REALISING FORMAL GROUPS

N. P. STRICKLAND

Abstract. We show that a large class of formal groups can be realised functorially by even periodic ring spectra.

1. Introduction

Let FG be the category of formal groups (of the sort usually considered in algebraic topology) over affine schemes. Thus, an object of FG consists of a pair \((G, S)\), where \(S\) is an affine scheme, \(G\) is a formal group scheme over \(S\), and a coordinate \(x\) can be chosen such that \(O_G \cong O_S[x]\) as \(O_S\)-algebras. A morphism from \((G_0, S_0)\) to \((G_1, S_1)\) is a commutative square

\[
\begin{array}{ccc}
G_0 & \rightarrow & G_1 \\
\downarrow & & \downarrow \\
S_0 & \rightarrow & S_1
\end{array}
\]

such that the induced map \(G_0 \rightarrow p^*G_1\) is an isomorphism of formal group schemes over \(S_0\).

Next, recall that an even periodic ring spectrum is a commutative and associative ring spectrum \(E\) such that \(E^1 = 0\) and \(E^2\) contains a unit (which implies that \(E \cong \Sigma^2 E\) as spectra). Here we are using the usual notation \(E^k = E^k(\text{point}) = \pi_{-k} E\). We write EPR for the category of even periodic ring spectra.

Given an even periodic ring spectrum \(E\), we can form the scheme \(S_E := \text{spec}(E^0)\) and the formal group scheme \(G_E = \text{spf}(E^0 \mathbb{CP}^\infty)\) over \(S_E\). This construction gives rise to a functor \(\Gamma: \text{EPR}^{op} \rightarrow \text{FG}\).

It is a natural problem to try to define a realisation functor \(R: \text{FG} \rightarrow \text{EPR}^{op}\) with \(\Gamma R(G, S) \cong (G, S)\), or at least to do this for suitable subcategories of FG. For example, if we let LFG denote the category of Landweber exact formal groups, and put \(\text{LEPR} = \{E \in \text{EPR} \mid \Gamma(E) \in \text{LFG}\}\), one can show that the functor \(\Gamma: \text{LEPR}^{op} \rightarrow \text{LFG}\) is an equivalence; this is essentially due to Landweber, but details of this formulation are given in [4, Proposition 8.43]. Inverting this gives a realisation functor for LFG, and many well-known spectra are constructed using this. In particular, this gives various different versions of elliptic cohomology, based on various universal families of elliptic curves over rings such as \(\mathbb{Z}[\frac{1}{p}, c_4, c_6]\).

In Definition 3.13, we will introduce a full subcategory \(\text{GFG} \subseteq \text{FG}\) of “good” formal groups. Among many other things, this contains all formal groups over fields of characteristic not equal to 2, and all formal groups over rings of the
form $\mathbb{Z}[1/n]$ with $n$ an even integer (this is proved as Theorem 3.14). Most formal groups in GFG are not Landweber exact. Our main result is as follows.

**Theorem 1.1.** There is a realisation functor $R: \text{GFG} \to \text{EPR}$, with $\Gamma R \simeq 1: \text{GFG} \to \text{GFG}$.

The results of [5] give all the required objects in EPR; the new content of the theorem is the analysis of morphisms. Here we explain the formal part of the construction; in Section 4 we will give additional details and prove that we have the required properties. The functor $R$ actually arises as $UV^{-1}$ for a pair of functors $GFG \xrightarrow{V} \mathcal{E} \xrightarrow{U} \text{EPR}$ in which $V$ is an equivalence. The definition of $\mathcal{E}$ involves the periodic bordism spectrum $MP = \bigvee_{n \in \mathbb{Z}} \Sigma^nMU$. We will verify in the appendix that this can be constructed as a strictly commutative ring, so we have a topological category $\text{Mod}_0$ of $MP$-modules. We write $\text{DMod}_0$ for the derived category, and $\text{EPA}_0$ for the category of even periodic commutative ring objects in $\text{DMod}_0$. The unit map $\eta: S \to MP$ gives a functor $\eta^*: \text{EPA}_0 \to \text{EPR}$, and the objects of the category $\mathcal{E}$ are the objects $E \in \text{EPA}_0$ for which the associated formal group law is “stably realisable” in a sense to be explained later. The morphism set $\mathcal{E}(E_0, E_1)$ is a subset of $\text{EPR}(\eta^*E_0, \eta^*E_1)$, the functor $\Gamma: \mathcal{E} \to \text{GFG}$ is given by $\Gamma$, and the functor $\Gamma: \mathcal{E} \to \text{EPR}$ is given by $\eta^*$. We say that a map $f: \eta^*E_0 \to \eta^*E_1$ in EPR is good if there is a commutative ring object $A$ in the derived category of $MP \wedge MP$-modules together with maps $f': E_0 \to (1 \land \eta)^*A$ and $f'': (\eta \land 1)^*A \to E_1$ in $\text{EPA}_0$ such that $f''$ is an equivalence and $f$ is equal to the composite

