MULTICURVES AND EQUIVARIANT COBORDISM

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Abstract. Let A be a finite abelian group. We set up an algebraic framework for studying A-equivariant complex-orientable cohomology theories in terms of a suitable kind of equivariant formal groups. We compute the equivariant cohomology of many spaces in these terms, including projective bundles (and associated Gysin maps), Thom spaces, and infinite Grassmannians.

1. Introduction

Let A be a finite abelian group. In this paper, we set up an algebraic framework for studying A-equivariant complex-orientable cohomology theories in terms of a suitable kind of formal groups. In part, this is a geometric reformulation of earlier work of Cole, Greenlees, Kriz and others on equivariant formal group laws [3, 4, 5, 10]. However, the theory of divisors, residues and duality for multicurves is new, and forms a substantial part of the present paper. Although we focus on the finite case, many results can be generalised to compact abelian Lie groups. On the other hand, we have evidence that nonabelian groups will need a completely different theory.

We now briefly recall the nonequivariant theory, using the language of formal schemes developed in [15]. Let E be an even periodic cohomology theory, and put $S = \text{spec}(E^0)$ and $C = \text{spf}(E^0\mathbb{C}P^\infty_+) = \varinjlim_n \text{spec}(E^0\mathbb{C}P^n_+)$. The basic facts are

(a) C is a formal group scheme over S.
(b) If we forget the group structure, then C is isomorphic to the formal affine line $\widehat{\mathbb{A}}^1_S$ as a formal scheme over S; in other words, C is a formal curve over S.
(c) For many interesting spaces X, the formal scheme $\text{spf}(E^0X)$ has a natural description as a functor of C; for example, we have $\text{spf}(E^0BU(d)) = C^d/\Sigma_d = \text{Div}_d^+(C)$, the formal scheme of effective divisors of degree d on C.

Now let $\mathcal{U} = \mathcal{U}_A$ be a complete A-universe, and let $\mathcal{S}_A$ be the category of A-spectra indexed on $\mathcal{U}$ (as in [11]). Consider an A-equivariant commutative ring spectrum $E \in \mathcal{S}_A$ that is periodic and orientable in a sense to be made precise later. In this context, the right analogue of $\mathbb{C}P^\infty$ is the projective space $P\mathcal{U}$. This has an evident A-action. We put $S = \text{spec}(E^0)$ and $C = \text{spf}(E^0P\mathcal{U})$. This is again a formal group scheme over S, but it is no longer a formal curve. This appears to create difficulties with (c) above, because we no longer have a good hold on C^d/Σ_d or a good theory of divisors on C.

Our first task is to define the notion of a *formal multicurve* over S, and to show that C is an example of this notion. Later we will develop an extensive theory of formal multicurves and their divisors, and show that many statements about generalized cohomology can be made equivariant by replacing curves with multicurves.

2. Multicurves

Definition 2.1. Let $X = \text{spf}(R)$ be a formal scheme, and let Y be a subscheme of X. We say that Y is a *regular hypersurface* if $Y = \text{spf}(R/J)$ for some ideal $J = I_Y \leq R$ that is a free module of rank one over R. Equivalently, there should be a regular element $f \in R$ such that the vanishing locus $V(f) = \text{spf}(R/f)$ is precisely Y.

Let $S = \text{spec}(k)$ be an affine scheme.

Definition 2.2. A *formal multicurve* over S is a formal scheme C over S such that

(a) $C = \text{spf}(R)$ for some formal ring R

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(b) There exists a regular element $y \in R$ such that for all $k \geq 0$, the ideal Ry^k is open in R, and R/y^k is a finitely generated free module over $\mathcal{O}_S$, and $R = \varprojlim_k R/y^k$.
(c) The diagonal subscheme $\Delta \subset C \times_S C$ is a regular hypersurface.

A generator d for the ideal I_Δ will be called a *difference function* for C (because $d(a,b) = 0$ iff $(a,b) \in \Delta$ iff $a = b$). We will choose a difference function d, but as far as possible we will express our results in a form independent of this choice. An element y as in (b) will be called a *good parameter* on C.

Remark 2.3. If S is a formal scheme, then we can write $S = \varinjlim_\alpha S_\alpha$ for some filtered system of affine schemes, and formal schemes over S are the same as compatible systems of formal schemes over the S_α by [15, Proposition 4.27]. In the rest of this paper, we will generally work over an affine base but will silently use this result to transfer definitions and theorems to the case of a formal base where necessary.

The formal affine line $\widehat{\mathbb{A}}^1_S = \operatorname{spf}(k[\![x]\!])$ is a formal multicurve, and the category of formal multicurves is closed under disjoint union. Conversely, condition (c) implies that the module $\Omega^1_{C/S} = I_\Delta/I_\Delta^2$ is free of rank one over $R = \mathcal{O}_C$, so formal multicurves may be thought of as being smooth and one-dimensional. Similarly, if y is a good parameter then R is a finitely generated projective module over $k[\![y]\!]$, which means that C admits a finite flat map to $\widehat{\mathbb{A}}^1_S$, again indicating a one-dimensional situation. If k is an algebraically closed field, we shall see later that every small formal multicurve over S is a finite disjoint union of copies of $\widehat{\mathbb{A}}^1_S$.

Remark 2.4. Note that I_Δ is the kernel of the multiplication map $\mu\colon R\widehat{\otimes}R \to R$, which is split by the map $a \mapsto a \otimes 1$. It follows that I_Δ is topologically generated by elements of the form $a \otimes b - ab \otimes 1$. We also see by similar arguments that for any ideal $J \leq R$, the kernel of the multiplication map $(R/J) \otimes (R/J) \to R/J$ is just the image of I_Δ and thus is generated by d.

Definition 2.5. A *formal multicurve group* over S is a formal multicurve over S with a commutative group structure.

In the presence of a group structure, axiom (c) can be modified.

Definition 2.6. Let C be a commutative formal group scheme over S. A *coordinate* on C is a regular element $x \in \mathcal{O}_C$ whose vanishing locus is the zero-section. Clearly, such an x exists iff the zero-section is a regular hypersurface.

Remark 2.7. If x is a coordinate, then so is the function $\overline{x}$ defined by $\overline{x}(a) = x(-a)$.

Proposition 2.8. *Let C be a formal group scheme over S satisfying axioms (a) and (b) in Definition 2.2. Then C is a formal multicurve iff the zero-section $S \to C$ is a regular hypersurface. More precisely, if x is a coordinate on C, then the function $d(a,b) = x(b-a)$ defines a difference function, and if d is a difference function, then the function $x(b) = d(0,b)$ is a coordinate.*

The proof relies on the following basic lemma.

Lemma 2.9. *Let C be a formal multicurve, and let $f\colon X \to C$ be any map of schemes. Then the function $d'(x,b) = d(f(x),b)$ on $X \times_S C$ is regular in $\mathcal{O}_{X\times_S C}$.*

Proof. We have a short exact sequence as follows:

$$R\widehat{\otimes}R \xrightarrow{\times d} R\widehat{\otimes}R \xrightarrow{\mu} R.$$

We regard $R\widehat{\otimes}R$ as a module over R via the map $t \mapsto t \otimes 1$. The map μ is then R-linearly split by the map $t \mapsto t \otimes 1$, so the sequence remains exact after applying the functor $\mathcal{O}_X\widehat{\otimes}_R(-)$. The resulting sequence is just

$$\mathcal{O}_{X\times_S C} \xrightarrow{\times d'} \mathcal{O}_{X\times_S C} \to \mathcal{O}_X,$$

which proves the lemma. □

Corollary 2.10. *Let $C \xrightarrow{q} S$ be a formal multicurve, and let $S \xrightarrow{u} C$ be a section. Then the subscheme $uS \subset C$ is a regular hypersurface, and the ideal I_{uS} is generated by the function $d'(c) = d(u(q(c)),c)$, or equivalently*

$$d' = (C = S \times_S C \xrightarrow{u\times 1} C \times_S C \xrightarrow{d} \mathbb{A}^1).$$

Proof. Take $X = S$ in the lemma. □

Proof of Proposition 2.8. First suppose that the zero section is a regular hypersurface, so we can choose a coordinate x. It follows easily from axiom (b) that $R \simeq \prod_{k=0}^{\infty} \mathcal{O}_S$ as topological $\mathcal{O}_S$-modules, so $R\widehat{\otimes}R = \prod_{k=0}^{\infty} R$ as R-modules, so $1 \otimes x$ is a regular element in $R \otimes R$. If we regard x as a function on C, this says that the function $x_1 \colon (a,b) \mapsto x(b)$ is a regular element of $\mathcal{O}_{C \times_S C}$, whose vanishing locus is precisely the closed subscheme where $b = 0$. The map $s \colon (a,b) \mapsto (a, b-a)$ is an automorphism of $C \times_S C$, and $s^* x_1$ is the function $d(a,b) = x(b-a)$. As s is an automorphism, we see that d is regular and its vanishing locus is the subscheme where $a = b$, or in other words the diagonal.

The converse is the case $u = 0$ of Corollary 2.10. □

To formulate the definition of an equivariant formal group, we need some basic notions about divisors.

Definition 2.11. A *divisor* on C is a scheme of the form $D = \operatorname{spec}(\mathcal{O}_C/J)$, where J is an open ideal generated by a single regular element, and $\mathcal{O}_C/J$ is a finitely generated projective module over $\mathcal{O}_S$. Thus D is a regular hypersurface in C and is finite and very flat over S. Strictly speaking, we should refer to such subschemes as *effective divisors*, but we will have little need for more general divisors in this paper.

If $D_i = \operatorname{spf}(R/J_i)$ is a divisor for $i = 0, 1$ then we put $D_0 + D_1 := \operatorname{spf}(R/(J_0 J_1))$, which is easily seen to be another divisor.

The *degree* of D is the rank of $\mathcal{O}_D$ over k. Note that this need not be constant, but that S can be split as a finite disjoint union of pieces over which D has constant degree.

If T is a scheme over S, then a divisor on C over T means a divisor on the formal multicurve $T \times_S C$ over T.

Note that if D is an effective divisor of degree one, then the projection $D \xrightarrow{\pi} S$ is an isomorphism, so the map $S \xrightarrow{\pi^{-1}} D \subset C$ is a section of C. Conversely, if $u \colon S \to C$ is a section, then (by Corollary 2.10) the image uS is a divisor of degree one, which is conventionally denoted by $[u]$.

In the case of ordinary formal curves, it is well-known that there is a moduli scheme $\operatorname{Div}_d^+(C)$ for effective divisors of degree d on C, and that it can be identified with the symmetric power C^d/Σ_d. Analogous facts are true for multicurves, but much more difficult to prove. We will return to this in Section 14.

Let A be a finite abelian group (with the group operation written additively). We write A^* for the dual group $\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Definition 2.12. Let $X = \operatorname{spf}(R)$ be a formal scheme, and let $Y = \operatorname{spf}(R/J)$ be a closed formal subscheme. We say that X is a *formal neighbourhood* of Y if R is isomorphic to $\varprojlim_m R/J^m$ as a topological ring, or equivalently $X = \varinjlim_m \operatorname{spf}(R/J^m)$, which essentially means that every point in X is infinitesimally close to Y.

Definition 2.13. An *A-equivariant formal group* or *A-efg* over a scheme S is a formal multicurve group C over S, together with a homomorphism $\phi \colon A^* \to C$, such that C is the formal neighbourhood of the divisor

$$[\phi(A^*)] := \sum_{\alpha \in A^*} [\phi(\alpha)] \subset C.$$

Remark 2.14. The notation $\phi \colon A^* \to C$ really means that ϕ is a homomorphism from A^* to the group of sections of the projection $C \to S$. Equivalently, we have a group scheme

$$A^* \times S = \coprod_{\alpha \in A^*} S = \operatorname{spec}(\prod_{\alpha \in A^*} \mathcal{O}_S)$$

over S, and ϕ gives a homomorphism $A^* \times S \to C$ of group schemes over S.

Now choose a coordinate x on C, and put $d(a,b) = x(b-a)$. For any $\alpha \in A^*$ we have a function x_α on C defined by $x_\alpha(a) = x(a - \phi(\alpha)) = d(\phi(\alpha), a)$. More precisely, x_α is the composite

$$C = S \times_S C \xrightarrow{\phi(\alpha) \times_S 1} C \times_S C \xrightarrow{\text{subtract}} C \xrightarrow{x} \mathbb{A}^1.$$

The vanishing locus of x_α is the divisor $[\phi(\alpha)]$, so the vanishing locus of the product $y := \prod_\alpha x_\alpha$ is the divisor $[\phi(A^*)]$. We see using Corollary 2.10 that y is a regular element in $\mathcal{O}_C$. The final condition in Definition 2.13 says that y is topologically nilpotent. It is not hard to deduce that y is a good parameter on C.

Proposition 2.15. *Let f be a monic polynomial of degree $d > 0$ over $\mathcal{O}_S$, and let R be the completion of $\mathcal{O}_S[x]$ at f. Then the scheme $C = \mathrm{spf}(R) = \varinjlim_k V(f^k) \subseteq \mathbb{A}^1_S$ is a formal multicurve.*

Proof. Condition (b) is clear, because $\{x^i \mid i < dj\}$ is a basis for $R/(f^j)$ over $\mathcal{O}_S$. Next, observe that

$$R \simeq \mathcal{O}_S[\![y]\!][x]/(f(x) - y) = \mathcal{O}_S[\![y]\!]\{x^i \mid i < d\},$$

so

$$R\widehat{\otimes}R \simeq \mathcal{O}_S[\![y_0, y_1]\!][x_0, x_1]/(f(x_0) - y_0, f(x_1) - y_1) = \mathcal{O}_S[\![y_0, y_1]\!]\{x_0^i x_1^j \mid i, j < d\}.$$

It is clear that $y_1 - y_0$ is not a zero-divisor in this ring, and $x_1 - x_0$ divides $y_1 - y_0$ so it is also not a zero-divisor. It is not hard to check that the multiplication map $R\widehat{\otimes}R \to R$ induces an isomorphism $(R\widehat{\otimes}R)/(x_1 - x_0) \simeq R$, and it follows that $x_1 - x_0$ generates I_Δ, and thus that (c) holds. □

Definition 2.16. We say that a formal multicurve C over S is *embeddable* if it has the form $\varinjlim_k V(f^k)$ as above for some monic polynomial f.

Lemma 2.17. *Suppose that $k = \mathcal{O}_S$ is an algebraically closed field, and that C is a formal multicurve over S. Then C is a finite disjoint union of copies of $\widehat{\mathbb{A}}^1_S$, and is embeddable.*

Proof. Let $y \in R$ be a good parameter. Then the ring $\overline{R} := R/y$ is a finite-dimensional algebra over the field k, so it splits as a finite product of local algebras. As R is complete at (y) we can lift this splitting to R, which splits C as a disjoint union, say $C = C_1 \amalg \ldots \amalg C_r$. It is easy to see that each C_i is a formal multicurve. Put $R_i = \mathcal{O}_{C_i}$, so $R = R_1 \times \ldots \times R_r$. Let y_i be the component of y in R_i and put $\overline{R}_i = R_i/y_i$, so $\overline{R} = \overline{R}_1 \times \ldots \times \overline{R}_r$. Moreover, $\overline{R}_i$ is local, with maximal ideal $\mathfrak{m}_i$ say. As k is algebraically closed we see that $\overline{R}_i/\mathfrak{m}_i = k$. This gives an augmentation $u_i^* \colon R_i \to k$, or equivalently a section $u_i \colon S \to C_i$. It follows from Corollary 2.10 that the kernel of u_i^* is generated by a single regular element, say x_i. This means that the image of x_i in $\overline{R}_i$ generates $\mathfrak{m}_i$, so $\overline{R}_i \simeq k[x_i]/x_i^m$ for some m, and thus y_i divides x_i^m. On the other hand, we clearly have $u^*(y_i) = 0$ so x_i divides y_i. It is now easy to check that $R_i = k[\![x_i]\!]$, so $C_i \simeq \widehat{\mathbb{A}}^1_S$.

Finally, as k is algebraically closed, it is certainly infinite, so we can choose distinct elements $\lambda_1, \ldots, \lambda_r \in k$ say. If we put $f(x) = \prod_i (x - \lambda_i)$ we find that the completion of $k[x]$ at (f) is isomorphic to $\prod_{i=1}^r k[\![x]\!]$ and thus to $\mathcal{O}_C$. This proves that C is embeddable. □

3. Differential forms

We next recall some basic ideas about differential forms, and record some formulae that will be useful later in our study of residues.

Given a formal multicurve C over S, we put

$$\Omega = \Omega^1_{C/S} = I_\Delta/I_\Delta^2,$$

and call this the module of differential forms on C.

We also put $\Delta_2 = \mathrm{spf}(\mathcal{O}_{C \times_S C}/I_\Delta^2)$, and regard this as the second-order infinitesimal neighbourhood of Δ in $C \times_S C$. In these terms, Ω is the module of functions on Δ_2 that vanish on Δ.

Given a difference function $d \in I_\Delta$, we let α be the image of d in Ω; this generates Ω freely as a module over $\mathcal{O}_C$, so we can regard Ω as a trivialisable line bundle on C.

For any function $f \in \mathcal{O}_C$, we write $\mathbf{d}f$ for the image of $1 \otimes f - f \otimes 1$ in Ω, or equivalently the function $(a, b) \mapsto f(b) - f(a)$ on Δ_2. As usual, we have the Leibniz rule

$$\mathbf{d}(fg) = f\mathbf{d}(g) + g\mathbf{d}(f).$$

Now suppose that C has a commutative group structure. In particular, this gives a zero-section $Z \subset C$, and we write $Z_2 = \mathrm{spec}(\mathcal{O}_C/I_Z^2)$ and

$$\omega = I_Z/I_Z^2 = \{\text{ functions on } Z_2 \text{ that vanish on } Z\ \}.$$

The map $b \mapsto (0, b)$ gives an inclusion $Z_2 \to \Delta_2$ and thus a map $\Omega \to \omega$, which in turn gives an isomorphism $\Omega|_Z = \omega$ of line bundles on S. The image of $\mathbf{d}f$ under this map is the element $\mathbf{d}_0 f$ corresponding to the function $b \mapsto f(b) - f(0)$ on Z_2. If x is a coordinate on C, then $\mathbf{d}_0 x$ generates ω freely as a module over $\mathcal{O}_S$.

Next, for any function $f \in \mathcal{O}_C$ we define a function $\mathbf{D}f$ on Δ_2 by

$$(\mathbf{D}f)(a, b) = f(b - a) - f(0).$$

This construction gives a map $\mathbf{D}\colon \mathcal{O}_C \to \Omega$. If x is a coordinate then $\mathbf{D}x$ is the restriction of the usual difference function $d(a,b) = x(b-a)$ to Δ_2, so it is a generator of Ω.

It is easy to see that $\mathbf{D}f$ depends only on $\mathbf{d}_0 f$, and thus that $\mathbf{D}$ induces an $\mathcal{O}_S$-linear inclusion $\omega \to \Omega$, right inverse to the restriction map $\Omega \to \Omega|_Z = \omega$. A differential form is said to be *invariant* if it lies in the image of this map.

By extension of scalars, we obtain an $\mathcal{O}_C$-linear map $\mathcal{O}_C \otimes_{\mathcal{O}_S} \omega \to \Omega$, sending $f \otimes \mathbf{d}_0 g$ to $f\mathbf{D}g$. In particular, it sends $f \otimes \mathbf{d}_0 x$ to $f\mathbf{D}x$, and so is an isomorphism.

4. Equivariant projective spaces

We now start to build a connection between multicurves and A-equivariant topology (where A is a finite abelian group). Naturally, this involves the generalised cohomology of the projective spaces of representations of A. In this section, we assemble some facts about the homotopy theory of such projective spaces.

For $\alpha \in A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ we write L_α for $\mathbb{C}$ with A acting by $a.z = e^{2\pi i\alpha(a)}z$. In particular, L_0 has trivial action, and $L_\alpha \otimes L_\beta = L_{\alpha+\beta}$. For any finite-dimensional representation V, we put

$$V[\alpha] = \{v \in V \mid av = e^{2\pi i\alpha(a)}v \text{ for all } a \in A\}.$$

It is well-known that $V = \bigoplus_\alpha V[\alpha]$ and $\mathrm{Hom}_{\mathbb{C}[A]}(V,W) = \bigoplus_\alpha \mathrm{Hom}_{\mathbb{C}}(V[\alpha], W[\alpha])$. It follows that if there exists an equivariant linear embedding $V \to W$, then the space of such embeddings is connected, giving a canonical homotopy class of maps $PV \to PW$ of projective spaces.

We write $\mathcal{U}[\alpha] = L_\alpha \otimes \mathbb{C}^\infty$, and $\mathcal{U} = \mathcal{U}_A = \bigoplus_\alpha \mathcal{U}[\alpha]$, so $\mathcal{U}$ is a complete A-universe. We write $P\mathcal{U}$ for the projective space associated to $\mathcal{U}$, which has a natural A-action. By the previous paragraph, for any finite-dimensional representation V, there is a canonical map $PV \to P\mathcal{U}$ up to homotopy. Similarly, the space of equivariant linear isometries $\mathcal{U} \otimes \mathcal{U} \to \mathcal{U}$ is contractible, which gives a canonical homotopy class of maps $P\mathcal{U} \times P\mathcal{U} \to P\mathcal{U}$, making $P\mathcal{U}$ an abelian group up to equivariant homotopy. We can choose a conjugate-linear equivariant automorphism $\chi\colon \mathcal{U} \to \mathcal{U}$, and the resulting map $P\mathcal{U} \to P\mathcal{U}$ is the negation map for our group structure.

It is well-known that $P\mathcal{U}$ is the classifying space for equivariant complex line bundles. More precisely, for any A-space X, we write $\mathrm{Pic}_A(X)$ for the group of isomorphism classes of equivariant complex line bundles over X. Let T denote the tautological line bundle over $P\mathcal{U}$, so $T \in \mathrm{Pic}(P\mathcal{U})$. Then for any A-space X, the construction $[f] \mapsto [f^*T]$ gives a group isomorphism $[X, P\mathcal{U}]^A \simeq \mathrm{Pic}_A(X)$. Note that we regard T as the universal example; some other treatments in the literature use the dual bundle $T^* = \mathcal{O}(1)$ instead.

Note that A acts by scalars on $\mathcal{U}[\alpha]$, and thus acts as the identity on $P\mathcal{U}[\alpha] \subset P\mathcal{U}$. Moreover, the map $L \mapsto L_\alpha \otimes L$ gives a homeomorphism $\mathbb{C}P^\infty = P(\mathbb{C}^\infty) \to P\mathcal{U}[\alpha]$. Using this, we have a homeomorphism $(P\mathcal{U})^A = A^* \times \mathbb{C}P^\infty$, and thus a bijection $\pi_0((P\mathcal{U})^A) = A^*$, which is easily seen to respect the natural group structures. Thus, the group structure on $P\mathcal{U}$ gives a translation action (up to homotopy) of A^* on $P\mathcal{U}$. We write $\tau_\alpha\colon P\mathcal{U} \to P\mathcal{U}$ for translation by an element $\alpha \in A^*$.

For various purposes we will need to use an A-fixed basepoint in $P\mathcal{U}$. We have embeddings $L_\alpha \to \mathcal{U}[\alpha] \to \mathcal{U}$, and PL_α is an A-fixed point. Any other fixed point lies in the same component of $(P\mathcal{U})^A$ as PL_α for some α, so it can be replaced by PL_α for most purposes. Moreover, the map τ_α gives a homotopy equivalence of pairs $(P\mathcal{U}, PL_\beta) \to (P\mathcal{U}, PL_{\alpha+\beta})$. Where not otherwise stated, we use PL_0 as the basepoint.

Proposition 4.1. *Let V, W and X be unitary representations of A, where V and W have finite dimension and X is a colimit of finite-dimensional subrepresentations. Put $U = V \oplus W \oplus X$. Then there is a homotopy-commutative diagram as follows, in which the maps marked q are the obvious collapses, the maps marked j are the obvious inclusions, and δ is the diagonal map.*

$$\begin{array}{ccccc}
PU & & \xrightarrow{\quad\delta\quad} & & PU \times PU \\
\big\downarrow{\scriptstyle q_{V\oplus W}} & & & & \big\downarrow{\scriptstyle q_V \wedge q_W} \\
PU/P(V\oplus W) & \xrightarrow[\overline{\delta}]{} & P(V\oplus X)/PV \wedge P(W\oplus X)/PW & \xrightarrow[j\wedge j]{} & PU/PV \wedge PU/PW
\end{array}$$

Moreover, if $\dim(X) = 1$ *then $\overline{\delta}$ is just the standard homeomorphism*

$$S^{\mathrm{Hom}(X,V\oplus W)} = S^{\mathrm{Hom}(X,V)} \wedge S^{\mathrm{Hom}(X,W)}.$$

All maps and homotopies are natural for isometric embeddings of V, W and X.

Remark 4.2. The above diagram gives a map

$$\overline{\delta}^*: E^*(P(V\oplus X),PV)\otimes E^*(P(W\oplus X),PW)\to E^*(PU,P(V\oplus W)).$$

In his unpublished thesis [3], Cole writes $a*b$ for $\overline{\delta}^*(a\otimes b)$. The idea of using this construction seems to be original to that thesis; our approach differs only in being somewhat more geometric.

Proof. Assume for the moment that X is finite-dimensional. We start by defining a map

$$\overline{\gamma}\colon PU/P(V\oplus W)\to PU/PV\wedge PU/PW,$$

which will be homotopic to $(j\wedge j)\circ\overline{\delta}$. For $u=(v,w,x)\in U^\times := U\setminus\{0\}$ we put

$$s=s(u)=(\|w\|-\|v\|)/(\|v\|+\|w\|+\|x\|).$$

Note that $s(u)\in[-1,1]$, and $s(\lambda u)=s(u)$ for all $\lambda\in\mathbb{C}^\times$, and $s(u)\geq 0$ iff $\|w\|\geq\|v\|$. We next define $\alpha,\beta\colon U^\times\to U$ by

$$\alpha(v,w,x)=\begin{cases}((1-s)v,sw,x) & \text{if } s\geq 0\\ (v,0,x) & \text{if } s\leq 0\end{cases}$$

$$\beta(v,w,x)=\begin{cases}(0,w,x) & \text{if } s\geq 0\\ (-sv,(1+s)w,x) & \text{if } s\leq 0.\end{cases}$$

Note that $\alpha(\lambda u)=\lambda\alpha(u)$ and similarly for β.

We claim that when $u\neq 0$, the line joining u to $\alpha(u)$ never passes through 0 (so in particular $\alpha(u)\neq 0$). Indeed, if $s\leq 0$, then the points on the line have the form (v,tw,x) for $0\leq t\leq 1$. Thus, the line can only pass through zero if $v=x=0$. The relation $s\leq 0$ means that $\|w\|\leq\|v\|=0$, so $w=0$ as well, contradicting the assumption that $u\neq 0$. In the case $s>0$, the points on the line have the form $((1-ts)v,(1-t+ts)w,x)$. As $s>0$ and $0\leq t\leq 1$ we have $1-t+ts>0$. For the line to pass through zero we must thus have $x=w=0$, and the relation $s\geq 0$ means that $\|v\|\leq\|w\|=0$, again giving a contradiction. Similarly, the line from u to $\beta(u)$ never passes through 0.

It follows that α and β induce self-maps of PU that are homotopic to the identity, so the map $\gamma=(\alpha,\beta)\colon PU\to PU\times PU$ is homotopic to the diagonal map δ.

Next, note that if $u\in V\oplus W$, then for $s\geq 0$ we have $\gamma(u)\in U\times W$, and for $s\leq 0$ we have $\gamma(u)\in V\times U$. It follows that the induced map on projective spaces has

$$\gamma(P(V\oplus W))\subseteq(PU\times PW)\cup(PV\times PU),$$

so there is an induced map

$$\overline{\gamma}\colon PU/P(V\oplus W)\to PU/PV\wedge PU/PW.$$

As γ is homotopic to δ, we see that $\overline{\gamma}\circ q_{V\oplus W}\simeq(q_V\wedge q_W)\circ\delta$.

To construct the map $\overline{\delta}$, we need a slightly different model. Clearly

$$PU\setminus P(V\oplus W)=(V\times W\times X^\times)/\mathbb{C}^\times=(V\times W\times S(X))/S^1,$$

and $PU/P(V\oplus W)$ is the one-point compactification of this. Similarly, $P(V\oplus X)/PV\wedge P(W\oplus X)/PW$ is the one-point compactification of the space $(V\times S(X))/S^1\times(W\times S(X))/S^1$. We can thus define $\overline{\delta}$ by giving a proper map

$$V\times W\times S(X)\to V\times S(X)\times W\times S(X)$$

with appropriate equivariance. The map in question just sends (v,w,x) to (v,x,w,x).

If X is one-dimensional and $(v,x)\in V\times S(X)$ then we have a linear map $\alpha\colon X\to V$ given by $\alpha(x)=v$, which does not change if we multiply (v,x) by an element of S^1. This gives a homeomorphism $(V\times S(X))/S^1=\operatorname{Hom}(X,V)$, and thus $P(V\oplus X)/PV=S^{\operatorname{Hom}(X,V)}$. It is easy to see that with this identification, $\overline{\delta}$ is just the standard homeomorphism

$$S^{\operatorname{Hom}(X,V\oplus W)}=S^{\operatorname{Hom}(X,V)}\wedge S^{\operatorname{Hom}(X,W)}.$$

We now show that $(j\wedge j)\circ\overline{\delta}\simeq\overline{\gamma}$. Put

$$T=\{((v_0,w_0,x_0),(v_1,w_1,x_1))\in U^2\mid\|(w_0,x_0)\|=\|(v_1,x_1)\|=1\},$$

so that $PU/PV\wedge PU/PW$ is the one-point compactification of $T/(S^1\times S^1)$. Define maps

$$\theta_t\colon V\times W\times S(X)\to T$$

for $0 \leq t \leq 1$ by

$$\theta_t(v,w,x) = \begin{cases} \left(\frac{((1-st)v,stw,x)}{\|(stw,x)\|},(0,w,x)\right) & \text{if } s \geq 0 \\ \left((v,0,x),\frac{(-stv,(1+st)w,x)}{\|(-stv,x)\|}\right) & \text{if } s \leq 0, \end{cases}$$

where $s = (\|w\| - \|v\|)/(\|v\| + \|w\| + \|x\|)$ as before. We claim that these maps are proper. To see this, put

$$\nu((v_0,w_0,x_0),(v_1,w_1,x_1)) = \max(\|v_0\|,\|w_1\|),$$

and $T_k = \{t \in T \mid \nu(t) \leq k\}$. It is easy to see that every compact subset of T is contained in some T_k, so it will be enough to show that $\theta_t^{-1}T_k$ is compact. In the case $s \geq 0$ we have $0 \leq 1 - st \leq 1$ and $\|(stw,x)\| \geq \|x\| = 1$ so $\|((1-st)v/\|(stw,x)\|)\| \leq \|v\| \leq \|w\|$, so $\nu(\theta_t(v,w,x)) = \|w\|$. Similarly, when $s \leq 0$ we have $\nu(\theta_t(v,w,x)) = \|v\|$, so in general $\nu(\theta_t(v,w,x)) = \max(\|v\|,\|w\|)$. It follows immediately that θ_t is proper, and we get an induced family of maps

$$\theta_t \colon PU/P(V \oplus W) \to PU/PV \wedge PU/PW.$$

It is easy to see that $\theta_0 = (j \wedge j) \circ \overline{\delta}$ and $\theta_1 = \overline{\gamma}$. The proposition follows easily (for the case where X has finite dimension).

If X has infinite dimension, we apply the above to all finite dimensional subrepresentations of X. We see by inspection that all constructions pass to the colimit, so the conclusion is valid for X itself. □

By an evident inductive extension, we obtain the following:

Corollary 4.3. *Let $L_1,\dots,L_d$ be one-dimensional representations of A, and let X be as above. Put $Y = \bigoplus_i L_i$ and $U = Y \oplus X$. Then there is a homotopy-commutative diagram as follows:*

$$\begin{array}{ccccc} PU & & \xrightarrow{\quad\delta\quad} & & PU^r \\ \downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle q} \\ PU/PY & \xrightarrow[\overline{\delta}]{} & \bigwedge_i P(L_i \oplus X)/PL_i & \xrightarrow[j]{} & \bigwedge_i PU/PL_i \end{array}$$

Moreover, if $\dim(X) = 1$ *then $\overline{\delta}$ is just the standard homeomorphism*

$$S^{\mathrm{Hom}(X,Y)} = \bigwedge_i S^{\mathrm{Hom}(X,L_i)}. \quad \square$$

We conclude with some further miscellaneous observations about the space $P\mathcal{U}$.

Proposition 4.4. *The space $P\mathcal{U}$ is equivariantly equivalent to $F(EA_+,\mathbb{C}P^\infty)$ (where $\mathbb{C}P^\infty$ is the usual space with trivial A-action). Equivalently, $P\mathcal{U}$ is the second space in the Borel cohomology spectrum $F(EA_+,H)$, so $[X,P\mathcal{U}]^A = H^2(X_{hA})$ for any A-space X. Moreover, the space $\Omega P\mathcal{U}$ is equivariantly equivalent to S^1 with the trivial action.*

Proof. There is an evident inclusion $\mathbb{C}P^\infty = P(\mathcal{U}^A) \to (P\mathcal{U})^A \to P\mathcal{U}$. This is a nonequivariant equivalence, and so gives an equivariant equivalence $F(EA_+,\mathbb{C}P^\infty) \to F(EA_+,P\mathcal{U})$. On the other hand, the collapse map $EA_+ \to S^0$ gives a map $j \colon P\mathcal{U} \to F(EA_+,P\mathcal{U}) \simeq F(EA_+,\mathbb{C}P^\infty)$. We claim that this is an equivalence. Indeed, if we take fixed points for a subgroup $A_0 \leq A$ we get a map $A_0^* \times \mathbb{C}P^\infty \to F((BA_0)_+,\mathbb{C}P^\infty)$ of commutative H-spaces. It is clear that

$$\pi_k(A_0^* \times \mathbb{C}P^\infty) = \begin{cases} A_0^* & \text{if } k = 0 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$\pi_k F((BA_0)_+,\mathbb{C}P^\infty) = [\Sigma^k(BA_0)_+,K(\mathbb{Z},2)] = H^{2-k}BA_0.$$

This clearly vanishes for $k > 2$ and gives $\mathbb{Z}$ for $k = 2$. Standard arguments with the coefficient sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ give $H^1BA_0 = 0$ and $H^2BA_0 = A_0^*$, showing that $\pi_*F((BA_0)_+,\mathbb{C}P^\infty)$ is abstractly isomorphic to $\pi_*(A_0^* \times \mathbb{C}P^\infty)$. With a little more work one sees that the isomorphism is induced by j, and the first part of the proposition follows.

