The sectional category of spherical fibrations

Don Stanley
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Abstract

We give homological conditions which determine sectional category, secat, for rational spherical fibrations. In the odd dimensional case the secat is the least power of the Euler class which is trivial. In the even dimensional case secat is one when a certain homology class in twice the dimension of the sphere is $-1$ times a square. Otherwise secat is two. We apply our results to construct a fibration $p$ such that $\text{secat}(p) = 2$ and genus($p$) = $\infty$. We also observe that secat, unlike cat, can decrease in a field extension of $\mathbb{Q}$.

1 Introduction

The study of sectional category, or secat, goes back at least to Krasnosel’skii [8] and Yang [14] who studied genus and B-index respectively. These can be considered as special cases of secat. Still the main reference for secat is the paper of Švarc [11] (who also used the term genus). Let $p : E \to B$ be a fibration. Then secat($p$) is the least number of open subsets of $B$ over which $p$ has a section that it takes to cover $B$. If $E \simeq \ast$ then $\text{secat}(p) = \text{cat}(B) + 1$ so we see that secat is a generalization of LS category. secat also has many other applications which include critical point theory and embedding theory (see [11]).

This paper is concerned with the secat of fibrations with fibre a sphere. The secat of such fibrations has been previously studied [11]. By restricting ourselves to the rational case we can completely solve the problem of determining secat.
We work in the category of spaces having the homotopy type of a CW-complex [9]. For any map \( p \) we will always let \( \text{Fib}(p) \) denote the homotopy fibre of \( p \). We give another definition of secat. They were shown to be equivalent by Švarc [11].

**Definition 1.1** Let \( p(n) : \ast_B^n E \to B \) be the \( n \)-fold fibrewise join of \( p \) with itself (see [7]). If \( B \) is a point we denote \( \ast^n_B E \) by \( \ast^n E \). Then \( \text{sec}(p) \leq n \) if and only if \( p(n) \) has a section.

The fact that this definition is equivalent to the open set one follows from [7] together with two facts. The fact that CW-complexes have the homotopy type of paracompact spaces and the fact that secat of a fibration induced by a homotopy equivalence is the same as secat of the original fibration. Observe that our definition of \( \ast^n E \) coincides with the usual one. We state a few facts about the \( n \)-fold fibrewise join.

**Proposition 1.2**

1) \( p(n) \) is natural in \( p \). In other words a diagram

\[
\begin{array}{ccc}
E & \longrightarrow & E' \\
p \downarrow & & \downarrow p' \\
B & \longrightarrow & B'.
\end{array}
\]

gives us a diagram

\[
\begin{array}{ccc}
\ast_B^n E & \longrightarrow & \ast_B^n E' \\
p(n) \downarrow & & \downarrow p'(n) \\
B & \longrightarrow & B'.
\end{array}
\]

If the first diagram is a pullback then so is the second.

2) \( \text{Fib}p(n) \simeq \ast^n \text{Fib}p \)
Proof: 1) is proved by Doeraene [2]. 2) follows from 1) when we let \( E \simeq \text{Fib}p \) and \( B \simeq * \).

We will use the following fact about \text{secat}. It is due to Švarc.

**Theorem 1.3** Let \( p : E \to B \) be a fibration. Assume \((\ker H^*(p))^r \leq 0\). Then \( \text{secat} \ p \geq r \).

**Proof:** See [11] or [7].

We also need to use the Lusternik–Schnirelmann category of a map.

**Definition 1.4** Let \( f : X \to Y \) be a map. Then

\[
\text{cat}(f) = \min \{|\{U_i \subset X| \cup U_i = X, \ f|u_i \simeq *\}| - 1.
\]

We define \( \text{cat}(X) = \text{cat}(\text{id}_X) \). The following facts about \( \text{cat}(f) \) are easy to prove.

**Proposition 1.5** Let \( X, Y, Z \) be spaces and \( f', f : X \to Y, \ g : Y \to Z \) be maps. Then \( \text{cat}(gf) \leq \text{cat}(Y) \). If there exists a cell decomposition of \( X \) with \( r \) cells then \( \text{cat}(X) \leq r - 1 \). If \( f \simeq f' \) then \( \text{cat}(f) = \text{cat}(f') \).