$$\eta^*E_0 \xrightarrow{\eta^*f'} (\eta \land \eta)^*A \xrightarrow{\eta^*f''} \eta^*E_1.$$  

The morphisms in the category $\mathcal{E}$ are just the good maps. To prove Theorem 1.1, we need to show that

1. The composite of two good maps is good, so $\mathcal{E}$ really is a category.
2. For any map $\Gamma(\eta^*E_0) \to \Gamma(\eta^*E_1)$ of good formal groups, there is a unique good map $\eta^*E_0 \to \eta^*E_1$ inducing it, so that $V$ is full and faithful.
3. For any good formal group $(G, S)$ there is an object $E \in \text{EPA}_0$ such that $\Gamma(\eta^*E) \simeq (G, S)$, so $V$ is essentially surjective.

To prove statement (1), we need to construct modules over the $k$-fold smash power of $MP$. It will be most efficient to do this for all $k$, and use cosimplicial ideas to organize the functors between the various module categories.

Almost everywhere in this paper, we will assume that 2 has been inverted; we do not know whether there are any good formal groups in which this is not the case. We found in [5] that when 2 is not inverted, there are nontrivial obstructions to various realisation problems, which can only be calculated with considerable labour. We may return to this in future work.

2. **Preliminaries**

2.1. **Differential forms.** Let $(G, S)$ be a formal group, and let $I \subseteq \mathcal{O}_G$ be the augmentation ideal. Recall that the cotangent space of $G$ at zero is the module $\omega_G = I/I^2$. If $x$ is a coordinate on $G$ that vanishes at zero, then we write $dx$ for the image of $x$ in $I/I^2$, and note that $\omega_G$ is freely generated over $\mathcal{O}_S$ by $dx$. We
define a graded ring \( D(G, S)^* \) by
\[
D(G, S)^k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \omega^\otimes(-k/2) & \text{if } k \text{ is even}, \end{cases}
\]

Here the tensor products are taken over \( \mathcal{O}_S \), and \( \omega^\otimes n \) means the dual of \( \omega^\otimes |n| \) when \( n < 0 \). Where convenient, we will convert to homological gradings by the usual rule: \( D(G, S)_k = D(G, S)^{-k} \).

Now let \( E \) be an even periodic ring spectrum with \( \Gamma(E) = (G, S) \). We then have \( \mathcal{O}_G = E^0 \mathbb{C} \mathbb{P}^\infty \) and \( I = E^0 \mathbb{C} \mathbb{P}^\infty \) and one checks easily that the inclusion \( S^2 = \mathbb{C} \mathbb{P}^1 \to \mathbb{C} \mathbb{P}^\infty \) gives an isomorphism \( \omega_G = I/I^2 = E^0 S^2 = E^{-2} \). Using the periodicity of \( E \), we see that this extends to a canonical isomorphism \( D(\Gamma(E))^* \cong E^* \).

It also follows from this analysis (or from more direct arguments) that a map \( f: E_0 \to E_1 \) in EPR is a weak equivalence if and only if \( \pi_0 f \) is an isomorphism.

2.2. Groups and laws. We define a category FGL as follows. The objects are pairs \((R, F)\), where \( R \) is a commutative ring, and \( F \) is a formal group law over \( R \). A morphism from \((R_0, F_0)\) to \((R_1, F_1)\) consists of a ring homomorphism \( \phi: R_0 \to R_1 \), together with a formal power series \( f(x) \in R_1[[x]] \) satisfying \( f(0) = 0 \) and \( f'(0) \in R_1^* \) and
\[
(\phi_*, F_0)(f(x), f(y)) = f(F_1(x, y)) \in R_1[[x, y]].
\]
Here \( \phi_*, F_0 \) means the FGL over \( R_1 \) obtained by applying \( \phi \) to the coefficients of \( F_0 \). The composition is given by \((\phi_1, f_1) \circ (\phi_0, f_0) = (\phi_1 \phi_0, f_2)\), where \( f_2(x) = (\phi_1 \phi_0)(f_1(x)) \).