We now see that

$$\Omega P\mathcal{U} \simeq \Omega F(EA_+,\mathbb{C}P^\infty) = F(EA_+,\Omega\mathbb{C}P^\infty) = F(EA_+,S^1).$$

As above we find that

$$\pi_k(F(EA_+, S^1)^{A_0}) = H^{1-k}BA_0 = \begin{cases} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the obvious map $S^1 \to F(EA_+, S^1)$ is an equivariant equivalence. □

Proposition 4.5. *Let T be the tautological line bundle over $P\mathcal{U}$, and let $S(T^n)$ be the unit circle bundle in the n'th tensor power of T. Then $S(T^n)$ is equivariantly equivalent to $F(EA_+, B(\mathbb{Z}/n))$.*

Proof. It is well-known that in the case $A = 0$ we have $S(T^n) = B(\mathbb{Z}/n) = K(\mathbb{Z}/n, 1)$. In the general case, note that $S(T^n)$ consists of pairs (L, u) where $L \in P\mathcal{U}$ and $u \in L^n$ and $\|u\| = 1$. Suppose that (L, u) is fixed by a subgroup $A_0 \leq A$. We see that A_0 acts on L by some character $\alpha \in A_0^*$, so A_0 acts on u by $n\alpha$, but u is fixed so $n\alpha = 0$. Given that $n\alpha = 0$, we see that every point in L^n is fixed by A_0. Using this, we see that $S(T^n)^{A_0} = A_0^*[n] \times B(\mathbb{Z}/n)$, where $A_0^*[n]$ denotes the subgroup of points of order n in A_0^*. Using this, we find that $\pi_* S(T^n)^{A_0} = H^{1-*}(BA_0; \mathbb{Z}/n)$, and the claim follows by the same method as in the previous proposition. □

Proposition 4.6. *Put $\mathcal{F} = \{B \leq S^1 \times A \mid B \cap S^1 = \{1\}\}$, which is a family of subgroups of $S^1 \times A$. Then the unit sphere $S(\mathcal{U})$ is a model for $E\mathcal{F}$, and so $P\mathcal{U} = (E\mathcal{F})/S^1$.*

Proof. First, we let $S^1 \subset \mathbb{C}^\times$ act on $S(\mathcal{U})$ by multiplication, and let A act in the usual way. These actions commute and so give an action of $S^1 \times A$. We need only check that $S(\mathcal{U})$ has the characterizing property of $E\mathcal{F}$, or in other words that $S(\mathcal{U})^B$ is contractible for $B \in \mathcal{F}$ and empty for $B \notin \mathcal{F}$. If $B \in \mathcal{F}$ then $B \cap S^1$ is trivial so B is the graph of a homomorphism $\phi \colon A_0 \to S^1$ for some subgroup $A_0 \leq A$. Put

$$\mathcal{V} = \{v \in \mathcal{U} \mid a.v = \phi(a)^{-1}v \text{ for all } a \in A_0\},$$

so $S(\mathcal{U})^B = S(\mathcal{V})$. As $\mathcal{U}$ is a complete A_0-universe, we see that $\mathcal{V}$ is infinite dimensional, and so $S(\mathcal{V})$ is contractible as required. On the other hand, as S^1 acts freely on $S(\mathcal{U})$, it is clear that $S(\mathcal{U})^B = \emptyset$ whenever $B \notin \mathcal{F}$. □

5. Equivariant orientability

Now let E be a commutative A-equivariant ring spectrum. We next need to formulate suitable notions of orientability and periodicity for E, and deduce consequences for the rings E^*PV. Our results differ from those of [3] only in minor points of technical detail. We start by introducing some auxiliary ideas.

Definition 5.1. Let R be an E-algebra spectrum, and M a module spectrum over R. We say that M is a free R-module if it is equivalent as an R-module to a wedge of (unsuspended) copies of R, or equivalently, there is a family of elements $e_i \in \pi_0 M$ such that the resulting maps $\bigoplus_i [\Sigma^n A/B_+, R] \to [\Sigma^n A/B_+, M]$ are isomorphisms for all $n \in \mathbb{Z}$ and all $B \leq A$. We say that such elements e_i are *universal generators* for for $\pi_0 M$ over $\pi_0 R$. We will often leave the identification of R and M implicit. For example, if we say that an element e is a universal generator for $E^0(X, Y)$ over $E^0 X$, we are referring to the case $R = F(X_+, E)$ and $M = F(X/Y, E)$.

Definition 5.2. Let E an A-equivariant ring spectrum, and consider a class $x \in E^0(P\mathcal{U}, PL_0)$. For any $\alpha \in A^*$ we can embed $L_\alpha \oplus L_0$ in $\mathcal{U}$, and thus restrict x to get a class $u_{L_\alpha} \in E^0(P(L_\alpha \oplus L_0), PL_0) = \widetilde{E}^0 S^{L_\alpha}$. This in turn gives an E-module map $m_\alpha \colon \Sigma^{L_\alpha} E \to E$.

We say that x is a *complex coordinate* for E if for all α the map m_α is an equivalence, or equivalently u_{L_α} generates $\Sigma^{-L_\alpha} E$ as an E-module. We say that E is *periodically orientable* if it admits such a coordinate. We say that E is *evenly orientable* if in addition, we have $\pi_1 E = 0$.

From now on, we assume that E is periodically orientable. We choose a complex coordinate x, but as far as possible we state our results in a form independent of this choice. We write $\overline{x} = \chi^* x$, where $\chi \colon P\mathcal{U} \to P\mathcal{U}$ is the negation map for the group structure. It is easy to see that this is again a coordinate.

Recall that for any line bundle L over X, there is an essentially unique map $f_L \colon X \to P\mathcal{U}$ with $f^*T \simeq L$ (where T is the tautological bundle over $P\mathcal{U}$). We define the *Euler class* of L by

$$e(L) = f_{L^*}^*(x) = f_L^*(\overline{x}).$$

Thus, the element $x \in E^0 P\mathcal{U}$ is the Euler class of T^*, and $\overline{x}$ is the Euler class of T.

Remark 5.3. There is some inconsistency in the literature about whether $e(L)$ should be $f_L^*(x)$ or $f_L^*(\overline{x})$. The convention adopted here is the opposite of that used in [15], but I believe that it is more common in other work and has some technical advantages. The conventions used elsewhere in this paper are fixed by the following requirements.

(a) We have $e(V\oplus W)=e(V)e(W)$.

(b) The Euler class of V is the restriction of the Thom class in $\widetilde{E}^0X^V$ to the zero section $X\subset X^V$.

Our substitute for the nonequivariant theory of Chern classes will be more abstract, so we will not need sign conventions. The rôle normally played by the Chern polynomial $\sum_{i+j=\dim(V)}\pm c_ix^j$ will be played by a certain element f_V; if $A=0$ and $V=\bigoplus_i L_i$ then $f_V=\prod_i(x+_F e(L_i))$.

Next note that we can define

$$x_\alpha:=\tau_{-\alpha}^*x\in E^0(P\mathcal{U},PL_\alpha).$$

Because

$$(\tau_{-\alpha}^*T)^*=(L_{-\alpha}\otimes T)^*=L_\alpha\otimes T^*=\mathrm{Hom}(T,L_\alpha),$$

we have $x_\alpha=e(\mathrm{Hom}(T,L_\alpha))$. If L is a one-dimensional representation isomorphic to L_α, we also use the notation x_L for x_α. We can identify $E^0(P(L_\beta\oplus L_\alpha),PL_\alpha)$ with $\widetilde{E}^0S^{L_{\beta-\alpha}}$, and we find that x_α restricts to $u_{\beta-\alpha}$, which is a universal generator.

Now consider a finite-dimensional representation V of A. We have a canonical homotopy class of embeddings $PV\to P\mathcal{U}$, and thus a well-defined group $E^0(P\mathcal{U},PV)$. We can write V as $\bigoplus_{i=1}^d L_i$, and Corollary 4.3 gives a map

$$P\mathcal{U}/PV\to\bigwedge_i P\mathcal{U}/PL_i$$

compatible with the diagonal. Using this, we can pull back $x_{L_1}\wedge\ldots\wedge x_{L_d}$ to get a class $x_V\in E^0(P\mathcal{U},PV)$ that maps to $\prod_i x_{\alpha_i}$ in $E^0P\mathcal{U}$. Note that for any representation W containing V we can choose an embedding $W\to\mathcal{U}$ and pull back x_V along the resulting map $PW\to P\mathcal{U}$ to get a class in $E^0(PW,PV)$, which we again denote by x_V.

Lemma 5.4. *Let $V\leq W$ be complex representations of A, with $\dim(W/V)=1$. Then x_V is a universal generator for $E^0(PW,PV)$.*

Proof. Write $V=L_1\oplus\ldots\oplus L_d$ as before, and $X=W\ominus V$, so $W=V\oplus X$ and $PW/PV=S^{\mathrm{Hom}(X,V)}=\bigwedge_i S^{\mathrm{Hom}(X,L_i)}$. Because x is a complex coordinate, we know that $x_{L_i}\in E^0(PW,PL_i)$ restricts to a universal generator v_i of $S^{\mathrm{Hom}(X,L_i)}$. It follows from Corollary 4.3 that $x_V=\prod_i v_i\in\widetilde{E}^0S^{\mathrm{Hom}(X,V)}=E^0(PW,PV)$, and this is easily seen to be a universal generator. □

Corollary 5.5. *Let $0=U_0<U_1<\ldots<U_d=U$ be representations of A with $\dim(U_i)=i$. Then $\{x_{U_i}\mid i<d\}$ is a universal basis for E^0PU over E^0.*

Proof. This follows by an evident induction from the lemma. □

Remark 5.6. As $\overline{x}$ is another coordinate, it gives rise to another universal basis $\{\overline{x}_{U_i}\mid i<d\}$ for E^0PU, which is sometimes more convenient.

We record separately some easy consequences that are independent of the choice of flag $\{U_i\}$:

Proposition 5.7. *Let U be a d-dimensional representation of A. Then*

(a) *$F(PU_+,E)$ is a free module of rank d over E.*

(b) *If $U=V\oplus W$ then the restriction map $F(PU_+,E)\to F(PV_+,E)$ is split surjective. The kernel is a free module of rank one over $F(PW_+,E)$, generated by x_V.* □

We now put $S=\mathrm{spec}(E^0)$ and $R=E^0P\mathcal{U}$ and $C=\mathrm{spf}(R)$. We must show that C is an equivariant formal group over S.

We first exhibit a topological basis for R. We can list the elements of A^* as

$$A^*=\{\alpha_0=0,\alpha_1,\ldots,\alpha_{n-1}\}$$

(where $n=|A|$), and then define α_k for all $k\geq 0$ by $\alpha_{ni+j}=\alpha_j$. We then have an evident filtration

$$0=V_0<V_1<V_2<\ldots<\mathcal{U}=\varinjlim_k V_k$$

where $V_k = \bigoplus_{j<k} L_{\alpha_j}$. If we put $e_k = x_{V_k}$ we find that $\{e_i \mid 0 \le i < k\}$ is a universal basis for E^0PV_k, and it follows by an evident limiting argument that $\{e_i \mid i \ge 0\}$ is a universal topological basis for $E^0P\mathcal{U}$, giving an isomorphism $F(P\mathcal{U}_+, E) = \prod_k E$. If we put $y = x_{\mathbb{C}[A]} = x_{V_n} = e_n$, it is easy to see that $e_{ni+j} = y^i e_j$, and it follows that $E^0P\mathcal{U}$ is a free module over $E^0[\![y]\!]$ with basis $\{e_i \mid i < n\}$. Thus, conditions (a) and (b) in Definition 2.2 are satisfied.

Next, we have

$$F(P\mathcal{U}_+^2, E) = F(P\mathcal{U}_+, F(P\mathcal{U}_+, E)) = F(P\mathcal{U}_+, \prod_j E) = \prod_{i,j} E.$$

By working through the definitions, we deduce that the elements $e_i \otimes e_j$ form a universal topological basis for $E^0(P\mathcal{U} \times P\mathcal{U})$, so $E^0(P\mathcal{U} \times P\mathcal{U}) = R\widehat{\otimes}R$, so $\operatorname{spf}(E^0(P\mathcal{U} \times P\mathcal{U})) = C \times_S C$. As $P\mathcal{U}$ is an commutative group up to equivariant homotopy, we now see that C is a commutative formal group scheme over S.

Now note that e_1 is just the coordinate x, and this divides e_k for all $k > 0$. In particular it divides y, which is a regular element in R, so x is also a regular element. It is also now easy to see x generates the ideal $E^0(P\mathcal{U}, PL_0)$, which is just the augmentation ideal in the Hopf algebra R, so the vanishing locus of x is the zero-section in C. Thus x is a coordinate on C, showing (via Proposition 2.8) that C is in fact a formal multicurve group.

Next, recall that $\pi_0((P\mathcal{U})^A) = A^*$, which gives a map $A^* \to P\mathcal{U}$ of groups up to homotopy, and thus a map $\phi\colon A^* \to C$ of formal group schemes. By working through the definitions, we see that the image of the section $\phi(\alpha)$ is the closed subscheme $\operatorname{spec}(E^0PL_\alpha) = \operatorname{spec}(R/x_\alpha)$, so the divisor $D := \sum_\alpha [\phi(\alpha)]$ is

$$\operatorname{spec}(R/\prod_\alpha x_\alpha) = \operatorname{spec}(R/y) = E^0P\mathbb{C}[A].$$

As y is topologically nilpotent, we see that any function on C that vanishes on D is topologically nilpotent, so C is a formal neighbourhood of D. We have thus proved the following result:

Theorem 5.8. *Let E be a periodically orientable A-equivariant ring spectrum. Then the scheme $C := \operatorname{spf}(E^0P\mathcal{U})$ is an A-equivariant formal group over $S := \operatorname{spec}(E^0)$.* □

Remark 5.9. We have $I_0 = \{f \in \mathcal{O}_C \mid f(0) = 0\} = E^0(P\mathcal{U}, P\mathbb{C})$, and thus $I_0^2 = E^0(P\mathcal{U}, P(\mathbb{C} \oplus \mathbb{C}))$, and thus

$$\omega = I_0/I_0^2 = E^0(P(\mathbb{C} \oplus \mathbb{C}), P\mathbb{C}) = \widetilde{E}^0 S^2 = \pi_2 E.$$

6. Simple examples

Let $\widehat{C}$ be a nonequivariant formal group over a scheme S, so $\widehat{C}$ is the formal neighbourhood of its zero section. For any finite abelian group A, we can of course let $\phi\colon A^* \to \widehat{C}$ be the zero map, and this gives us an A-equivariant formal group. More generally, *any* homomorphism $A^* \to \widehat{C}$ will give an A-efg, although often there will not be any homomorphisms other than zero.

Now suppose that $\widehat{C}$ is the formal group associated to a nonequivariant even periodic ring spectrum $\widehat{E}$. We then have an A-equivariant ring spectrum $E = F(A_+, \widehat{E})$ (which the Wirthmüller isomorphism also identifies with $A_+ \wedge \widehat{E}$). This satisfies $E^*X = \widehat{E}^* \operatorname{res}(X)$, where $\operatorname{res}\colon \mathcal{S}_A \to \mathcal{S}_0$ is the restriction functor. It follows easily that E is periodically orientable, and that the associated equivariant formal group is just $\widehat{C}$, equipped with the zero map $\phi\colon A^* \to \widehat{C}$ as above.

For a slightly more subtle construction, suppose we allow S to be a formal scheme, and assume that some prime p is topologically nilpotent in $\mathcal{O}_S$. Suppose also that the formal group $\widehat{C}$ has finite height n. Put $S' = \operatorname{Hom}(A^*, \widehat{C})$; it is well-known that $\mathcal{O}_{S'}$ is a free module of rank $|A|^n$ over $\mathcal{O}_S$, so S' is finite and flat over S. By definition, S' is the universal example of a formal scheme T over S equipped with a homomorphism from A^* to the group of maps $T \to \widehat{C}$ of formal schemes over S, or equivalently the group of sections of $T \times_S C$ over T. If we put $C' = S' \times_S \widehat{C}$, there is thus a tautological map $\phi\colon A^* \to C'$. Here C' is an ordinary formal group over S' and thus is the formal neighbourhood of its zero section. It follows that (C', ϕ) is automatically an A-equivariant formal group over S'.

Now suppose we have a $K(n)$-local even periodic ring spectrum $\widehat{E}$. We give the ring $\pi_0\widehat{E}$ the natural topology as in [8, Section 11] — in most cases of interest, this is the same as the I_n-adic topology. We then put $S = \operatorname{spf}(\pi_0\widehat{E})$ and $\widehat{C} = \operatorname{spf}(\widehat{E}^0\mathbb{C}P^\infty)$, which gives an ordinary formal group of height n over S. Let EA denote a contractible space with free A-action, and put $E = F(EA_+, \widehat{E})$. This is a commutative A-equivariant ring spectrum, with $E^*X = \widehat{E}^*X_{hA}$, where

X_{hA} denotes the homotopy orbit space or Borel construction. In particular, we have $E^0(\text{point}) = \widehat{E}^0BA$, and it is well-known that this is canonically isomorphic to $\mathcal{O}_{\text{Hom}(A^*,\widehat{C})}$. Next, observe that we have an A-equivariant inclusion $P\mathcal{U}[0] \to P\mathcal{U}$, which is nonequivariantly a homotopy equivalence, so the map $EA \times P\mathcal{U}[0] \to EA \times P\mathcal{U}$ is an equivariant homotopy equivalence. It follows that $E^*P\mathcal{U} = E^*P\mathcal{U}[0] = \widehat{E}^*(BA \times \mathbb{C}P^\infty) = \widehat{E}^*BA \otimes_{\widehat{E}^*} \widehat{E}^*\mathbb{C}P^\infty$, and thus that $\text{spf}(E^0P\mathcal{U}) = \text{Hom}(A^*,\widehat{C}) \times \widehat{C}$. This shows that the equivariant formal group associated to E is just the pullback $C' = S' \times_S \widehat{C}$ as discussed above.

7. Formal groups from algebraic groups

We now show how to pass from algebraic groups (in particular, elliptic curves or the multiplicative group) to equivariant formal groups.

7.1. **The multiplicative group.** Let $S = \text{spec}(k)$ be a scheme, and consider the group scheme $\mathbb{G}_m \times S = \text{spec}(k[u,u^{-1}])$ over S. Suppose we are given a homomorphism ϕ from $A^* \times S$ to $\mathbb{G}_m \times S$ of group schemes over S, or equivalently a homomorphism $\phi\colon A^* \to k^\times$ of abstract groups. We can then form the divisor

$$D = \sum_\alpha [\phi(\alpha)] = \text{spec}(k[u^{\pm 1}]/y),$$

where $y = \prod_\alpha (1 - u/\phi(\alpha))$. It is convenient to observe that u is invertible in $k[u]/y$ and thus in $k[u]/y^m$ for all m, so D can also be described as $\text{spec}(k[u]/y)$. We then define C to be the formal neighbourhood of D in $\mathbb{G}_m \times S$, so

$$C = \varinjlim_m \text{spec}(k[u]/y^m) = \text{spf}(k[u]^\wedge_y),$$

which is an embeddable formal multicurve. It is easy to see that this is a subgroup of $\mathbb{G}_m \times S$ and is an equivariant formal group, with coordinate $x = 1 - u$.

The universal example of a ring with a map $A^* \to k^\times$ is $k = \mathbb{Z}[A^*]$, which can be identified with the representation ring $R(A)$. Thus, the universal example of a scheme S with a map $A^* \times S \to \mathbb{G}_m \times S$ as above is $S = \text{Hom}(A^*, \mathbb{G}_m) = \text{spec}(R(A))$. We can apply the above construction in this tautological case to get an equivariant formal group C over $\text{Hom}(A^*, \mathbb{G}_m)$. Explicitly, if we let $v_\alpha \in \mathbb{Z}[A^*]$ be the basis element corresponding to $\alpha \in A^*$ and put $y = \prod_\alpha (1 - uv_{-\alpha}) \in \mathbb{Z}[A^*][u]$, then $C = \text{spf}(\mathbb{Z}[A^*][u]^\wedge_y)$.

Theorem 7.1 (Cole-Greenlees-Kriz)**.** *The A-efg associated to the equivariant complex K-theory spectrum K_A is isomorphic to the A-efg C over* $\text{Hom}(A^*, \mathbb{G}_m)$ *constructed above.*

Proof. This is just a geometric restatement of [4, Section 6]. It is proved by identifying $K^*_A P\mathcal{U}$ with $K^*_{A\times S^1} E\mathcal{F}_+$ (where $\mathcal{F} = \{B \leq A \times S^1 \mid B \cap S^1 = \{1\}\}$ as in Proposition 4.6) and applying a suitable completion theorem. ∎

7.2. **Elliptic curves.** We now carry out the same program with the multiplicative group replaced by an elliptic curve (with some technical conditions assumed for simplicity). Suppose that we are given a ring k and an element $\lambda \in k$, and that 2, λ and $1-\lambda$ are invertible in k. Let $\widetilde{C}$ be the elliptic curve given by the homogeneous cubic $y^3 = x(x-z)(x-\lambda z)$, so the zero element is $O = [0:1:0]$, and the points $P := [0:0:1]$, $Q := [1:0:1]$ and $R := [\lambda:0:1]$ are the three points of exact order two in $\widetilde{C}$. Define rational functions t and r on $\widetilde{C}$ by $t([x:y:z]) = x/y$ and $r([x:y:z]) = z/y$. One checks that the subscheme $U = \widetilde{C} \setminus \{P,Q,R\}$ is the affine curve with equation $r = t(t-r)(t-\lambda r)$, and that on U, the function t has a simple zero at O and no other poles or zeros.

Now let A be an abelian group of odd order n, and let $\phi\colon A^* \to \widetilde{C}$ be a homomorphism. Define $V = \bigcap_\alpha (U + \phi(\alpha))$, which is an affine open subscheme of U.

Lemma 7.2. *For each $\beta \in A^*$, the section $\phi(\beta)\colon S \to \widetilde{C}$ actually lands in V.*

Proof. We first show that for all $\gamma \in A^*$, the section $\phi(\gamma)$ lands in U. Put $D = [P] + [Q] + [R]$, so $U = \widetilde{C} \setminus D$. Let T be the closed subscheme of points $s \in S$ where $\phi(\gamma)(s) \in D$; we must show that $T = \emptyset$. As n is odd and D is the divisor of points of exact order 2, we see that multiplication by n is the identity on D, but of course $n.\phi(\alpha) = O$. We conclude that over T we have $O \in D$. As 2 is invertible in k we know that O and D are disjoint, so $T = \emptyset$ as required.

We now apply this to $\gamma = \beta - \alpha$ to deduce that $\phi(\beta) \in U + \phi(\alpha)$. This holds for all α, so $\phi(\beta) \in V$ as claimed. ∎

We now define C to be the formal neighbourhood of the divisor $D = \sum_\alpha [\phi(\alpha)]$ in V. If we put $s(a) = \prod_\alpha t(a - \phi(\alpha))$ then $s \in \mathcal{O}_V$ and the vanishing locus of s is just D, so we have $\mathcal{O}_C = (\mathcal{O}_V)^\wedge_s$. Using this, we see that C is an equivariant formal group, with coordinate t and good parameter s.

Now suppose instead that we are given a curve $\widetilde{C}$ over S as above, but not the map $\phi\colon A^* \to \widetilde{C}$. We can then consider the scheme $S_1 = \operatorname{Hom}(A^*, \widetilde{C})$, which is easily seen to be a closed subscheme of $\operatorname{Map}(A^*, U)$ and thus affine. We can thus pull back $\widetilde{C}$ to get a curve $\widetilde{C}_1$ over S_1 equipped with a tautological map $\phi\colon A^* \to \widetilde{C}_1$, and we can carry out the previous construction to get an equivariant formal group C_1 over S_1. This should be associated to some kind of A-equivariant elliptic cohomology theory. It is not hard to construct a suitable theory if $\mathcal{O}_S$ is a $\mathbb{Q}$-algebra; see [18], for example. For more general base schemes, little is known.

8. Equivariant formal groups of product type

A simple class of A-efg's can be constructed as follows. Let $\widehat{C}$ be an ordinary, nonequivariant formal group, and let B be a subgroup of A. We then have a formal multicurve $C := B^* \times \widehat{C}$ and a homomorphism

$$\phi := (A^* \xrightarrow{\text{res}} B^* \xrightarrow{\text{inc}} B^* \times \widehat{C} = C),$$

giving an A-efg. Equivariant formal groups of this kind are said to be of *product type.*

Proposition 8.1. *An A-efg (C, ϕ) is of product type iff for every character $\alpha \in A^*$ with $\phi(\alpha) \neq 0$ in C (or equivalently, $x(\phi(\alpha)) \neq 0$ in $\mathcal{O}_S$), the element $x(\phi(\alpha))$ is invertible in $\mathcal{O}_S$. (This is easily seen to be independent of the choice of coordinate.)*

Proof. First suppose that for all α with $\phi(\alpha) \neq 0$, the element $x(\phi(\alpha))$ is invertible. The kernel of ϕ is a subgroup of A^*, so it necessarily has the form $\operatorname{ann}(B)$ for some $B \leq A$, so ϕ factors as $A^* \xrightarrow{\text{res}} B^* \xrightarrow{\psi} C$ for some ψ. By assumption, $x(\psi(\beta))$ is invertible for all $\beta \in B^* \setminus \{0\}$.

Let $\widehat{C} = \{a \in C \mid x(a) \text{ is nilpotent }\}$ be the formal neighbourhood of 0 in C, and define $\sigma\colon B^* \times \widehat{C} \to C$ by $\sigma(\beta, a) = \psi(\beta) + a$. We need to show that σ is an isomorphism. For this, we define $x_\beta(a) = x(a - \psi(\beta))$ and $y = \prod_{\beta \in B^*} x_\beta$ and $R = \mathcal{O}_C$. From the definition of an equivariant formal group, we know that $R = R^\wedge_y$, and it is clear that

$$\mathcal{O}_{B^* \times \widehat{C}} = \prod_\beta R^\wedge_{x_\beta}.$$

It will thus suffice to show that the natural map

$$R^\wedge_y \to \prod_\beta R^\wedge_{x_\beta}$$

is an isomorphism. This will follow from the Chinese Remainder Theorem if we can check that the ideal $(x_\beta(a), x_\gamma(a))$ contains 1 whenever $\beta \neq \gamma$. This is clear because modulo that ideal, we have $\psi(\beta) = a = \psi(\gamma)$, so $\psi(\beta - \gamma) = 0$, so $x(\psi(\beta - \gamma)) = 0$, but $x(\psi(\beta - \gamma))$ is invertible by assumption. Thus, C is of product type, as claimed.

Conversely, suppose that C is of product type. The vanishing locus of x is contained in $\{0\} \times \widehat{C}$, so x must be invertible on $(B^* \setminus \{0\}) \times \widehat{C}$. It follows immediately that when $\phi(\alpha) \neq 0$ we have $\phi(\alpha) \in (B^* \setminus \{0\}) \times \widehat{C}$ and so $x(\phi(\alpha))$ is invertible, as required. ☐

Corollary 8.2. *Every A-equivariant formal group over a field is of product type.*

Proof. This is immediate from the proposition. ☐

We next show how groups of product type occur in topology. For this we need to use the geometric fixed point functors $\overline{\phi}^B : \mathcal{S}_A \to \mathcal{S}_0$ for $B \leq A$. The definition and properties of these functors will be recalled in Section 10.

Theorem 8.3. *Let $\widehat{K}$ be a nonequivariant even periodic cohomology theory, with associated formal group $\widehat{C}$ over S, and let B be a subgroup of A. Define a cohomology theory K^* on $\mathcal{S}_A$ by $K^*X = \widehat{K}^*\overline{\phi}^B X$. Then K is evenly periodic, and the associated equivariant formal group is just $B^* \times \widehat{C}$ over S.*

Proof. Note that $\overline{\phi}^B S^V = S^{V^B}$ for any virtual complex representation V, and that $\overline{\phi}^B \Sigma^\infty X = \Sigma^\infty X^B$ for any based A-space X. It follows that $\pi_1 K = \pi_1 \widehat{K} = 0$ and that the periodicity isomorphism $F(S^{2n}, \widehat{K}) = \widehat{K}$ gives an isomorphism

$$K^*(X_+ \wedge S^V) = \widehat{K}^*(X_+^B \wedge S^{V^B}) = \widehat{K}^* X_+^B = K^* X_+$$

of modules over $K^* X_+$. This implies that K is evenly periodic, with $K^0(\text{point}) = \widehat{K}^0(\text{point})$ and thus $\text{spec}(K^0(\text{point}))$ is the base scheme S for $\widehat{C}$. We also have

$$K^0 P\mathcal{U} = \widehat{K}^0(P\mathcal{U})^B = \widehat{K}^*(B^* \times \mathbb{C}P^\infty) = \mathcal{O}_{B^* \times \widehat{C}},$$

so the equivariant formal group associated to K is just $B^* \times \widehat{C}$ as claimed. □

Example 8.4. Let $\widehat{K} = \widehat{K}(p,n)$ be the two-periodic version of Morava K-theory at a prime p, with height n. We define an equivariant theory $K = K(p,n,B)$ as above; this is called *equivariant Morava K-theory.* In [17] we present evidence that these theories deserve this name, because they play the expected rôle in equivariant analogues of the Hopkins-Devinatz-Smith nilpotence theorems, among other things. The same paper also explains the representing object for the theory K, and shows that we have natural isomorphisms as follows:

$$K_*(X \wedge Y) = K_*(X) \otimes_{K_*} K_*(Y)$$
$$K^* X = \text{Hom}_{K_*}(K_* X, K_*).$$

9. Equivariant formal groups over rational rings

We next prove the equivariant analogue of the well-known fact that all formal groups over a $\mathbb{Q}$-algebra are additive. We write $\widehat{G}_a$ for the ordinary additive formal group over S. If we consider formal schemes over S as functors in the usual way, this sends an $\mathcal{O}_S$-algebra R to the set $\text{Nil}(R)$ of nilpotents in R. Given a free module L of rank one over $\mathcal{O}_S$ (or equivalently, a trivialisable line bundle over S), we can instead consider the functor $R \mapsto L \otimes_{\mathcal{O}_S} \text{Nil}(R)$, which we denote by $L \otimes \widehat{G}_a$. This gives a formal group over S, noncanonically isomorphic to $\widehat{G}_a$. If C is a formal multicurve group over S, then the cotangent spaces to the fibres give a trivialisable line bundle ω_C on S. This is easily seen to be the same as $\omega_{\widehat{C}}$, where $\widehat{C}$ is the formal neighbourhood of zero, as usual. From now on we just write ω for this module. If S lies over $\text{spec}(\mathbb{Q})$ then the theory of logarithms for ordinary formal groups gives a canonical isomorphism $\widehat{C} \to \omega^{-1} \otimes \widehat{G}_a$.

Theorem 9.1. *Let (C, ϕ) be an A-equivariant formal group over a scheme S, such that the integer $n = |A|$ is invertible in $\mathcal{O}_S$. Then there is a canonical decomposition $S = \coprod_{B \leq A} S_B$, and a corresponding decomposition*

$$C \simeq \coprod S_B \times_S \widehat{C} \times B^*,$$

where $\widehat{C}$ is the formal neigbourhood of 0 in C. Moreover, if $\mathcal{O}_S$ is a $\mathbb{Q}$-algebra than $\widehat{C} \simeq \omega_C^{-1} \otimes \widehat{G}_a$ and so

$$C \simeq \coprod S_B \times_S (\omega_C^{-1} \otimes \widehat{G}_a) \times B^*.$$

Proof. Put $n = |A|$, and choose a coordinate x on C. For formal reasons we have $x(a+b) = x(a) + x(b) \pmod{x(a)x(b)}$ as functions on C^2, and it follows that $x(na) = f(a)x(a)$ for some function f on C with $f(0) = n$. Let $C[n]$ denote the closed subscheme of points of order n in C, so $C[n] = \{a \in C \mid f(a)x(a) = 0\} = \text{spf}(\mathcal{O}_C/(f.x))$. Note that S is embedded as the zero section in C with $\mathcal{O}_S = \mathcal{O}_C/x$, so in $\mathcal{O}_S$ we have $f = f(0) = n \in \mathcal{O}_S \supseteq \mathbb{Q}$, so f is invertible mod x, so $1 \in (f) + (x)$. By the Chinese remainder theorem, the scheme $C[n]$ splits as $S \amalg T$, where $T = \text{spf}(\mathcal{O}_C/f)$. Note that x is zero on S and invertible on T.

Now consider the map ϕ from A^* to the group of sections of $C[n]$ over S. Suppose that for each $\alpha \in A^*$ we either have $\phi(\alpha)(S) \subseteq T$ (and so $x(\phi(\alpha))$ is invertible) or $\phi(\alpha)(S) \subseteq S$ (so $x(\phi(\alpha)) = 0$); it then follows immediately from Proposition 8.1 that C is of product type. In general, however, it is not true that $\phi(\alpha)(S) \subseteq T$ or $\phi(\alpha)(S) \subseteq S$; instead, we can just pull back the splitting $C[n] = S \amalg T$ along the map $\phi(\alpha)\colon S \to C$ to get a splitting $S = S_\alpha \amalg T_\alpha$ with $\phi(\alpha)(S_\alpha) \subseteq S$ and $\phi(\alpha)(T_\alpha) \subseteq T$. Next, for $U \subseteq A^*$ we put

$$M_U = \bigcap_{\alpha \in U} S_\alpha \cap \bigcap_{\alpha \notin U} T_\alpha.$$

It is clear that $S = \coprod_U M_U$, and that $M_U = \emptyset$ unless U is a subgroup of A^*. Thus, if we put $S_B = M_{\mathrm{ann}(B)}$, we have a splitting $S = \coprod_B S_B$, and a corresponding splitting $C = \coprod_B C_B$, where C_B is an A-efg over S_B. It is now easy to see that $C_B = S_B \times_S \widehat{C} \times B^*$ as required. The rational statement now follows from the nonequivariant theory. □

Remark 9.2. Nonequivariantly, one knows that rational spectra are determined by their homotopy groups. This gives a classification of rational even periodic cohomology theories, as follows. Let $\mathcal{E}$ denote the category of pairs (S, L), where S is an affine scheme over $\mathbb{Q}$ and L is a trivialisable line bundle over S. The morphisms from (S_0, L_0) to (S_1, L_1) are pairs (f, g) where $f\colon S_0 \to S_1$ and g is an isomorphism $L_0 \to f^*L_1$ of line bundles over S_0. Let $\mathcal{E}'$ be the category of pairs $(S, \widehat{C})$, where S is as before and $\widehat{C}$ is a (nonequivariant) formal group over S, with morphisms defined in the analogous way. Let $\mathcal{E}''$ be the category of even periodic rational ring spectra. Then there is a contravariant equivalence $\mathcal{E}'' \to \mathcal{E}'$ sending E to $(\mathrm{spec}(E^0), \mathrm{spf}(E^0\mathbb{C}P^\infty))$, and a covariant equivalence $\mathcal{E}' \to \mathcal{E}$ sending $(S, \widehat{C})$ to $(S, \omega_{\widehat{C}})$, so the composite sends E to $(\mathrm{spec}(E^0), E^{-2})$. If E maps to (S, L) then $E^m X = \prod_n H^{m+2n}(X; L^n)$.

