-4t} w^4
$$

$$
\equiv \epsilon_{(n-4t+2,4t)} (1 + tw^4)
$$

$$
+ \binom{n-4t+6}{4} \epsilon_{(n-4t+6,4t-4)} w^4.
$$

Therefore

$$
t \epsilon_{(n-4t+2,4t)} \equiv \binom{n-4t+6}{4} \epsilon_{(n-4t+6,4t-4)} \pmod{2}.
$$

Setting $t = 2k + 1$ proves the following result:

\textbf{Lemma 8.15}

\textit{Suppose that $n \equiv 2$ (modulo 4).}

\textit{Then, modulo 2,}

$$
\epsilon_{(n-8k-2,8k+4)} \equiv \begin{cases} 
0 & \text{if } n \equiv 6 \pmod{8}, \\
\epsilon_{(n-8k+2,8k)} & \text{if } n \equiv 2 \pmod{8}.
\end{cases}
$$

\textbf{8.16} Suppose that $n \equiv 1$ (modulo 4). Then $\epsilon_{(n+2-2j,2j)} = \epsilon_{(n+1-2j,2j+1)}$ for all $j$ and $\epsilon_{(n+1-2^\beta,2^\beta,1)} = \epsilon_{(n+2-2^\beta,2^\beta)} = \epsilon_{(2^\beta,2^\beta+2)}$ and $\epsilon_{(n-2^\beta,2^\beta+1,1)} = 0$. Substituting these identities into the congruence of §2 when $j = 2^\beta + 1$ (with $\beta \geq 3$) we obtain the following relation in $\mathbb{Z}/2[w]/(w^8)$:
\[ \epsilon_{(n+1-2^\beta,2^\beta+1)} \]
\[ \equiv (1 + w + w^3)^e \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w^4)^{(n-1)/4} w^4 \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{2n+5} w^5 \]
\[ + (1 + w + w^3 + w^7)^n e_{(n-2^\beta+2,2^\beta)} (1 + w)^{n+1} \]
\[ + (1 + w + w^3)^n+2 e_{(n-2^\beta+2,2^\beta)} (1 + w)^{n+1} w \]
\[ + (1 + w + w^3)^n e_{(n-2^\beta+2,2^\beta)} (1 + w)^{n+1} w^4 \]
\[ \equiv (w^4 + w^5 + w^7) e_{(n-2^\beta+2,2^\beta)} \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (w^5 + w^6 + w^7) \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w)^{n+1} ((1 + w + w^3)^n+4 + w^7) \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (w(1 + w)^{2n+3} + w^4 (1 + w)^{2n+2} + w^7 (1 + w)^{2n+1}) \]
\[ + (1 + w + w^3) e_{(n-2^\beta+2,2^\beta)} (1 + w^2) w^4 \]
\[ \equiv (w^4 + w^6 + w^7) e_{(n-2^\beta+2,2^\beta)} \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^3 + w^5 + w^7) \]
\[ + \epsilon_{(n-2^\beta+2,2^\beta)} (w + w^2 + w^5 + w^6 + w^4 + w^7) \]
\[ + (w^2 + w^3 + w^5 + w^4 + w^5 + w^7) e_{(n-2^\beta+2,2^\beta)} \]
\[ \equiv e_{(n-2^\beta+2,2^\beta)} (1 + w^2 + w^3 + w^5 + w^6 + w^7). \]

Hence we have show that
\[ 0 \equiv e_{(n-2^\beta+2,2^\beta)} (w^2 + w^3 + w^5 + w^6 + w^7) \in \mathbb{Z}/2[w]/(w^8) \]

which implies that \( e_{(n-2^\beta+2,2^\beta)} = 0 \) and proves the following result:

**Proposition 8.17**

*Theorem ?? is true when \( n \equiv 1 \pmod{4} \).*

**8.18** Suppose that \( n \equiv 3 \pmod{4} \). Then \( e_{(n+2-2j,2j)} = e_{(n+1-2j,2j+1)} \) for all \( j, \) \( e_{(n+1-2j,2j,1)} = e_{(n+2-2j,2j)} \) and \( e_{(n-2j,2j+2)} = e_{(n-2j,2j+1,1)} = 0. \) Substituting these identities into the congruence of ?? (as simplified in ??) when \( j = 2^\beta + 3 \) (with \( \beta \geq 3 \)) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):
\[0 \equiv (1 + w + w^3)^{n+4} \epsilon_{(n-2^3-1,2^3+2,1)} (1 + w)^{n-1} w^4 + (1 + w)^{2n} \epsilon_{(n-2^3-2,2^3+3,1)} w^6 \]
\[\equiv (w^4 + w^5 + w^7) \epsilon_{(n-2^3-1,2^3+2,1)} + \epsilon_{(n-2^3-2,2^3+3,1)} w^6.\]