Given \((R, F)\) \in FGL we can make \( R[[x]] \) into a Hopf algebra over \( R \) (in a suitably completed sense) by defining
\[
\psi(x) = F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]] = R[[x]] \hat{\otimes} R[[x]].
\]
This makes the formal scheme \( \text{spec}(R) \times \hat{\mathbb{A}}^1 = \text{spec}(R[[x]]) \) into a formal group over \( \text{spec}(R) \). This construction gives a functor \( Q: \text{FGL}^{op} \to \text{FG} \), which is easily seen to be full and faithful.

If \((G, S)\) \in FG then we can choose a coordinate \( x \) on \( G \) that vanishes at zero, and we find that there is a unique FGL \( F \) over \( \mathcal{O}_S \) such that
\[
\psi(x) = F(x \otimes 1, 1 \otimes x) \in \mathcal{O}_S[[x \otimes 1, 1 \otimes x]].
\]
We can regard \( x \) as a map \( G \to \hat{\mathbb{A}}^1 \), and as such it induces an isomorphism \( G \to Q(\mathcal{O}_S, F) \). It follows that \( Q \) is essentially surjective and thus an equivalence.

2.3. Periodic bordism. Consider the homology theory \( MP_*(X) = MU_*(X) \otimes \mathbb{Z}[u, u^{-1}] \), where \( u \) has homological degree 2 (and thus cohomological degree –2). This is represented by the spectrum \( MP = \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU \), with an evident ring structure. It is well-known that \( MU \) is an \( E_\infty \) ring spectrum; see for example [3, Section IX]. It is also shown there that \( MU \) is an \( H^2_\infty \) ring spectrum, which means (as explained in [3, Remark VII.2.9]) that \( MP \) is an \( H_\infty \) ring spectrum; this is weaker than \( E_\infty \) in theory, but usually equivalent in practise. As one would expect, \( MP \) is actually an \( E_\infty \) ring spectrum; a proof is given in the appendix. It follows from [2, Proposition II.4.3] that one can construct a model for \( MP \) that is a strictly
commutative ring spectrum (or "S-algebra"). We may also assume that it is a
cofibrant object in the category of all strictly commutative ring spectra.

For typographical convenience, we write $MP(r)$ for the $(r+1)$-fold smash power
$MP \wedge \ldots \wedge MP$, which is again a strictly commutative ring. The spectra $MP(r)$
fit together into a cosimplicial object in the usual way. In the category of strictly
commutative ring spectra, the coproduct is the smash product. It follows formally
that the smash product of cofibrant objects is cofibrant, so in particular the objects
$MP(r)$ are all cofibrant.

The formal group $\Gamma(MP)$ has a canonical coordinate $\bar{x}$, giving a canonical formal
group law $F$ and an isomorphism $Q(\pi_0 MP, F) = \Gamma(MP)$. By a well-known theorem
of Quillen, this formal group law is universal: for any FGL $F$ over any ring $R$, there
is a unique map $c[F] : \pi_0 MP \to R$ carrying $\bar{F}$ to $F$. In this situation, we use the
notation $(R, F)$ to refer to $R$ with the $\pi_0 MP$-algebra structure obtained from $c[F]$.

Next consider $MP_0 MP = \pi_0 MP(1)$. There are two maps $1 \wedge \eta, \eta \wedge 1 : MP \to
MP(1)$, giving rise to formal group laws $\tilde{F}_0, \tilde{F}_1$ over $\pi_0 MP(1)$. These arise from two
derent coordinates on the same formal group, so there is a canonical isomorphism
$\tilde{b} : \tilde{F}_1 \to \tilde{F}_0$ defined by a power series
$\tilde{b}(x) = \sum_{k \geq 0} \tilde{b}_k x^{k+1} \in (\pi_0 MP(1))[x]$. It is
well-known that this is universal in the following sense: given any ring $R$ and any
isomorphism $b : F_1 \to F_0$ of formal group laws over $R$, there is a unique ring map
c[$F_0, b$] : $\pi_0 MP(1) \to R$ carrying $\tilde{F}_1$ to $F_1$ and $\tilde{b}$ to $b$. (This can be deduced
from the corresponding result for $MU$, but note that in our periodic setting we get
all isomorphisms, not just the strict ones.) We use the notation $(R, F_0, b)$ to refer to $R$ with the $\pi_0 MP(1)$-algebra structure obtained from this map. It is clear
by construction that