Now let $\mathbb{Q}\mathcal{S}_A$ denote the category of rational A-spectra. One knows that the functors $\overline{\phi}^B\colon \mathcal{S}_A \to \mathcal{S}_0$ induce an equivalence $\mathbb{Q}\mathcal{S}_A \to \prod_{B \le A} \mathbb{Q}\mathcal{S}_0$. (Note here that because A is abelian, there are no nontrivial Weyl groups or conjugacies between subgroups; we have used this to simplify the usual statement.) In particular, any evenly periodic rational equivariant cohomology theory E^* has the form

$$E^m X = \prod_B E_B^m \overline{\phi}^B X = \prod_{B,n} H^{m+2n}(\overline{\phi}^B X; \omega_B^n)$$

for some family $\{E_B^*\}_{B \le A}$ of nonequivariant even periodic rational theories, with associated formal groups $\widehat{C}_B$ and line bundles ω_B over $S_B = \mathrm{spec}(E_B^0)$. By taking $X = 1$ and then $X = P\mathcal{U}$ we find that $S := \mathrm{spec}(E^0) = \coprod_B S_B$ and

$$C := \mathrm{spf}(E^0 P\mathcal{U}) = \coprod_B B^* \times \widehat{C}_B = \coprod S_B \times (\omega_B^{-1} \otimes \widehat{G}_a) \times B^*.$$

In other words, the topological picture is perfectly parallel to the algebraic one.

The following slight extension can easily be proved in the same way.

Corollary 9.3. *Let (C, ϕ) be an A-equivariant formal group over a scheme S, such that $\mathcal{O}_S$ is an algebra over $\mathbb{Z}_{(p)}$. There is of course a unique splitting $A = A_0 \times A_1$, where A_0 is a p-group and p does not divide $|A_1|$. Let $C_0 \subseteq C$ be the formal neighbourhood of $[\phi(A_0^*)]$, and let $\phi_0\colon A_0^* \to C_0$ be the restriction of ϕ. Then there is a canonical decomposition $S = \coprod_{B \le A_1} S_B$, and a corresponding decomposition*

$$C \simeq \coprod S_B \times_S C_0 \times B^*,$$

such that over S_B, the map ϕ is the product of ϕ_0 and the restriction map $A_1^* \to B^*$. □

10. Equivariant formal groups of pushout type

We next consider a slightly different generalization of the notion of a group of product type.

Definition 10.1. Suppose we have a subgroup $B \le A$ and a formal multicurve group C', with a map $\phi'\colon (A/B)^* \to C'$ making it an A/B-equivariant formal group. There is an evident embedding $(A/B)^* \to A^*$, which we can use to form a pushout

$$\begin{array}{ccc} (A/B)^* & \xrightarrow{\phi'} & C' \\ \downarrow & & \downarrow \\ A^* & \xrightarrow[\phi]{} & C. \end{array}$$

If we choose a transversal T to $(A/B)^*$ in A^*, then the underlying scheme of C is just $\coprod_{\alpha \in T} C$. This implies that the formation of the pushout is compatible with base change, and that C is an A-equivariant formal group. Formal groups constructed in this way are said to be of *pushout type.* (The case where $\phi' = 0$ evidently gives groups of product type.)

We next examine how formal groups of this kind can arise in equivariant topology. For this, we need to recall the various different change of group functors and fixed-point functors for A-spectra.

Given a homomorphism $\zeta\colon B \to A$, there is a pullback functor $\zeta^*\colon \mathcal{S}_A \to \mathcal{S}_B$, which preserves smash products and function spectra. (Note that if ζ is not injective, then $\zeta^*\mathcal{U}_A$ is not a complete B-universe, so the definition of ζ^* contains an implicit change of universe.) If ζ is the inclusion of a subgroup then ζ^* is called *restriction* and written res^A_B. This functor has a left adjoint written $X \mapsto A \wedge_B X$, and a right adjoint written $X \mapsto F_B(A_+, X)$. These two adjoints are actually isomorphic, by the generalized Wirthmüller isomorphism [11, Theorem II.6.2].

If ζ is the projection $A \to A/B$ then ζ^* is called *inflation*. This has a right adjoint functor $\lambda^B\colon \mathcal{S}_A \to \mathcal{S}_{A/B}$, which we call the Lewis-May fixed point functor. The adjunction is discussed in [11, Section II.7]; there $\lambda^B X$ is written X^B. One can check that the following square commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\lambda^C} & \mathcal{S}_{A/C} \\ {\scriptstyle \mathrm{res}^A_B}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{A/C}_{B/C}} \\ \mathcal{S}_B & \xrightarrow[\lambda^C]{} & \mathcal{S}_{B/C}. \end{array}$$

It will be convenient to write

$$\overline{\lambda}^B = \mathrm{res}^{A/B}_0 \lambda^B = \lambda^B \,\mathrm{res}^A_B\colon \mathcal{S}_A \to \mathcal{S}_0.$$

The usual equivariant homotopy groups of X are defined by $\pi^B_* X = \pi_* \overline{\lambda}^B X$. The functors λ^B and $\overline{\lambda}^B$ do not preserve smash products, and there is no sense in which λ^B acts as the identity on B-fixed objects.

Lewis and May also introduce another functor $\phi^B\colon \mathcal{S}_A \to \mathcal{S}_{A/B}$, called the *geometric fixed point functor*. To explain the definition, let V be a complex representation of A. We write χ_V for the usual inclusion $S^0 \to S^V$, which can be regarded as an element of the $R(A)$-graded homotopy ring $\pi_* S^0$ in dimension $-V$. It is easily seen to be zero if $V^A \neq 0$, but it turns out to be nonzero otherwise. It is also clear that $\chi_{V\oplus W} = \chi_V \chi_W$.

By dualizing the standard cofibration $S(V)_+ \to S^0 \xrightarrow{\chi_V} S^V$, we see that $D(S(V)_+)$ deserves to be called S^0/χ_V. On the other hand, we have

$$S^0[\chi_V^{-1}] = \varinjlim (S^0 \xrightarrow{\chi_V} S^V \xrightarrow{\chi_V} S^{2V} \to \dots) = S^{\infty V}.$$

It follows that for any $X \in \mathcal{S}_A$, the spectrum $X[\chi_V^{-1}] = X \wedge S^{\infty V}$ is a Bousfield localization of X, or more specifically, the finite localization away from the thick ideal generated by S^0/χ_V. There is another characterization as follows. Let $\mathcal{F}$ be the family of those subgroups $A' \leq A$ such that $V^{A'} \neq 0$, and let $\mathcal{C}$ be the thick ideal generated by $\{A/A'_+ \mid A' \in \mathcal{F}\}$. It is not hard to see that $(S^{\infty V})^{A'}$ is contractible for $A' \in \mathcal{F}$, and equivalent to S^0 for $A' \notin \mathcal{F}$. It is well-known that up to homotopy there is a unique space with these properties, denoted by $\widetilde{E}\mathcal{F}$, and that $X \wedge \widetilde{E}\mathcal{F}$ is the finite localization of X away from $\mathcal{C}$. It follows that $\mathcal{C}$ is the same as the thick ideal generated by S^0/χ_V.

Now fix a subgroup $B \leq A$, and take

$$V = V_{A,B} := \mathbb{C}[A] \ominus (\mathbb{C}[A]^B) = \bigoplus_{\alpha \in A^* \setminus \mathrm{ann}(B)} L_\alpha.$$

In this context, we write $\chi_{A,B}$ for χ_V, and we also write χ_A for $\chi_{A,A}$. We also put $\mathcal{F}[B] = \{C \leq A \mid B \not\leq C\}$, and note that $\widetilde{E}\mathcal{F}[B] = S^{\infty V}$. The geometric fixed-point functor $\phi^B\colon \mathcal{S}_A \to \mathcal{S}_{A/B}$ is defined by

$$\phi^B X = \lambda^B(X[\chi_{A,B}^{-1}]) = \lambda^B(X \wedge \widetilde{E}\mathcal{F}[B]).$$

(In [11] the functor ϕ^B is actually defined in a different way, but the above description is proved as Theorem II.9.8). Let $\pi\colon A \to A/B$ be the projection. One can check that ϕ^B preserves smash products [11, Proposition 9.12], the composite

$$\mathcal{S}_{A/B} \xrightarrow{\pi^*} \mathcal{S}_A \xrightarrow{\phi^B} \mathcal{S}_{A/B}$$

is the identity [11, Proposition 9.10], and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\phi^C} & \mathcal{S}_{A/C} \\ {\scriptstyle \mathrm{res}^A_B}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{A/C}_{B/C}} \\ \mathcal{S}_B & \xrightarrow[\phi^C]{} & \mathcal{S}_{B/C}. \end{array}$$

Moreover, for any A-space X we have $\phi^B\Sigma^\infty X = \Sigma^\infty X^B$ [11, Corollary 9.9]. It will be convenient to write

$$\overline{\phi}^B = \mathrm{res}_0^{A/B}\phi^B = \phi^B \mathrm{res}^A_B : \mathcal{S}_A \to \mathcal{S}_0.$$

This again preserves smash products, and it is known that a spectrum $X \in \mathcal{S}_A$ satisfies $X = 0$ iff $\overline{\phi}^B X = 0$ in $\mathcal{S}_0$ for all $B \leq A$. We will also need the following property:

Lemma 10.2. *Suppose that $B \leq A$, and write $\chi = \chi_{A,B}$. Then for $X, Y \in \mathcal{S}_A$ there are natural equivalences*

$$\lambda^B F(X, Y[\chi^{-1}]) = \lambda^B F(X[\chi^{-1}], Y[\chi^{-1}]) = F(\phi^B X, \phi^B Y).$$

Proof. First note that the map $W \to W[\chi^{-1}]$ is an equivalence iff W is concentrated over B as defined in [11, page 109]. Let $\mathcal{C}$ be the category of such W, so we have functors $\phi^B = \lambda^B : \mathcal{C} \to \mathcal{S}_{A/B}$ and $\psi : \mathcal{S}_{A/B} \to \mathcal{C}$ given by $\psi(Z) = (\pi^* Z)[\chi^{-1}]$. We see from [11, Corollary II.9.6] that ϕ^B and ψ are mutually inverse equivalences, and it follows that

$$[X, Y[\chi^{-1}]]^B_* = [X[\chi^{-1}], Y[\chi^{-1}]]^B_* = [\phi^B X, \phi^B Y]^{A/B}_*.$$

Now consider $W \in \mathcal{S}_{A/B}$ and replace X by $(\pi^* W) \wedge X$ in the above. We deduce that

$$[W, \lambda^B F(X, Y[\chi^{-1}])]^{A/B}_* = [W, \lambda^B F(X[\chi^{-1}], Y[\chi^{-1}])]^{A/B}_* = [W, F(\phi^B X, \phi^B Y)]^{A/B}_*.$$

The claim now follows by the Yoneda lemma. □

Theorem 10.3. *Let E' be an A/B-equivariant periodically orientable ring spectrum, with associated equivariant formal group $(A/B)^* \xrightarrow{\phi'} C'$. Let $\pi : A \to A/B$ be the projection, and put $E = (\pi^* E')[\chi_A^{-1}]$. Then E is an A-equivariant periodically orientable ring spectrum, and for all $X \in \mathcal{S}_A$ we have*

$$E_* X = E'_* \phi^B X$$
$$E^* X = (E')^* \phi^B X.$$

Moreover, the formal group associated to E is the pushout of C' along the inclusion $(A/B)^* \to A^*$.

Proof. Because π^* preserves smash products, it is clear that $\pi^* E'$ is a commutative A-equivariant ring spectrum, and so the same is true of E. We saw earlier that $\phi^B \pi^* = 1$, so $\phi^B E = E'$. Also, we have $E \wedge X = E \wedge X[\chi_A^{-1}]$, so

$$\lambda^B(E \wedge X) = \phi^B(E \wedge X) = \phi^B(E) \wedge \phi^B(X) = E' \wedge \phi^B(X).$$

We can apply $\lambda^{A/B}$ to this to see that $\lambda^A(E \wedge X) = \lambda^{A/B}(E' \wedge \phi^B(X))$, and by applying π_* we deduce that $E_* X = E'_* \phi^B X$.

For the corresponding statement in cohomology, we see using Lemma 10.2 that $\lambda^B F(X, E) = F(\phi^B X, \phi^B E) = F(\phi^B X, E')$. We again apply the functor $\pi_* \lambda^{A/B}(-)$ to see that $E^* X = E'^* \phi^B X$, as claimed.

In particular, if X is an A-space we have $\phi^B \Sigma^\infty X = \Sigma^\infty X^B$ and so $E^* X = E'^* X^B$. Thus, if we put $S = \mathrm{spec}(E'^0(\mathrm{point}))$, then S is also the same as $\mathrm{spec}(E^0(\mathrm{point}))$. We next consider the space PV, where V is a representation of A. We can split V into isotypical parts for the action of B, say $V = \bigoplus_\beta V[\beta]$, where $V[\beta]$ is a sum of representations L_α with $\alpha|_B = \beta$. We then have $(PV)^B = \coprod_\beta PV[\beta]$, and so $E^* PV = \prod_\beta E'^* PV[\beta]$. Using this, it is easy to see that E is periodically orientable. Next, consider the space $P\mathcal{U}_A$, so $(P\mathcal{U}_A)^B = \coprod_\beta P(\mathcal{U}_A[\beta])$. The space $P(\mathcal{U}_A[0])$ is canonically identified with $P\mathcal{U}_{A/B}$, so $\mathrm{spf}(E'^0 P\mathcal{U}_A[0]) = C'$. For $\beta \neq 0$, we can choose $\tilde{\beta} \in A^*$ extending β, and then tensoring with $L_{-\tilde{\beta}}$ gives an equivalence $\theta : P\mathcal{U}_A[\beta] \simeq P\mathcal{U}_{A/B}$. If we change $\tilde{\beta}$ by an element $\gamma \in (A/B)^*$, then θ changes by the automorphism $\tau_{-\gamma}$ of $P\mathcal{U}_{A/B}$. Using this, it is not hard to identify the curve $C = \mathrm{spf}(E^0 P\mathcal{U}_A) = \coprod \mathrm{spf}(E'^0 P\mathcal{U}_A[\beta])$ with the pushout of C' along the map $(A/B)^* \to A^*$. □

11. Equivariant Morava E-theory

Let $\widehat{C}_0$ be the standard p-typical formal group of height n over $S_0 = \text{spec}(\mathbb{F}_p)$. We write $\widehat{K}$ for the two-periodic Morava K-theory spectrum whose associated formal group is $\widehat{C}_0$, so $\widehat{K}_* = \mathbb{F}_p[u^{\pm 1}]$ with $|u| = 2$. This formal group has a universal deformation $\widehat{C}_1$ over $S_1 := \text{spf}(\mathbb{Z}_p[\![u_1, \dots, u_{n-1}]\!])$. We write $\widehat{E}$ for the corresponding Landweber-exact cohomology theory, and refer to it as Morava E-theory. Now suppose we have a finite abelian group A and a subgroup B. We define $C_0 = B^* \times \widehat{C}_0$, which is an A-efg of product type over S_0, associated to the equivariant Morava K-theory $K^*X := \widehat{K}^*\overline{\phi}^B X$. We can also define an A/B-equivariant cohomology theory by $X \mapsto \widehat{E}^*X_{h(A/B)}$, as in Section 6. The associated equivariant formal group is $\widehat{C}_2 = \widehat{C}_1 \times_{S_1} S$ over S, where $S = \text{Hom}((A/B)^*, \widehat{C}_1)$. We then perform the construction in Section 10. This gives an A-equivariant theory $E = E(p, n, B)$, defined by

$$E^*X = \widehat{E}^*((\phi^B X)_{h(A/B)}),$$

whose associated equivariant formal group is the pushout of $\widehat{C}_2$ along the inclusion $(A/B)^* \to A^*$. We write C for this pushout, and we refer to E as *equivariant Morava E-theory*. In [17] we give some evidence that this name is reasonable, related to the theory of Bousfield classes and nilpotence. Here we give a further piece of evidence, based on formal group theory.

We first note that S_0 is a closed subscheme of S_1, which is in turn a closed subscheme of $S = \text{Hom}((A/B)^*, \widehat{C}_1)$ (corresponding to the zero homomorphism). The restriction of C to S_1 is just $B^* \times \widehat{C}_1$, and the restriction of this to S_0 is just C_0. The inclusion $C_0 \to C$ corresponds to a ring map $\mathcal{O}_C \to \mathcal{O}_{C_0}$, or equivalently $E^0 P\mathcal{U} \to K^0 P\mathcal{U}$. It can be shown that this comes from a natural map $E^*X \to K^*X$ of cohomology theories. Indeed, there is certainly a nonequivariant map $q\colon \widehat{E} \to \widehat{K}$. Moreover, up to homotopy there is a unique map $A/B \to E(A/B)$ of A/B-spaces, which gives a natural map

$$\text{res}(Y) = (A/B_+ \wedge Y)/(A/B) \to (E(A/B)_+ \wedge Y)/(A/B) = Y_{h(A/B)}$$

for A/B-spectra Y. If $Y = \phi^B X$ then $\text{res}(Y) = \overline{\phi}^B X$ and so we get a map

$$E^*X = \widehat{E}^*(\phi^B X)_{h(A/B)} \to \widehat{E}^*\overline{\phi}^B \xrightarrow{q_*} \widehat{K}^*\overline{\phi}^B X = K^*X,$$

as required.

Definition 11.1. A *deformation* of the A-efg C_0 over S_0 consists of an A-efg C' over a base S' together with a commutative square

$$\begin{array}{ccc} C_0 & \xrightarrow{\tilde{f}} & C' \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow[f]{} & S' \end{array}$$

such that

(a) f is a closed inclusion, and S' is a formal neighbourhood of $f(S_0)$
(b) $\tilde{f}$ induces an isomorphism $C_0 \to f^*C$ of A-efg's over S_0.

If C' and C'' are deformations, a morphism between them means a commutative square

$$\begin{array}{ccc} C' & \xrightarrow{\tilde{g}} & C'' \\ \downarrow & & \downarrow \\ S' & \xrightarrow[g]{} & S'' \end{array}$$

such that $\tilde{g}$ induces an isomorphism $C' \to g^*C''$ of A-efg's over S'. A *universal deformation* means a terminal object in the category of deformations.

As mentioned previously, the formal group $\widehat{C}_1$ associated to $\widehat{E}$ is the universal deformation of the formal group $\widehat{C}_0$ associated to $\widehat{K}$. Equivariantly, we have the following analogue.

Theorem 11.2. *The A-equivariant formal group C (associated to equivariant Morava E-theory) is the universal deformation of C_0 (associated to equivariant Morava K-theory).*

Proof. Suppose we have an A-efg (C', ϕ') over S' equipped with maps $(f, \tilde{f})$ making it a deformation of C_0. We will identify S_0 with $f(S_0)$ and thus regard it as a closed subscheme of S'. Similarly, we regard C_0 as the closed subscheme $C'|_{S_0}$ of C'. Note that S' is a formal neighbourhood of S_0, and it follows that C' is a formal neighbourhood of C_0. We choose a coordinate x' on C', and note that it restricts to give a coordinate on C_0.

Now let $\widehat{C}'$ denote the formal neighbourhood of the zero section in C'. We have $(\widehat{C}')|_{S_0} = \widehat{C}_0$, so we can regard $\widehat{C}'$ as a deformation of the ordinary formal group $\widehat{C}_0$. As $\widehat{C}_1$ is the universal deformation of $\widehat{C}_0$, this gives us a pullback square

$$\begin{array}{ccc} \widehat{C}' & \xrightarrow{\tilde{g}} & \widehat{C}_1 \\ \downarrow & & \downarrow \\ S' & \xrightarrow[g]{} & S_1. \end{array}$$

Next, suppose we have $\alpha \in (A/B)^* \subset A^*$, giving a section $\phi'(\alpha)$ of C' and an element $x'(\phi'(\alpha)) \in \mathcal{O}_{S'}$. As $C'|_{S_0} = C_0 = B^* \times \widehat{C}_0$ and $\alpha|_B = 0$ we have $\phi'(\alpha)|_{S_0} = 0$, so $x'(\phi'(\alpha))$ maps to 0 in $\mathcal{O}_{S_0}$. As S' is a formal neighbourhood of S_0, it follows that $x'(\phi'(\alpha))$ is topologically nilpotent in $\mathcal{O}_{S'}$, and thus that $\phi'(\alpha)$ is actually a section of $\widehat{C}'$. Thus, $\tilde{g} \circ \phi'$ gives a map $(A/B)^* \to \widehat{C}_1$, which is classified by a map $h\colon S' \to \mathrm{Hom}((A/B)^*, \widehat{C}_1) = S$. The maps $\tilde{g}$ and h combine to give a map

$$\tilde{h}\colon \widehat{C}' \to h^*\widehat{C} = h^*(\widehat{C}_1 \times_{S_1} S) = g^*\widehat{C}_1.$$

This can be regarded as an isomorphism of A/B-equivariant formal groups.

Next, the decomposition $C_0 = B^* \times \widehat{C}_0 = \coprod_{\beta \in B^*} \widehat{C}_0$ gives orthogonal idempotents $e_\beta \in \mathcal{O}_{C_0}$ with $\sum_\beta e_\beta = 1$. As C' is a formal neighbourhood of C_0, these can be lifted to orthogonal idempotents in $\mathcal{O}_{C'}$, giving a decomposition $C' = \coprod_\beta C'_\beta$ say. One can check that $C'_\beta = \phi'(\alpha) + \widehat{C}'$ for any $\alpha \in A^*$ with $\alpha|_B = \beta$, and it follows that C' is just the pushout of the map $\phi'\colon (A/B)^* \to \widehat{C}'$ and the inclusion $(A/B)^* \to A^*$. It follows in turn that $\tilde{h}$ extends to give an isomorphism $C' \to g^*C$, and thus a morphism $C' \to C$ of deformations. All steps in this construction are forced, so one can check that this morphism is unique. This means that C is the universal deformation of C_0, as claimed. □

12. A completion theorem

Suppose we have an A-equivariant formal group (C, ϕ), and a subgroup $B \leq A$, giving a subgroup $(A/B)^* \leq A^*$. Let S_0 be the closed subscheme of S where $\phi((A/B)^*) = 0$. Equivalently, if we put $e_\alpha = x(\phi(-\alpha))$ and $J = (e_\alpha \mid \alpha \in (A/B)^*)$, then $S_0 = V(J) = \mathrm{spec}(\mathcal{O}_S/J)$. If we put $C_0 = S_0 \times_S C$ then ϕ induces a map $\psi\colon B^* \times S_0 \to C_0$ making C_0 into a B-equivariant formal group over S_0. Next, we put $S_1 = \varinjlim_m \mathrm{spec}(\mathcal{O}_S/J^m) = \mathrm{spf}((\mathcal{O}_S)^\wedge_J)$, the formal neighbourhood of S_1 in S, and $C_1 = S_1 \times_S C$. This is an A-equivariant formal group over S_1 for which $\phi((A/B)^*)$ is infinitesimally close to 0.

Now suppose that C comes from an A-equivariant periodically orientable theory E. We would like to interpret C_0 and C_1 topologically.

Proposition 12.1. *Let E_0 be the B-spectrum $\mathrm{res}^A_B(E)$, representing the theory $E^*(A \times_B Y)$ for B-spaces Y. Let C'_0/S'_0 be the associated B-equivariant formal group. Then there is a map $S'_0 \to S_0$ (which may or may not be an isomorphism) and an isomorphism $C'_0 = C_0 \times_{S_0} S'_0$.*

Proof. We have $S'_0 = \mathrm{spec}(\pi_0 E_0) = \mathrm{spec}(E^0 A/B)$, so there is a natural map $S'_0 \to S$. Moreover, we have $P\mathcal{U}_B \simeq \mathrm{res}^A_B P\mathcal{U}_A$, which gives an isomorphism $A \times_B P\mathcal{U}_B \simeq A/B \times P\mathcal{U}_A$ and thus

$$E^0_0 P\mathcal{U}_B \simeq E^0(A/B \times P\mathcal{U}_A) = E^0(A/B) \otimes_{E^0} E^0 P\mathcal{U}_A.$$

This shows that the formal group for E_0 is just $C'_0 := C \times_S S'_0$. All that is left is to check that the map $S'_0 \to S$ factors through S_0, so C'_0 can also be described as $C_0 \times_{S_0} S'_0$. To see this, note that ϕ comes from the inclusion $j\colon A^* = \pi_0^A P\mathcal{U} \to P\mathcal{U}$, so the corresponding map ϕ'_0 over S'_0 comes from the map $1 \times j\colon (A/B) \times A^* \to (A/B) \times P\mathcal{U}$. Using the isomorphism

$$\begin{aligned}[][(A/B) \times A^*, (A/B) \times P\mathcal{U}]^A &= [A^*, (A/B) \times P\mathcal{U}]^B \\ &= \mathrm{Map}(A^*, \pi_0^B((A/B) \times P\mathcal{U})) \\ &= \mathrm{Map}(A^*, (A/B) \times B^*)\end{aligned}$$

we see that the restriction of $(1 \times j)$ to $(A/B) \times (A/B)^*$ is null, so that $\phi'_0((A/B)^*) = 0$ as claimed. ∎

If E is the complex K-theory spectrum KU_A, then we saw earlier that $S = \mathrm{Hom}(A^*, \mathbb{G}_m)$ and so

$$S_0 = \{\phi \in \mathrm{Hom}(A^*, \mathbb{G}_m) \mid \phi((A/B)^*) = 0\} = \mathrm{Hom}(B^*, \mathbb{G}_m).$$

On the other hand, it is well-known that $KU^*_A(A \times_B Y) = KU^*_B Y$ so $E_0 = KU_B$ so $S'_0 = \mathrm{Hom}(B^*, \mathbb{G}_m) = S_0$. A similar argument works for theories of the form $E^*X = \widehat{E}^*X_{hA}$ where $\widehat{E}$ is $K(n)$-local as in Section 6, in which case we have $S = \mathrm{Hom}(A^*, \widehat{C})$ and $S_0 = S'_0 = \mathrm{Hom}(B^*, \widehat{C})$. At the other extreme, for theories of the form $E^*X = \widehat{E}^*(\mathrm{res}^A_0(X))$, we have $S_0 = S$ and $S'_0 = (A/B) \times S$.

We next consider C_1. Recall that there is an A-space $E[\leq B]$ characterised by the property that $E[\leq B]^C$ is contractible for $C \leq B$ and empty for $C \not\leq B$. The first approximation would be to consider the ring spectrum $F(E[\leq B]_+, E)$. However, as S_1 is a formal scheme rather than an affine scheme, we need a pro-spectrum rather than a spectrum. The solution is to define $F_\bullet(X_+, E)$ to be the pro-system of ring spectra $F(X_{\alpha+}, E)$, where X_α runs over finite subcomplexes of X, and to put $E_1 = F_\bullet(E[\leq B]_+, E)$. The desired description of $E^*_1 P\mathcal{U}$ is a kind of completion theorem in the style of Atiyah-Segal, so we expect to need finiteness hypotheses. However, with these hypotheses, we have an exact result rather than an approximate one as in the previous proposition.

Theorem 12.2. *Suppose that $E^*(point)$ is a Noetherian ring, and that $E^*(A/C)$ is finitely generated over it for all $C \leq A$. Then the A-equivariant formal group associated to E_1 is C_1.*

Proof. This is essentially taken from [7]. Choose generators $\alpha_1, \dots, \alpha_r$ for $(A/B)^*$, let L_i be the one-dimensional representation corresponding to α_i and let χ_i denote the inclusion $S^0 \to S^{L_i}$. There is a canonical Thom class u_i in $E^0 S^{L_i}$, and $\chi_i^*(u_i)$ is the Euler class $e_i = x(\phi(-\alpha_i))$. One checks easily that the space $P := \prod_i S(\infty L_i)$ is a model for $E[\leq B]$, and the spaces $T(m) := \prod_i S(mL_i)$ form a cofinal system of finite subcomplexes, so E_1 is equivalent to the tower of ring spectra $F(T(m)_+, E) = D(T(m)_+) \wedge E$. Next, by taking the Spanier-Whitehead dual of the cofibration $S(mL_i)_+ \to S^0 \xrightarrow{\chi_i^m} S^{mL_i}$, we see that $D(S(mL_i)_+)$ deserves to be called S/χ_i^m, and so $D(T(m)_+)$ deserves to be called $S/(\chi_1^m, \dots, \chi_r^m)$. This suggests that $\pi_*(E \wedge D(T(m)_+))$ should be E_*/J_m, where $J_m = (u_1^m, \dots, u_r^m) \leq E_*$. Unfortunately, there are correction terms. More precisely, the cofibration displayed above gives a two-stage filtration of $D(S(mL_i)_+)$ for each i, and by smashing these together we get a $(r+1)$-stage filtration of $D(T(m)_+)$, and thus a spectral sequence converging to $\pi_*(E \wedge D(T(m)_+))$. The first page is easily seen to be the Koszul complex for the sequence $e_1^m, \dots, e_r^m$, so the bottom line of the second page is E_*/J_m, and the remaining lines are higher Koszul homology groups. The filtrations are compatible as m varies, so we get a spectral sequence in the abelian category of pro-groups converging to $\pi_* E_1$. In the second page, the bottom line is the tower $\{E_*/J_m\}_{m \geq 0}$, and the remaining lines are pro-trivial by [7, Lemma 3.7]. It follows that $\pi_* E_1 \simeq \{E_*/J_m\}$ as pro-groups, and so the formal scheme corresponding to $\pi_0 E_1$ is $\varinjlim_m \mathrm{spec}(E^0/J_m) = \varinjlim_m \mathrm{spec}(E^0/J^m) = S_1$. We now replace E by $F(P(n.\mathbb{C}[A])_+, E)$ and then take the limit as n tends to infinity to conclude that $\mathrm{spf}(E^0_1 P\mathcal{U}) = C \times_S S_1 = C_1$ as claimed. ∎

Remark 12.3. Using the same circle of ideas one proves that the kernel of the map $E^0/J \to E^0(A/B)$ is nilpotent, so the map $S'_0 \to S_0$ is dominant; compare [7, Theorem 1.4].

13. A counterexample

Here we exhibit a $\mathbb{Z}/2$-equivariant formal group C with a number of unusual properties, which are only possible because the base scheme S is not Noetherian. The phenomena described here are the main obstruction to our understanding of the equivariant Lazard ring.

For any A-equivariant formal group (C, ϕ), there is a natural map $\psi \colon A^* \times \widehat{C} \to C$ given by $\psi(\alpha, a) = \phi(\alpha) + a$. As C is a formal neighbourhood of $[\phi(A^*)]$, it is natural to expect that ψ should be an epimorphism, or equivalently that the map $\psi^* \colon \mathcal{O}_C \to \prod_\alpha \mathcal{O}_{\widehat{C}}$ should be injective. The key feature of the example to be constructed here is that ψ^* is not in fact injective.

Start with $k_0 = \mathbb{F}_2[e]$, let M be the module $\mathbb{F}_2[e^{\pm 1}]/\mathbb{F}_p[e]$, and let k be the square-zero extension $k_0 \oplus M$. More explicitly, k is generated over k_0 by elements $u_1, u_2, \dots$ subject to $eu_{i+1} = u_i$ (with u_0 interpreted as 0) and $u_i u_j = 0$. Put $S = \mathrm{spec}(k)$.

Next, let R be the completion of $k[x]$ at the element $y = x^2 + ex$, so $R = k[\![y]\!]\{1, x\}$, and put

$$C = \mathrm{spf}(R) = \{x \in \mathbb{A}^1_S \mid x^2 + ex \text{ is nilpotent } \}.$$

This is a subgroup of $\mathbb{A}^1_S$ under addition. In the corresponding Hopf algebra structure on R, the elements x and y are both primitive. There is a homomorphism $\phi\colon \mathbb{Z}/2 \to C$ sending 0 to 0 and 1 to e. The corresponding divisor is just R/y, and as R is complete at y, we deduce that (C, ϕ) is an equivariant formal group.

Next, we can define maps $\lambda_0, \lambda_a \colon R \to k[\![t]\!]$ by

$$\begin{aligned}\lambda_0(x) &= t\\ \lambda_a(x) &= t + e\\ \lambda_0(y) &= \lambda_a(y) = t^2 + te.\end{aligned}$$

The map $\psi^*\colon \mathcal{O}_C \to \prod_\alpha \mathcal{O}_{\widehat{C}}$ is just the map $(\lambda_0, \lambda_a)\colon R \to k[\![t]\!] \times k[\![t]\!]$. Now consider the element

$$f = \sum_{k\geq 0} u_{1-2^{k+1}} y^{2^k} \in R.$$

We then have

$$\begin{aligned}\lambda_0(f) &= \sum_{k\geq 0} u_{1-2^{k+1}} (t^2 + et)^{2^k}\\ &= \sum_{k\geq 0} u_{1-2^{k+1}} t^{2^{k+1}} + \sum_{k\geq 0} u_{1-2^{k+1}} e^{2^k} t^{2^k}\\ &= \sum_{k\geq 0} u_{1-2^{k+1}} t^{2^{k+1}} + \sum_{k\geq 0} u_{1-2^k} t^{2^k}\\ &= u_{1-2^0} t^{2^0} = u_0 t = 0.\end{aligned}$$

We also have $\lambda_a(f) = 0$ by the same argument, so $\psi^*(f) = 0$.

14. Divisors

We now return to the purely algebraic theory of formal multicurves and their divisors.

Recall that a divisor on C is a regular hypersurface $D \subseteq C$ such that $\mathcal{O}_D$ is a finitely generated projective module over $\mathcal{O}_S$, which is discrete in the quotient topology. We also make the following temporary definition; one of our main tasks in this section is to show (in Proposition 14.15) that it is equivalent to the preceeding one. (For divisors of degree one, this follows from Corollary 2.10.)

Definition 14.1. A *weak divisor* on C is a closed subscheme $D \subset C$ that is finite and very flat over S (so $\mathcal{O}_D$ is a discrete finitely generated projective module over $\mathcal{O}_S$). Thus, a weak divisor $D = \mathrm{spf}(R/J)$ is a divisor iff the ideal J is open and generated by a regular element. If y is a good parameter on C, we note that J is open iff $y^N \in J$ for $N \gg 0$.

If $D_0 = \mathrm{spf}(R/J_0)$ is a divisor and $D_1 = \mathrm{spf}(R/J_1)$ is a weak divisor then one checks that the scheme $D_0 + D_1 := \mathrm{spf}(R/(J_0 J_1))$ is again a weak divisor.

Definition 14.2. Now suppose we have a map $q\colon T \to S$ of schemes which is finite and very flat, so that $\mathcal{O}_T$ is a discrete finitely generated projective module over $\mathcal{O}_S$. If $g \in \mathcal{O}_T$ then multiplication by g gives an $\mathcal{O}_S$-linear endomorphism μ_g of $\mathcal{O}_T$, whose determinant we denote by $N_q(g)$ or $N_{T/S}(g)$.

Definition 14.3. Fix a difference function d on C. For any weak divisor D on C over, we can regard d by restriction as a function on $D \times_S C$. We also have a projection $q\colon D \times_S C \to C$, and we put

$$f_D = N_q(d) = N_{(D\times_S C)/C}(d) \in \mathcal{O}_C.$$

We will eventually show that $D = \mathrm{spf}(\mathcal{O}_C/f_D)$.