Therefore
\[\epsilon_{(n-2^3-1,2^3+2,1)} = 0 = \epsilon_{(n-2^3-2,2^3+3,1)}.
\]

Substituting the previous identities plus these new ones into the congruence of $\S 7$ (as simplified in $\S 7$) when $j = 2^3 + 4$ (with $\beta \geq 3$) we obtain the following
relation in $\mathbb{Z}/2[w]/(w^8)$:

$$
\epsilon_{(n-2^3,2^3+4)} \\
\equiv \epsilon_{(n-2^3-4,2^3+6)}(w^6 + w^7) \\
+ (1 + w + w^3)^{n+2}\epsilon_{(n-2^3-4,2^3+6)}(1 + w)^n w^4 \\
+ (1 + w + w^3)^{n+4}\epsilon_{(n-2^3-2,2^3+4)}(1 + w)^n w^2 \\
+ \left( \begin{array}{c}
  n + 1 \\
  4 \\
\end{array} \right) \epsilon_{(n-2^3+2,2^3)}(w^6 + w^7) \\
+ (1 + w + w^3 + w^7)^{n+2}\epsilon_{(n-2^3-2,2^3+4)}(1 + w)^n w^2 \\
+ (1 + w + w^3)^{n+2} \left( \begin{array}{c}
  n + 2 \\
  4 \\
\end{array} \right) \epsilon_{(n-2^3+2,2^3)}(1 + w)^n w^4 \\
\equiv \epsilon_{(n-2^3-4,2^3+6)}(w^6 + w^7) \\
+ (w^4 + w^7)\epsilon_{(n-2^3-4,2^3+6)} \\
+ (w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^5 + w^7)\epsilon_{(n-2^3-2,2^3+4)} \\
+ \left( \begin{array}{c}
  n + 1 \\
  4 \\
\end{array} \right) \epsilon_{(n-2^3+2,2^3)}(w^6 + w^7) \\
+ ((1 + w + w^3)^{n+2}(1 + w)^n w^2 + w^7)\epsilon_{(n-2^3-2,2^3+4)} \\
+ (w^4 + w^5 + w^7 + w^5 + w^6) \left( \begin{array}{c}
  n + 2 \\
  4 \\
\end{array} \right) \epsilon_{(n-2^3+2,2^3)} \\
\equiv \epsilon_{(n-2^3-4,2^3+6)}(w^4 + w^6) \\
+ (w^2 + w^3 + w^4 + w^6 + w^7)\epsilon_{(n-2^3-2,2^3+4)} \\
+ (1 + w^2 + w^4 + w^6 + w^3 + w^4 + w^7)\epsilon_{(n-2^3-2,2^3+4)} \\
+ \left( \begin{array}{c}
  n + 1 \\
  4 \\
\end{array} \right) \epsilon_{(n-2^3+2,2^3)}(w^6 + w^7)
Therefore we have proved the following result:

**Lemma 8.19**

*If* $n \equiv 3 \pmod{4}$ *then* $\epsilon_{(n-2^{\alpha} - 4, 2^{\alpha} + 6)} = 0$ *and*

$$\epsilon_{(n-2^{\alpha} - 3, 2^{\alpha} + 4, 1)} = \epsilon_{(n-2^{\alpha} - 2, 2^{\alpha} + 4)} \equiv \binom{n+2}{4} \epsilon_{(n-2^{\alpha} + 2, 2^{\alpha})} \pmod{2}.$$ 

**8.20** We continue to suppose that $n \equiv 3 \pmod{4}$. By Proposition ?? we have

$$0 \equiv \epsilon_{(n-2^{\alpha} - 4, 2^{\alpha} + 5, 1)} + \epsilon_{(n-2^{\alpha} - 3, 2^{\alpha} + 4, 1)} + \epsilon_{(n-2^{\alpha} - 2, 2^{\alpha} + 4)} \pmod{2}$$

and

$$0 \equiv \epsilon_{(n-2^{\alpha} - 3, 2^{\alpha} + 4, 1)} + \epsilon_{(n-2^{\alpha} - 2, 2^{\alpha} + 4)} \pmod{2}$$

so that $\epsilon_{(n-2^{\alpha} - 4, 2^{\alpha} + 5, 1)} = 0$.