$$(\pi_0 MP \xrightarrow{\pi_0 (1 \wedge \eta)} \pi_0 MP(1) \xrightarrow{c[F_0, b]} R) = c[F_0]$$

$$(\pi_0 MP \xrightarrow{\pi_0 (\eta \wedge 1)} \pi_0 MP(1) \xrightarrow{c[F_0, b]} R) = c[F_1].$$

The same circle of ideas shows that $\pi_0 MP(1) = \pi_0 MP[\tilde{b}_0, \tilde{b}_1, \ldots] [\tilde{b}_0^{-1}]$ (where
$\pi_0 MP(1)$ is regarded as an algebra over $\pi_0 MP$ using the map $\eta \wedge 1$).

More generally, we find that $\pi_0 MP(r)$ is the universal example of a ring equipped
with $r + 1$ different formal group laws $(F_0, \ldots, F_r)$ and a chain of isomorphisms
$F_0 \leftarrow \ldots \leftarrow F_r$ between them, and that $\pi_0 MP(r)$ is obtained from a polynomial
algebra on countably many generators over $\pi_0 MP$ by inverting $r$ of the generators.

One also checks that $\pi_1 MP(r) = 0$ for all $r$. The module $\pi_2 MP(r)$ is free of
rank one over $\pi_0 MP(r)$, and there are a number of different natural choices of
generator. The most natural way around this is to use the formalism of Section 2.1,
and regard $\pi_2 MP(r)$ as $\omega_{G_{MP(r)}}^*$.

2.4. Module categories. We write $\text{Mod}_r$ for the category of $MP(r)$-modules (in
the strict sense, not the homotopical one). Note that a map $f : A_0 \to A_1$ of strictly
commutative ring spectra gives a functor $f^*$ : $\text{Mod}_{A_1} \to \text{Mod}_{A_0}$, which is just the
identity on the underlying spectra (and thus preserves weak equivalences). It follows
easily that for any two maps $A_0 \xrightarrow{\delta_0} A_1 \xrightarrow{\delta_1} A_2$, the functor $f^* g^*$ is actually equal
(not just naturally isomorphic or naturally homotopy equivalent) to $(gf)^*$. Thus,
the categories $\text{Mod}_r$ fit together to give a simplicial category $\text{Mod}_*$. 
Remark 2.1. For us, a simplicial category means a simplicial object in the category of categories. Elsewhere in the literature, the same phrase is sometimes used to refer to categories enriched over the category of simplicial sets, which is a rather different notion.

Next, we write $\text{DMod}_r$, the derived category of $\text{Mod}_r$, as in [2, Chapter III]. As usual, there are two different models for a category such as $\text{DMod}_r$:

(a) One can take the objects to be the cofibrant objects in $\text{Mod}_r$, and morphisms to be homotopy classes of maps; or

(b) One can use all objects in $\text{Mod}_r$ and take morphisms to be equivalence classes of “formal fractions”, in which one is allowed to invert weak equivalences.

We will use model (b). This preserves the strong functoriality mentioned previously, and ensures that $\text{DMod}_r$ is again a simplicial category.

We also write $\text{EPA}_r$ for the category of even periodic commutative ring objects in $\text{DMod}_r$, giving another simplicial category. (Note that periodicity is actually automatic, because $MP(r)$ is itself periodic.) We reserve the letter $\eta$ for the unit map $S \to MP$. We generally write $\zeta: MP \to E$ for the unit map of an object in $\text{EPA}_0$, and $\xi$ for unit maps in $\text{EPA}_1$.

If $E \in \text{EPA}_0$ then the canonical coordinate $\tilde{x}$ on $\Gamma(MP)$ gives a coordinate $\tilde{\zeta}, \tilde{x}$ on $\Gamma(\eta^*E)$ and thus a formal group law $F$ over $\pi_0E$, with $c[F] = \pi_0\zeta; \pi_0MP \to \pi_0E$, and a canonical isomorphism $\Gamma(\eta^*F) \simeq Q(\pi_0E, F)$. A map $f: \eta^*E_0 \to \eta^*E_1$ in $\text{EPR}$ gives a map $\Gamma(f): \Gamma(\eta^*E_1) \to \Gamma(\eta^*E_0)$ in FG, corresponding to a map $(\pi_0f, b): (\pi_0E_0, F_0) \to (\pi_0E_1, F_1)$ in FGL; the power series $b \in (\pi_0E_1)[[t]]$ is characterised by the fact that $f_\ast x_0 = b(x_1)$ in $E_0^0 \mathbb{C}P^\infty$.