Remark 14.4. Consider the case where C is an ordinary formal group, with coordinate x and associated formal group law F. We then have $\mathcal{O}_C = \mathcal{O}_S[\![x]\!]$ and $\mathcal{O}_{C\times C} = \mathcal{O}_S[\![x_0, x_1]\!]$, and $d = x_1 -_F x_0$. If D has the form $\sum_i [u_i]$ for some family of sections u_i, then we have elements $a_i = x(u_i) \in \mathcal{O}_S$ and we will see that $f_D = \prod_i (x -_F a_i)$. This is a unit multiple of the Chern polynomial $g_D = \prod_i (x - a_i)$, and it is familiar that $D = \mathrm{spf}(\mathcal{O}_C/g_D)$, so $D = \mathrm{spf}(\mathcal{O}_C/f_D)$ also. In the multicurve case, one can still define g_D (as the norm of the function $(a, b) \mapsto x(b) - x(a)$) and we find that it is divisible by f_D, but g_D/f_D need not be invertible so $\mathcal{O}_C/g_D \neq \mathcal{O}_D$.

Lemma 14.5. *Let R be a ring, P a finitely generated projective R-module, and α an automorphism of P. Then α is injective iff $\det(\alpha)$ is a regular element.*

Proof. After localising we may assume that $P = R^d$ for some d, and α is represented by a $d \times d$ matrix A. If $\det(A)$ is regular, the equation $\operatorname{adj}(A)A = \det(A)I_d$ implies immediately that α is injective. Conversely, suppose that α is injective. As P is flat, it follows that $\alpha^{\otimes d} : P^{\otimes d} \to P^{\otimes d}$ is also injective. It is easy to check with bases that $\lambda^d P$ is naturally isomorphic to the image of the antisymmetrisation map $P^{\otimes d} \to P^{\otimes d}$. In particular, it embeds naturally in $P^{\otimes d}$, and it therefore follows that $\lambda^d \alpha$ is injective. On the other hand, $\lambda^d P$ is an invertible R-module, so $\operatorname{End}(\lambda^d P) = R$, and $\lambda^d(\alpha) = \det(\alpha)$ under this isomorphism. It follows that $\det(\alpha)$ is regular as claimed. □

Corollary 14.6. *For any weak divisor D on C, the element $f_D \in \mathcal{O}_C$ is regular.*

Proof. Take $R = \mathcal{O}_C$ and $P = \mathcal{O}_{D \times_S C}$ and $\alpha = \mu_d$. We know from Lemma 2.9 that α is injective, and the claim follows. □

Lemma 14.7. *Let $q : T \to S$ be finite and very flat, and let g be a function on T. If there is a section $u : S \to T$ such that $g \circ u = 0$ then $N_q(g) = 0$.*

Proof. Put $J = \ker(u^* : \mathcal{O}_T \to \mathcal{O}_S)$, so $g \in J$. We have a short exact sequence of $\mathcal{O}_S$ modules $J \to \mathcal{O}_T \xrightarrow{u^*} \mathcal{O}_S$, which is split by the map $q^* : \mathcal{O}_S \to \mathcal{O}_T$. The sequence is preserved by μ_f, and $\mu_f(\mathcal{O}_T) = \mathcal{O}_T.f \leq J$ so the induced map on the cokernel is zero. Zariski-locally on S we can choose bases adapted to the short exact sequence and it follows easily that $\det(\mu_f) = 0$ as claimed. □

Corollary 14.8. *The function $f_D \in \mathcal{O}_C$ vanishes on D.*

Proof. We have $f_D|_D = N_{q'}(d)$, where $q' : D \times_S D \to D$ is the projection on the second factor. The diagonal map $\delta : D \to D \times_S D$ is a section of q' with $d \circ \delta = 0$, so $N_{q'}(d) = 0$. □

Lemma 14.9. *If $D = D_0 + D_1$ (where D_0, D_1 are divisors) and $g \in \mathcal{O}_D$ then*

$$N_{D/S}(g) = N_{D_0/S}(g) N_{D_1/S}(g).$$

Proof. Put $R = \mathcal{O}_C$, and let the ideals corresponding to D_i be $J_i = (f_i)$ for $i = 0, 1$. We then have a short exact sequence of $\mathcal{O}_D$-modules as follows:

$$\mathcal{O}_{D_0} = R/f_0 \xrightarrow{\times f_1} \mathcal{O}_D = R/(f_0 f_1) \to \mathcal{O}_{D_1} = R/f_1.$$

This is splittable, because $\mathcal{O}_{D_1}$ is projective over $\mathcal{O}_S$. The map μ_g preserves the sequence, and it follows easily that $\det(\mu_g) = \det(\mu_g|\mathcal{O}_{D_0}) \det(\mu_g|\mathcal{O}_{D_1})$, as required. □

Corollary 14.10. *If $D = D_0 + D_1$ as above then $f_D = f_{D_0} f_{D_1}$.*

Proof. Just change base to C and take $g = d$. □

Lemma 14.11. *Suppose that D is a weak divisor of degree r, that D' is a divisor of degree r', and that $D' \subseteq D$. Then $D = D' + D''$ for some weak divisor D'' of degree $r - r'$.*

Proof. Put $J = I_D$ and $J' = I_{D'}$. As D' is a genuine divisor, we have $J' = Rf'$ for some regular element $f' \in R$. As $D' \subseteq D$, we have $J \leq J'$. Put $J'' = \{g \in R \mid f'g \in J\} \geq J$. We then have a short exact sequence

$$R/J'' \xrightarrow{\times f'} R/J \to R/J'.$$

As R/J and R/J' are projective modules of ranks r and r' over k, it follows that R/J'' is a projective module of rank $r - r'$. Thus, the scheme $D'' := \operatorname{spf}(R/J'')$ is a weak divisor. From the definition of J'' we have $J'J'' \leq J$. Conversely, if $h \in J$ then certainly $h \in J' = Rf'$ so $h = gf'$ for some $g \in R$. From the definitions we have $g \in J''$, so $h \in J''J'$. This shows that $J = J'J''$ and so $D = D' + D''$. □

Definition 14.12. Let D be a weak divisor of constant degree r. A *full set of points* for D is a list $u_1, \ldots, u_r$ of sections of S such that $D = \sum_i [u_i]$. If there exists a full set of points, it is clear that D is actually a genuine divisor. (This concept is due to Drinfeld, and is explained and used extensively in [9].)

Proposition 14.13. *If $u_1, \ldots, u_r$ is a full set of points for D, then $N_{D/S}(g) = \prod_i g(u_i)$ for any function g on D. Moreover, we have $f_D(a) = \prod_i d(a, u_i)$, and so $\mathcal{O}_D = \mathcal{O}_C/f_D$.*

Proof. As the projection $[u_i] \to S$ is an isomorphism, we see that $N_{[u_i]/S}(g) = g(u_i)$. The first claim follows easily using from Lemma 14.9 by induction on r. It follows similarly from Corollary 14.10 that $f_D(a) = \prod_i d(a, u_i)$. As d is a difference function we have $\mathcal{O}_{[u_i]} = \mathcal{O}_C/d(a, u_i)$ and so $\mathcal{O}_D = \mathcal{O}_C/f_D$ as claimed. □

Lemma 14.14. *Let D be a weak divisor of constant degree r. Then there is a finite, very flat scheme T over S such that the weak divisor $T \times_S D$ on $T \times_S C$ has a full set of points (and so is genuine).*

Proof. By an evident induction, it suffices to show that after very flat base change we can split D as $[u] + D''$ for some section u and some weak divisor D''. It is enough to find a section $u\colon S \to D$, for then $[u] \subseteq D$ and we can apply the previous lemma. For this we can simply pull back along the projection map $D \to S$ (which is very flat by assumption) and then the diagonal map $D \to D \times_S D$ gives the required "tautological" section. □

Proposition 14.15. *Every weak divisor is a genuine divisor.*

Proof. Let $D = \mathrm{spf}(R/J)$ be a weak divisor. We may assume without loss that it has constant degree r. We know from Corollary 14.6 and Corollary 14.8 that f_D is regular in R and lies in J; we need only show that it generates J. It is enough to do this after faithfully flat base change, so by Lemma 14.14 we may assume that we have a full set of points. Proposition 14.13 completes the proof. □

15. Embeddings

Let C be a nonempty formal multicurve over a scheme S. In this section we study embeddings of S in the affine line $\mathbb{A}^1_S = \mathbb{A}^1 \times S$. If q is the given map $C \to S$, then any map $C \to \mathbb{A}^1_S$ of schemes over S has the form (x, q) for some $x\colon C \to \mathbb{A}^1$, or equivalently $x \in \mathcal{O}_C$.

Now choose a difference function d on C. Given $x \in \mathcal{O}_C$, we can define $x'\colon C \times_S C \to \mathbb{A}^1$ by $x'(a, b) = x(b) - x(a)$. Equivalently, x' is the element $1 \otimes x - x \otimes 1$ in $\mathcal{O}_{C \times_S C} = \mathcal{O}_C \widehat{\otimes}_{\mathcal{O}_S} \mathcal{O}_C$. It is clear that x' vanishes on the diagonal, and thus is divisible by d, say $x' = \theta(x)d$ for some $\theta(x) \in \mathcal{O}_{C \times_S C}$. This element $\theta(x)$ is unique, because d is not a zero-divisor.

Proposition 15.1. *Let $C \xrightarrow{q} S$ be a nonempty formal multicurve. A map $(x, q)\colon C \to \mathbb{A}^1_S$ is injective if and only if $\theta(x)$ is invertible. If so, then (x, q) induces an isomorphism $C \to \varinjlim_k V(f^k) \subset \mathbb{A}^1_S$ for some monic polynomial $f \in \mathcal{O}_S[t]$, showing that C is embeddable.*

Proof. Put $X = \{(a, b) \in C \times_S C \mid x(a) = x(b)\} = V(x') = V(\theta(x)d)$. We see that x is injective if and only if $V(x') = \Delta = V(d)$, if and only if $d = ux' = u\theta(x)d$ for some $u \in \mathcal{O}_{C \times_S C}$. As d is not a zero divisor, this holds if and only if $\theta(x)$ is invertible.

If so, we may assume without loss that $d = x'$. Choose a good parameter y, so $\mathcal{O}_C/y$ has constant rank r over $\mathcal{O}_S$ for some r. Put $D = \mathrm{spec}(R/y)$, let $p\colon C \times_S D \to C$ be the projection, and put $z = N_p(x')$. The proof of Proposition 14.15 shows that z is a unit multiple of y.

We next claim that $\{1, x, \ldots, x^{r-1}\}$ is a basis for $R/y = R/z$ over k, and that $z = f(x)$ for a unique monic polynomial f of degree r. It is enough to check this after faithfully flat base change, so we may assume that $D = [u_0] + \ldots + [u_{r-1}]$ for some list of sections u_i of D. If we put $a_i = x(u_i)$ we see that $z = \prod_i (x - a_i)$. If we put $e_i = \prod_{j<i}(x - a_j)$ we also find that $\{e_0, \ldots, e_{n-1}\}$ is a basis for R/z. As $e_i = x^i +$ lower terms , we also find that $\{1, \ldots, x^{n-1}\}$ is a basis as claimed.

The rest of the proposition follows easily from this. □

Now suppose we have an arbitrary element $x \in \mathcal{O}_C$. Given a map $u\colon S' \to S$ we get a multicurve $C' := S' \times_S C$ over S' and a function $x' = (C' \to C \xrightarrow{x} \mathbb{A}^1) \in \mathcal{O}_{C'}$.

Lemma 15.2. *There is a basic open subscheme $U \subset S$ such that $(x', q')\colon C' \to \mathbb{A}^1_{S'}$ is an embedding if and only if $u\colon S' \to S$ factors through U.*

Proof. Choose a good parameter y on C and put $D = \mathrm{spec}(\mathcal{O}_C/y)$. Put $w = N_{D \times_S D/S}(\theta(x)) \in \mathcal{O}_S$. We see that w is invertible in $\mathcal{O}_S$ if and only if $\theta(x)$ is invertible in $\mathcal{O}_{D \times_S D}$. As $\mathcal{O}_C$ is complete at y, we see that $\theta(x)$ is invertible in $\mathcal{O}_{C \times_S C}$ if and only if it is invertible in $\mathcal{O}_{D \times_S D}$. Given this, it is clear that the scheme $U = \mathrm{spec}(\mathcal{O}_S[1/w])$ has the stated property. □

Corollary 15.3. *Let C be a formal multicurve over S. Then there is a faithfully flat map $S' \to S$ such that the pullback $C' := S' \times_S C$ is embeddable.*

Proof. Put $R = \mathcal{O}_C$ and $k = \mathcal{O}_S$, and let y be a very good parameter on C. Let P be the continuous dual of R, which is a projective module of countable rank over k. We have $P\widehat{\otimes}_k R \simeq \mathrm{Hom}_k^{\mathrm{cts}}(R, R)$, so there is an element $x \in P\widehat{\otimes}_k R$ corresponding to the identity map 1_R. The scheme $M := \mathrm{Map}_S(C, \mathbb{A}^1)$ is the spectrum of the symmetric algebra $k[P]$, with the tautological map $M \times_S C \to \mathbb{A}^1$ corresponding to the element $x \in P\widehat{\otimes}_k R \subset k[P]\widehat{\otimes}_k R = \mathcal{O}_{M\times_S C}$. As in the lemma, there is a largest open subscheme $S' \subseteq M$ where this tautological map gives an embedding $S' \times_S C \to \mathbb{A}^1_{S'}$. Note that $M = \mathrm{spec}(k[P])$ is flat over S and S' is open in M, it is again flat over S. It is clear by construction that $S' \times_S C$ has a canonical embedding in $\mathbb{A}^1_{S'}$. All that is left is to check that the map $u\colon S' \to S$ is *faithfully* flat. It will suffice to show that u is surjective on geometric points, and this follows easily from Lemma 2.17. □

16. Symmetric powers of multicurves

In this section, we study the formal schemes C^r/Σ_r, or in other words the symmetric powers of C. As usual, we write $R = \mathcal{O}_C$ and $k = \mathcal{O}_S$. We choose a good parameter y on C, and a basis $\{e_0, \dots, e_{n-1}\}$ for R/y. We then put $e_{ni+j} = y^i e_j$, which gives a topological basis $\{e_i \mid i \geq 0\}$ for R over k and thus an isomorphism $R \simeq \prod_i k$ of topological k-modules. We write

$$\begin{aligned}
R_r &= R\widehat{\otimes}_k \dots \widehat{\otimes}_k R \\
S_r &= R_r^{\Sigma_r} \\
\overline{R} &= k[\![y]\!] \\
\overline{R}_r &= \overline{R}\widehat{\otimes}_k \dots \widehat{\otimes}_k \overline{R} = k[\![y_1, \dots, y_r]\!] \\
u_i &= \text{ } i\text{'th elementary symmetric function of } y_1, \dots, y_r \\
\overline{S}_r &= \overline{R}_r^{\Sigma_r} = k[\![u_1, \dots, u_r]\!] \\
C^r &= C \times_S \dots \times_S C = \mathrm{spf}(R_r) \\
C^r/\Sigma_r &= \mathrm{spf}(S_r) \\
\overline{C} &= \mathrm{spf}(\overline{R}) \\
\overline{C}^r &= \overline{C} \times_S \dots \times_S \overline{C} = \mathrm{spf}(\overline{R}_r) \\
\overline{C}^r/\Sigma_r &= \mathrm{spf}(\overline{S}_r).
\end{aligned}$$

Here we have topologized $\overline{R}_r$, S_r and $\overline{S}_r$ as closed subrings of R_r. We clearly have a commutative square of topological rings as shown on the left below, and thus a commutative square of formal schemes as shown on the right.

$$\begin{array}{ccc}
R_r & \hookleftarrow & S_r \\
\uparrow & & \uparrow \\
\overline{R}_r & \hookleftarrow & \overline{S}_r
\end{array}
\qquad\qquad
\begin{array}{ccc}
C^r & \twoheadrightarrow & C^r/\Sigma_r \\
\downarrow & & \downarrow \\
\overline{C}^r & \twoheadrightarrow & \overline{C}^r/\Sigma_r
\end{array}$$

We next exhibit topological bases for the above rings. Put

$$\begin{aligned}
A &= \mathbb{N}^r \\
\overline{A} &= (n\mathbb{N})^r = \{\alpha \in A \mid \alpha_i = 0 \pmod{n} \text{ for all } i\} \\
B &= \{\beta \in \mathbb{N}^\infty \mid \sum_{i=0}^{\infty} \beta_i = r\} \\
\overline{B} &= \{\beta \in B \mid \beta_i = 0 \text{ whenever } i \neq 0 \pmod{n}\}.
\end{aligned}$$

Next, for $\alpha \in A$ we put

$$e_\alpha = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_r} \in R_r.$$

Note that $e_{n\alpha} = \prod_{i=1}^r y_i^{\alpha_i} \in \overline{R}_r$, and $e_{n\alpha+\alpha'} = e_{n\alpha} e_{\alpha'}$.

Now define $\tau\colon A \to B$ by $\tau(\alpha)_j = |\{i \mid \alpha_i = j\}|$. This gives bijections $A/\Sigma_r = B$ and $\overline{A}/\Sigma_r = \overline{B}$. For $\beta \in B$, we put

$$e'_\beta = \sum_{\tau(\alpha)=\beta} e_\alpha \in S_r.$$

It is clear that

- $\{e_\alpha \mid \alpha \in A\}$ is a topological basis for R_r over k, giving an isomorphism $R_r = \prod_A k$.
- $\{e_\alpha \mid \alpha \in \overline{A}\}$ is a topological basis for $\overline{R}_r$.
- $\{e'_\beta \mid \beta \in B\}$ is a topological basis for S_r.
- $\{e'_\beta \mid \beta \in \overline{B}\}$ is a topological basis for $\overline{S}_r$.

Of course, the monomials in the symmetric functions u_i give another topological basis for $\overline{S}_r$ over k.

Proposition 16.1. *If $S' = \text{spec}(k')$ is any scheme over S, and $C' = S' \times_S C$ (considered as a multicurve over S') then $(C')^r/\Sigma_r = S' \times_S (C^r/\Sigma_r)$. The schemes $\overline{C}$, $\overline{C}^r$ and $\overline{C}^r/\Sigma_r$ are also compatible with base change in the same sense.*

Proof. Put $R' = \mathcal{O}_{C'} = k'\widehat{\otimes}_k R = \prod_{i\in\mathbb{N}} k'$, and $R'_r = \mathcal{O}_{(C')^r} = R'\widehat{\otimes}_{k'} \dots \widehat{\otimes}_{k'} R' = \prod_{\alpha\in A} k'$, and $S'_r = \mathcal{O}_{(C')^r/\Sigma_r} = \prod_{\beta\in B} k'$. This is clearly the same as $k'\widehat{\otimes}_k S_r$, so $(C')^r/\Sigma_r = (C^r/\Sigma_r) \times_S S'$. The same argument works for the other claims. □

We next need to formulate and prove various compatibility statements for the topologies on the rings considered above.

Definition 16.2. Let A be a linearly topologised ring, and let M be a topological module over A. We say that M is *topologically free of rank r* if it is isomorphic to A^r (with the product topology) as a topological module.

Definition 16.3. Let A be a linearly topologised ring, and let B be a closed subring (with the subspace topology). We write $I \leq_O A$ to indicate that I is an open ideal in A. We say that B is *neat* if for every open ideal $J \leq_O B$, the ideal JA is open in A.

Remark 16.4. As B has the subspace topology, we see that $\{I \cap B \mid I \leq_O A\}$ is a basis of open ideals in B. It follows that B is neat iff $(I \cap B)A$ is open in A whenever $I \leq_O A$. If so, then (using the inclusion $(I \cap B)A \leq I$) we see that $\{(I \cap B)A \mid I \leq_O A\}$ is a basis of open ideals in A.

Remark 16.5. Suppose that we start with a linear topology on B. We can then give A a linear topology by declaring $\{AJ \mid J \leq_O B\}$ to be a basis of open ideals in A. By regarding B as a subspace of A, we obtain a new linear topology on B, which may or may not be the same as the old one. Now suppose that A is faithfully flat over B. It follows that $A/JA = A \otimes_B B/J$ is faithfully flat over B/J, and in particular that the map $B/J \to A/JA$ is injective, so $J = (JA) \cap B$. Using this we see that the two topologies on B are the same, and that B is neat in A.

In particular, if A is topologically free of finite rank over B, then B is neat in A. Conversely, if A is free of finite rank over B and B is neat, it is easy to see that A is topologically free.

Proposition 16.6. (a) *R_r is topologically free of rank n^r over $\overline{R}_r$*
(b) *$\overline{R}_r$ is topologically free of rank $r!$ over $\overline{S}_r$*
(c) *R_r is topologically free of rank $n^r r!$ over $\overline{S}_r$*
(d) *S_r is topologically free of rank n^r over $\overline{S}_r$*
(e) *R_r is a finitely generated module over S_r, and S_r is neat in R_r.*

Moreover, in each of the four rings there is a finitely generated ideal J such that $\{J^m \mid m \geq 0\}$ is a basis of open ideals.

The proof will follow after a number of lemmas. In Corollary 16.14, we will extend part (e) by proving that R_r is a projective module of rank $r!$ over R_r.

Lemma 16.7. *Suppose we have a ring A and elements $a_1, \dots, a_n \in A$, and we put $I_m = (a_1^m, \dots, a_n^m)$. Then $I_1^{n(m-1)+1} \leq I_m \leq I_1^m$, so the topology defined by the ideals I_m is the same as that defined by the ideals I_1^m.*

Proof. The ideal I_1^m is generated by the monomials $\prod_i a_i^{v_i}$ for which $\sum_i v_i = m$. It is clear from this that $I_m \leq I_1^m$, and thus that I_1^m is open in the topology defined by the ideals I_k. Now suppose we have a monomial $\prod_i a_i^{v_i}$ that is not contained in I_m. This means that $v_i \leq m-1$ for all i, and thus $\sum_{i=1}^n v_i < n(m-1)+1$. By the contrapositive, we see that $I_1^{n(m-1)+1} \leq I_m$, so I_m is open in the topology defined by the ideals I_1^k. □

Lemma 16.8. *Let A be a linearly topologised ring with a continuous action of a finite group G. Suppose that there exists a finitely generated G-invariant ideal $I = (a_1, \dots, a_n)$ such that*

$\{I^m \mid m \geq 0\}$ is a basis of open ideals. Then A^G is neat in A. Moreover, if A is faithfully flat over A^G then $\{(I^G)^m \mid m \geq 0\}$ is a basis of open ideals in A^G.

Proof. Put $r = |G|$. For any $a \in A$, put $\phi_a(t) = \prod_{g \in G}(t - g.a)$, so $\phi_a(a) = 0$. If J is any G-invariant ideal containing a we see that $\phi_a(t) \in t^r + J^G[t]$, so the equation $\phi_a(a) = 0$ gives $a^r \in A.J^G$. Thus, all elements of J are nilpotent modulo $A.J^G$. If J is finitely generated we deduce that there exists $s > 0$ with $J^s \leq A.J^G$.

Now apply this with $J = I^m$; we see that $A.(I^m)^G$ contains I^{ms} for some s, and thus is open. This shows that A^G is neat in A.

Now suppose that A is faithfully flat over A^G. We claim that $(I^G)^m$ is open in A^G. Indeed, the above shows that for large j we have $I^j \leq A.I^G$. It follows that $I^{jm} \leq (I^G A)^m = A.(I^G)^m$. It is also clear that $A.(I^{jm})^G \leq I^{jm}$, so $A.(I^{jm})^G \leq A.(I^G)^m$. By faithful flatness, for any ideals $J, J' \leq A^G$ we have $A.J \leq A.J'$ iff $J \leq J'$. We deduce that $(I^{jm})^G \leq (I^G)^m$. The ideal $(I^{jm})^G = I^{jm} \cap A^G$ is open in the subspace topology, so the same is true of $(I^G)^m$. We also have $(I^G)^m \leq (I^m)^G$ and the ideals $(I^m)^G$ form a basis of neighbourhoods of 0; it follows that the same is true of the ideals $(I^G)^m$. □

Corollary 16.9. *Let A, I and G be as in the lemma, and let H be a subgroup of G. Suppose that the inclusion $A^H \to A$ is faithfully flat, and that I^H is finitely generated. Then A^G is neat in A^H.*

Proof. The lemma (with G replaced by H) tells us that $\{(I^H)^m \mid m \geq 0\}$ is a basis of open ideals in A^H. As I^H is finitely generated, the same is true of $(I^H)^m$, say $(I^H)^m = (b_1, \ldots, b_n)$. Consider the polynomial $\phi_{b_i}(t) = \prod_g (t - g.b_i)$ as in the proof of the lemma. As $b_i \in (I^H)^m \subseteq I^m$, we see that $\phi_{b_i}(t) \in t^r + (I^m)^G[t]$. Using the relation $\phi_{b_i}(b_i) = 0$ we see that $b_i^r \in (I^m)^G A^H$, so

$$(I^H)^{m(n(r-1)+1)} \leq (b_1^r, \ldots, b_n^r) \leq (I^m)^G A^H,$$

so $(I^m)^G A^H$ is open in A^H. As the ideals $(I^m)^G$ are a basis of open ideals in A^G, we deduce that A^G is neat as claimed. □

Lemma 16.10. *Suppose that $A = k[y_1, \ldots, y_r]$, with the evident action of $G = \Sigma_r$, and with topology determined by the powers of the ideal $I = (y_1, \ldots, y_r)$. Let H be a subgroup of G of the form $\Sigma_{r_1} \times \ldots \times \Sigma_{r_k}$, with $r = r_1 + \ldots + r_k$. Then*

(a) *A is topologically free of rank $|G| = r!$ over A^G*
(b) *A is topologically free of rank $|H| = \prod_i r_i!$ over A^H*
(c) *A^H is topologically free of rank $|G/H|$ over A^G*
(d) *The topology on A^H (resp. A^G) is determined by powers of the ideal I^H (resp. I^G), which is finitely generated.*

Proof. It is well-known that $A^G = k[u_1, \ldots, u_r]$, where u_i is the i'th elementary symmetric function in the variables y_i. Similarly, we have $A^H = k[v_1, \ldots, v_r]$, where $v_1, \ldots, v_{r_1}$ are the elementary symmetric functions of $y_1, \ldots, y_{r_1}$, and $v_{r_1+1}, \ldots, v_{r_1+r_2}$ are the elementary symmetric functions of $y_{r_1+1}, \ldots, y_{r_1+r_2}$ and so on. By considering the maps

$$A^G \to A^H \to A \to A/I = k,$$

we see that $I^G = (u_1, \ldots, u_r)$ and $I^H = (v_1, \ldots, v_r)$, so in particular these ideals are finitely generated.

We next claim that A is algebraically free of rank $|H|$ over A^H. Everything is compatible with base change, so it will be enough to prove this when $k = \mathbb{Z}$. In this case, all the rings involved are Noetherian domains with unique factorisation and the claim is a standard piece of invariant theory. Similarly, we see that A and A^H are algebraically free of the indicated ranks over A^G, and so the inclusions $A^G \to A^H \to A$ are faithfully flat.

Using Lemma 16.8 and Corollary 16.9 we deduce that the inclusions $A^G \to A^H \to A$ are neat. A neat extension that is an algebraically free module is always topologically free, which proves (a), (b) and (c). We have seen that I^G and I^H are finitely generated, and the rest of (d) follows from Lemma 16.8. □

Proof of Proposition 16.6. Claim (a) is clear. Claim (b) follows from part (a) of Lemma 16.10 by passing to completions, and part (c) follows immediately from (a) and (b).

For claim (d), put

$$A' = \{\alpha \in A \mid \alpha_i < n \text{ for all } i\}$$
$$B' = A'/\Sigma_r = \{\beta \in B \mid \beta_j = 0 \text{ for all } j \geq n\}.$$

For $\beta \in B'$ we put $H_\beta = \prod_i \Sigma_{\beta_i} \leq \Sigma_r$, so $\tau^{-1}\{\beta\} \simeq \Sigma_r/H_\beta$. As A' is a basis for R_r over $\overline{R}_r$, we deduce that $S_r = R_r^{\Sigma_r}$ is isomorphic to $\bigoplus_\beta \overline{R}_r^{H_\beta}$ as a module over $\overline{S}_r$. It will thus suffice to show that $k[\![y_1,\dots,y_r]\!]^{H_\beta}$ is topologically free of rank $|\Sigma_r/H_\beta|$ over $k[\![y_1,\dots,y_r]\!]^{\Sigma_r}$, and this follows from part (c) of Lemma 16.10 by passing to completions.

For part (e), note that R_r is finitely generated over $\overline{S}_r$ and thus is certainly finitely generated over the larger ring S_r. Neatness follows from Lemma 16.8.

Finally, we must show that for each of our rings there is a finitely generated ideal J whose powers determine the topology. For $\overline{R}_r$, we can obviously take J to be the ideal $\overline{I}_r := (y_1,\dots,y_r)$. Lemma 16.8 tells us that for $\overline{S}_r$ we can use the ideal $\overline{J}_r := \overline{I}_r^{\Sigma_r} = (u_1,\dots,u_r)$. For S_r (which is topologically free over $\overline{S}_r$) we can therefore use the ideal $J_r := \overline{J}_r S_r$. Similarly, for R_r we can use the ideal $I_r = \overline{I}_r R_r$. $\square$

Lemma 16.11. *If the curve C is embeddable, then R_r is topologically free of rank $r!$ over S_r.*

Proof. We may assume that

$$C = \operatorname{spf}(k[x]^\wedge_{f(x)}) = \varinjlim_m \operatorname{spec}(k[x]/f(x)^m)$$

for some monic polynomial $f(x)$. Put $A = k[x_1,\dots,x_r]$, and give this the topology determined by the powers of the ideal $I = (f(x_1),\dots,f(x_r))$, so $C^r = \operatorname{spf}(A^\wedge_I)$. The evident action of $G := \Sigma_r$ on A is continuous, and A is free of rank $r!$ over A^G. We see from Lemma 16.8 that A^G is neat in A, so A is topologically free over A^G of rank $r!$, and the claim follows by passing to completions. $\square$

Lemma 16.12. *Let A be a ring, M a finitely generated A-module, and B a faithfully flat A-algebra. Suppose that $B \otimes_A M$ is a free B-module of rank s. Then M is a projective A-module of the same rank.*

Proof. First, we claim that if $\mathfrak{m}$ is a maximal ideal in A with residue field $K = A/\mathfrak{m}$, then $\dim_K(K \otimes_A M) = s$. Indeed, by faithful flatness there exists a prime ideal $\mathfrak{n} \leq B$ with $\mathfrak{n} \cap A = \mathfrak{m}$. Using Zorn's lemma we can find a maximal element of the set of all such ideals $\mathfrak{n}$, and this is easily seen to be a maximal ideal in B. It follows that the residue field $L = B/\mathfrak{n}$ is a field extension of K, so

$$\dim_K(K \otimes_A M) = \dim_L(L \otimes_K K \otimes_A M) = \dim_L(L \otimes_B (B \otimes_A M)),$$

which is evidently equal to s.

We now choose a finite generating set $\{m_1,\dots,m_t\}$ for M. For each subset $S \subseteq \{1,\dots,t\}$ with $|S| = s$, we let $f_S \colon A^S \to M$ be the map $\underline{a} \mapsto \sum_s a_s m_s$, and we let P_S and Q_S be the kernel and cokernel of f_S.

Next, we put $I_S = \operatorname{ann}(Q_S) \leq A$. If $\mathfrak{m}$ is maximal as before, we claim that there exists S such that $I_S \not\leq \mathfrak{m}$. Indeed, as $\dim_K(K \otimes M) = s$, we can certainly choose S such that $\{m_i \mid i \in S\}$ gives a basis for $K \otimes_A M$. It follows that $K \otimes_A Q_S = 0$, or equivalently that $\mathfrak{m}Q_S = Q_S$. The module Q_S is generated by the elements m_j for $j \notin S$, so we can find elements $u_{jk} \in \mathfrak{m}$ for each $j,k \notin S$ such that $m_j = \sum_k u_{jk} m_k$. Let U be the square matrix with entries u_{jk} and put $u = \det(I - U)$. As in [13], we see that $u = 1 \pmod{\mathfrak{m}}$ and $u \in I_S$, so $I_S \not\leq \mathfrak{m}$ as claimed.

It follows from this claim that $\sum_S I_S$ is not contained in any maximal ideal, so $\sum_S I_S = A$. We can thus choose $a_S \in I_S$ with $\sum_S a_S = 1$. It follows that $\operatorname{spec}(A)$ is the union of the basic open subschemes $D(a_S) = \operatorname{spec}(A[a_S^{-1}])$.

We have $a_S Q_S = 0$ and so $Q_S[a_S^{-1}] = 0$, so the map f_S becomes surjective after inverting a_S. It follows that the resulting map $1 \otimes f_S \colon B[a_S^{-1}]^S \to B[a_S^{-1}] \otimes_A M$ is also surjective. Here both source and target are free modules of the same finite rank over $B[a_S^{-1}]$, so our map must in fact be an isomorphism. As $B[a_S^{-1}]$ is faithfully flat over $A[a_S^{-1}]$, we deduce that f_S actually gives an isomorphism $A[a_S^{-1}]^s \to M[a_S^{-1}]$. This shows that M is locally free of rank s, and thus is projective. $\square$

Corollary 16.13. *Let k be a ring, and let A be a formal k-algebra whose topology is defined by the powers of a single open ideal J (so $A = \varprojlim_m A/J^m$). Let M be a finitely generated A-module such that $M = \varprojlim_m M/J^m M$. Let k' be a faithfully flat k-algebra, and put $A' = k' \widehat{\otimes}_k A$ and $M' = k' \widehat{\otimes}_k M = A' \widehat{\otimes}_A M$. Suppose that M' is a free module of rank s over A'; then M is a projective module of rank s over A.*

Proof. First, note that the map $A/J^m \to A'/J^mA' = k' \otimes_k A/J^m$ is a faithfully flat extension of discrete rings. We can thus apply the lemma and deduce that M/J^mM is a finitely generated projective module of rank s over A/J^m.