Adding this to the previous congruences and setting $j = 2^{\alpha} + 5$ in §?? (with
\(\beta \geq 3\) we obtain the following relation in \(\mathbb{Z}/2[w]/(w^8)\):

\[
\epsilon_{(n-2^\beta-2^2\beta^2+4)} \equiv (1 + w + w^3 + w^7)^{n+4}\epsilon_{(n-2^\beta-2^2\beta^2+4)}(1 + w)^n - 3 \\
+ (1 + w + w^3)^{n+2}\epsilon_{(n-2^\beta-2^2\beta^2+4)}(1 + w)^n - 3w \\
+ (1 + w + w^3)^3 w^4 \binom{n+1}{4} \epsilon_{(n-2^\beta+2^2\beta)} \\
+ (w^5 + w^6) \binom{n+2}{5} \epsilon_{(n-2^\beta+2^2\beta)} \\
+ (1 + w + w^3)^3 \epsilon_{(n-2^\beta-3^2\beta^2+4,1)} w^4 \\
\equiv (1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^3 + w^5 + w^7 + w^7)\epsilon_{(n-2^\beta-2^2\beta^2+4)} \\
+ (w + w^2 + w^5 + w^6 + w^4)\epsilon_{(n-2^\beta-2^2\beta^2+4)} \\
+ (w^4 + w^5 + w^7 + w^6 + w^7) \binom{n+1}{4} \epsilon_{(n-2^\beta+2^2\beta)} \\
+ (w^5 + w^6) \binom{n+2}{5} \epsilon_{(n-2^\beta+2^2\beta)} \\
+ (w^4 + w^5 + w^7 + w^5 + w^6)\epsilon_{(n-2^\beta-3^2\beta^2+4,1)} \\
\equiv (1 + w^4 + w^5 + w^6)\epsilon_{(n-2^\beta-2^2\beta^2+4)} + w^4 \binom{n+1}{4} \epsilon_{(n-2^\beta+2^2\beta)}.
\]

This congruence proves the following result:

**Proposition 8.21**

*If \(n \equiv 3\) (modulo 4) then \(\epsilon_{(n-2^\beta-2^2\beta^2+4)} = 0\).*

*If \(n \equiv 3\) (modulo 8) then \(\epsilon_{(n-2^\beta+2^2\beta)} = 0\) so that Theorem ?? is true when \(n \equiv 3\) (modulo 8).*

8.22 Now assume that \(n \equiv 7\) (modulo 8) so that \(\epsilon_{(n+2^2-2^2\beta^2,2j)} = \epsilon_{(n+1-2^2\beta^2,2j+1)}\) for all \(j\), \(\epsilon_{(n+1-2^2\beta^2,1)} = \epsilon_{(n+2-2^2\beta^2,2)}\) and \(\epsilon_{(n-2^\beta,2^2\beta+2)} = \epsilon_{(n-2^\beta,2^2\beta+1,1)} = 0\). Also 

\[
\epsilon_{(n-2^\beta-4^2\beta^2+6)} = 0 = \epsilon_{(n-2^\beta-3^2\beta^2+4,1)} = \epsilon_{(n-2^\beta-4^2\beta^2+5,1)} = \epsilon_{(n-2^\beta-1^2\beta^2+2,1)} = \epsilon_{(n-2^\beta-2^2\beta^2+3,1)} = \epsilon_{(n-2^\beta-2^2\beta^2+2,1)}.
\]
Substituting these identities into the congruence of \( \S ?? \) when \( j = 2^\beta + 7 \) — trying \( j = 2^\beta + 6 \) yields nothing new — (with \( \beta \geq 3 \)) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):

\[
0 = \epsilon_{(n-2^\beta-5,2^\beta+7)} = \epsilon_{(n-2^\beta-6,2^\beta+7,1)} w^6.
\]

This congruence, together with Proposition \( ?? \), yields \( \epsilon_{(n-2^\beta-6,2^\beta+7,1)} = 0 = \epsilon_{(n-2^\beta-5,2^\beta+6,1)} \).