3. Basic Realisation Results

Let $R$ be a strictly commutative ring spectrum that is even and periodic, such that $R_0$ is an integral domain. The main examples will be $R = MP(r)$ for $r \geq 0$. Let $\mathcal{D}$ be the derived category of $R$-modules, and let $\mathcal{R}$ be the category of commutative ring objects $A \in \mathcal{D}$ such that $\pi_1A = 0$ and $\pi_0A$ has no 2-torsion. We also write $\mathcal{R}_0$ for the category of commutative algebras over $\pi_0R$ without 2-torsion, We say that an object $A_0 \in \mathcal{R}_0$ is strongly realisable if there exists an object $A \in \mathcal{R}$ and an isomorphism $A_0 \to \pi_0A$ such that for any $B \in \mathcal{R}$, the induced map $\mathcal{R}(A, B) \to \mathcal{R}_0(A_0, \pi_0B)$ is an isomorphism.

The results of [5] provide a good supply of strongly realisable rings, except that we need a little translation between the even periodic framework and the usual graded framework. Suppose that $A_0 \in \mathcal{R}_0$, and put $T = \text{spec}(A_0)$. We have a unit map $\eta: \pi_0R \to A_0$ and thus a map $\text{spec}(\eta): T \to S_R$; we can pull back the formal group $G_R$ along this to get a formal group $H := \text{spec}(\eta)^*G_R$ over $T$. From this we get a map $\eta_*: R_* = D(G_R, S_R)_* \to D(H, T)_*$, which agrees with $\eta$ in degree zero. Indeed, if we choose a generator $u$ of $R_2$ over $R_0$, then $\eta_*$ is just the map $R_0[u, u^{-1}] \to A_0[u, u^{-1}]$ obtained in the obvious way from $\eta$. It is easy to check that $A_0$ is strongly realisable (as defined in the previous paragraph) iff $D(H, T)_*$ is strongly realisable over $R_*$ (as defined in [5]).

Definition 3.1. A short ordinal is an ordinal $\lambda$ of the form $n.\omega + m$ for some $n, m \in \mathbb{N}$. A regular sequence in a ring $R_0$ is a system of elements $(x_\alpha)_{\alpha < \lambda}$ for some short ordinal $\lambda$ such that $x_\alpha$ is not a zero-divisor in the ring $S^{-1}R_0/(x_\beta \mid \beta < \alpha)$. 
An object \( A_0 \in \mathcal{R}_0 \) is a \textit{localised regular quotient} (or LRQ) of \( R_0 \) if \( A_0 = (S^{-1}R_0)/I \) for some subset \( S \subset R_0 \) and some ideal \( I \leq S^{-1}R_0 \) that can be generated by a regular sequence.

\textbf{Remark 3.2.} We have made a small extension of the usual notion of a regular sequence, to ensure that any LRQ of an LRQ of \( R_0 \) is itself an LRQ of \( R_0 \); see Lemma 3.9.

\textbf{Proposition 3.3.} If \( A_0 \) is an LRQ of \( R_0 \) in which 2 is invertible, then it is strongly realisable.

\textit{Proof.} This is essentially [5, Theorem 2.6], translated into a periodic setting as explained above. Here we are using a slightly more general notion of a regular sequence, but all the arguments can be adapted in a straightforward way. The main point is that any countable limit ordinal has a cofinal sequence, so homotopy colimits can be constructed using telescopes in the usual way. Andrey Lazarev has pointed out a lacuna in [5]: it is necessary to assume that the elements \( x_\alpha \) are all regular in \( S^{-1}R_0 \) itself, which is not generally automatic. However, we are assuming that \( R_0 \) is an integral domain so this issue does not arise. \qed

\textbf{Proposition 3.4.} Suppose that

- \( A \) and \( B \) are strong realisations of \( A_0 \) and \( B_0 \)
- The ring \( A_0 \otimes_{R_0} B_0 \) has no 2-torsion
- The natural map \( A_0 \otimes_{R_0} B_0 \rightarrow (A \wedge_R B)_0 \) is an isomorphism.