Next, as M is finitely generated, we can choose an epimorphism $f\colon A^t \to M$ for some t. Let X_m be the set of A-module maps $g\colon M/J^mM \to (A/J^m)^t$ such that the induced map

$$M/J^mM \xrightarrow{g} (A/J^m)^t \xrightarrow{f} M/J^mM$$

is the identity. As M/J^mM is projective over A/J^m, we see that X_m is nonempty. There is an evident projection $\pi_m\colon X_m \to X_{m-1}$, which we claim is surjective. Indeed, given $g \in X_{m-1}$ we can use the projectivity of M/J^mM again to see that there exists a map $h\colon M/J^mM \to (A/J^m)^t$ lifting g. Let δ be the determinant of the resulting map $fh\colon M/J^mM \to M/J^mM$, so $\delta \in A/J^m$. Because $g \in X_{m-1}$, we see that δ maps to 1 in A/J^{m-1}. As the kernel of the projection $A/J^{m-1} \to A/J^m$ is nilpotent, it follows that δ is a unit, so fh is an isomorphism. After replacing h by $h(fh)^{-1}$ we may assume that $fh = 1$, so $h \in X_n$ and $\pi(h) = g$. It follows that $\varprojlim_m X_m \neq \emptyset$, and this gives a map $g\colon M \to A^t$ with $fg = 1$. Thus, M is a retract of a free module, and hence is projective. □

Corollary 16.14. *R_r is a projective module of rank $r!$ over S_r, so the projection $C^r \to C^r/\Sigma_r$ is a finite, faithfully flat map of degree $r!$.*

Proof. In Corollary 16.13, we take $A = S_r$ and $M = R_r$. We know from Proposition 16.6 that the topology on S_r is determined by powers of the ideal $J_r = (u_1, \dots, u_r)$, and that S_r is neat in R_r. This means that the given topology on R_r is determined by the ideals $J_r^m R_r$. As R_r is complete, we deduce that $R_r = \varprojlim_r R_r/J_r^m R_r$. We next take $k' = \mathcal{O}_{S'}$ to be a faithfully flat extension of k such that the curve $C' = S' \times_S C$ is embeddable; this is possible by Corollary 15.3. Using Lemma 16.11, we see that M' is topologically free of rank $r!$ over A', so we can apply Corollary 16.13 and deduce that R_r is projective over S_r. □

17. Classification of divisors

Our main task in this section is to prove the following result.

Theorem 17.1. *Let C be a formal multicurve over a scheme S. Then for formal schemes S' over S, there is a natural bijection between divisors of degree r on $S' \times_S C$ and maps $S' \to C^r/\Sigma_r$ over S.*

Construction 17.2. We must first construct a universal example. We start by putting $\Delta_i = \{(a_1, \dots, a_r, b) \in C^{r+1} \mid b = a_i\}$, which is a divisor of degree one on C over C^r. If we define $d_i(\underline{a}, b) = d(a_i, b)$ then $\mathcal{O}_{\Delta_i} = R_{r+1}/d_i$. Now put $\delta_r = \prod_i d_i \in R_{r+1}$ and $\widetilde{D}_r = \sum_i \Delta_i = \operatorname{spf}(R_{r+1}/\delta_r)$, which is a divisor of degree r on C over C^{r+1}. On the other hand, we note that $\delta_r \in R_{r+1}^{\Sigma_r} = S_r \widehat{\otimes}_k R$, so we can define $D_r = \operatorname{spf}((S_r \widehat{\otimes} R)/\delta_r)$, which is a closed formal subscheme of $C^r/\Sigma_r \times_S C$. It is clear that $R_r \otimes_{S_r} \mathcal{O}_{D_r} = \mathcal{O}_{\widetilde{D}_r}$, which is free of rank r over R_r. We know from Corollary 16.14 that R_r is faithfully flat over S_r, and it follows from Lemma 16.12 that $\mathcal{O}_{D_r}$ is a projective module of rank r over S_r. Moreover, the relevant ideal is generated by the regular element δ_r, so D_r is a divisor on C over C^r/Σ_r.

Now put $Q_r = C^r/\Sigma_r$ for brevity. As in Section 16, we choose a topological basis $\{e_i\}$ for $\mathcal{O}_C$, and use it to construct a topological basis $\{e'_\beta \mid \beta \in B\}$ for $\mathcal{O}_{Q_r}$. We then put $M = \operatorname{spec}(\mathbb{Z}[t_0, t_1, \dots])$, and put $g = \sum_i t_i e_i$, regarded as a function on $M \times Q_r \times C$. We then put

$$h = N_{M\times D_r/M\times Q_r}(g) \in \mathcal{O}_{M\times Q_r} = \prod_\beta \mathcal{O}_S[t_i \mid i \geq 0] e'_\beta.$$

We claim that h is actually equal to $\sum_\beta t^\beta e'_\beta$. Indeed, although this is an equation in $\mathcal{O}_{M\times Q_r}$, it will suffice to prove it in the larger ring $\mathcal{O}_{M\times C^r}$. In that context, we can describe h as $N_{M\times\widetilde{D}_r/M\times C^r}(g)$. Now let $\pi_j\colon C^r \to C$ be the j'th projection. Using Proposition 14.13 we see that $h = \prod_j \pi_j^* g = \prod_j \sum_i t_i \pi_j^* e_i$. Expanding this out gives

$$h = \sum_{\alpha\in A} t^{\tau(\alpha)} e_\alpha = \sum_{\beta\in B} \left(t^\beta \sum_{\tau(\alpha)=\beta} e_\alpha \right) = \sum_{\beta\in B} t^\beta e'_\beta$$

as claimed.

Now suppose we have a map $c\colon S' \to Q_r$ over S, and $D = c^*D_r$ over S'. We deduce easily that

$$N_{M\times D/M\times S'}(g) = \sum_\beta t^\beta c^*(e'_\beta).$$

This shows that $c^*(e'_\beta)$ depends only on D, and $\{e'_\beta \mid \beta \in B\}$ is a topological basis for S_r, so the ring map $c^*\colon S_r \to \mathcal{O}_{S'}$ depends only on D, so the map $c\colon S' \to C^r/\Sigma_r$ depends only on c. We record this formally as follows:

Proposition 17.3. *Suppose we have two maps $c_0, c_1\colon S' \to C^r/\Sigma_r$ over S, and that $c_0^*D_r = c_1^*D_r$ as divisors over S'. Then $c_0 = c_1$.* □

Proof of Theorem 17.1. Let S' be a scheme over S, and let A be the set of maps $S' \to C^r/\Sigma_r$ over S, and let B be the set of divisors of degree r on C over S'. The construction $c \mapsto c^*D_r$ gives a map $\phi\colon A \to B$, which is injective by Proposition 17.3. To show that ϕ is surjective, suppose we have a divisor $D \in B$. We can choose a faithfully flat map $q\colon T \to S'$ such that q^*D has a full set of points, say $\underline{u} = (u_1, \dots, u_r)$. We deduce that q^*D is the pullback of $\widetilde{D}_r$ along the map $T \xrightarrow{u} C^r$, and thus is the pullback of D_r along the composite $c = (T \xrightarrow{u} C^r \to C^r/\Sigma_r)$. Now let $q_0, q_1 : T \times_{S'} T \to T$ be the two projections, so $qq_0 = qq_1$. Note that

$$(cq_0)^*D_r = q_0^*c^*D_r = q_0^*q^*D = (qq_0)^*D,$$

and similarly

$$(cq_1)^*D_r = q_1^*c^*D_r = q_1^*q^*D = (qq_1)^*D.$$

As $qq_0 = qq_1$ we see that $(cq_0)^*D = (cq_1)^*D$, and so (by Proposition 17.3) we have $cq_0 = cq_1$. By faithfully flat descent, we have $c = \overline{c}q$ for a unique map $\overline{c}\colon S' \to C^r/\Sigma_r$. We then have $q^*\overline{c}^*D_r = c^*D_r = q^*D$, and using the faithful flatness of q, we deduce that $D = \overline{c}^*D_r = \phi(\overline{c})$. This shows that ϕ is also surjective, and thus a natural bijection. □

Definition 17.4. In the light of Theorem 17.1, it makes sense to write $\mathrm{Div}_r^+(C)$ for C^r/Σ_r. The evident projection

$$C^r/\Sigma_r \times_S C^s/\Sigma_s = C^{r+s}/(\Sigma_r \times \Sigma_s) \to C^{r+s}/\Sigma_{r+s}$$

gives a map

$$\sigma_{r,s}\colon \mathrm{Div}_r^+(C) \times_S \mathrm{Div}_s^+(C) \to \mathrm{Div}_{r+s}^+(C).$$

It is easy to check that this classifies addition of divisors, in the following sense: if we have divisors $D = u^*D_r$ and $D' = v^*D_s$ on C over S', then $D + D' = w^*D_{r+s}$, where

$$w = (S' \xrightarrow{(u,v)} \mathrm{Div}_r^+(C) \times_S \mathrm{Div}_s^+(C) \xrightarrow{\sigma} \mathrm{Div}_{r+s}^+(C)).$$

We put $\mathrm{Div}^+(C) = \coprod_r \mathrm{Div}_r^+(C)$. As one would expect, this is the free abelian monoid scheme generated by C; see [15, Section 6.2] for technical details.

Definition 17.5. Now suppose that C has an abelian group structure, written additively. We can then define $\tilde{\mu}\colon C^r \times_S C^s \to C^{rs}$ by

$$\tilde{\mu}(a_0, \dots, a_{r-1}; b_0, \dots, b_{s-1})_{i+rj} = a_i + b_j$$

(for $0 \le i < r$ and $0 \le j < s$). The composite

$$C^r \times_S C^s \xrightarrow{\tilde{\mu}} C^{rs} \xrightarrow{q} C^{rs}/\Sigma_{rs}$$

is invariant under $\Sigma_r \times \Sigma_s$, so we get an induced map

$$\mu_{r,s}\colon \mathrm{Div}_r(C) \times_S \mathrm{Div}_s(C) \to \mathrm{Div}_{rs}(C).$$

If we have divisors $D = u^*D_r$ and $D' = v^*D_s$ on C over S', then we define $D * D'$ to be the divisor w^*D_{rs}, where

$$w = (S' \xrightarrow{(u,v)} \mathrm{Div}_r^+(C) \times_S \mathrm{Div}_s^+(C) \xrightarrow{\mu} \mathrm{Div}_{rs}^+(C)).$$

We call this the *convolution* of D and D'. This operation makes $\mathrm{Div}^+(C)$ into a semiring. If we have full sets of points, say $D = \sum_i [a_i]$ and $D' = \sum_j [b_j]$ then $D * D'$ is just $\sum_{i,j}[a_i + b_j]$.

Proposition 17.6. *Let D and D' be divisors on C over S. Then there exists a closed subscheme $T \subseteq S$ such that for any scheme S' over S, we have $S' \times_S D \le S' \times_S D'$ iff the map $S' \to S$ factors through T.*

Proof. As $\mathcal{O}_D$ is finitely generated and projective over $\mathcal{O}_S$, we can choose an embedding $i\colon \mathcal{O}_D \to \mathcal{O}_S^N$ of $\mathcal{O}_S$-modules, and a retraction $r\colon \mathcal{O}_S^N \to \mathcal{O}_D$. We then have $i(f_{D'}) = (a_1, \dots, a_N)$ for some elements $a_j \in \mathcal{O}_S$, and we put $J = (a_1, \dots, a_N)$ and $T = \operatorname{spec}(\mathcal{O}_S/J)$. We find that a map $S' \to S$ factors through T iff J maps to 0 in $\mathcal{O}_{S'}$, iff $f_{D'}$ maps to 0 in $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_D$, iff $S' \times_S D \subseteq S' \times_S D'$. □

Proposition 17.7. *Let D be a divisor on C over S, and suppose that $r \geq 0$. Then there is a scheme $\operatorname{Sub}_r(D)$ over S such that maps $S' \to \operatorname{Sub}_r(D)$ over S biject with divisors $D' \leq S' \times_S D$ of degree r.*

Proof. Over the formal scheme $\operatorname{Div}_r^+(C)$ we have both the originally given divisor D and the universal divisor D_r. We let $\operatorname{Sub}_r(D)$ denote the largest closed subscheme of $\operatorname{Div}_r^+(C)$ where D_r is contained in D (which makes sense by Proposition 17.6). It is formal to check that this has the required property. □

Proposition 17.8. *Let D be a divisor on C over S, and suppose that $r \geq 0$. Then there is a scheme $P_r(D)$ over S such that maps $S' \to P_r(D)$ over S biject with lists $(u_1, \dots, u_r)$ of sections of C over S' such that $\sum_i [u_i] \leq S' \times_S D$.*

Proof. Over the formal scheme C^r we have both the originally given divisor D and the divisor $\widetilde{D}^r$. We let $P_r(D)$ denote the largest closed subscheme of C^r where $\widetilde{D}_r$ is contained in D (which makes sense by Proposition 17.6). It is formal to check that this has the required property. □

Remark 17.9. Suppose that D has degree r. Then $P_r(D)$ classifies r-tuples for which $\sum_i [u_i] \leq D$, but by comparing degrees we see that this means that $\sum_i [u_i] = D$. Thus, $P_r(D)$ classifies full sets of points for D.

Lemma 17.10. *Suppose we have ring maps $A \to B \to C$, and C is a projective module of degree $m > 0$ over B, and also a projective module of degree $nm > 0$ over A. Then B is a projective module of degree n over A.*

Proof. We can use the second copy of B to make $\operatorname{Hom}_A(B, B)$ into a B-module. For any B-module N there is an evident map $\operatorname{Hom}_A(B, B) \otimes_B N \to \operatorname{Hom}_A(B, N)$. This is evidently an isomorphism if N is a free module of finite rank, and thus (by taking retracts) also when N is projective of finite rank over B. In particular, we have $\operatorname{Hom}_A(B, B) \otimes_B C = \operatorname{Hom}_A(B, C)$. As C is also projective over A, the same kind of argument shows that

$$\operatorname{Hom}_A(B, C) = \operatorname{Hom}_A(B, A) \otimes_A C = (\operatorname{Hom}_A(B, A) \otimes_A B) \otimes_B C.$$

It follows that $(\operatorname{Hom}_A(B, A) \otimes_A B) \otimes_B C = \operatorname{Hom}_A(B, B) \otimes_B C$. More precisely, there is a natural map

$$\alpha\colon \operatorname{Hom}_A(B, A) \otimes_A B \to \operatorname{Hom}_A(B, B),$$

given by $\alpha(\phi \otimes b)(b') = \phi(b')b$. By working through the above argument more carefully, we see that $\alpha \otimes_B 1_C$ is an isomorphism. However, C is faithfully flat over B so α itself must be an isomorphism. In particular, we see that 1_B lies in the image of α, so $1_B = \sum_{i=1}^N \alpha(\phi_i \otimes b_i)$ for some maps $\phi_i\colon B \to A$ and elements $b_i \in B$. This means that for all $b \in B$ we have $b = \sum_i \phi_i(b) b_i$. We can use the elements ϕ_i to give a map $\phi\colon B \to A^N$, and the elements b_i to give a map $\beta\colon A^N \to B$. We find that $\beta\phi = 1$, which proves that B is projective. It is now clear that the rank must be n. □

Proposition 17.11. *Let D be a divisor of degree s on C over S, and suppose that $0 \leq r \leq s$. Then there are natural maps $P_r(D) \xrightarrow{p} \operatorname{Sub}_r(D) \xrightarrow{q} S$ which are finite and very flat, with $\deg(p) = r!$ and $\deg(q) = s!/(r!(s-r)!)$ (so $\deg(qp) = s!/(s-r)!$).*

Proof. Over $P_r(D)$ we have tautological sections $u_1, \dots, u_r$ of C giving a divisor $D'_r := \sum_i [u_i]$ on C. This is contained in $(qp)^*D$, so we can form the divisor $D''_r := (qp)^*D - D'_r$, which has degree $s - r$ over $P_r(D)$. It is easy to identify $P_{r+1}(D)$ with D''_r, so $\deg(P_{r+1}(D) \to S) = (s-r)\deg(P_r(D) \to S)$. By an evident induction, we see that the map pq is finite and very flat, with degree $s!/(s-r)!$, as claimed.

Next, let $\overline{D}$ be the tautological divisor of degree r on C over $\operatorname{Sub}_r(D)$. We can then form the scheme $P_r(\overline{D})$, which classifies full sets of points on $\overline{D}$. As above, we see that the map $P_r(\overline{D}) \to \operatorname{Sub}_r(D)$ is finite and very flat, with degree $r!$. We claim that $P_r(\overline{D}) = P_r(D)$. Indeed, a map $S' \to P_r(\overline{D})$ over S corresponds to a map $S' \to \operatorname{Sub}_r(D)$, together with a lifting to $P_r(\overline{D})$. Equivalently, it corresponds to a divisor of degree r contained in $S' \times_S D$, together with sections

$u_1,\dots,u_r\colon S'\to C$ giving a full set of points for that divisor. The full set of points determines the divisor, so it is equivalent to just give sections u_i with $\sum_i[u_i]\le S'\times_S D$, or equivalently, a map $S'\to P_rD$ over S. The claim follows by Yoneda's lemma. It follows that the map p is finite and very flat, with degree $r!$. We can now apply Lemma 17.10 to see that q is finite and very flat, with degree $s!/(r!(s-r)!)$. □

Proposition 17.12. *For the universal divisor D_s over $\mathrm{Div}_s^+(C)$ we have*

$$\mathrm{Sub}_r(D_s)=\mathrm{Div}_r^+(C)\times_S\mathrm{Div}_{s-r}^+(C)$$
$$P_r(D_s)=C^r\times_S\mathrm{Div}_{s-r}^+(C).$$

Proof. Let S' be a scheme over S. Then a map $S'\to\mathrm{Sub}_r(D_s)$ over S corresponds to a map $S'\to\mathrm{Div}_r^+(C)$, together with a lifting to $\mathrm{Sub}_r(D_s)$. Equivalently, it corresponds to a divisor D of degree s on C over S', together with a subdivisor $D'\le D$ of degree r. Given such a pair (D,D'), we have another divisor $D''=D-D'$, which has degree $s-r$. There is evidently a bijection between pairs (D,D') as above, and pairs (D',D'') where D' and D'' are arbitrary divisors of degrees r and $s-r$. These pairs correspond in turn to maps $S'\to\mathrm{Div}_r^+(C)\times_S\mathrm{Div}_{s-r}^+(C)$ over S. The first claim follows by Yoneda's lemma, and the second claim can be proved in the same way. □

Corollary 17.13. *We have $D_s=\mathrm{Div}_{s-1}^+(C)\times_S C=C^s/\Sigma_{s-1}$.*

Proof. Take $r=1$, and observe that $P_1(D_s)=\mathrm{Sub}_1(D_s)=D_s$ and $\mathrm{Div}_1^+(C)=C$. □

18. Local structure of $\mathrm{Div}_d^+(C)$

Let C be a formal multicurve over a base S. In the nonequivariant case, we know that $\mathrm{Div}_n^+(C)\simeq\mathrm{spf}(\mathcal{O}_S[\![c_1,\dots,c_n]\!])=\widehat{\mathbb{A}}_S^n$, so $\mathrm{Div}_n^+(C)$ is a formal affine space of dimension n over S. Equivariantly, this is not even true when $n=1$. However, we will show in this section that $\mathrm{Div}_n^+(C)$ is still a "formal manifold", in the sense that the formal neighbourhood of any point is isomorphic to $\widehat{\mathbb{A}}_S^n$, at least up to a slight twisting. Later we will apply this to calculate $E^0BU(V)$, where $BU(V)$ is the simplicial classifying space of the unitary group of a representation V of A.

We state the result more formally as follows.

Theorem 18.1. *Let $C=\mathrm{spf}(R)$ be a formal multicurve over $S=\mathrm{spec}(k)$, with a difference function d. Let $s\colon S\to\mathrm{Div}_n^+(C)$ be a section, classifying a divisor $D=\mathrm{spf}(R/J)\subset C$. Then the formal neighbourhood of sS in $\mathrm{Div}_n^+(C)$ is isomorphic to the formal neighbourhood of zero in $\mathrm{Map}_S(D,\mathbb{A}_S^1)$ (by an isomorphism that depends on the choice of d).*

The rest of this section constitutes a more detailed explanation and a proof.

We first examine the two formal schemes that are claimed to be isomorphic. We put $A_0=(R^{\widehat{\otimes}n})^{\Sigma_n}$ and $X_0=\mathrm{spf}(A_0)=\mathrm{Div}_n^+(C)$. The section s corresponds to a k-algebra map $A_0\to k$, with kernel K say. We put $A=(A_0)^\wedge_K$ and $X=\mathrm{spf}(A)$. This is the formal neighbourhood of sS in $\mathrm{Div}_n^+(C)$.

Now consider the scheme $Y_0=\mathrm{Map}_S(D,\mathbb{A}_S^1)$. For any scheme T over S, the maps $T\to Y_0$ over S are (essentially by definition) the maps $D\times_S T\to\mathbb{A}^1$ of schemes, or equivalently the elements in the ring $\mathcal{O}_D\otimes_k\mathcal{O}_T$. These biject with the maps $\mathcal{O}_D^\vee=\mathrm{Hom}_k(\mathcal{O}_D,k)\to\mathcal{O}_T$ of k-modules, or with the maps $B_0=\mathrm{Sym}_k[\mathcal{O}_D^\vee]\to\mathcal{O}_T$ of k-algebras. Thus, we have $Y_0=\mathrm{spec}(B_0)$. We let B be the completion of B_0 at the augmentation ideal, and put $Y=\mathrm{spf}(B)$, which is the formal neighbourhood of the zero section in Y_0. Of course B_0 is just the direct sum of all the symmetric tensor powers of $\mathcal{O}_D^\vee$, and B is the direct product of the same terms. If $\mathcal{O}_D$ is free over k (rather than just projective) then B is isomorphic to $k[\![c_1,\dots,c_n]\!]$; in the general case, it should be regarded as a slight twist of this. Note that maps $T\to Y$ over S biject with k-linear maps $\mathcal{O}_D^\vee\to\mathrm{Nil}(\mathcal{O}_T)$, or equivalently elements of $\mathcal{O}_D\otimes_k\mathrm{Nil}(\mathcal{O}_T)$. Note also that a choice of generators $x_1,\dots,x_r$ for $\mathcal{O}_D^\vee$ gives a split surjection $k[\![x_1,\dots,x_r]\!]\to B$.

There is an evident map

$$\alpha\colon\ \mathrm{Div}_n^+(C)=X_0\to Y_0=\mathrm{Map}_S(D,\mathbb{A}_S^1),$$

sending the section s' classifying a divisor D' to the function $(f_{D'})|_D\colon D\to\mathbb{A}^1$. This clearly sends s itself to zero, so it sends the formal neighbourhood of s to the formal neighbourhood of zero, so it gives a map $\alpha\colon X\to Y$. We shall show that this is an isomorphism.

Note that because D and D' have the same degree, we have $s' = s$ if and only if $f_{D'}$ is divisible by f_D, if and only if $\alpha(s') = 0$. This shows that the kernel K of the map $s^*\colon A_0 \to k$ is generated by the image under α^* of the augmentation ideal in B_0. In particular, we see that K is finitely generated.

Because $\mathcal{O}_D = R/J$ is projective over k, we can choose a k-submodule $P \leq R$ such that $R = P \oplus J$. It follows that the map $P \to R \to \mathcal{O}_D$ is an isomorphism, with inverse ξ say.

Lemma 18.2. *Let $I \leq k$ be a finitely generated ideal with $I^m = 0$, and let $g \in R$ be such that $g = f_D \pmod{IR}$. Then g is a regular element, the ideal Rg is open, and we have $R = Rg \oplus P$.*

Proof. A standard topological basis for R gives an isomorphism $R = \prod_i k$, and using the fact that I is finitely generated we see that $I^j R = (IR)^j = \prod_i I^j$. We thus have a finite filtration of R with quotients $\prod_i I^j/I^{j+1}$.

Now consider the k-linear self-map of R given by $\lambda(qf_D + r) = qg + r$ for $q \in R$ and $r \in P$. This is easily seen to induce the identity map on the quotients of the above filtration, so it is an isomorphism. It follows easily that g is regular and $R = Rg \oplus P$.

As D is a divisor, we know that Rf_D is open. Thus, for any good parameter y we have $y^l \in Rf_D$ for large l, say $y^l = uf_D$. We also know that $f_D = g + h$ for some $h \in IR$, so $y^l = uh \pmod{g}$. As $I^m = 0$ we have $y^{lm} = u^m h^m = 0 \pmod{g}$, so Rg is also open. □

We now define a map β from sections of Y to sections of X. A section of Y is an element $r \in \mathrm{Nil}(k)\mathcal{O}_D$. As $\mathcal{O}_D$ is finitely generated we have $r \in I\mathcal{O}_D$ for some finitely generated ideal $I \leq \mathrm{Nil}(k)$, and by finite generation this satisfies $I^m = 0$ for some m. We can thus apply the lemma to the function $g = f_D + \xi(r) \in R$ and conclude that the subscheme $D' = \mathrm{spec}(R/g)$ is a divisor of degree n, classified by a section s' of X_0 say. Over the subscheme $\mathrm{spec}(k/I) \subset S$ it clearly coincides with s, so $(s')^*(K) \leq I$, so $(s')^*(K^m) = 0$. This shows that s' is actually a section of X, as required. We can thus define $\beta(r) = s'$.

In order to define a map $\beta\colon Y \to X$ of formal schemes over S, we need to define maps β_T from sections of Y over T to sections of X over T, naturally for all schemes T over S. For this we just replace C by $T \times_S C$, P by $\mathcal{O}_T \otimes_k P$ and follow the same procedure.

We now define another map $\alpha'\colon X \to Y$. It will again be sufficient to do this for sections defined over S. Let s' be a section of X, classifying a divisor D'. Put $I = (s')^*K \leq k$; this is finitely generated because K is, and nilpotent because s' lands in X. Over $\mathrm{spec}(k/I)$ we have $D' = D$, so $f_{D'} = f_D \pmod{IR}$. The lemma tells us that $R = Rf_{D'} \oplus P$, so there are unique elements $h \in R$ and $p \in P$ such that $f_D = hf_{D'} - p$. By reducing modulo I we see that $h = 1 \pmod{IR}$ and $p \in IP$. We let r be the image of p in $R/f_D = \mathcal{O}_D$, so $r \in I\mathcal{O}_D$ and $\xi(r) = p$. The map $\alpha'\colon X \to Y$ is defined by $\alpha'(s') = r$. Note that h is invertible so $f_{D'}$ is a unit multiple of $f_D + p = f_D + \xi(r)$, so $D' = \mathrm{spec}(R/(f_D + \xi(r))) = \beta(r)$. This shows that $\beta\alpha' = 1$.

In the other direction, suppose we start with $r \in I\mathcal{O}_D$ and put $D' = \mathrm{spec}(R/(f_D + \xi(r)))$ (corresponding to $\beta(r)$). There is then a unique element $p \in P$ congruent to $-f_D$ modulo $f_{D'}$, and $\alpha'\beta(r)$ is by definition the image of p in $\mathcal{O}_D$. It is clear that $-f_D$ is congruent to $\xi(r)$ modulo $f_D + \xi(r)$, which is a unit multiple of $f_{D'}$, so $p = \xi(r)$ and $\alpha'\beta(r) = r$. This shows that $\alpha'\beta = 1$, so α' and β are isomorphisms.

We actually started by claiming that the (slightly more canonical) map α is an isomorphism. As β is an isomorphism, it suffices to check that the map $\alpha\beta\colon Y \to Y$ is an isomorphism, or that $(\alpha\beta)^*$ is an automorphism of $\mathcal{O}_Y = B$. As B is the completed symmetric algebra of a finitely generated projective module, it will suffice to show that $(\alpha\beta)^*$ is the identity modulo the square of the augmentation ideal. By base-change to the universal case, it will suffice to show that $\alpha\beta(r) = r$ whenever $r \in I\mathcal{O}_D$ with $I^2 = 0$. Given such an r, we form the divisor $D' = \mathrm{spec}(R/(f_D + \xi(r)))$ corresponding to $\beta(r)$, and observe that $f_{D'} = u(f_D + \xi(r))$ for some $u \in R^\times$. As $f_{D'} = f_D \pmod{IR}$ we must have $u = 1 \pmod{IR}$. As $\xi(r) \in IR$ and $I^2 = 0$ we have $u\xi(r) = \xi(r)$ and so $f_{D'} = \xi(r) \pmod{f_D}$, so $\alpha\beta(r) = r$ as claimed.

19. Generalised homology of Grassmannians

Consider a periodically orientable theory E with associated equivariant formal group $C = \mathrm{spf}(E^0 P\mathcal{U})$ over $S = \mathrm{spec}(E^0)$. Let $G_r\mathcal{U}$ be the space of r-dimensional subspaces of $\mathcal{U}$, and put $G\mathcal{U} = \coprod_{r=0}^{\infty} G_r\mathcal{U}$. Here we reprove the following result from [5].

Theorem 19.1 (Cole, Greenlees, Kriz)**.** *There are natural isomorphisms*

$$\begin{aligned} E_*G_r\mathcal{U} &= (E_*P\mathcal{U})^{\otimes r}_{\Sigma_r} \\ E^*G_r\mathcal{U} &= ((E^*P\mathcal{U})^{\widehat{\otimes} r})^{\Sigma_r} \\ \operatorname{spf}(E^0G_r\mathcal{U}) &= C^r/\Sigma_r. \end{aligned}$$

We first introduce some additional structure. For any complex inner product space V, we put

$$R_0(V) = \Sigma^V GV_+ = \bigvee_r \Sigma^V G_rV_+.$$

Using the evident maps $G_rU \times G_sV \to G_{r+s}(U \oplus V)$ we get maps $\mu_{U,V}\colon R_0(U) \wedge R_0(V) \to R_0(U \oplus V)$. We also have inclusions $\eta_U\colon S^U = \Sigma^U G_0U_+ \to R_0(U)$. These maps make R_0 into a commutative and associative ring in the category of orthogonal prespectra [12]. All this works equivariantly in an obvious way. The weak homotopy type of R_0 is

$$R_0 \simeq \varinjlim_{U \ll \mathcal{U}} \Sigma^{-U}\Sigma^\infty R_0(U) = \varinjlim_{U \ll \mathcal{U}} \Sigma^\infty GU_+ = \Sigma^\infty G\mathcal{U}_+ = \Sigma^\infty \left(\coprod_r G_r\mathcal{U}\right)_+ .$$

We write Q_rR_0 for the subfunctor $V \mapsto \Sigma^V G_rV_+$, so that $R_0 = \bigvee_r Q_rR_0$ and $Q_rR_0 \simeq G_r\mathcal{U}_+$. In particular, we have $Q_0R_0 = S^0$ and $Q_1R_0 = P\mathcal{U}_+$. This gives a map $E_*P\mathcal{U} \to E_*R_0$ and thus a ring map $\operatorname{Sym}_{E_*} E_*P\mathcal{U} \to E_*R_0$. The theorem says that this is an isomorphism. For the proof, we need some intermediate spectra. For any representation W, we put

$$\begin{aligned} Q_rR_W(V) &= \Sigma^V G_rV^{\operatorname{Hom}(T,W)} \\ R_W(V) &= \bigvee_r Q_rR_W(V) = \Sigma^V GV^{\operatorname{Hom}(T,W)}. \end{aligned}$$

This again gives a commutative orthogonal ring spectrum, with weak homotopy type $R_W \simeq G\mathcal{U}^{\operatorname{Hom}(T,W)}$. In the case $W = 0$ we recover R_0 as before. An inclusion $W \to W'$ gives a ring map $i\colon R_W \to R_{W'}$. In particular, we have a ring map $R_0 \to R_W$, whose fibre we denote by J_W. This is weakly equivalent to the stable fibre of the zero section $G\mathcal{U}_+ \to G\mathcal{U}_+^{\operatorname{Hom}(T,W)}$, and thus is the sphere bundle of the bundle $\operatorname{Hom}(T,W)$ over $G\mathcal{U}$.

Next, recall that there is an isometric embedding $\mathcal{U} \oplus W \to \mathcal{U}$, and that the space of such embeddings is connected. We have

$$Q_1R_W = P\mathcal{U}^{\operatorname{Hom}(T,W)} = P(\mathcal{U} \oplus W)/PW \simeq P\mathcal{U}/PW.$$

Using this, we have a diagram as follows, in which the rows are cofibrations:

$$\begin{array}{ccccc} PW_+ & \longrightarrow & P\mathcal{U}_+ & \longrightarrow & P\mathcal{U}/PW \\ \downarrow & & \downarrow & & \downarrow \\ J_W & \longrightarrow & R_0 & \longrightarrow & R_W. \end{array}$$

The map $P\mathcal{U}/PW \to R_W$ gives a ring map

$$\theta_W\colon \operatorname{Sym}_{E_*} E_*(P\mathcal{U}, PW) \to E_*R_W.$$

Theorem 19.2. *The above maps θ_W are isomorphisms.*

The proof will be given after some preparatory results.

First, suppose we have representations W and L with $\dim(L) = 1$. We put $W' = W \oplus L$ and investigate the fibre of the map $R_W \to R_{W'}$. We may assume that $W' \leq \mathcal{U}$, and then we have a map

$$S^{\operatorname{Hom}(L,W)} = PL^{\operatorname{Hom}(T,W)} \subseteq P\mathcal{U}^{\operatorname{Hom}(T,W)} \to R_W,$$

which we denote by $b_{W,L}$. Multiplication by $b_{W,L}$ gives a map $\Sigma^{\operatorname{Hom}(L,W)}R_W \to R_W$, which we again denote by $b_{W,L}$. (Note that this sends $\Sigma^{\operatorname{Hom}(L,W)}Q_rR_W$ into $Q_{r+1}R_W$, or in other words, it increases internal degrees by one.)

Proposition 19.3. *The sequence*

$$\Sigma^{\operatorname{Hom}(L,W)}R_W \xrightarrow{b_{W,L}} R_W \to R_{W \oplus L} = R_{W'}$$

is a cofibration.

Proof. This is a special case of the following fact. Suppose we have a space X with vector bundles U and V. Let $S(U)$ for the unit sphere bundle in U, and $D(U)$ for the unit disc bundle, so X^U is homeomorphic to $D(U)/S(U)$. We can pull back V along the projection $q\colon S(U) \to X$ and thus form the Thom space $S(U)^{q^*V}$. It is not hard to see that there is a cofibration

$$S(U)^{q^*V} \to X^V \to X^{U\oplus V}.$$

We will apply this with $X = G\mathcal{U}$ and $V = \mathrm{Hom}(T, W)$ and $U = \mathrm{Hom}(T, L)$, so that $X^V = R_W$ and $X^{U\oplus V} = R_{W'}$. To prove the proposition, we need to identify $S(U)^{q^*V}$ with $\Sigma^{\mathrm{Hom}(L,W)} R_W$.

To do this, observe that $S(U)$ is the space of pairs (M, α) where M is a finite-dimensional subspace of $\mathcal{U}$ and $\alpha\colon M \to L$ is a linear map of norm one. As L has dimension one, we find that α can be written as the composite of the orthogonal projection $M \to M \ominus \ker(\alpha)$ with an isometric isomorphism $M \ominus \ker(\alpha) \to L$. Using this, we identify $S(U)$ with the space of pairs (N, β), where N is a finite-dimensional subspace of $\mathcal{U}$ and $\beta\colon L \to \mathcal{U} \ominus N$ is an isometric embedding; the correspondence is that $M = N \oplus \beta(L)$ and

$$\alpha = (N \oplus \beta(L) \xrightarrow{\text{proj}} \beta(L) \xrightarrow{\beta^{-1}} L).$$

We can thus define a map $k\colon S(U) \to G\mathcal{U}$ by $k(N, \beta) = N$ (or equivalently, $k(M, \alpha) = \ker(\alpha)$). This makes $S(U)$ into an equivariant fibre bundle over $G\mathcal{U}$. The fibre over a point $N \in G\mathcal{U}$ is the space $\mathcal{L}(L, \mathcal{U}\ominus N)$, which is well-known to be contractible, and the contraction can be chosen to be equivariant with respect to the stabilizer of N in A. It follows that k is an equivariant equivalence.