**Proof:** See [7].

### 2 Odd dimensional fibre

Let \( F \xrightarrow{i} E \xrightarrow{p} B \) be a fibration sequence such that \( E \) and \( B \) are simply connected and of finite type. Assume \( F \simeq S^{2n+1} \). We work over the rationals. This means that all spaces and cohomology are rational. In this section we show that \( \text{secat}(p) \) is the smallest \( r \) such that \( \ker(H^*(p))^r = 0 \) (Corollary 2.4).

Notice that \( F \) is a \( K(\mathbb{Q}, 2n+1) \). So it follows from the theory of relative Postnikov systems [13] that \( p \) is a principle fibration. (This can also be seen by looking at Sullivan models.) So \( p \) is the inclusion of the fibre of a fibration

\[
f : B \to K(\mathbb{Q}, 2n+2).
\]
In this situation we have the following theorem of Schwartz.

**Theorem 2.1** \( \text{secat}(p) = \text{cat}(f) + 1 \).

**Proof:** See [11] Theorem 19” or [7].

**Definition 2.2** \( f : B \rightarrow K(\mathbb{Q}, 2n + 2) \) corresponds to \( \alpha \in H^{2n+2}(B) \). We call \( \alpha \) the Euler class of \( p \). Since \( B \) is simply connected this is the same as the usual Euler class.

**Theorem 2.3** Let \( \alpha \) be the Euler class of \( p \). Let \( r \) be the least integer such that \( r = 0 \). Then \( \text{secat}(p) = r \).

**Proof:** Assume \( r = 0 \). Then \( f \) factors through the fibre \( G \) of the map

\[
K(\mathbb{Q}, 2n + 2) \rightarrow K(\mathbb{Q}, (2n + 2)r)
\]

which represents \( \alpha^r \in H^{(2n+2)r}(K(\mathbb{Q}, 2n + 2)) \). Of course \( G \) is just the \((2n + 2)(r - 1)\) skeleton of \( K(\mathbb{Q}, 2n + 2) \) and so \( \text{cat}(G) \leq r - 1 \). (To see \( \text{cat}(G) \leq r - 1 \) calculate that \( H^*(G) \cong P(a)/(a^r) \). So \( G \) can be built with \( r \) cells and so \( \text{cat}(G) \leq r - 1 \) by Proposition 1.5.) Therefore by Proposition 1.5 \( \text{cat}(f) \leq \text{cat}(G) \leq r - 1 \). So \( \text{secat}(p) \leq r \). But \( \alpha^{r-1} \neq 0 \) so by Theorem 1.3 \( \text{secat}(p) \geq r \). We conclude that \( \text{secat}(p) = r \). \( \square \)

Another way to phrase the result of the theorem is:

**Corollary 2.4** \( \text{secat}(p) \) is the smallest \( r \) such that \( (\ker H^*(p))^r = 0 \).

**Proof:** Assume \( (\ker H^*(p))^r = 0 \) but \( (\ker H^*(p))^{r-1} \neq 0 \). Then by Theorem 1.3 \( \text{secat}(p) \geq r \). But the \( r \)-th power of the Euler class is trivial so \( \text{secat}(p) \leq r \). \( \square \)

### 3 Even dimensional fibre

Again let \( F \xrightarrow{i} E \xrightarrow{p} B \) be a fibration sequence such that \( E \) and \( B \) are simply connected and of finite type. Assume \( F \simeq S^{2n} \). Again we work over the rationals. This time we need to make use of Sullivan models. For information on Sullivan models see [5], [12] or [10]. \((\Lambda V, d)\) denotes the free
commutative differential graded algebra on a graded vector space $V$ with differential $d$. For a graded set $(a(1), \ldots, a(n))$, $(\Lambda(a(1), \ldots, a(n)), d)$ denotes $(\Lambda V, d)$ where $V$ is the vector space with basis $(a(1), \ldots, a(n))$. For vector spaces $V$ and $W$, $(\Lambda V \otimes \Lambda W, d)$ is the same as $(\Lambda(V \oplus W), d)$ with the added assumption that $d(V) \subset \Lambda V$. For convenience we will use the same notation for a map and a model of the map.