Then \( A \wedge_R B \) is a strong realisation of \( A_0 \otimes_{R_0} B_0 \).

\textit{Proof.} This follows from [5, Corollary 4.5]. \qed

\textbf{Proposition 3.5.} If \( A_0 \in \mathcal{R}_0 \) is strongly realisable, and \( B_0 \) is an algebra over \( A_0 \) that is free as a module over \( A_0 \), then \( B_0 \) is also strongly realisable.

\textit{Proof.} This follows from [5, Proposition 4.13]. \qed

\textbf{Proposition 3.6.} Suppose that \( R_0 \) is a polynomial ring in countably many variables over \( \mathbb{Z} \), that \( A_0 \in \mathcal{R}_0 \), and that \( A_0 = \mathbb{Z}[1/n] \) as a ring (for some \( n \)). Then \( A_0 \) is an LRQ of \( R_0 \), and thus is strongly realisable if \( n \) is even.

\textit{Proof.} Choose a system of polynomial generators \( \{x_k \mid k \geq 0\} \) for \( R_0 \) over \( \mathbb{Z} \). Put \( a_k = \eta(x_k) \in A_0 = \mathbb{Z}[1/n] \) and \( y_k = x_k - a_k \in R_0[1/n] \). It is clear that \( R_0[1/n] = \mathbb{Z}[1/n][y_k \mid k \geq 0] \), that the elements \( y_k \) form a regular sequence generating an ideal \( I \), say, and that \( A_0 = R_0[1/n]/I \). \qed

\textbf{Proposition 3.7.} Suppose that \( R_0 \) is a polynomial ring in countably many variables over \( \mathbb{Z} \), that \( A_0 \in \mathcal{R}_0 \), and that \( A_0 \) is a field (necessity of characteristic different from 2). Then \( A_0 \) is a free module over an LRQ of \( R_0 \), and thus is strongly realisable.

\textit{Proof.} For notational simplicity, we assume that \( A_0 \) has characteristic \( p > 2 \); the case of characteristic 0 is essentially the same.

Choose a set \( X \) of polynomial generators for \( R_0 \) over \( \mathbb{Z} \). Let \( K \) be the subfield of \( A_0 \) generated by the image of \( \eta \), or equivalently by \( \eta(X) \). We can choose a subset \( Y \subseteq X \) such that \( \eta(Y) \) is a transcendence basis for \( K \) over \( F_p \). This means that the subfield \( L_0 \) of \( K \) generated by \( \eta(Y) \) is isomorphic to the rational function field \( F_p(Y) \), and that \( K \) is algebraic over \( L_0 \). Put \( S = \mathbb{Z}[Y] \setminus (p\mathbb{Z}[Y]) \), so \( L_0 = \)}
$$(S^{-1}Z[Y])/p.$$ Next, list the elements of $X \setminus Y$ as \{x_1, x_2, \ldots\}, and let $L_k$ be the subfield of $K$ generated by \{x_i | i \leq k\}. (We will assume that $X \setminus Y$ is infinite; if not, the notation changes slightly.) As $x_k$ is algebraic over $L_{k-1}$, there is a monic polynomial $f_k(t) \in L_{k-1}[t]$ with $L_k = L_{k-1}[x_k]/f_k(x_k)$. As $L_{k-1}$ is a quotient of the ring $P_{k-1} := S^{-1}Z[x_1, \ldots, x_{k-1}]$, we can choose a monic polynomial $g_k(t) \in P_{k-1}[t]$ lifting $f_k$, and put $z_k := g_k(x_k) \in P_k \subseteq S^{-1}R_0$. It is not hard to check that the sequence $(p_i, z_1, z_2, \ldots)$ is regular in $S^{-1}R_0$, and that $(S^{-1}R_0)/(z_i \mid i > 0) = K$, so $K$ is an LRQ of $R_0$. It is clear that $A_0$ is free over the subfield $K$. □

**Definition 3.8.** Let $A_0$ be a commutative algebra over $\pi_0 MP$ with no 2-torsion. Regard $\pi_0 MP$ as a subring of $\pi_0 MP(r)$ via any one of the $r+1$ obvious ring maps $\eta_0, \ldots, \eta_r : MP \to MP(r)$. We say that $A_0$ is stably realisable if, given any map $\pi_0 MP(r) \to A_0$ extending the given map $\pi_0 MP \to A_0$, the resulting $\pi_0 MP(r)$-algebra is strongly realisable. (Using the symmetric group action on $MP(r)$, we see that this does not depend on which $\eta_i$ we use.)