To understand the inverse of k, recall that $L \leq W' \leq \mathcal{U}$, so we can put

$$Y = \{N \in G\mathcal{U} \mid N \text{ is orthogonal to } L\} = G(\mathcal{U} \ominus L).$$

Define $j\colon Y \to X$ by $j(N) = N \oplus L$, and then define $\tilde{\jmath}\colon Y \to S(U)$ by

$$\tilde{\jmath}(N) = (N \oplus L, \text{proj}\colon N \oplus L \to L),$$

so $q\tilde{\jmath} = j$. Clearly $k\tilde{\jmath}\colon Y = G(\mathcal{U} \ominus L) \to G\mathcal{U} = X$ is just the map induced by the inclusion $\mathcal{U}\ominus L \to \mathcal{U}$. As the space of linear isometries between any two complete A-universes is equivariantly contractible, we see that this inclusion is an equivariant equivalence. As the same is true of k, we deduce that $\tilde{\jmath}$ is also an equivariant equivalence. We can thus identify $S(U)$ with Y and $q\colon S(U) \to X$ with $j\colon Y \to X$. It follows that we can identify q^*V with j^*V, but the fibre of j^*V over a point $N \in Y$ is $\mathrm{Hom}(j(N), W) = \mathrm{Hom}(L, W) \oplus \mathrm{Hom}(N, W)$, so

$$Y^{j^*V} = \Sigma^{\mathrm{Hom}(L,W)} Y^{\mathrm{Hom}(T,W)} \simeq \Sigma^{\mathrm{Hom}(L,W)} R_W.$$

This gives a cofibration $\Sigma^{\mathrm{Hom}(L,W)} R_W \to R_W \to R_{W'}$, and one can check from the definitions that the first map is just multiplication by $b_{W,L}$. □

Now choose a complete flag

$$0 = W_0 < W_1 < \ldots < \mathcal{U},$$

where $\dim_{\mathbb{C}} W_i = i$ and $\mathcal{U} = \varinjlim_i W_i$. Put $R(i) = R_{W_i}$, so we have maps

$$R_0 = R(0) \to R(1) \to R(2) \to \ldots.$$

Put $L_i = W_{i+1} \ominus W_i$ and $U_i = \mathrm{Hom}(L_i, W_i)$ and $b_i = b_{W_i, L_i}$, so we have a cofibration

$$\Sigma^{U_i} R(i) \xrightarrow{b_i} R(i) \to R(i+1).$$

Lemma 19.4. *Suppose that $B \leq A$, and split W as $\bigoplus_{\beta\in B^*} W[\beta]$ in the usual way. Then*

$$\overline{\phi}^B R_W = \bigwedge_{\beta\in B^*} R_{W[\beta]},$$

where as before

$$W[\beta] = \{w \in W \mid bw = e^{2\pi i\beta(b)} w \text{ for all } b \in B\},$$

and so the connectivity of $(\overline{\phi}^B R_W)/S^0$ is at least $\min_\beta(2\dim_{\mathbb{C}}(W[\beta]) - 1)$.

Proof. We have

$$\overline{\phi}^B G\mathcal{U} = (G\mathcal{U})^B = \{\ B\text{-invariant subspaces of } \mathcal{U}\ \}.$$

Any complex representation U of B splits as $\bigoplus_\beta U[\beta]$, so a subspace $U \leq \mathcal{U}$ is invariant iff it is the direct sum of its intersections with the subspaces $\mathcal{U}[\beta]$. It follows that

$$(G\mathcal{U})^B = \prod_\beta G\mathcal{U}[\beta] \simeq \prod_\beta G\mathbb{C}^\infty.$$

We have a tautological bundle $T[\beta]$ over $G\mathcal{U}[\beta]$, and the bundle $\mathrm{Hom}_{\mathbb{C}B}(T,W)$ over $(G\mathcal{U})^B$ is the external direct sum of the bundles $\mathrm{Hom}_{\mathbb{C}}(T[\beta],W[\beta])$. The Thom complex $G\mathcal{U}[\beta]^{\mathrm{Hom}(T[\beta],W[\beta])}$ is just $R_{W[\beta]}$, and it follows that $\left[(G\mathcal{U})^{\mathrm{Hom}(T,W)}\right]^B$ is just the smash product of these factors, as claimed.

For the last statement, note that if X is a space and U is a vector bundle of real dimension d over X, then X^U is always $(d-1)$-connected. Now let $\mathcal{V}$ be a complex universe, and V a complex vector space of finite dimension d. The bundle $\mathrm{Hom}(T,V)$ over $G_r\mathcal{V}$ has real dimension $2rd$, so $\mathrm{conn}(Q_rR_V)\geq 2rd-1$, and

$$\mathrm{conn}(R_V/S^0)=\mathrm{conn}(\bigvee_{r>0} Q_rR_V)\geq 2d-1.$$

The claim follows easily. □

Corollary 19.5. $\varinjlim_i R(i)=S^0$.

Proof. The unit map $S^0\to Q_0R(i)$ is an isomorphism for all i, so $\varinjlim_i Q_0R(i)=S^0$. It will thus suffice to show that $\varinjlim_i R(i)/S^0=0$, or equivalently that the spectrum $\overline{\phi}^B(\varinjlim_i R(i)/S^0)=\varinjlim_i((\overline{\phi}^B R(i))/S^0)$ is nonequivariantly contractible for all B. As $\mathcal{U}$ is a complete universe, we have $\dim W_i[\beta]\to\infty$ as $i\to\infty$ for all β, so $\mathrm{conn}(\overline{\phi}^B R(i)/S^0)\to\infty$, and the claim follows. □

We now let E be a periodically oriented theory, with orientation x say. This gives a universal generator u_i for $\widetilde{E}_0S^{U_i}$, and a basis $\{c_i\mid i\geq 0\}$ for $\widetilde{E}_0P\mathcal{U}$. Put

$$ER(i)=\mathrm{Sym}_{E_*}E_*(P\mathcal{U},PW_i)=E_*[c_j\mid j\geq i]=ER(0)/(c_k\mid k<i),$$

and let $Q_rER(i)$ be the submodule generated by monomials of weight r (where each generator c_j is considered to have weight one). We then have maps

$$\theta_i=\theta_{W_i}\colon ER(i)\to E_*R(i),$$

which restrict to give maps

$$\theta_{ir}\colon Q_rER(i)\to E_*Q_rR(i).$$

The elements u_i and c_i are related as follows: the inclusion $PL_i\to P\mathcal{U}$ gives an inclusion $S^{U_i}\to P\mathcal{U}^{\mathrm{Hom}(T,W_i)}\simeq P\mathcal{U}/PW_i$, and the image of u_i under this map is the same as the image of c_i under the evident quotient map $P\mathcal{U}\to P\mathcal{U}/PW_i$. It follows that the cofibration $\Sigma^{U_i}R(i)\xrightarrow{b_i}R(i)\to R(i+1)$ gives rise to a cofibration

$$E\wedge R(i)\xrightarrow{c_i}E\wedge R(i)\to E\wedge R(i+1),$$

which restricts to give a cofibration

$$E\wedge Q_{r-1}R(i)\xrightarrow{c_i}E\wedge Q_rR(i)\to E\wedge Q_rR(i+1).$$

Proposition 19.6. *The maps θ_{ir} are isomorphisms for all i and r.*

Proof. The maps θ_{j0} and θ_{j1} are visibly isomorphisms, so we may assume inductively that $\theta_{j,r-1}$ is an isomorphism for all j. The cofibration displayed above gives a diagram $D(i)$ as follows:

$$\begin{array}{ccccc}
Q_{r-1}ER(i) & \overset{c_i}{\rightarrowtail} & Q_rER(i) & \twoheadrightarrow & Q_rER(i+1) \\
{\scriptstyle\theta_{i,r-1}}\downarrow\simeq & & \downarrow{\scriptstyle\theta_{i,r}} & & \downarrow{\scriptstyle\theta_{i+1,r}} \\
E_*Q_{r-1}R(i) & \underset{c_i}{\longrightarrow} & E_*Q_rR(i) & \underset{q_{ir}}{\longrightarrow} & E_*Q_rR(i+1)
\end{array}$$

We first prove that θ_{ir} is surjective for all i. Let $\Theta(i)$ be the image of θ_{ir}, so the claim is that $\Theta(i)=E_*Q_rR(i)$. For $j\geq i$ we write $K(j)$ for the kernel of the map $E_*Q_rR(i)\to E_*Q_rR(j)$. Clearly $K(i)=0\leq\Theta(i)$, and by chasing the diagram $D(j)$ we see that if $K(j)\leq\Theta(i)$ then $K(j+1)\leq\Theta(i)$ also. Corollary 19.5 now tells us that $E_*Q_rR(i)=\bigcup_j K(j)\leq\Theta(i,r)$ as required.

We now see that in $D(j)$, the vertical maps are surjective, so q_{jr} is surjective. As the bottom row is part of a long exact sequence and the right hand map is surjective, we conclude that the bottom row is actually a short exact sequence. Using the snake lemma, we conclude that the induced map $\ker(\theta_{jr})\to\ker(\theta_{j+1,r})$ is an isomorphism. It follows that for any $m>j$, the map

$\ker(\theta_{jr}) \to \ker(\theta_{mr})$ is an isomorphism. However, we have $\ker(\theta_{jr}) \leq Q_r ER(j)$, and it is also clear that when $r > 0$, any element of $Q_r ER(j)$ maps to zero in $Q_r ER(m)$ for $m \gg 0$. It follows that $\ker(\theta_{jr})$ must be zero, so θ_{jr} is an isomorphism as claimed. □

Proof of Theorem 19.2. Given any subrepresentation $W < \mathcal{U}$, we can choose our flag $\{W_i\}$ such that $W = W_i$ for some i. The theorem then follows from Proposition 19.6. □

20. Thom isomorphisms and the projective bundle theorem

Let E be a periodically orientable A-equivariant cohomology theory, with associated equivariant formal group (C, ϕ) over S. For any A-space X, we will write $X_E = \mathrm{spf}(E^0 X)$.

Now let V be an equivariant complex vector bundle over X. We write PV for the associated bundle of projective spaces, and X^V for the Thom space (so $X^V = P(V \oplus \mathbb{C})/PV$). In this section, we will give a Thom isomorphism and a projective bundle theorem to calculate $\widetilde{E}^* X^V$ and $E^* PV$.

First, it is well-known that equivariant bundles of dimension r over X are classified by homotopy classes of A-maps $X \to G_r \mathcal{U}$. We saw above that $E^0 G_r \mathcal{U} = S_r$, and moreover the standard topological basis $\{e'_\beta\}$ for S_r is dual to a universal basis for $E_0 G_r \mathcal{U}$. It follows that $(G_r \mathcal{U})_E = C^r / \Sigma_r = \mathrm{Div}_r^+(C)$.

Now let T denote the tautological bundle over $G_r \mathcal{U}$. It is not hard to identify the projective bundle $PT \to G_r \mathcal{U}$ with the addition map

$$G_{r-1}\mathcal{U} \times P\mathcal{U} = G_{r-1}\mathcal{U} \times G_1 \mathcal{U} \to G_r \mathcal{U},$$

and thus to identify $E^0 PT$ with $S_{r-1} \widehat{\otimes} R = \mathcal{O}_{C^r/\Sigma_{r-1}}$, so $PT_E = C^r/\Sigma_{r-1}$. On the other hand, we can use Corollary 17.13 to identify C^r/Σ_{r-1} with the universal divisor D_r over C^r/Σ_r.

Now suppose we have a vector bundle V over X, classified by a map $c \colon X \to G_r \mathcal{U}$, so $c^* T \simeq V$. The map c is then covered by a map $\tilde{c} \colon PV \to PT$, which gives a map $\tilde{c}^* \colon \mathcal{O}_{D_r} = E^0 PT \to E^0 PV$. We can combine this with the evident map $E^* X \to E^* PV$ to get a map

$$\theta_{X,V} \colon \mathcal{O}_{D_r} \otimes_{S_r} E^* X \to E^* PV.$$

Theorem 20.1. *For any X and V as above, the map $\theta_{X,V}$ is an isomorphism (and so $E^* PV$ is a projective module of rank r over $E^* X$).*

Proof. We first examine the simplifications that occur when V admits a splitting $V = L_1 \oplus \ldots \oplus L_r$, where each L_i is a line bundle. In this case, the classifying map $X \to G_r \mathcal{U}$ factors through $P\mathcal{U}^r$, so the map $S_r \to E^0 X$ factors through R_r. As $\mathcal{O}_{D_r} \otimes_{S_r} R_r = \mathcal{O}_{\widetilde{D}_r}$, we see that $\theta_{X,V}$ is the composite of an isomorphism with a map

$$\theta'_{X,V} \colon \mathcal{O}_{\widetilde{D}_r} \otimes_{R_r} E^* X \to E^* PV.$$

Next, choose a coordinate x on C and define a difference function $d(a, b) = x(b - a)$ as usual. Define a function d_i on C^{r+1} by

$$d_i(a_1, \ldots, a_r, b) = d(a_i, b) = x(b - a_i),$$

as in Construction 17.2. We then put $c_i = \prod_{j<i} d_j$. It is easy to see that $\{c_1, \ldots, c_r\}$ is a basis for $\mathcal{O}_{\widetilde{D}}$ over R_r, so $\theta'_{X,V}$ is just the map $(E^* X)^r \to E^* PV$ given by $(t_1, \ldots, t_r) \mapsto \sum_i t_i c_i$.

Now consider the case where X is a point, so V is just a representation of A. In this case there is always a splitting $V = L_1 \oplus \ldots \oplus L_r$ as above, where $L_i = L_{\alpha_i}$ for some $\alpha_i \in A^*$. In this case the image of c_i in $E^0 PV$ is just the element $x_{U_{i-1}}$ from Corollary 5.5, so the map $\theta_{X,V}$ is an isomorphism.

More generally, suppose that X is arbitrary but V is a constant bundle, with fibre given by a representation $W = \sum_i L_{\alpha_i}$ say. As the elements x_{U_i} form a universal basis for $E^* PW$, we see that $E^* PV = E^* X \otimes_{E^*} E^* PW = \bigoplus_i E^* . x_{U_i}$ and it follows easily that $\theta_{X,V}$ is again an isomorphism.

Now consider the case $X = A/B$ for some $B \leq A$. It is easy to see that any bundle over X has the form $A \times_B W_0$ for some representation W_0 of B. However, as A is a finite abelian group, we can find a representation W of A such that $W|_B = W_0$, and it follows that $A \times_B W_0$ is isomorphic to the constant bundle $A/B \times W$. It follows that $\theta_{X,V}$ is again an isomorphism.

Now let X and V be arbitrary, and suppose we can decompose X as the union of two open sets X_0 and X_1, with intersection X_2. Suppose we know that the maps $\theta_{X_i,V}$ are isomorphisms for $i = 0, 1$ and 2; we claim that $\theta_{X,V}$ is also an isomorphism. Indeed, the decomposition gives a Mayer-Vietoris sequence involving $E^* X$. We can tensor this by the projective module $\mathcal{O}_{D_r}$ over S_r, and it will remain exact. Alternatively, we can pull back the decomposition to get a decomposition of PV, and obtain another Mayer-Vietoris sequence. The θ maps are easily seen to be compatible with these sequences, so the claim follows by the five lemma.

We can now now prove that $\theta_{X,V}$ is an isomorphism for all X and V, by induction on the number of cells and passage to colimits. □

Corollary 20.2. *If V is an equivariant vector bundle over X, then the formal scheme $D(V) := PV_E$ is a divisor on C over X_E, of degree equal to the dimension of V. Moreover, we have $D(V \oplus W) = D(V) + D(W)$ and $D(V \otimes W) = D(V) * D(W)$.*

Proof. The first statement is clear from the theorem. We need only check the equation $D(V \oplus W) = D(V) + D(W)$ in the universal case, where $X = G_r\mathcal{U} \times G_s\mathcal{U}$. As the map $P\mathcal{U}^{r+s} \to G_r\mathcal{U} \times G_s\mathcal{U}$ induces a faithfully flat map $C^{r+s} \to C^r/\Sigma_r \times_S C^s/\Sigma_s$, it suffices to check that equation for the obvious bundles over $P\mathcal{U}^{r+s}$, in which case it is clear. A similar approach works for convolution of divisors. □

We next consider the Thom isomorphism.

Definition 20.3. Let C be a formal multicurve group over S, with zero section $\zeta\colon S \to C$. Given a divisor D on C over S, we let J_D denote the kernel of the restriction map $\mathcal{O}_C \to \mathcal{O}_D$, which is a free module of rank one over $\mathcal{O}_C$. We also use the map $\zeta^*\colon \mathcal{O}_C \to \mathcal{O}_S$ to make $\mathcal{O}_S$ into a module over $\mathcal{O}_C$, and we define $L(D) = \mathcal{O}_S \otimes_{\mathcal{O}_C} J_D$, which is a free module of rank one over $\mathcal{O}_S$, or equivalently a trivialisable line bundle over S. We call this the *Thom module* for D. More generally, given a scheme S' over S and a divisor D on C over S', we obtain a trivialisable line bundle $L(D)$ over S'.

Remark 20.4. Note that $\ker(\zeta^*) = J_{[0]}$ and that $J_D J_{[0]} = J_{D+[0]}$. It follows that

$$L(D) = J_D/J_{D+[0]} = \ker(\mathcal{O}_{D+[0]} \to \mathcal{O}_D).$$

Remark 20.5. If we fix a coordinate x and put $d(a,b) = x(b-a)$, we get a generator f_D for J_D as in Definition 14.3, and thus a generator $u_D = 1 \otimes f_D$ for $L(D)$, which we call the *Thom class.* However, these generators are not completely canonical because of the choice of coordinate.

We also define the *Euler class* e_D to be the element $f_D(0) = \zeta^* f_D \in \mathcal{O}_S$. Note that if $D = [u]$ for some section u, then $f_D(a) = x(a-u)$ and so $e_D = x(-a) = \overline{x}(a)$.

Remark 20.6. For any two divisors D and D', we have $J_{D+D'} = J_D J_{D'}$, which can be identified with $J_D \otimes_{\mathcal{O}_C} J_{D'}$ (because J_D and $J_{D'}$ are each generated by a single regular element). It follows that $L(D+D') = L(D) \otimes_{\mathcal{O}_S} L(D')$. In terms of a coordinate, we have $u_{D+D'} = u_D \otimes u_{D'}$ and $e_{D+D'} = e_D e_{D'}$.

Theorem 20.7. *Let V be an equivariant complex bundle over a space X, giving a divisor $D(V) = PV_E$ on C over X_E as in Corollary 20.2 and thus a free rank one module $L(D(V))$ over E^0X. Then there is a natural isomorphism $\widetilde{E}^0X^V = L(D(V))$ (and $\widetilde{E}^*X^V = L(D(V)) \otimes_{E^0X} E^*X$). Moreover, if we choose a coordinate and thus obtain a Thom class $u_{D(V)}$ as in Remark 20.5, then this gives a* universal *generator for $\widetilde{E}^0X$.*

Proof. Consider the cofibration $P(V) \to P(V \oplus \mathbb{C}) \to X^V$. Using Theorem 20.1 we see that

$$\begin{aligned} E^*P(V) &= E^*X \otimes_{E^0X} \mathcal{O}_{D(V)} \\ E^*P(V \oplus \mathbb{C}) &= E^*X \otimes_{E^0X} \mathcal{O}_{D(V \oplus \mathbb{C})} = E^*X \otimes_{E^0X} \mathcal{O}_{D(V)+[0]}. \end{aligned}$$

As the map $\rho\colon \mathcal{O}_{D(V)+[0]} \to \mathcal{O}_{D(V)}$ is a split surjection of E^0X-modules, we see that the long exact sequence of the cofibration splits into short exact sequences. As $\ker(\rho) = L(D(V))$, we see that $\widetilde{E}^*X^V = L(D(V)) \otimes_{E^0X} E^*X$. By looking in degree zero, we see that $\widetilde{E}^0X^V = L(D(V))$. As this isomorphism is natural in X, it is easy to see that the generator is universal. □

Remark 20.8. If we have two bundles V and V', the above results give

$$\begin{aligned} \widetilde{E}^0X^{V \oplus V'} &= L(D(V \oplus V')) = L(D(V) + D(V')) \\ &= L(D(V)) \otimes_{E^0X} L(D(V')) = \widetilde{E}^0X^V \otimes_{E^0} \widetilde{E}^0X^{V'}. \end{aligned}$$

One can check that this isomorphism $\widetilde{E}^0X^V \otimes \widetilde{E}^0X^{V'} = \widetilde{E}^0X^{V \oplus V'}$ is induced by the usual diagonal map $X^{V \oplus V'} \to X^V \wedge X^{V'}$.

Definition 20.9. We write u_V for $u_{D(V)}$, and call this the Thom class of V. We also write e_V for $e_{D(V)}$, and call this the Euler class of V. (Using Remark 20.5, we see that this is consistent with the definition for line bundles given in Section 5.)

It is easy to see that the Euler class is the pullback of the Thom class along the zero section $X \to X^V$, and that $e_{V\oplus W} = e_V e_W$.

Now suppose that $s \leq r = \dim(V)$, and consider the space $P_r(V)$ consisting of all tuples $(x; L_1, \ldots, L_s)$ where $x \in X$ and the L_i are orthogonal lines in V_x. Recall also that $P_r(D(V))$ is the classifying scheme for r-tuples $(u_1, \ldots, u_r)$ of sections of C such that $\sum_i [u_i] \leq D(V)$, as in Proposition 17.8.

Proposition 20.10. *There is a natural isomorphism $P_r(V)_E = P_r(D(V))$.*

Proof. For each i we have a line bundle over $P_r(V)$ whose fibre over $(x, L_1, \ldots, L_r)$ is L_i. This is classified by a map $P_r(V) \to P\mathcal{U}$, which gives rise to a map $u_i \colon P_r(V)_E \to C$. The direct sum of these line bundles corresponds to the divisor $[u_1] + \ldots + [u_r]$. This direct sum is a subbundle of V, so $[u_1] + \ldots + [u_r] \leq D(V)$. This construction therefore gives us a map $P_r(V)_E \to P_r(D(V))$.

In the case $r = 1$ we have $P_1(V) = PV$ and $P_1(D(V)) = D(V)$ so the claim is that $(PV)_E = D(V)$, which is true by definition. In general, suppose we know that $P_{r-1}(V)_E = P_{r-1}(D(V))$. We can regard $P_r(V)$ as the projective space of the bundle over $P_{r-1}(V)$ whose fibre over a point $(x, L_1, \ldots, L_{r-1})$ is the space $V_x \ominus (L_1 \oplus \ldots \oplus L_{r-1})$. It follows that $P_r(V)_E$ is just the divisor $D(V) - ([u_1] + \ldots + [u_{r-1}])$ over $P_{r-1}(D(V))$, which is easily identified with $P_r(D(V))$. The proposition follows by induction. □

We next consider the Grassmannian bundle

$$G_r(V) = \{(x, W) \mid x \in X \ , \ W \leq V_x \text{ and } \dim(W) = r\}.$$

Proposition 20.11. *There is a natural isomorphism $G_r(V)_E = \mathrm{Sub}_r(D(V))$.*

Proof. Let T denote the tautological bundle over $G_r(V)$. This is a rank r subbundle of the pullback of V so we have a degree r subdivisor $D(T)$ of the pullback of $D(V)$ over $G_r(V)_E$. This gives rise to a map $f \colon G_r(V)_E \to \mathrm{Sub}_r(D(V))$, so if we put $A = \mathcal{O}_{\mathrm{Sub}_r(D(V))}$ we get a map $f^* \colon A \to E^0 G_r(V)$, and we must show that this is an isomorphism. Now consider the tautological divisor $\overline{D}$ of degree r over $\mathrm{Sub}_r(D(V))$. As the module $B = \mathcal{O}_{P_r\overline{D}}$ is faithfully flat over A, it will suffice to show that the map $f^* \colon B \to B \otimes_A E^0 G_r(V)$ is an isomorphism. However, we saw in the proof of Proposition 17.11 that $P_r\overline{D} = P_r D(V) = (P_r V)_E$, so $B = E^0 P_r V$. If we let T be the tautological bundle over $G_r V$, it is easy to see that $P_r T = P_r V$ and so $B = E^0 P_r T = \mathcal{O}_{P_r D(T)}$. It is also easy to see that $D(T) = f^* \overline{D}$, so $P_r D(T) = f^* P_r \overline{D}$, and so

$$B = \mathcal{O}_{P_r D(T)} = E^0 G_r(V) \otimes_A \mathcal{O}_{P_r \overline{D}} = E^0 G_r(V) \otimes_A B,$$

as required. □

We conclude this section with a consistency check that will be useful later.

Definition 20.12. Given a one-dimensional complex vector space L and an arbitrary complex vector space V, we define $\rho \colon PV \to P(L \otimes V)$ by $\rho(M) = L \otimes M$. This is evidently a homeomorphism. If V has the form $V = \mathrm{Hom}(L, W) = L^* \otimes W$ then we identify $L \otimes V$ with W in the obvious way, and thus obtain a homeomorphism $\rho \colon P(\mathrm{Hom}(L, W)) \to PW$. All this clearly works equivariantly, and fibrewise for vector bundles.

Proposition 20.13. *Let X be a space equipped with two complex vector bundles V and W. Let $p \colon PV \to X$ be the projection, and let T be the tautological bundle over PV, so we have a bundle $\mathrm{Hom}(T, p^*W)$ over PV. Then there is a natural homeomorphism*

$$P(V \oplus W)/PW = PV^{\mathrm{Hom}(T, p^*W)}.$$

Proof. Put $U = \mathrm{Hom}(T, p^*W))$. We will construct a diagram as follows:

$$\begin{array}{ccccccc}
PU & \xrightarrow[\simeq]{\rho} & P(p^*W) & & \xrightarrow{\mathrm{proj}} & & PV \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & & & \downarrow{\scriptstyle i_2} \\
P(\mathbb{C} \oplus U) & \xrightarrow[\rho]{\simeq} & P(T \oplus p^*W) & \rightarrowtail & P(p^*(V \oplus W)) & \xrightarrow[\mathrm{proj}]{} & P(V \oplus W)
\end{array}$$

First note that the obvious map $\mathbb{C} \to \operatorname{Hom}(T, T)$ is an isomorphism (because T has dimension one), so

$$\mathbb{C} \oplus U = \operatorname{Hom}(T, T) \oplus \operatorname{Hom}(T, p^*W) = \operatorname{Hom}(T, T \oplus p^*W).$$

Given this, it is clear that we have homeomorphisms ρ as indicated; this gives the left hand half of the diagram, and shows that the cofibre of i_0 is homeomorphic to that of i_1.

Next, observe that T is a subbundle of p^*V so $T \oplus p^*W \leq p^*(V \oplus W)$, so $P(T \oplus p^*W) \subseteq P(p^*(V \oplus W))$ as indicated. There is also an obvious projection $P(p^*(V \oplus W)) \to P(V \oplus W)$, giving the right hand rectangle in the diagram. Note also that $P(p^*W) = p^*PV = PV \times_X PW$.

We next consider in more detail the map $P(T \oplus p^*W) \to P(V \oplus W)$, which we shall call τ. A point in $P(T \oplus p^*W)$ consists of a triple (x, L, M), where $x \in X$ and L is a one-dimensional subspace of V_x and M is a one-dimensional subspace of $L \oplus W_x$. We have $\tau(x, L, M) = (x, M) \in P(V \oplus W)$. Suppose we start with a point $(x, M) \in P(V \oplus W)$. If $M \in PW_x$ then it is clear that $\tau^{-1}\{(x, M)\} = PV_x \times \{M\}$. On the other hand, if $M \notin PW_x$ then the image of M under the projection $V_x \oplus W_x \to V_x$ is a one-dimensional subspace $L \leq V_x$, and the point (x, L, M) is the unique preimage of (x, M) under τ. This means that the rectangle is a pullback, in which the horizontal maps are surjective. Using this, we see that τ induces a homeomorphism from the cofibre of i_1 to that of i_2.

The cofibre of i_0 is PV^U, and the cofibre of i_2 is $P(V \oplus W)/PW$, so these are homeomorphic as claimed. ☐

As a corollary of the above, we have $E^0(P(V \oplus W), PW) = \widetilde{E}^0 PV^{\operatorname{Hom}(T, p^*W)}$. We can use the projective bundle theorem and the Thom isomorphism to calculate both sides in terms of divisors, and they are not obviously the same. Nonetheless, there is an isomorphism between them that can be constructed by pure algebra, as explained in the following result.

Proposition 20.14. *Let $C = \operatorname{spf}(R)$ be a formal multicurve group over $S = \operatorname{spec}(k)$, equipped with two divisors $P = \operatorname{spec}(R/K)$ and $Q = \operatorname{spec}(R/L)$. Define an automorphism ρ of $P \times_S C$ by $\rho(a, b) = (a, b + a)$ (so $\rho^{-1}(a, b) = (a, b - a)$) and let $p \colon P \to S$ be the projection. Then there is a natural isomorphism*

$$L(\rho^{-1}(p^*Q)) = L/KL = \ker(\mathcal{O}_{P+Q} \to \mathcal{O}_Q).$$

Moreover, if we have a coordinate x and use it to define a difference function and Thom classes, then the above isomorphism sends the Thom class in $L(\rho^{-1}(p^*Q))$ to the element $f_Q \in L/KL$.

Remark 20.15. In the last part of the statement, it is important that we are using the generator f_Q defined as a norm as in Definition 14.3. As explained in Remark 14.4, in the nonequivariant case, this is different from the Chern polynomial which is more usually used as a generator.

Proof. Write Z for the scheme $P \times 0$ and Δ for the image of the diagonal map $P \to P \times_S P$. Both of these can be regarded as divisors on the multicurve $P \times_S C$ over P, and it is clear that $\rho^{-1}(\Delta) = Z$, and so $\rho^{-1}(\Delta + p^*Q) = Z + \rho^{-1}(p^*Q)$. It is also clear that $\Delta \leq p^*P$ and so $\Delta + p^*Q \leq p^*(P + Q)$. There are evident projection maps $p^*Q \to Q$ and $p^*(P + Q) \to P + Q$. All this fits together into the following diagram.

$$\begin{array}{ccccccc}
\rho^{-1}(p^*Q) & \xrightarrow[\simeq]{\rho} & p^*Q & \longrightarrow\!\!\!\!\to & & & Q \\
\Big\downarrow {\scriptstyle i_0} & & \Big\downarrow {\scriptstyle i_1} & & & & \Big\downarrow {\scriptstyle i_2} \\
Z + \rho^{-1}(p^*Q) & \xrightarrow[\rho]{\simeq} & \Delta + p^*Q & \rightarrowtail & p^*(P+Q) & \longrightarrow\!\!\!\!\to & P + Q
\end{array}$$

The kernel of the ring map i_0^* is (essentially by definition) the Thom module $L(\rho^{-1}(p^*Q))$. As ρ is an isomorphism, it induces an isomorphism $\ker(i_1^*) \simeq \ker(i_0^*)$. It will thus be enough to show that the map $\ker(i_2^*) \to \ker(i_1^*)$ is also an isomorphism. To be more explicit, write $A = \mathcal{O}_P = R/K$ and $B = \mathcal{O}_Q = R/L$. Let I be the ideal in $A \otimes R = \mathcal{O}_{P \times_S C}$ defining the closed subscheme Δ, so I is generated by the images of elements $1 \otimes a - a \otimes 1$ for $a \in R$. The right hand half of the above diagram then gives the following diagram of rings and ideals:

$$\begin{array}{ccc}
\dfrac{A\otimes R}{A\otimes L} & \xleftarrow{\;f_0\;} & \dfrac{R}{L} \\
{\scriptstyle i_1^*}\big\uparrow & & \big\uparrow{\scriptstyle i_2^*} \\
\dfrac{A\otimes R}{I.(A\otimes L)} & \xleftarrow{\;f_1\;} & \dfrac{R}{KL} \\
\big\uparrow & & \big\uparrow \\
\dfrac{A\otimes L}{I.(A\otimes L)} & \xleftarrow[\;f_2\;]{} & \dfrac{L}{KL}
\end{array}$$

The maps f_i have the form $a \mapsto 1\otimes a$, and we must show that f_2 is an isomorphism. Now choose a generator g for the ideal L, giving an isomorphism $L \simeq R$ of R-modules. This gives isomorphisms $(A\otimes L)/I.(A\otimes L) \simeq (A\otimes R)/I$ and $L/KL \simeq R/K = A$, in terms of which f_2 becomes the ring map $A \to (A\otimes R)/I$ given by $a \mapsto (a\otimes 1 + I) = (1\otimes a + I)$. This corresponds to the projection $\Delta \to B$, which is evidently an isomorphism as required. □

Remark 20.16. In the context of Proposition 20.13, we can take $P = D(V)$ and $Q = D(W)$. We find that $D(\mathrm{Hom}(T, q^*V)) = \rho^{-1}(q^*P)$, and the diagram in the proof of Proposition 20.14 can be identified with that in the proof of Proposition 20.13. It follows that the isomorphism $L(\rho^{-1}(q^*P)) = \ker(\mathcal{O}_{P+Q} \to \mathcal{O}_P)$ obtained by applying $E^0(-)$ to Proposition 20.13 is the same as that given by Proposition 20.14.

21. Duality

Let $D = \mathrm{spf}(R/I)$ be a divisor of degree r on C. In this section we will prove that $\mathrm{Hom}_S(\mathcal{O}_D, \mathcal{O}_S)$ is a free module of rank one over $\mathcal{O}_D$, which means that $\mathcal{O}_D$ is a Poincaré duality algebra over $\mathcal{O}_S$. More precisely, we will identify $\mathrm{Hom}_S(\mathcal{O}_D, \mathcal{O}_S)$ with a subquotient of the module of meromorphic differential forms on C. In the case where C is embeddable, the duality is given by a kind of residue. It is therefore reasonable to define the residue map in the general case so that this continues to hold.

21.1. **Abstract duality.** It will be convenient to start by considering a more abstract situation. Fix a ground ring k, and write $\mathrm{Hom}(M,N)$ for $\mathrm{Hom}_k(M,N)$ and $M\otimes N$ for $M\otimes_k N$. Let A be a k-algebra that is a finitely generated projective module of rank r over k, and write

$$\begin{aligned} M^\vee &= \mathrm{Hom}(M,k) = \mathrm{Hom}_k(M,k) \\ N^* &= \mathrm{Hom}_A(N,A). \end{aligned}$$

If M is an A-module, then we make $M^\vee$ an A-module by the usual rule $(a\phi)(m) = \phi(am)$.

Now Let I be the kernel of the multiplication map $\mu\colon A\otimes A \to A$, and let J be the annihilator of I in $A\otimes A$, and put $B = (A\otimes A)/J$. Assume that I and J are both principal.

Given a k-linear map $\phi\colon A \to k$, we get an A-linear map $1\otimes\phi\colon A\otimes A \to A$, so we can define

$$\tilde{\phi} = \theta_0(\phi) = (1\otimes\phi)|_J \colon J \to A.$$

This construction gives a map $\theta_0\colon A^\vee \to J^*$.

Theorem 21.1. *The A-modules $A^\vee$ and J are both free of rank one (but without canonical generator) and the map $\theta_0\colon A^\vee \to J^*$ is an A-linear isomorphism.*

The rest of this section constitutes the proof.