Substituting these identities into the congruence of \( \S ?? \) when \( j = 2^\beta + 8 \) (with \( \beta \geq 3 \)) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):

\[
\epsilon_{(n-2^\beta-6,2^\beta+8)}
\equiv \epsilon_{(n-8,2^\beta+10)} w^6
\]
\[
+(w^4 + w^5 + w^7) \epsilon_{(n-2^\beta-8,2^\beta+10)} (1 + w + w^2 + w^3)
\]
\[
+(w^2 + w^6) \epsilon_{(n-2^\beta-6,2^\beta+8)} + (1 + w + w^3 + w^7 + w + w^2 + w^4) \epsilon_{(n-2^\beta-6,2^\beta+8)}
\]
\[
\equiv (w^4 + w^6 + w^7) \epsilon_{(n-2^\beta-8,2^\beta+10)}
\]
\[
+(1 + w^3 + w^4 + w^6 + w^7) \epsilon_{(n-2^\beta-6,2^\beta+8)}.
\]

This congruence shows that \( \epsilon_{(n-2^\beta-6,2^\beta+8)} = 0 = \epsilon_{(n-2^\beta-8,2^\beta+10)} \).

Substituting these identities into the congruence of \( \S ?? \) when \( j = 2^\beta + 12 - j = 2^\beta + 9, j = 2^\beta + 10, 2^\beta + 11 \) yield nothing since we already know that \( \epsilon_{(n-2^\beta-7,2^\beta+8,1)} = 0 = \epsilon_{(n-2^\beta-8,2^\beta+9,1)} = \epsilon_{(n-2^\beta-8,2^\beta+10,1)} \) (with \( \beta \geq 3 \)) we obtain the following relation in \( \mathbb{Z}/2[w]/(w^8) \):

\[
\epsilon_{(n-2^\beta-10,2^\beta+12)}
\equiv \epsilon_{(n-2^\beta-12,2^\beta+14)} (w^6 + w^7)
\]
\[
+(1 + w + w^3) \epsilon_{(n-2^\beta-12,2^\beta+14)} (1 + w + w^2 + w^3) w^4
\]
\[
+(1 + w + w^3) \epsilon_{(n-2^\beta-10,2^\beta+12)} (w^2 + w^6)
\]
\[
+(1 + w + w^3 + w^7) \epsilon_{(n-2^\beta-10,2^\beta+12)} (1 + w)^5
\]
\[
\equiv \epsilon_{(n-2^\beta-12,2^\beta+14)} (w^4 + w^6)
\]
\[
+(1 + w^6 + w^4 + w^7) \epsilon_{(n-2^\beta-10,2^\beta+12)}.
\]

From this we see that \( \epsilon_{(n-2^\beta-12,2^\beta+14)} = 0 = \epsilon_{(n-2^\beta-10,2^\beta+12)} \). In fact, by induction this line of argument shows that when \( n \equiv 7 \pmod{8} \) \( \epsilon_{(n-\beta+2,\beta)} = 0 \) except when \( j = 2^\beta \) and that \( \epsilon_{(n-j+1,\beta,1)} = 0 \) except when \( j = 2^\beta \) in which case \( \epsilon_{(n-2^\beta+1,2^\beta+1)} = \epsilon_{(n-2^\beta+1,2^\beta+2)} \)
8.23 Completion of the proof of Theorem ?? when \( n \) is odd

If \( n \equiv 7 \) (modulo 8) then \( \epsilon_{(n+2−2^j,2^j)} = \epsilon_{(n+1−2^j,2^j,1)} \) are the only possibly non-zero coefficients of length two or three. By Remark ??, in \( \mathbb{Z}/2[w]/(w^{2n−3n−6}) \), we obtain for \( k \geq 3 \)

\[
\epsilon_{(n+2−2^j,2^j)} = \sum_j \epsilon_{(n+2−j,j)} = \sum_{(i_1,i_2,\ldots,i_k+1,i_{k+2}),i_1+i_2+i_{k+1}+i_{k+2}=n+2} \epsilon_{(i_1,i_2,0,\ldots,0,i_{k+1},i_{k+2})} (w^{2k−1})^{2i_{k+1}+4i_{k+2}} (1 + w^{2k−1} + w^{2k−1} + \ldots)^{2^j−2^k+n+2+2i_2+(2^j−2)i_{k+1}+(2^j−2)i_{k+2}} = \epsilon_{(n+2−2^j,2^j)} (1 + +w^{2k−1} + O(w^{2k−1})) + O((w^{2k−1})^2).
\]

Considering the coefficient of \( w^{2k−1} \) shows that \( \epsilon_{(n+2−2^j,2^j)} = 0 \) when \( n \equiv 7 \) (modulo 8).

The completion of the induction proof of Theorem ?? when \( n \equiv 1 \) (modulo 4) or \( n \equiv 3 \) (modulo 8) was accomplished in Propositions ?? and ??, respectively.

\[\square\]

References