**Lemma 3.9.** An LRQ of an LRQ is an LRQ.

**Proof.** Suppose that $B = (S^{-1}A)/(x_\alpha \mid \alpha < \lambda)$ and $C = (T^{-1}B)/(y_\beta \mid \beta < \mu)$, where $\lambda$ and $\mu$ are short ordinals, and the $x$ and $y$ sequences are regular in $S^{-1}A$ and $T^{-1}B$ respectively. Let $T'$ be the set of elements of $A$ that become invertible in $T^{-1}B$; clearly $S \subseteq T'$ and $T^{-1}B = ((T')^{-1}A)/(x_\alpha \mid \alpha < \lambda)$. As $(T')^{-1}A$ is a localisation of $S^{-1}A$ and localisation is exact, we see that $x$ is a regular sequence in $(T')^{-1}A$ as well. After multiplying by suitable elements of $T'$ if necessary, we may assume that $y_\beta$ lies in the image of $A$ (this does not affect regularity, as the elements of $T'$ are invertible). We then put $z_\alpha = x_\alpha$ for $\alpha < \lambda$, and let $z_{\lambda + \beta}$ be any preimage of $y_\beta$ in $A$ for $0 \leq \beta < \mu$. This gives a regular sequence in $(T')^{-1}A$ indexed by $\lambda + \mu$, such that $C = ((T')^{-1}A)/(z_\gamma \mid \gamma < \lambda + \mu)$ as required. □

**Proposition 3.10.** If $A_0$ is an LRQ of $\pi_0 MP$ in which 2 is invertible, then $A_0$ is stably realisable.

**Proof.** We know from Section 2.3 that $\pi_0 MP(r)$ is a polynomial ring in countably many variables over $\pi_0 MP$, in which $r$ of the variables have been inverted, so we can write

$$\pi_0 MP(r) = \pi_0 MP[x_1, x_2, \ldots][x_1^{-1}, \ldots, x_r^{-1}].$$

Suppose that $A_0 = (S^{-1}\pi_0 MP)/I$. Put

$$B_0 = A_0[x_1, x_2, \ldots][x_1^{-1}, \ldots, x_r^{-1}],$$

which is evidently an LRQ of $\pi_0 MP(r)$. Let $f : \pi_0 MP(r) \to A_0$ be any map extending the given map $\pi_0 MP \to A_0$, and put $a_k = f(x_k) \in A_0$, and $y_k = x_k - a_k \in B_0$. Clearly $B_0$ is a localisation of $A_0[y_k \mid k > 0]$, the sequence of $y$'s is regular in $B_0$, and $B_0/(y_k \mid k > 0) = A_0$ as $\pi_0 MP(r)$-algebras. It follows that $A_0$ is an LRQ of an LRQ, and thus an LRQ, over $\pi_0 MP(r)$. It is thus strongly realisable as required. □

**Corollary 3.11.** If $A_0$ is an algebra over $\pi_0 MP$ that is just $\mathbb{Z}[1/n]$ as a ring for some even integer $n$, then $A_0$ is stably realisable. □

**Proposition 3.12.** Let $A_0$ be an algebra over $\pi_0 MP$ that is a field of characteristic not equal to 2. Then $A_0$ is stably realisable.

**Proof.** This is immediate from Proposition 3.7. □
It will be convenient to restate the above results in slightly different language.

**Definition 3.13.** Let $F$ be a formal group law over a ring $R$. We say that $F$ is strongly (resp. stably) realisable if $R$, regarded as an algebra over $\pi_0 MP$ by the classifying map of $F$, is strongly (resp. stably) realisable.

Let $(G, S)$ be a formal group. We say that a coordinate $x$ on $G$ is strongly (resp. stably) realisable if the associated formal group law over $O_S$ is strongly (resp. stably) realisable. We say that $G$ is good if it admits a stably realisable coordinate. We write GFG for the category of good formal groups.