**Lemma 3.1** There exists a model for $E$ of the form

$$(\Lambda V \otimes \Lambda(a, b), d)$$

such that $da = 0$ and $db = a^2 + \alpha$ for some cycle $\alpha \in \Lambda V$. Also the inclusion

$$p : (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda(a, b), d)$$

models the map $p$.

**Proof:** $(\Lambda(a, b), d)$ with $db = a^2$ and $da = 0$ is a model of $F$. Therefore there exists a model $(\Lambda V \otimes \Lambda(a, b), d)$ for $E$. First we show that we can assume $da = 0$. There exist $\delta, \gamma \in \Lambda V$ such that

1. $db = a^2 + \gamma a + \delta$

So

2. $0 = db = 2 ada + ad \gamma + \text{terms without } a$

Therefore $da = -\frac{1}{2}d \gamma$. So by changing basis in a way compatible with $p$ we can assume $da = 0$. In the new basis there still exists $\delta, \gamma$ so that (1) holds. Now (2) implies $d \gamma = 0$. So again we can change basis compatibly with $p$ so that $db = a^2 + \delta$ for some $\delta \in \Lambda V$. \qed

**Lemma 3.2** $\text{secat}(p) \leq 2$.

**Proof:** We have a commutative diagram

$$
\begin{array}{ccc}
(\Lambda V, d) & \xrightarrow{p'} & (\Lambda V \otimes \Lambda(a), d) \\
\downarrow & & \downarrow \text{include} \\
(\Lambda V, d) & \xrightarrow{p} & (\Lambda V \otimes \Lambda(a, b), d).
\end{array}
$$

5
Since \( da = 0 \) secat\((p') = 1 \). Taking fibrewise joins we get a commutative diagram

\[
\begin{array}{ccc}
(V,d) & \xrightarrow{p'(2)} & (V \otimes \Lambda(a',\ldots),d) \\
\downarrow & & \downarrow h \\
(V,d) & \xrightarrow{p(2)} & (V \otimes \Lambda(a'),d).
\end{array}
\]

Remember that \( \text{Fib } p \simeq S^{2n} \). Therefore \( \text{Fib } p(2) \simeq S^{4n+1} \). Also the induced map \( \text{Fib } (p') \rightarrow \text{Fib}(p) \) is an \( H_{2n} \) isomorphism. It follows that \( h(a') = a' \). Since \( da' = 0 \) and there is an \( r : (V \otimes \Lambda(a'),d) \rightarrow (V,d) \) such that \( rp(2) = id \). Therefore \( da' = 0 \) and there is an \( r : (V \otimes \Lambda(a'),d) \rightarrow (V,d) \) such that \( rp(2) = id \). So secat\((p) \leq 2 \).

**Theorem 3.3** Let \( p : E \rightarrow B \) be a fibration with fibre \( S^{2n} \). Let \( p : (AV,d) \rightarrow (AV \otimes \Lambda(a,b),d) \) be a model of \( p \). Let \( db = a^2 + \alpha \), where \( \alpha \) is a cycle in \( AV \). Then secat\((p) = 1 \) if \( \alpha \) represents \(-r^2\) for some \( r \in H^*B \). Otherwise secat\((p) = 2 \).

**Proof:** The statement about the differential of \( b \) is just Lemma 3.1. Let \( r \in H^*B \) and assume that \( \alpha \) represents \(-r^2\). Let \( \beta \in AV \) represent \( r \). Then there exists \( \gamma \) in \( AV \) such that \( d\gamma = \beta^2 + \alpha \). We define \( r : (AV \otimes \Lambda(a,b),d) \rightarrow (AV,d) \) by

\[
\begin{align*}
r|_{AV} &= \text{id} \\
r(a) &= \beta \\
r(b) &= \gamma
\end{align*}
\]

Clearly \( rp = id \). So secat\((p) = 1 \).