Lemma 21.2. *The map θ_0 is A-linear, and the adjoint map $\theta_1\colon J \to A^{*\vee}$ is an isomorphism.*

Proof. First suppose that $a \in A$ and $\phi \in A^\vee$ and $u \in J$; we must show that $(1\otimes a\phi)(u) = a((1\otimes\phi)(u))$. From the definitions we have $(1\otimes a\phi)(u) = (1\otimes\phi)((1\otimes a)u)$, and $(1\otimes\phi)((a\otimes 1)u) = a((1\otimes\phi)(u))$, so it will suffice to show that $(1\otimes a)u = (a\otimes 1)u$. This holds because $1\otimes a - a\otimes 1 \in I$ and $IJ = 0$. We now see that θ_0 is A-linear, which allows us to define the adjoint map $\theta_1\colon J \to A^{*\vee}$ by $\theta_1(u)(\phi) = \theta_0(\phi)(u) = (1\otimes\phi)(u)$.

Next, as A is k-projective, we have $A \otimes A = \mathrm{Hom}(A^\vee, A)$. If an element $u \in A \otimes A$ corresponds to a map $\alpha\colon A^\vee \to A$, then $(1 \otimes a)u$ corresponds to the map $x \mapsto a\alpha(x)$, and $(a \otimes 1)u$ corresponds to the map $x \mapsto \alpha(ax)$. It follows that $(1 \otimes a - a \otimes 1)u = 0$ iff $\alpha(ax) = a\alpha(x)$ for all $x \in A$. As I is generated by elements of the form $1 \otimes a - a \otimes 1$, we find that

$$A^{\vee *} = \mathrm{Hom}_A(A^\vee, A) = \mathrm{ann}(I, \mathrm{Hom}(A^\vee, A)) = \mathrm{ann}(I, A \otimes A) = J.$$

One can check that the isomorphism arising from this argument is just θ_1. □

Remark 21.3. It follows immediately that if $A^\vee$ has an inverse as an A-module, then that inverse must be J, and θ_0 must be an isomorphism.

We now define $\eta_0, \eta_1 : A \to A \otimes A$ by

$$\eta_0(a) = a \otimes 1$$
$$\eta_1(a) = 1 \otimes a.$$

We regard $A \otimes A$ as an A-algebra (and thus I, J and B as A-modules) via the map η_0.

Lemma 21.4. *The A-modules I and B are projective, both with rank $r-1$. Moreover, J is free of rank one as an A-module.*

Proof. As A is projective over k with rank r, we see that $A \otimes A$ is projective over A with rank r. There is a short exact sequence $I \to A \otimes A \xrightarrow{\mu} A$, that is A-linearly split by η_0. It follows that I is projective over A, with rank $r-1$. As I is principal with annihilator J, we see that $I \simeq (A \otimes A)/J = B$ as A-modules, so B is also projective of rank $r-1$. It follows that the short exact sequence $J \to A \otimes A \to B$ is A-linearly split, and so J is projective of rank one. It is also a principal ideal and thus a cyclic module, so it must in fact be free of rank one. □

Next, write λ^t for the t'th exterior power functor, and observe that η_1 induces a k-linear map

$$\hat{\eta}_1 : \lambda^{r-1} A \to A \otimes \lambda^{r-1} A = \lambda_A^{r-1}(A \otimes A),$$

and the projection $q\colon A \otimes A \to B$ induces a map $\lambda^{r-1}(q)\colon \lambda_A^{r-1}(A \otimes A) \to \lambda_A^{r-1} B$. We define $\psi\colon \lambda^{r-1} A \to \lambda_A^{r-1} B$ to be the composite of these two maps.

Lemma 21.5. *The map $\psi\colon \lambda^{r-1} A \to \lambda_A^{r-1} B$ is an isomorphism.*

Proof. One can see from the definitions that the image of ψ generates $\lambda_A^{r-1} B$ as an A-module. We will start by showing that the image is itself an A-submodule, so ψ must be surjective.

First, we show that $\lambda^{r-1} A$ has a natural structure as an A-module. Indeed, there is an evident multiplication $\nu\colon A \otimes \lambda^{r-1} A \to \lambda^r A$, which induces an isomorphism $\nu^\# : \lambda^{r-1} A \simeq \mathrm{Hom}(A, \lambda^r A)$. The A-module structure on A gives an A-module structure on $\mathrm{Hom}(A, \lambda^r A)$ which can thus be transported to $\lambda^{r-1} A$. More explicitly, there is a unique bilinear operation $*\colon A \otimes \lambda^{r-1} A \to \lambda^{r-1} A$ such that $a \wedge (b * u) = (ab) \wedge u$ for all $a, b \in A$ and $u \in \lambda^{r-1} A$.

This in turn gives an $A \otimes A$-module structure on the group $A \otimes \lambda^{r-1} A = \lambda_A^{r-1}(A \otimes A)$, by the formula $(a \otimes b) * (c \otimes u) = (ac) \otimes (b * u)$. It follows that

$$\hat{\eta}_1(b * u) = \eta_1(b) * \hat{\eta}_1(u).$$

We next claim that $\lambda_A^{r-1} B$ is a quotient $A \otimes A$-module of $\lambda_A^{r-1}(A \otimes A)$, and is annihilated by I. Indeed, we certainly have an A-module structure and $(A \otimes A)/I = A$ so it will suffice to show that the map $\lambda^{r-1}(q)\colon \lambda_A^{r-1}(A \otimes A) \to \lambda_A^{r-1} B$ annihilates $I * \lambda_A^{r-1}(A \otimes A)$. To see this, choose an element e that generates J (so $Ie = 0$). Using the splittable exact sequence $J \to A \otimes A \xrightarrow{q} B$, we see that there is a commutative diagram as follows, in which χ is an isomorphism.

$$\begin{array}{ccc} \lambda_A^{r-1}(A \otimes A) & \xrightarrow{e \wedge (-)} & \lambda_A^r(A \otimes A) \\ {\scriptstyle \lambda^{r-1}(q)} \downarrow & \nearrow_{\chi,\ \simeq} & \\ \lambda_A^{r-1} B & & \end{array}$$

Now for $u \in \lambda_A^{r-1}(A \otimes A)$ we have $e \wedge (I * u) = (Ie) \wedge u = 0$, so $\lambda^{r-1}(q)(I * u) = 0$ as required.

We can now apply the map $\lambda^{r-1}(q)$ to the equation $\hat{\eta}_1(b * u) = \eta_1(b) * \hat{\eta}_1(u)$ to see that our map $\psi\colon \lambda^{r-1} A \to \lambda_A^{r-1} B$ is A-linear. In particular, the image of ψ is an A-submodule, and thus ψ is surjective as explained previously.

Next, note that A is a projective k-module of rank r, so the same is true of $\lambda^{r-1}A$. On the other hand, B is a projective A-module of rank $r-1$, so $\lambda_A^{r-1}B$ is a projective A-module of rank one, and thus also a projective k-module of rank r. Thus ψ is a surjective map between projective k-modules of the same finite rank, so it is necessarily an isomorphism. □

Example 21.6. It is illuminating to see how this works out in the case where $A = k[x]/f(x)$, where $f(x) = \sum_{i=0}^r a_i x^{r-i}$ is a monic polynomial of degree r. We write x_0 for $x \otimes 1$ and x_1 for $1 \otimes x$, so $A \otimes A = k[x_0, x_1]/(f(x_0), f(x_1))$. Put

$$d(x_0, x_1) = x_1 - x_0$$
$$e(x_0, x_1) = (f(x_1) - f(x_0))/(x_1 - x_0) = \sum_{i+j<r} a_{r-i-j-1} x_0^i x_1^j.$$

One checks that I is generated by d and J is generated by e. Put

$$u_i = (-1)^i x^0 \wedge \ldots \wedge \widehat{x^i} \wedge \ldots \wedge x^{r-1},$$

so $\{u_0, \ldots, u_{r-1}\}$ is a basis for $\lambda^{r-1}A$ over k. If we put $v = x^0 \wedge \ldots \wedge x^{r-1} \in \lambda^r A$, then $x^i \wedge u_j = \delta_{ij} v$, so $x_1^i \wedge \hat{\eta}_1(u_j) = \delta_{ij}\hat{\eta}_1(v)$. Using this, we find that

$$\chi\psi(u_j) = e \wedge \hat{\eta}_1(u_j) = (\sum_{i=k}^{r-1} a_{i-k} x^{r-1-i})\hat{\eta}_1(v).$$

This means that the matrix of the map $\chi\psi\colon \lambda^{r-1}A \to \lambda_A^r(A \otimes A)$ (with respect to the obvious bases) is triangular, with ones on the diagonal. The map $\chi\psi$ is thus an isomorphism, and the same is true of χ, so ψ is an isomorphism as expected.

Proof of Theorem 21.1. First, suppose we have a map $\phi\colon A \to k$. It is well-known that there is a unique derivation i_ϕ of the exterior algebra $\lambda^* A$ whose effect on $\lambda^1 A$ is just the map $\lambda^1 A = A \xrightarrow{\phi} k = \lambda^0 A$ (this is called *interior multiplication* by ϕ). We write $\zeta(\phi)$ for the map $i_\phi\colon \lambda^r A \to \lambda^{r-1}A$. This construction gives a map $\zeta\colon A^\vee \to \mathrm{Hom}(\lambda^r A, \lambda^{r-1}A)$. If we have a basis for A then we find that ζ sends the obvious basis of $A^\vee$ to the obvious basis for $\mathrm{Hom}(\lambda^r A, \lambda^{r-1}A)$ (up to sign), so ζ is an isomorphism. In general, A need only be projective over k but we can still choose a basis Zariski-locally on spec(k) and the argument goes through. It follows that ζ is always an isomorphism.

Next, as mentioned above, the short exact sequence $J \to A \otimes A \to B$ gives $J \otimes_A \lambda_A^{r-1}B = \lambda_A^r(A \otimes A) = A \otimes \lambda^r A$. As the modules J, $\lambda_A^{r-1}B$ and $A \otimes \lambda^r A$ are all dualisable, we deduce that

$$\begin{aligned} J^* &= \mathrm{Hom}_A(A \otimes \lambda^r A, \lambda_A^{r-1}B) \\ &= \mathrm{Hom}(\lambda^r A, \lambda_A^{r-1}B) \\ &= \mathrm{Hom}(\lambda^r A, \lambda^{r-1}A) \\ &= A^\vee. \end{aligned}$$

In particular, we see that $A^\vee$ is an invertible A-module, so Remark 21.3 tells us that the map θ_0 must be an isomorphism. (In fact, the above chain of identifications implicitly constructs an isomorphism $\theta_0'\colon A^\vee \to J^*$, and one could presumably check directly that $\theta_0' = \theta_0$, but we have not done so.) □

Definition 21.7. The isomorphism $J^* = A^\vee$ gives $J^{*\vee} = A^{\vee\vee} = A$; we let $\epsilon\colon J^* \to k$ be the element of $J^{*\vee}$ corresponding to $1 \in A$ under this isomorphism. Equivalently, ϵ is the unique map such that $\epsilon(\theta_0(\phi)) = \epsilon((1 \otimes \phi)|_J) = \phi(1)$ for all $\phi \in A^\vee$.

We will prove later that in cases arising from topology, the map ϵ can be identified with a Gysin map. We conclude this section with an algebraic characterisation of ϵ that will be the basis of the proof.

Construction 21.8. Given $\lambda \in J^*$ and $a \in A$ we can define a map $m(a \otimes \lambda)\colon J \to A \otimes A$ by $e \mapsto a \otimes \lambda(e)$. This map is A-linear if we use the second copy of A to make $A \otimes A$ into an A-module. Thus, this construction gives a map $m\colon A \otimes J^* \to \mathrm{Hom}_A(J, A \otimes A)$, and as A is projective over k, this is easily seen to be an isomorphism. Under the inverse of this isomorphism, the inclusion $J \to A \otimes A$ corresponds to an element $u \in A \otimes J^*$. Lemma 21.11 will give a more concrete description of this element.

Proposition 21.9. *The map $\epsilon\colon J^* \to k$ is such that $(1\otimes\epsilon)(u) = 1$ (where u is as constructed above). Moreover, ϵ is the unique map with this property.*

The proof will follow after some discussion.

It is convenient to choose a generator $e = \sum_i a_i \otimes b_i$ for J, and a dual generator η for J^*, so $\eta(e) = 1$. We then put $\psi = \theta_0^{-1}(\eta) \in A^\vee$; this is the unique map $\psi\colon A \to k$ such that $(1\otimes\psi)(e) = 1$.

Lemma 21.10. *We have $\psi(a) = \epsilon(a\eta)$ for all $a \in A$.*

Proof. Using the A-linearity of θ_0 and the fact that $\epsilon\theta_0(\phi) = \phi(1)$, we see that

$$\epsilon(a\eta) = \epsilon\theta_0(a\psi) = (a\psi)(1) = \psi(a).$$

□

Lemma 21.11. *The element u in Construction 21.8 is given by*

$$u = e.(1\otimes\eta) = \sum_i a_i \otimes b_i\eta.$$

Proof. The element $v := e.(1\otimes\eta)$ corresponds to the map $i = m(v)\colon J \to A\otimes A$ given by $i(x) = \sum_i a_i \otimes b_i\eta(x)$. In particular, we have $i(e) = \sum_i a_i \otimes b_i = e$, so i is the inclusion, so $v = u$. □

Proof of Proposition 21.9. Recall that ϵ is the image of 1 under an isomorphism $A \simeq A^{\vee\vee} \simeq J^{*\vee}$, so it certainly generates $J^{*\vee}$. It will thus suffice to check that $(1\otimes(t\epsilon))(u) = t$ for all $t \in A$. The calculation is as follows:

$$\begin{aligned}
(1\otimes(t\epsilon))(u) &= \sum_i a_i\epsilon(tb_i\eta) \\
&= \sum_i a_i\psi(tb_i) \\
&= (1\otimes\psi)((1\otimes t)e) \\
&= (1\otimes\psi)((t\otimes 1)e) \\
&= t.(1\otimes\psi)(e) \\
&= t.
\end{aligned}$$

The first equality is Lemma 21.11, and the second is Lemma 21.10. The fourth equality holds because $(1\otimes t)e = (t\otimes 1)e$, and the last equality is essentially the definition of ψ. □

21.2. **Duality for divisors.** Now consider a divisor $D = \mathrm{spec}(A)$ on a multicurve $C = \mathrm{spf}(R)$ over a scheme $S = \mathrm{spec}(k)$. In this section, we explain and prove the following theorem.

Theorem 21.12. *For any divisor $D = \mathrm{spec}(\mathcal{O}_C/I_D)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S) = (I_D^{-1}/\mathcal{O}_C)\otimes_{\mathcal{O}_C}\Omega.$$

(The right hand side consists of meromorphic differential forms whose polar divisor is less than or equal to D, modulo holomorphic differential forms; it is easily seen to be free of rank one over $\mathcal{O}_D$.) Moreover, if a map $\phi\colon \mathcal{O}_D \to \mathcal{O}_S$ corresponds to a meromorphic form μ, then $\phi(1) = \mathrm{res}(\mu)$.

The proof is postponed to the end of the section. The last part of the theorem is not yet meaningful, as we have not defined residues. The definition will be such as to make the claim trivial, but we will also check that the definition is compatible with the usual one in the case of embeddable multicurves.

The first step in proving the theorem is to show that the theory in the previous section is applicable.

Definition 21.13. Let X be a scheme over S, with closed subschemes Y and Z. Suppose that $\mathcal{O}_X$, $\mathcal{O}_Y$ and $\mathcal{O}_Z$ are all finitely generated projective modules over $\mathcal{O}_S$. We say that Y and Z are *perfectly complementary* if

(a) the ideals I_Y and I_Z are principal.
(b) $\mathrm{ann}(I_Y) = I_Z$ and $\mathrm{ann}(I_Z) = I_Y$.

Lemma 21.14. *If D_0 and D_1 are divisors on a multicurve C, then D_0 and D_1 are perfectly complementary in $D_0 + D_1$.*

Proof. Put $A_i = \mathcal{O}_{D_i}$; this is a finitely generated projective module over $\mathcal{O}_S$, and has the form $\mathcal{O}_C/f_i$ for some regular element $f_i \in \mathcal{O}_C$. We also put $B = \mathcal{O}_C/(f_0f_1) = \mathcal{O}_{D_0+D_1}$, so $A_i = B/f_i$.

Suppose that $gf_1 = 0 \pmod{f_0f_1}$. Then $(g - hf_0)f_1 = 0$ for some $h \in \mathcal{O}_C$, but f_1 is regular, so $g = hf_0 = 0 \pmod{f_0}$. This shows that the annihilator of f_1 in B is generated by f_0, and by symmetry, the annihilator of f_0 is generated by f_1, and this proves the lemma. □

Corollary 21.15. *Let D be a divisor on a multicurve C. Then the diagonal subscheme $\Delta \subseteq D \times_S D$ and the subscheme $P_2D \subseteq D \times_S D$ are perfectly complementary.*

Proof. Let $q\colon D \to S$ be the projection. We can regard Δ and P_2D as divisors on the multicurve q^*C over D, with $D \times_S D = q^*D = \Delta + P_2D$, so the claim follows from the lemma. □

Corollary 21.16. *Let D be a divisor on a multicurve C, and put $J = \ker(\mathcal{O}_{D^2} \to \mathcal{O}_{P_2D})$. Then J is a free $\mathcal{O}_D$-module of rank one, and there is a natural isomorphism*

$$\theta_0\colon \operatorname{Hom}_{\mathcal{O}_D}(J, \mathcal{O}_D) \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S)$$

given by $\theta_0(\phi) = (1 \otimes \phi)|_J$.

Proof. Let I be the kernel of the multiplication map $\mathcal{O}_{D^2} = \mathcal{O}_D^{\otimes 2} \to \mathcal{O}_D$, so that $\mathcal{O}_\Delta = \mathcal{O}_{D^2}/I$. Note that I is principal, generated by any difference function on C. We see from Corollary 21.15 that J is the annihilator of I, and that J is also principal. We can thus apply Theorem 21.1 to get the claimed isomorphism. □

To proceed further, we need a better understanding of the ideal J.

Definition 21.17. For the rest of this section, we will use the following notation.

$$\begin{aligned}
k &= \mathcal{O}_S \\
R &= \mathcal{O}_C \\
A &= \mathcal{O}_D \\
R_2 &= R\widehat{\otimes}R \\
A_2 &= A \otimes A \\
\widetilde{I} &= \ker(R_2 \xrightarrow{\mu} R) \\
I &= \ker(A_2 \xrightarrow{\mu} A) \\
J &= \operatorname{ann}_{A_2}(I) \\
K &= I_D = \ker(R \to A) \\
K_2 &= K\widehat{\otimes}R + R\widehat{\otimes}K = \ker(R_2 \to A_2) \\
\widetilde{J} &= \{u \in R_2 \mid u\widetilde{I} \subseteq K_2\}.
\end{aligned}$$

We will also choose a difference function d on C (so $d \in \widetilde{I}$), and let α denote the image of d in Ω. We choose a generator f of K, and we let e denote the unique element of R_2 such that $1 \otimes f - f \otimes 1 = de$. We will check later that the image of e in A_2 is a generator of J. After that, we will write η for the dual generator of J^*, and ψ for the corresponding element of $A^\vee$.

Remark 21.18. It is clear that $\widetilde{J}$ is the preimage of J in R_2, so $J = \widetilde{J}/K_2$. As $\widetilde{I}$ is generated by a single regular element, one can check that $\widetilde{I} \cap K_2 = \widetilde{I}\widetilde{J}$.

Definition 21.19. We define $\chi_0\colon R \to \widetilde{I}$ by $\chi_0(a) = 1 \otimes a - a \otimes 1$, and observe that

$$\chi_0(ab) = (a \otimes 1)\chi_0(b) + (1 \otimes b)\chi_0(a).$$

Given a difference function d on C, we let $\xi(a) \in R_2$ denote the unique element such that $\chi_0(a) = \xi(a)d$. We again have

$$\xi(ab) = (a \otimes 1)\xi(b) + (1 \otimes b)\xi(a).$$

Lemma 21.20. *There is a unique map $\nu\colon K_2 \to K/K^2$ such that*

$$\begin{aligned}
\nu(a \otimes b) &= ab \text{ whenever } a \in K \text{ and } b \in R \\
\nu(a \otimes b) &= 0 \text{ whenever } a \in R \text{ and } b \in K.
\end{aligned}$$

Moreover, we have $\nu(K_2\widetilde{I}) = 0$.

Proof. Using a k-linear splitting of the sequence $K \to R \to A$, we see that $(K\widehat{\otimes}R) \cap (R\widehat{\otimes}K) = K\widehat{\otimes}K$, so we have

$$\frac{K_2}{R\widehat{\otimes}K} = \frac{K\widehat{\otimes}R}{K\widehat{\otimes}K}.$$

The multiplication map $\mu\colon R\widehat{\otimes}R \to R$ evidently induces a map $(K\widehat{\otimes}R)/(K\widehat{\otimes}K) \to K/K^2$. Putting this together, we get a map

$$\nu = \left(K_2 \to \frac{K_2}{R\widehat{\otimes}K} = \frac{K\widehat{\otimes}R}{K\widehat{\otimes}K} \xrightarrow{\mu} \frac{K}{K^2}\right).$$

It is clear that this is uniquely characterised by the stated properties. As ν is essentially given by μ on $K\widehat{\otimes}R$ and $\mu(\widetilde{I}) = 0$, we see that $\nu(\widetilde{I}.(K\widehat{\otimes}R)) = 0$. We also have $\nu(R\widehat{\otimes}K) = 0$ and so $\nu(\widetilde{I}.(R\widehat{\otimes}K)) = 0$, so $\nu(\widetilde{I}K_2) = 0$ as claimed. □

Proposition 21.21. *The map ξ induces a A-linear isomorphism $K/K^2 \to J$, so J is freely generated over A by the element $e = \xi(f)$. (Note however that this isomorphism is not canonical, because it depends on the choice of d.)*

Proof. Suppose that $a \in K$, so $\chi_0(a) \in K_2$. As $\widetilde{I} = R_2 d$ we see that $\xi(a)\widetilde{I} = R_2\chi_0(a) \subseteq \widetilde{I}$, so $\xi(K) \subseteq \widetilde{J}$, so we get an induced map $K \to \widetilde{J}/K_2 = J$. Using the product formula for ξ, we deduce that this map is R-linear and induces a map $K/K^2 \to J$. In the opposite direction, we define $\zeta\colon \widetilde{J} \to K/K^2$ by $\zeta(u) = \nu(ud)$. If $u \in K_2$ then $ud \in \widetilde{I}K_2$ so $\nu(ud) = 0$. Thus, ζ induces a map $J = \widetilde{J}/K_2 \to K/K^2$. It is easy to see that $\zeta\xi = 1\colon K/K^2 \to K/K^2$, and both J and K/K^2 are invertible A-modules, so ζ and ξ must be mutually inverse isomorphisms. It follows immediately that J is freely generated by e. □

Our next task is to reformulate the above isomorphism in a way that is independent of any choices.

Proposition 21.22. *Put*

$$\overline{\Omega} = \Omega|_D = \Omega \otimes_R A = \widetilde{I} \otimes_{R_2} A.$$

There is a natural *isomorphism $\chi\colon K/K^2 \to J \otimes_R \Omega = J \otimes_A \overline{\Omega}$ of A-modules, satisfying $\chi(a) = \xi(a) \otimes \alpha$, where α is the image of d in Ω. By adjunction, there is also a natural isomorphism $\chi'\colon J^* \to (K/K^2)^* \otimes_A \overline{\Omega} = (K/K^2)^* \otimes_R \Omega$.*

Proof. We have

$$J \otimes_A \overline{\Omega} = J \otimes_A (A \otimes_{R_2} \widetilde{I}) = (\widetilde{J}/K_2) \otimes_{R_2} \widetilde{I} = (\widetilde{I}\widetilde{J})/(\widetilde{I}K_2) = (\widetilde{I} \cap K_2)/(\widetilde{I}K_2).$$

We have seen that the map χ sends K to $\widetilde{I} \cap K_2$ and K^2 to $\widetilde{I}K_2$, so it induces a map $\chi\colon K/K^2 \to J \otimes_A \overline{\Omega}$, which is obviously canonical. One checks from the definitions that $\chi(a) = \xi(a) \otimes \alpha$, where ξ and α are defined in terms of a difference function d as above. As ξ is an isomorphism and $\overline{\Omega}$ is freely generated by α over A, we conclude that χ is an isomorphism. □

Our next task is to interpret the module $(K/K^2)^*$.

Definition 21.23. Let C be a formal multicurve over S. We say that an element $f \in \mathcal{O}_C$ is *divisorial* if it is not a zero-divisor, and $\mathcal{O}_C/f$ is a projective $\mathcal{O}_S$-module of finite rank. One can check that the set of divisorial elements is closed under multiplication, so we can invert it to get a new ring $\mathcal{M}_C$, whose elements we call *meromorphic functions*. We say that a meromorphic function is *divisorial* if it can be written as f/g, where f and g are divisorial elements of $\mathcal{O}_C$ (this can be seen to be compatible with the previous definition). A *fractional ideal* is an $\mathcal{O}_C$-submodule $I \leq \mathcal{M}_C$ that can be generated by a divisorial meromorphic function. The set of fractional ideals forms a group under multiplication, with $I^{-1} = \{f \in \mathcal{M}_C \mid fI \subseteq \mathcal{O}_C\}$.

Lemma 21.24. *There is a natural isomorphism $(K/K^2)^* = (K^{-1}/R)$. With respect to this, the generator f of K/K^2 is dual to the generator $1/f$ of K^{-1}/R.*

Proof. The multiplication map $K\widehat{\otimes}_R K^{-1} \to R$ induces a map $(K/K^2) \otimes_A (K^{-1}/R) \to A$, and thus a map $K^{-1}/R \to (K/K^2)^*$. This is easily seen to be an isomorphism. The statement about generators is clear. □

Proof of Theorem 21.12. Everything except the last part now follows immediately from Corollary 21.16, Proposition 21.22 and Lemma 21.24. The residue map will be defined in Definition 21.31, and then the last part of the theorem will be true by definition. □

We can make the theorem more explicit as follows.

Proposition 21.25. *The natural isomorphism*

$$A^\vee \xrightarrow{\theta_0} J^* \xrightarrow{\chi'} K^{-1}/R \otimes_A \Omega$$

sends ϕ to the element $((1 \otimes \phi)(e))/f \otimes \alpha$.

Proof. Using Proposition 21.21, we see that e generates J, so there is a unique element $\eta \in J^*$ with $\eta(e) = 1$. The natural isomorphism $K/K^2 \to J \otimes_A \Omega$ sends f to $e \otimes \alpha$, so the adjoint map $\chi' \colon J^* \to K^{-1}/R \otimes \Omega$ sends η to $1/f \otimes \alpha$, and thus sends $a\eta$ to $(a/f) \otimes \alpha$. Next, we certainly have $\theta_0(\phi) = a\eta$ for some $a \in A$, and by evaluating this equation on e we find that $a = (1 \otimes \phi)(e)$. It follows that

$$\chi'\theta_0(\phi) = (a/f) \otimes \alpha = ((1 \otimes \phi)(e))/f \otimes \alpha$$

as claimed. □

We next examine how this works in the case where C is embeddable, say $C = \mathrm{spf}(k[x]^\wedge_{(g)})$ for some monic polynomial g. We then have $K = Rf$ for some monic polynomial f that divides some power of g, and $A = R/K = k[x]/f$.

Definition 21.26. Suppose we have a ring k and an expression $\alpha = f(x)\mathbf{d}x = p(x)\mathbf{d}x/q(x)$, where p and q are polynomials with q monic; we then define the residue $\mathrm{res}(\alpha)$ as follows. Let R' denote the ring of series of the form $u(x) = \sum_{n=-\infty}^{N} a_n x^n$ for some finite N. Clearly $k[x] \subseteq R'$. Moreover, if $q(x)$ is a monic polynomial then we can write $q(x) = x^n r(1/x)$ for some polynomial $r(t)$ with $r(0) = 1$. It follows that $r(1/x)$ is invertible in R', and thus the same is true of $q(x)$, so we can expand out $f(x) = p(x)/q(x)$ as an element of R', say $f(x) = \sum_{n=-\infty}^{N} a_n x^n$. We then put

$$\mathrm{res}(\alpha) = a_{-1}.$$

Remark 21.27. In the case $k = \mathbb{C}$, one can check that $\mathrm{res}(\alpha)$ is the sum of the residues of α at all its poles, so the integral of α around any sufficiently large circle is $2\pi i \,\mathrm{res}(\alpha)$. By standard arguments, most formulae that hold when $k = \mathbb{C}$ will be valid for all k. In particular, we have

- $\mathrm{res}(f(x)\mathbf{d}x) = 0$ if f is a polynomial
- $\mathrm{res}(f'(x)\mathbf{d}x) = 0$ for any $f = p/q$ as above
- $\mathrm{res}(f'(x)\mathbf{d}x/f(x)) = \deg(p) - \deg(q)$ if $f = p/q$ for some monic polynomials p and q.

Lemma 21.28. *Suppose that $f(x) = p(x)/q(x)$ where q is monic of degree n, and*

$$p(x) = \sum_{i=0}^{n-1} b_i x^i \pmod{q(x)}.$$

Then $\mathrm{res}(f(x)\mathbf{d}x) = b_{n-1}$.

Proof. First, put $m(x) = \sum_{i=0}^{n-1} b_i x^i$, so $p(x) = m(x) + l(x)q(x)$ for some polynomial $l(x)$, so $f(x) = m(x)/q(x) + l(x)$. We have $\mathrm{res}(l(x)\mathbf{d}x) = 0$, so it will suffice to show that $\mathrm{res}(m(x)/q(x)\mathbf{d}x) = b_{n-1}$.

Next, write $q(x) = x^n r(1/x)$ as in the definition, and put $u(x) = 1/r(1/x)$, so $u(x) = \sum_{i=0}^{\infty} a_i x^{-i}$ for some coefficients $a_i \in k$ with $a_0 = 1$. We have

$$x^j/q(x) = x^{-n}u(x) = \sum_{i \geq 0} a_i x^{j-n-i},$$

so

$$\mathrm{res}(x^j \mathbf{d}x/q(x)) = \begin{cases} 0 & \text{if } 0 \leq j < n-1 \\ 1 & \text{if } j = n-1. \end{cases}$$

The claim follows immediately. □

Proposition 21.29. *If $A = k[x]/f(x)$, then the map*

$$(K^{-1}/R) \otimes \Omega \xrightarrow{(\chi')^{-1}} J^* \xrightarrow{\epsilon} k$$

is just the residue.

Proof. As in example 21.6, we put

$$\begin{aligned} f(x) &= \sum_{i=0}^{r} a_i x^{r-i} \\ d(x_0,x_1) &= x_1 - x_0 \\ \alpha &= \mathbf{d}x \in \Omega \\ e(x_0,x_1) &= (f(x_1)-f(x_0))/(x_1-x_0) = \sum_{i+j<r} a_{r-i-j-1} x_0^i x_1^j. \end{aligned}$$

This is of course compatible with the notation in Proposition 21.25, so $\chi'\theta_0(\phi) = (1\otimes\phi)(e)\mathbf{d}x/f(x)$. The map ϵ is characterised by the fact that $\epsilon(\theta_0(\phi)) = \phi(1)$, so it will suffice to check that $\operatorname{res}((1\otimes\phi)(e)\mathbf{d}x/f(x)) = \phi(1)$ for all $\phi \in A^\vee$.

Now let $\{\zeta_0,\ldots,\zeta_{r-1}\}$ be the basis for $A^\vee$ dual to the basis $\{x^0,\ldots,x^{r-1}\}$ for A. We then have $(1\otimes\zeta_j)(x_1^k) = \delta_{jk}$, and it follows that

$$(1\otimes\zeta_j)(e) = \sum_{i=0}^{r-1-j} a_{r-i-j-1} x^i.$$

Using Lemma 21.28 we deduce that $\operatorname{res}((1\otimes\zeta_j)(e)\mathbf{d}x/f(x)) = 0$ for $j>0$, whereas for $j=0$ we get a_0, which is 1 because the polynomial $f(x) = \sum_{i+j=r} a_j x^j$ is monic. On the other hand, we also have $\zeta_j(1) = \delta_{0j}$ by definition, so $\operatorname{res}((1\otimes\zeta_j)(e)\mathbf{d}x/f(x)) = \zeta_j(1)$ as required. □

Given this, it would be reasonable to define residues on multicurves using the maps ϵ. To make this work properly, we need to check that these maps are compatible for different divisors.

Proposition 21.30. *Suppose we have divisors $D_0 \subseteq D_1$, corresponding to ideals $K_1 \le K_0 \le R$. Let $j\colon K_0^{-1}/R \to K_1^{-1}/R$ be the evident inclusion, and let $q\colon A_1 = R/K_1 \to R/K_0 = A_0$ be the projection. Define $\delta_i\colon A_i^\vee \to k$ by $\delta_i(\phi) = \phi(1)$. Then the following diagram commutes:*

$$\begin{array}{ccccc} k & \xleftarrow{\ \delta_0\ } & A_0^\vee & \xrightarrow[\simeq]{\chi'\theta_0} & K_0^{-1}/R\otimes\Omega \\ \| & & \downarrow{\scriptstyle q^\vee} & & \downarrow{\scriptstyle j} \\ k & \xleftarrow[\ \delta_1\ ]{} & A_1^\vee & \xrightarrow[\chi'\theta_0]{\simeq} & K_1^{-1}/R\otimes\Omega \end{array}$$

Proof. As $q(1)=1$, it is clear that the left hand square commutes. For the right hand square, choose generators f_i for K_i and put $e_i = \xi(f_i)$, so that

$$\chi'\theta_0(\phi) = (1\otimes\phi)(e_i)/f_i\otimes\alpha$$

for $\phi\in A_i^\vee$.

As $D_0\subseteq D_1$, we have $f_1 = gf_0$ for some g, and so $\xi(f_1) = (g\otimes 1)\xi(f_0) + (1\otimes f_0)\xi(g)$, or in other words $e_1 = (g\otimes 1)e_0 + (1\otimes f_0)\xi(g)$.

Now suppose we have $\phi\in A_0^\vee$, so $(q^*\phi)(f_0) = 0$, so $(1\otimes q^*\phi)((1\otimes f_0)\xi(g)) = 0$. We then have

$$\begin{aligned} \chi'\theta_0 q^*(\phi) &= ((1\otimes q^*\phi)(e_1)/f_1)\otimes\alpha \\ &= (g(1\otimes\phi)(e_0))/(gf_0)\otimes\alpha \\ &= (1\otimes\phi)(e_0)/f_0\otimes\alpha \\ &= j\chi'\theta_0(\phi) \end{aligned}$$

as claimed. □

Definition 21.31. Define $\delta\colon \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_D,\mathcal{O}_S)\to\mathcal{O}_S$ by $\delta(\phi)=\phi(1)$. We let $\operatorname{res}\colon \mathcal{M}_C\otimes_{\mathcal{O}_C}\Omega\to\mathcal{O}_S$ be the unique map whose restriction to $I_D^{-1}\otimes_{\mathcal{O}_C}\Omega$ is the composite

$$I_D^{-1}\otimes_{\mathcal{O}_C}\Omega \to (I_D^{-1}/\mathcal{O}_C)\otimes_{\mathcal{O}_C}\Omega \simeq \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_D,\mathcal{O}_S)\xrightarrow{\delta}\mathcal{O}_S.$$

(This is well-defined, by Proposition 21.30, and compatible with the classical definition, by Proposition 21.29.)