Now assume secat\((p) = 1 \) Then there is a map

\[
r : (AV \otimes \Lambda(a,b),d) \rightarrow (AV,d)
\]

such that \( rp = id \).

Let \( r(a) = \beta \) and \( r(b) = \gamma \). Notice \( d\beta = 0 \). We also have the equation in \( AV \), \( \beta^2 + \alpha = rdb = drb = d\gamma \). Therefore \( \beta^2 + \alpha \simeq 0 \) and so \(-\beta^2 \simeq \alpha \). In other words \( \alpha \) represents \(-r^2\) for \( [\beta] = r \in H^*B \). The last statement of the theorem then follows easily from Lemma 3.2. \( \qed \)
Example: Consider the map

\[ p : (\Lambda(c), d) \rightarrow (\Lambda(c, a, b), d) \]

where \( da = 0 \) and \( db = a^2 + \alpha c^2 \). Then \( \text{secat}(p) = 1 \) if and only if \( \alpha = -k^2 \) for some \( k \in \mathbb{Q} \). This example makes it clear that \( \text{secat}(p) \) can decrease in a field extension. This is in contrast to the situation for LS category where Hess [6] showed that \( \text{cat} \) is independent of field extension of \( \mathbb{Q} \).

4 An application to genus.

In this section we apply the result of the last section together with a result of Gatsinzi [4] to construct maps \( f \) with \( \text{secat}(f) = 2 \) and \( \text{genus}(f) = \infty \). We first define genus and give a characterization of it in terms of the cat of the classifying map of the fibration. Let \( p : E \rightarrow B \) be fibration.

Definition:
\[
\text{genus}(p) = \min \{|U_i \subseteq B| \cup U_i = B \text{ and } p|_{U_i} \text{ is equivalent to a product fibration}| \}
\]

It is clear from the definition that \( \text{secat}(p) \leq \text{genus}(p) \).

We can give another characterization of genus which generalizes [11] Theorem 19” to non–principal fibrations. The method of proof is the same.

**Theorem 4.1** Let \( F \) denote the fibre of \( p \) and \( \phi : B \rightarrow \text{Baut } F \) its classifying map. Then \( \text{genus}(p) = \text{cat}(\phi) + 1. \)

**Proof:** It follows from [1] that \( \phi|_{U_i} : U_i \rightarrow \text{Baut } F \) is inessential if and only if \( p|_{U_i} \) is equivalent to a product fibration. The result then follows easily.

For the rest of this section we work over the rationals.

**Theorem 4.2**[4] There are maps
\[
f : K(Q, 4n) \rightarrow \text{Baut } S^{2n}
\]
that are non–trivial on \( \pi_* \)
Proof: This is a special case of a theorem in [4]. There it is shown that there is such a map for any Gottlieb element of any coformal space. \qed

Corollary 4.3 The fibrations $P$ corresponding to $f$ has genus($p$) = $\infty$.

Proof: The mapping theorem [3] implies that cat($f$) = $\infty$. (non–triviality implies injectivity in this case.) Theorem 4.1 then implies genus($p$) = $\infty$.

Theorem 4.4 The fibrations corresponding to non-trivial maps $f$ of Theorem 4.2 are represented by KS extensions of the form

$$
\Lambda(c) \rightarrow \Lambda(a, b, c)
$$

where $da = dc = 0$, $db = a^2 + \alpha c$ and $\alpha \in \mathbb{Q}^\times$. So secat($B$) = 2.

Proof: From Lemma 3.1 and for dimension reasons the fibration must be of the form stated in the theorem for some $\alpha \in \mathbb{Q}$. $\alpha \neq 0$ since otherwise the fibration would be trivial. That secat($p$) $\geq$ 2 follows directly from Theorem 3.3.

Remark: For every $r > 1$, consider the fibration $p$ represented by the KS extension

$$
\Lambda(c, e) \rightarrow \Lambda(c, e, a, b)
$$

where $de = c^r$, $dc = da = 0$ and $db = a^2 + \alpha c$, $\alpha \in \mathbb{Q}'$. It is not hard to see that secat($p$) = 2, genus($p$) = $r$.

References