Proposition 21.32. *For any $g,f\in\mathcal{O}_C$, if f is divisorial than*

$$\operatorname{res}((g/f)\mathbf{d}f) = \operatorname{trace}_{(\mathcal{O}_C/f)/\mathcal{O}_S}(g).$$

In particular, we have $\text{res}((1/f)\mathbf{d}f) = \dim_{\mathcal{O}_S}(\mathcal{O}_C/f)$. *Moreover, we also have*

$$\text{res}(\mathbf{d}(g/f)) = \text{res}\left(\frac{f\mathbf{d}(g) - g\mathbf{d}(f)}{f^2}\right) = 0.$$

Proof. Both facts are well-known for residues in the classical sense, so they hold whenever C is embeddable. Using Corollary 15.3, we deduce that they hold for a general multicurve C. We will give a more direct and illuminating proof for the first fact; we have not been able to find one for the second fact.

We use abbreviated notation as before, with $K = Rf$ so that $A = R/f$. The multiplication map $\mu\colon A_2 \to A$ restricts to give an A-linear map $\mu\colon J \to A$, or in other words an element of J^*. The trace map $\tau\colon A \to k$ can be regarded as an element of $A^\vee$. We claim that the elements τ, μ and $(\mathbf{d}f)/f$ correspond to each other under our standard isomorphisms

$$A^\vee \simeq J^* \simeq (K^{-1}/R) \otimes \Omega.$$

To see this, note that $(1 \otimes \tau)(u) = \text{trace}_{A_2/A}(u)$ for all $u \in A_2$. Using the splittable short exact sequence

$$I \to A_2 \xrightarrow{\mu} A,$$

we see that

$$(1 \otimes \tau)(u) = \text{trace}(I \xrightarrow{\times u} I) + \mu(u).$$

If $u \in J$ then multiplication by u is zero on I and we deduce that $(1 \otimes \tau)(u) = \mu(u)$. This shows that $\theta_0(\tau) = \mu$ as claimed.

Next, let $e = \xi(f)$ be the standard generator of J, and let η be the dual generator of J^*, so $\eta(e) = 1$. Using Proposition 21.25, we see that μ corresponds to the element $(\mu(e)/f) \otimes \alpha = (1/f) \otimes (\mu(e)\alpha)$ in $(K^{-1}/R) \otimes \Omega$. Now, the module $\Omega = \widetilde{I}/\widetilde{I}^2$ is originally a module over R_2 that happens to be annihilated by $\ker(\mu) = \widetilde{I}$, and so is regarded as a module over R via μ. Thus, $\mu(e)\alpha$ is just the same as $e\alpha$. Moreover, α is just the image of d in Ω, so $e\alpha$ is the image of $ed = \xi(f)d = 1 \otimes f - f \otimes 1$, and this image is by definition just $\mathbf{d}f$. Thus, $\mu \in J^*$ corresponds to $(1/f) \otimes \mathbf{d}f$ as claimed.

As the isomorphism $A^* \to (K^{-1}/R) \otimes \Omega$ is A-linear, we see that $g\tau$ maps to $(g/f)\mathbf{d}f$, so

$$\text{res}((g/f)\mathbf{d}f) = (g\tau)(1) = \tau(g),$$

as claimed. □

Remark 21.33. It is useful and interesting to reconcile this result with [16, Proposition 9.2]. There we have a p-divisible formal group $\widehat{C} = \text{spf}(R)$ of height n over a formal scheme $S = \text{spf}(k)$, where k is a complete local Noetherian ring of residue characteristic p, and we will assume for simplicity that k is torsion-free. Fix $m \geq 1$ and let $\psi\colon \widehat{C} \to \widehat{C}$ be p^m times the identity map. In this context the subgroup scheme $D := \widehat{C}[p^m] = \ker(\psi)$ is a divisor of degree p^{nm}, so the ring $\mathcal{O}_D$ is self-dual (with a twist) as before. Given any coordinate x, we note that $\mathcal{O}_D = R/\psi^*x$, so the meromorphic form $\alpha = \mathbf{D}(x)/(\psi^*x)$ is a generator of the twisting module $(K^{-1}/R) \otimes \Omega$. We claim that α is actually independent of x. Indeed, any other coordinate x' has the form $x' = (x + x^2q)u$ for some $u \in k^\times$ and $q \in R$. It follows that $\mathbf{d}_0(x') = u\mathbf{d}_0(x)$, so that $\mathbf{D}(x') = u\mathbf{D}(x)$. We also have $\psi^*(x') = u\psi^*(x) \pmod{\psi^*(x)^2}$, and it follows that $\psi^*(x')^{-1} = u^{-1}\psi^*(x)^{-1} \pmod{R}$, so $\mathbf{D}(x')/\psi^*(x') = \mathbf{D}(x)/\psi^*(x)$ mod holomorphic forms, as claimed. Thus, we have a canonical generator for $(K^{-1}/R) \otimes \Omega$ and thus a canonical generator for $A^\vee$, giving a Frobenius form on $\mathcal{O}_D$. The cited proposition says that this is the same as the Frobenius form coming from a transfer construction. As discussed in the preamble to that proposition, p^m times the transfer form is the same as the trace form. In view of Proposition 21.32, this means that $p^m\alpha = \mathbf{d}(\psi^*x)/(\psi^*x)$. In fact, this is easy to see directly. We know that $\mathbf{D}(x)$ generates Ω and agrees with $\mathbf{d}(x)$ at zero, so $\mathbf{d}(x) = (1 + xr)\mathbf{D}(x)$ for some function $r \in R$. It follows that

$$\mathbf{d}(\psi^*x) = \psi^*(\mathbf{d}(x)) = (1 + \psi^*(x)\psi^*(r))\psi^*(\mathbf{D}(x)).$$

As $\mathbf{D}(x)$ is invariant we have $\psi^*\mathbf{D}(x) = p^m\mathbf{D}(x)$. It follows that $\mathbf{d}(\psi^*x)/(\psi^*x) = p^m\mathbf{D}(x)/(\psi^*x) = p^m\alpha$ in $(K^{-1}/R) \otimes \Omega$, as claimed.

Remark 21.34. It should be possible to connect our treatment of residues with that of Tate [19]. However, Tate assumes that the ground ring k is a field, and it seems technically awkward to remove this hypothesis.

21.3. **Topological duality.** Consider a periodically orientable theory E, an A-space X, and an equivariant complex bundle V over X. To avoid some minor technicalities, we will assume that X is a finite A-CW complex; everything can be generalised to the infinite case by passage to (co)limits. Let C be the multicurve $\text{spf}(E^0(P\mathcal{U} \times X))$ over $S := \text{spec}(E^0X)$. We then have a divisor $D = D(V)$ on C, to which we can apply all the machinery in the previous section. In particular, this gives us a residue map

$$\text{res}\colon (I_D^{-1}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega \to \mathcal{O}_S.$$

On the other hand, if we let τ denote the tangent bundle along the fibres of PV, then there is a stable Pontrjagin-Thom collapse map $X_+ \to PV^{-\tau}$, giving a Gysin map

$$p_!\colon \widetilde{E}^0 PV^{-\tau} \to E^0X = \mathcal{O}_S.$$

Theorem 21.35. *There is a natural isomorphism* $\widetilde{E}^0PV^{-\tau} = (I_D^{-1}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega$*, which identifies the Gysin map with the residue map.*

This is an equivariant generalisation of a result stated by Quillen in [14]. Even in the nonequivariant case, we believe that there is no published proof. The rest of this section constitutes the proof of our generalisation. (The case of nonequivariant *ordinary* cohomology is easy, and is a special case of the result proved in [6].)

We retain our previous notation for rings, and write $P^2V = PV \times_X PV$, so

$$\begin{aligned} k &= \mathcal{O}_S = E^0X \\ R &= \mathcal{O}_C = E^0(P\mathcal{U} \times X) \\ A &= \mathcal{O}_D = E^0(PV) \\ R_2 &= R\widehat{\otimes}R = E^0(P\mathcal{U} \times P\mathcal{U} \times X) \\ A_2 &= A \otimes A = E^0(P^2V). \end{aligned}$$

Next, observe that P_2V is a subspace of P^2V, and Proposition 20.10 tells us that $E^0P_2V = \mathcal{O}_{P_2D} = A_2/J$, so $J = E^0(P^2V, P_2V)$. On the other hand, there is another natural description of $E^0(P^2V, P_2V)$, which we now discuss. Let T be the tautological bundle on PV, consider the vector bundles $T^\perp = V \ominus T$ and $U = \text{Hom}(T, T^\perp)$, and let $B^\circ U$ denote the open unit ball bundle in U. A point in $B^\circ U$ is a triple (x, L, α) where $x \in X$ and $L \in PV_x$ and $\alpha\colon L \to V_x \ominus L$, such that $\|\alpha(u)\| < \|u\|$ for all $u \in L \setminus \{0\}$. We can thus consider $\text{graph}(\alpha)$ and $\text{graph}(-\alpha)$ as one-dimensional subspaces of $L \times (V_x \ominus L) = V_x$, or in other words points of PV_x, so we have a map $\delta'\colon B^\circ U \to P^2V$ given by

$$\delta'(x, L, \alpha) = (\text{graph}(\alpha), \text{graph}(-\alpha)).$$

Proposition 21.36. *The map* δ' *is a diffeomorphism* $B^\circ U \to P^2V \setminus P_2V$*.*

Proof. First, we must show that $\delta'(x, L, \alpha) \notin P_2V$, or in other words that $\text{graph}(\alpha)$ is not perpendicular to $\text{graph}(-\alpha)$. For this, we choose a nonzero element $u \in L$, so $v_0 = u + \alpha(u) \in \text{graph}(\alpha)$ and $v_1 = u - \alpha(u) \in \text{graph}(-\alpha)$. It follows that $\langle v_0, v_1\rangle = \|u\|^2 - \|\alpha(u)\|^2$; this is strictly positive because $\|\alpha\| < 1$, so the lines are not orthogonal, as required. We therefore have a map $\delta'\colon B^\circ U \to P^2V \setminus P_2V$.

Any element of $P^2V \setminus P_2V$ has the form (x, M_0, M_1) where $x \in X$ and $M_0, M_1 \in PV_x$ and M_0 is not orthogonal to M_1. This means that we can choose elements $u_i \in M_i$ with $\|u_i\| = 1$ and such that $t := \langle u_0, u_1\rangle$ is a positive real number. One checks that the pair (u_0, u_1) is unique up to the diagonal action of S^1. Put $v = u_0 + u_1$ and $w = u_0 - u_1$. By Cauchy-Schwartz we have $t \le 1$, and by direct expansion we have

$$\begin{aligned} \langle v, w\rangle &= 0 \\ \langle v, v\rangle &= 2(1+t) > 0 \\ \langle w, w\rangle &= 2(1-t) < \langle v, v\rangle. \end{aligned}$$

We can thus put $L = \mathbb{C}v \in PV_x$ and define $\alpha\colon L \to L^\perp$ by $\alpha(zv) = zw$; these are clearly independent of the choice of pair (u_0, u_1). As $\|w\| < \|v\|$ we have $\|\alpha\| < 1$. As $v + \alpha(v) = 2u_0$ we have $\text{graph}(\alpha) = M_0$, and as $v - \alpha(v) = 2u_1$ we have $\text{graph}(-\alpha) = M_1$. It follows that the construction $(x, M_0, M_1) \mapsto (x, L, \alpha)$ gives a well-defined map $\zeta\colon P^2V \setminus P_2V \to B^\circ U$, with $\delta'\zeta = 1$. One can check directly that $\zeta\delta'$ is also the identity, so δ' is a diffeomorphism as claimed. ∎

Corollary 21.37. *The bundle U is the normal bundle to the diagonal embedding $\delta\colon PV \to P^2V$, there is a homeomorphism $P^2V/P_2V = PV^U$, and the quotient map $P^2V \to P^2V/P_2V$ can be thought of as the Pontrjagin-Thom collapse associated to δ.* □

Remark 21.38. There are easier proofs of this corollary if one is willing to be less symmetrical.

Now, the above corollary together with Proposition 21.22 and Section 3 gives

$$\widetilde{E}^0 PV^U = E^0(P^2V, P_2V) = J = K/K^2 \otimes_A \overline{\Omega}^* = K/K^2 \otimes_k \omega^\vee.$$

On the other hand, we have

$$U = \operatorname{Hom}(T, T^\perp) = \operatorname{Hom}(T, p^*V) \ominus \mathbb{C},$$

so (using the case $W = V$ of Proposition 20.13)

$$PV^U = \Sigma^{-2} PV^{\operatorname{Hom}(T,p^*V)} = \Sigma^{-2} P(V \oplus V)/PV.$$

Remark 5.9 tells us that $\widetilde{E}^0(S^{-2}) = \omega^\vee$, and it is clear that $E^0(P(V \oplus V), PV) = K/K^2$. We thus obtain

$$\widetilde{E}^0 PV^U = K/K^2 \otimes_k \omega^\vee$$

again. These two arguments apparently give two different isomorphisms $\widetilde{E}^0 PV^U \to K/K^2 \otimes_k \omega^\vee$, but one can show (using Remark 20.16) that they are actually the same.

We next recall some ideas about Gysin maps. We discuss the situation for manifolds, and leave it to the reader to check that everything works fibrewise for bundles of manifolds, at least in sufficient generality for the arguments below. Let $f\colon M \to N$ be an analytic map between compact complex manifolds. (It is possible to work with much less rigid data, but we shall not need to do so.) Let τ_M and τ_N be the tangent bundles of M and N, and let ν_f be the virtual bundle $f^*\tau_N - \tau_M$ over M. Then for any virtual bundle U over N, a well-known variant of the Pontrjagin-Thom construction gives a stable map $T(f,U)\colon N^U \to M^{f^*U+\nu_f}$, and thus a map $f_! = T(f,U)^*\colon \widetilde{E}^0 M^{f^*U+\nu_f} \to \widetilde{E}^0 N^U$. Using the ring map $f^*\colon E^0N \to E^0M$ we regard the source and target of $T(f,U)^*$ as E^0N-modules, and we find that $T(f,U)^*$ is E^0N-linear. We also find that $T(f,U)^*$ can be obtained from $T(f,0)^*$ by tensoring over E^0N with $\widetilde{E}^0 N^U$. Finally, we have a composition formula: given maps $M \xrightarrow{f} N \xrightarrow{g} P$, we have $\nu_{gf} = \nu_f + f^*\nu_g$ and

$$T(f,\nu_g) \circ T(g,0) = T(gf,0)\colon P^V \to M^{(gf)^*V+\nu_{gf}}.$$

Now consider the maps $M \xrightarrow{\delta} M^2 \xrightarrow{1\times\pi} M$, where π is the constant map from M to a point. We have $\nu_\delta = \tau_M$ and $\nu_\pi = -\tau_M$, so the transitivity formula says that the composite

$$M_+ \xrightarrow{1\wedge T(\pi,0)} M_+ \wedge M^{-\tau} \xrightarrow{T(\delta,\nu_{1\times\pi})} M_+$$

is the identity. Assuming a Künneth isomorphism, we get maps

$$E^0M \xrightarrow{\delta_!} E^0M \otimes \widetilde{E}^0 M^{-\tau} \xrightarrow{1\otimes\pi_!} E^0M,$$

whose composite is again the identity.

Now specialise to the case $M = PV$. As before we put $A = E^0PV$ and identify $\widetilde{E}^0 M^\tau$ with J, and the map $\delta_! = T(\delta,0)^*\colon \widetilde{E}^0 M^\tau \to E^0(M^2)$ with the inclusion $J \to A \otimes A$. We know that the map

$$\delta_! = T(\delta,\nu_{1\times\pi})^*\colon A = E^0M \to E^0M \otimes \widetilde{E}^0 M^{-\tau} = A \otimes J^*$$

is obtained from $T(\delta,0)^*$ by tensoring over $A \otimes A$ with $A \otimes J^*$. It follows easily that $\delta_!(1) = u \in A \otimes J^*$, where u is as in Construction 21.8. The equation $(1 \otimes \pi_!)\delta_! = 1$ now tells us that $(1 \otimes \pi_!)(u) = 1$. Proposition 21.9 now tells us that $\pi_! = \epsilon\colon J^* \to k$. This proves Theorem 21.35.

22. Further theory of infinite Grassmannians

Recall from Section 19 that $G\mathcal{U}$ denotes the space of finite-dimensional subspaces of $\mathcal{U}$, which is the natural equivariant generalization of the space $G\mathbb{C}^\infty = \coprod_{d\geq 0} BU(d)$. We know from Theorem 19.1 that $E_0G\mathcal{U}$ is the symmetric algebra over E_0 generated by $E_0P\mathcal{U} = \mathcal{O}_C^\vee$. It follows that

$$\operatorname{spec}(E_0G\mathcal{U}) = \operatorname{Map}(C, \mathbb{A}^1)$$
$$\operatorname{spf}(E^0G\mathcal{U}) = \operatorname{Div}_d^+(C).$$

In this section, we obtain similar results for spaces analogous to $\mathbb{Z} \times BU$, BU and BSU.

Definition 22.1. For any finite-dimensional A-universe U, we put $\mathbf{2}U = U \oplus U$. We write U_+ for $U \oplus 0$ and U_- for $0 \oplus U$ so $\mathbf{2}U = U_+ + U_-$. We put

$$\widetilde{G}(U) = G(\mathbf{2}U) = \coprod_{d=0}^{2\dim(U)} G_d(\mathbf{2}U);$$

a point $X \in \widetilde{G}(U)$ should be thought of as a representative of the virtual vector space $X - U_-$. We embed $G(U)$ in $\widetilde{G}(U)$ by $X \mapsto X \oplus U = X_+ + U_-$. We define $\widetilde{\dim}\colon \widetilde{G}(U) \to \mathbb{Z}$ by $\widetilde{\dim}(X) = \dim(X) - \dim(U)$, and $\widetilde{G}_d(U) = \{X \mid \widetilde{\dim}(X) = d\}$. Given an isometric embedding $j\colon U \to V$, we define $j_*\colon \widetilde{G}(U) \to \widetilde{G}(V)$ by $j_*(X) = (j \oplus j)(X) + W_-$, where $W = V \ominus j(U)$. There is an evident map $\sigma\colon \widetilde{G}(U) \times \widetilde{G}(V) \to \widetilde{G}(U \oplus V)$ sending (X, Y) to $X \oplus Y$; one checks that $\widetilde{\dim}(X \oplus Y) = \widetilde{\dim}(X) + \widetilde{\dim}(Y)$ and that the map σ is compatible in an obvious sense with the maps j_*.

If $\mathcal{U}$ is an infinite A-universe, we define $\mathbf{2}\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$ as before, and put $\widetilde{G}(\mathcal{U}) = \varinjlim_U \widetilde{G}(U)$, where U runs over finite-dimensional subspaces. Equivalently, $\widetilde{G}(\mathcal{U})$ is the space of subuniverses $\mathcal{V} < \mathbf{2}\mathcal{U}$ such that the space $\mathcal{V} \cap \mathcal{U}_-$ has finite codimension in $\mathcal{V}$ and also has finite codimension in $\mathcal{U}_-$. This is a natural analogue of the space $\mathbb{Z} \times BU$.

Proposition 22.2. *For any $B \leq A$ we have*

$$(G\mathcal{U})^B = \prod_{\beta \in B^*} G(\mathcal{U}[\beta]) = \operatorname{Map}(B^*, \coprod_d BU(d))$$
$$(\widetilde{G}\mathcal{U})^B = \prod_{\beta \in B^*} \widetilde{G}(\mathcal{U}[\beta]) = \operatorname{Map}(B^*, \mathbb{Z} \times BU)$$

where

$$\mathcal{U}[\beta] = \{u \mid b.u = \exp(2\pi i \beta(b))u \text{ for all } b \in B\}$$

is the β-isotypical part of $\mathcal{U}$. In each case, the first equivalence is A/B-equivariant, but the second is not.

Proof. For the first isomorphism, just note that $\mathcal{U}$ splits A-equivariantly as $\bigoplus_\beta \mathcal{U}[\beta]$, and a subspace $V < \mathcal{U}$ is B-invariant iff it is the direct sum of its intersections with the subspaces $\mathcal{U}[\beta]$. This gives an A/B-equivariant isomorphism $(G\mathcal{U})^B = \prod_\beta G(\mathcal{U}[\beta])$, and it is clear that $G(\mathcal{U}[\beta])$ is nonequivariantly a copy of $\coprod_d BU(d)$ so $(G\mathcal{U})^B = \operatorname{Map}(B^*, \coprod_d BU(d))$. The argment for $(\widetilde{G}\mathcal{U})^B$ is essentially the same. ☐

We next write $R^+A = \mathbb{N}[A^*] = \pi_0^A(G\mathcal{U})$ for the additive semigroup of honest representations of A, and $RA = \mathbb{Z}[A^*] = \pi_0^A(\widetilde{G}\mathcal{U})$ for the additive group of virtual representations. It is clear that the semigroup ring $E_0[R^+A]$ is a polynomial algebra over E_0 with one generator u_α for each character α, and the group ring $E_0[RA]$ is the Laurent series ring with the same generators. In other words, we have

$$E_0[R^+A] = E_0[u_\alpha \mid \alpha \in A^*]$$
$$E_0[RA] = E_0[u_\alpha, u_\alpha^{-1} \mid \alpha \in A^*] = E_0[R^+A][v^{-1}]$$

where $v = \prod_\alpha u_\alpha$. Note that

$$\operatorname{spec}(E_0[R^+A]) = \operatorname{Map}(A^*, \mathbb{A}^1)$$
$$\operatorname{spec}(E_0[RA]) = \operatorname{Map}(A^*, \mathbb{G}_m),$$

and the isomorphisms $R^+A = \pi_0^A G\mathcal{U}$ and $RA = \pi_0^A \widetilde{G}\mathcal{U}$ give maps $E_0[R^+A] \to E_0 G\mathcal{U}$ and $E_0[RA] \to E_0\widetilde{G}\mathcal{U}$.

Now let ϕ be the obvious isomorphism

$$\mathbb{C}[A] \oplus \mathcal{U} = \mathbb{C}[A] \oplus \mathbb{C}[A]^\infty \to \mathbb{C}[A]^\infty = \mathcal{U},$$

and define $\phi'\colon G_d\mathcal{U} \to G_{d+|A|}\mathcal{U}$ by $\phi'(X) = \phi(\mathbb{C}[A] \oplus X)$.

Proposition 22.3. *The space $\widetilde{G}\mathcal{U}$ is the telescope of the self-map ϕ' of $G\mathcal{U}$. We thus have*

$$E_0\widetilde{G}\mathcal{U} = v^{-1}E_0 G\mathcal{U} = E_0[RA] \otimes_{E_0[R^+A]} E_0 G\mathcal{U},$$

and so $\operatorname{spec}(E_0\widetilde{G}\mathcal{U}) = \operatorname{Map}(C, \mathbb{G}_m)$.

Proof. Put $\mathcal{U}' = \mathbb{C}[A][z, z^{-1}]$, and identify this with $\mathbf{2}\mathcal{U}$ by sending $(e_k, 0)$ to z^k and $(0, e_k)$ to z^{-k-1}. The standard embedding $G\mathcal{U} \to \widetilde{G}\mathcal{U}$ now sends X to $X \oplus \mathcal{U}_-$. It is easy to check that $\widetilde{G}\mathcal{U} = \varinjlim_k z^{-k}G\mathcal{U}$ on the nose, and that the inclusion $z^{-k}G\mathcal{U} \to z^{-k-1}G\mathcal{U}$ is isomorphic to the map ϕ'. The first claim follows, and the second claim is just the obvious consequence in homology. The tensor product description of $E_0\widetilde{G}\mathcal{U}$ gives us a pullback square

$$\begin{array}{ccc} \operatorname{spec}(E_0\widetilde{G}\mathcal{U}) & \longrightarrow & \operatorname{spec}(E_0G\mathcal{U}) = \operatorname{Map}(C, \mathbb{A}^1) \\ \downarrow & & \downarrow \\ \operatorname{Map}(A^*, \mathbb{G}_m) = \operatorname{spec}(E_0[RA]) & \longrightarrow & \operatorname{spec}(E_0[R^+A]) = \operatorname{Map}(A^*, \mathbb{A}^1). \end{array}$$

As C is a formal neighbourhood of the image of ϕ, we see that a map $C \xrightarrow{f} \mathbb{A}^1$ lands in $\mathbb{G}_m$ if and only if the composite $A^* \times S \xrightarrow{\phi} C \xrightarrow{f} \mathbb{A}^1$ lands in $\mathbb{G}_m$. Given this, we see that the pullback is just $\operatorname{Map}(C, \mathbb{G}_m)$ as required. □

We next introduce the analogue of BU.

Proposition 22.4. *There is a natural splitting $\widetilde{G}\mathcal{U} = \mathbb{Z} \times \widetilde{G}_0\mathcal{U}$, and we have $\operatorname{spec}(E_0\widetilde{G}_0\mathcal{U}) = \operatorname{Map}_0(C, \mathbb{G}_m)$ (the scheme of maps $f: C \to \mathbb{G}_m$ with $f(0) = 1$).*

Proof. We have already described an equivariant map $\widetilde{\dim}: \widetilde{G}\mathcal{U} \to \mathbb{Z}$, and defined $\widetilde{G}_0\mathcal{U} = \ker(\widetilde{\dim})$. We also have $(\widetilde{G}\mathcal{U})^A = \operatorname{Map}(A^*, \mathbb{Z} \times BU)$ so $\pi_0^A(\widetilde{G}\mathcal{U}) = \operatorname{Map}(A^*, \mathbb{Z}) = RA$, which gives an equivariant map $i: RA \to \widetilde{G}\mathcal{U}$ (where RA has trivial action). The composite $\widetilde{\dim} \circ i: RA \to \mathbb{Z}$ is just the usual augmentation map ϵ sending a virtual representation to its dimension. Thus, if we let $\eta: \mathbb{Z} \to RA$ be the unit map, then $i \circ \eta$ is a section of $\widetilde{\dim}$. As $\widetilde{G}\mathcal{U}$ is a commutative equivariant H-space, we can define a map $\delta: \widetilde{G}\mathcal{U} \to \widetilde{G}_0\mathcal{U}$ by $x \mapsto (i(\eta(\widetilde{\dim}(x))) - x)$, and we find that the resulting map $(\widetilde{\dim}, \delta): \widetilde{G}\mathcal{U} \to \mathbb{Z} \times \widetilde{G}_0\mathcal{U}$ is an equivalence. This is easily seen to be parallel to the splitting $\operatorname{Map}(C, \mathbb{G}_m) = \mathbb{G}_m \times \operatorname{Map}_0(C, \mathbb{G}_m)$ given by $f \mapsto (f(0), f(0)/f)$, which gives the claimed description of $\operatorname{spec}(E_0G\mathcal{U})$. □

Remark 22.5. There are two other possible analogues of BU. Firstly, one could take the colimit of the spaces $G_d\mathcal{U}$ using the maps $V \mapsto V \oplus \mathbb{C}$; the scheme associated to the corresponding space is then $\operatorname{Map}_0(C, \mathbb{A}^1)$, which classifies maps $f: C \to \mathbb{A}^1$ with $f(0) = 1$. Alternatively, one could take the colimit of the spaces $G_{d|A|}\mathcal{U}$ using the maps $V \mapsto V \oplus \mathbb{C}[A]$. This gives the scheme of maps $f: C \to \mathbb{G}_m$ for which $\prod_{\alpha \in A^*} f(\phi(\alpha)) = 1$. However, the space $\widetilde{G}_0\mathcal{U}$ described above is the one that occurs in Greenlees's definition of the spectrum kU_A, and is also the one whose Thom spectrum is MU_A.

We next introduce the analogue of BSU. For this, we need an analogue of the map $B\det: BU \to \mathbb{C}P^\infty$.

Definition 22.6. Given a universe U of finite dimension d, we put $\widetilde{F}U = \operatorname{Hom}(\lambda^d U_-, \lambda^d(\mathbf{2}U))$. We make this a functor as follows. Given an isometric embedding $j: U \to V$, we put $W = V \ominus jU$ and $e = \dim(W)$. As $j: U \to jU$ is an isomorphism and $\lambda^e W$ is one-dimensional, we have an evident isomorphism

$$\widetilde{F}U = \operatorname{Hom}(\lambda^d jU_- \otimes \lambda^e W_-, \lambda^d(\mathbf{2}jU) \otimes \lambda^e W_-).$$

The isomorphism $V = jU \oplus W$ gives an isomorphism $\lambda^d jU_- \otimes \lambda^e W = \lambda^{d+e} V$ and an embedding $\lambda^d(\mathbf{2}jU) \otimes \lambda^e W_- \to \lambda^{d+e}(\mathbf{2}V)$. Putting this together gives the required map $j_*: \widetilde{F}U \to \widetilde{F}V$.

There are also obvious maps $\widetilde{F}(U) \otimes \widetilde{F}(V) \to \widetilde{F}(U \oplus V)$, compatible with the above functorality. This gives maps $P\widetilde{F}(U) \times P\widetilde{F}(V) \to P\widetilde{F}(U \oplus V)$ of the associated projective spaces.

Next, recall that a point of $\widetilde{G}_0U$ is a d-dimensional subspace $X \leq \mathbf{2}U$. We define

$$\widetilde{\det}(X) = \operatorname{Hom}(\lambda^d U_-, \lambda^d X) \in P\widetilde{F}U.$$

One can check that this gives a natural map $\widetilde{\det}: \widetilde{G}_0 \to P\widetilde{F}$, with $\widetilde{\det}(X \oplus Y) = \widetilde{\det}(X) \otimes \widetilde{\det}(Y)$.

Finally, for our complete universe $\mathcal{U}$ we put $\widetilde{F}\mathcal{U} = \varinjlim_U \widetilde{F}U$, where U runs over the finite-dimensional subuniverses. It is easy to check that this is again a complete A-universe, and thus is unnaturally isomorphic to $\mathcal{U}$. The maps $\widetilde{\det}$ pass to the colimit to give an H-map $\widetilde{\det}\colon \widetilde{G}_0\mathcal{U} \to P\widetilde{F}\mathcal{U} \simeq P\mathcal{U}$. We write $S\widetilde{G}_0\mathcal{U}$ for the pullback of the projection $S(\widetilde{F}\mathcal{U}) \to P\widetilde{F}\mathcal{U}$ along the map $\widetilde{\det}$, or equivalently the space of pairs $(\mathcal{V}, u)$ where $\mathcal{V} \in \widetilde{G}_0\mathcal{U}$ and u is a unit vector in the one-dimensional space $\widetilde{\det}(\mathcal{V})$. As $S(\widetilde{F}\mathcal{U})$ is equivariantly contractible, this is just the homotopy fibre of $\widetilde{\det}$.

Proposition 22.7. *There is a natural splitting $\widetilde{G}_0\mathcal{U} = S\widetilde{G}_0\mathcal{U} \times P\mathcal{U}$ (which does not respect the H-space structure).*

Proof. It is enough to give a section of the H-map $\widetilde{\det}\colon \widetilde{G}_0\mathcal{U} \to P\mathcal{U}$. We can include $P\mathcal{U} = G_1\mathcal{U}$ in $\widetilde{G}_1\mathcal{U} \subset \widetilde{G}\mathcal{U}$ in the usual way, then use the projection $\widetilde{G}\mathcal{U} \to \widetilde{G}_0\mathcal{U}$ from Proposition 22.4. We find that the resulting composite $P\mathcal{U} \to P\mathcal{U}$ is actually minus the identity, but we can precompose by minus the identity to get the required section. □

Remark 22.8. Cartier duality identifies $\operatorname{spec}(E_0P\mathcal{U})$ with $\operatorname{Hom}(C, \mathbb{G}_m)$, and the proposition suggests that $\operatorname{spec}(E_0S\widetilde{G}_0\mathcal{U})$ should be the quotient $\operatorname{Map}_0(C, \mathbb{G}_m)/\operatorname{Hom}(C, \mathbb{G}_m)$. However, there are difficulties in interpreting this quotient, and it is in fact more useful to take a slightly different approach as in [1, 2]. We will not give details here.

Next, recall that Greenlees has defined an equivariant analogue of connective K-theory (denoted by kU_A) by the homotopy pullback square

$$\begin{array}{ccc} kU_A & \longrightarrow & F(EA_+, kU) \\ \big\downarrow & & \big\downarrow \\ KU_A & \longrightarrow & F(EA_+, KU). \end{array}$$

If $v \in \pi_2 kU$ is the Bott element then $kU[v^{-1}] = KU$ and $kU/v = H$. It is not hard to see that there is a corresponding element in $\pi_2^A kU_A$ with $kU_A[v^{-1}] = KU_A$ and $kU_A/v = F(EA_+, H)$.

Proposition 22.9. *The zeroth, second and fourth spaces of kU_A are $\widetilde{G}\mathcal{U}$, $\widetilde{G}_0\mathcal{U}$ and $S\widetilde{G}_0\mathcal{U}$ respectively.*

Proof. We take it as well-known that the zeroth space of KU_A is $\widetilde{G}\mathcal{U}$, and KU_A is two-periodic so this is also the $2k$'th space for all k. Let X_k denote the $2k$'th space of kU_A, so we have a homotopy pullback square

$$\begin{array}{ccc} X_k & \longrightarrow & F(EA_+, BU\langle 2k\rangle) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow \\ \widetilde{G}\mathcal{U} & \xrightarrow[j]{} & F(EA_+, \mathbb{Z} \times BU) \end{array}$$

(where $BU\langle 0\rangle$ is interpreted as $\mathbb{Z} \times BU$). In the case $k = 0$, the map i is the identity and so $X_0 = \widetilde{G}\mathcal{U}$. In the case $k = 1$, the map i is just the inclusion

$$F(EA_+, BU) \to \mathbb{Z} \times F(EA_+, BU) = F(EA_+, \mathbb{Z} \times BU)$$

and the map j sends $\widetilde{G}_k\mathcal{U}$ into $\{k\} \times F(EA_+, BU)$. It follows easily that $X_1 = \widetilde{G}_0\mathcal{U}$. In the case $k = 2$, we note that the cofibration $\Sigma^2 kU_A \xrightarrow{v} kU_A \to F(EA_+, H)$ gives a fibration $X_2 \to X_1 \to F(EA_+, K(\mathbb{Z}, 2))$. We know that $X_1 = \widetilde{G}_0\mathcal{U}$ and Proposition 4.4 that $F(EA_+, K(\mathbb{Z}, 2)) = P\mathcal{U}$. One can check that the resulting map $\widetilde{G}_0\mathcal{U} \to P\mathcal{U}$ is just $\pm\widetilde{\det}$, and so $X_2 = S\widetilde{G}_0\mathcal{U}$ as claimed. □

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