In these terms, our results give:

**Theorem 3.14.** Suppose that $R$ is a field of characteristic not equal to 2, or that $R = \mathbb{Z}[1/n]$ for some even integer $n$. Then every formal group law over $R$ is stably realisable, so every formal group over spec($R$) is good. \hfill $\Box$

4. **Proof of the main theorem**

Let $\mathcal{E}$ denote the class of objects $E \in EPA_0$ for which the formal group $\Gamma(\eta^*E)$ is good.

**Proposition 4.1.** For any good formal group $(G, S)$, there exists $E \in \mathcal{E}$ with $\Gamma(\eta^*E) \cong (G, S)$.

**Proof.** Choose a stably realisable coordinate $x$ on $G$, and let $F$ be the resulting formal group law over $O_S$. As $F$ is stably realisable, there exists $E \in EPA_0$ together with an isomorphism $\phi: \pi_0 E \to O_S$ of rings such that the composite

$$\pi_0 MP \xrightarrow{\pi_0 \zeta} \pi_0 E \xrightarrow{\phi} O_S$$

is just $c[F]$. If we let $F'$ be the formal group law over $\pi_0 E$ classified by $\pi_0 \zeta$ then $\Gamma(\eta^*E) = Q(\pi_0 E, F') \cong Q(O_S, F) \cong (G, S)$ as required. \hfill $\Box$

**Proposition 4.2.** Suppose we have objects $E_0, E_1 \in \mathcal{E}$, together with a map $p: \Gamma(\eta^*E_1) \to \Gamma(\eta^*E_0)$ in GFG. Then there is a unique good map $f: \eta^*E_0 \to \eta^*E_1$ such that $\Gamma(f) = p$.

**Proof.** For $i = 0, 1$ we let $x_i$ be the coordinate on $\Gamma(\eta^*E_i)$ supplied by the unit map $\zeta_i: MP \to E_i$ and let $F_i$ be the corresponding formal group law over $\pi_0 E_i$. The map $p$ gives us a map $\phi: \pi_0 E_0 \to \pi_0 E_1$ of rings together with an isomorphism $b: F_1 \to \phi_* F_0$ of formal group laws over $\pi_0 E_1$. We know that maps $MP \to \eta^*E_1$ in EPR biject with coordinates on $\Gamma(\eta^*E_1)$, so there is a unique map $\sigma: MP \to \eta^*E_1$ carrying the canonical coordinate $\tilde{x}$ on $\Gamma(MP)$ to $b(x_1)$. We let $\beta_0$ be the composite

$$MP(1) \xrightarrow{\sigma \wedge \zeta_1} (\eta^*E_1) \wedge (\eta^*E_1) \xrightarrow{\text{mult}} \eta^*E_1$$

(so $\beta_0 \in \text{EPR}(MP(1), \eta^*E_1)$) and we put

$$\beta = \pi_0 \beta_0: \pi_0 MP(1) \to \pi_0 E_1.$$

It is clear that $\beta = c[\phi_* F_0 \xrightarrow{b} F_1]$.

We next introduce a category $B = B(E_0, p, E_1)$ as follows. The objects are triples $(A, f', f'')$ where

(a) $A$ is an object of $EPA_0$, with unit map $\xi: MP(1) \to A$.

(b) $f'$ is a morphism $E_0 \to (1 \wedge \eta^*A)$ in $EPA_0$.

(c) $f''$ is an isomorphism $(\eta \wedge 1)^* A \to E_1$ in $EPA_0$. 

(d) The composite

\[ f = \theta(A, f', f'') := (\eta^* E_0 \xrightarrow{\eta^* f'} (\eta \land \eta)^* A \xrightarrow{\eta^* f''} \eta^* E_1) \]

satisfies \( \pi_0 f = \phi: \pi_0 E_0 \to \pi_0 E_1 \) and \( f_* x_0 = b(x_1) \in E_1^0 \mathbb{C} \mathbb{P}^\infty \). The morphisms from \((A, f', f'')\) to \((B, g', g'')\) in \( B \) are the isomorphisms \( u: A \to B \) in \( \text{EPA}_1 \) for which \(((1 \land \eta)^* u) f' = g' \) and \( g''((\eta \land 1)^* u) = f'' \).

The maps of the form \( \theta(A, f', f'') \) are precisely the good maps that induce \( p \), and isomorphic objects of \( B \) have the same image under \( \theta \). It will thus suffice to show that all objects of \( B \) are isomorphic.

We first claim that for any object \((A, f', f'')\) in \( B \), the composite

\[ (MP \xrightarrow{\zeta_0} \eta^* E_0 \xrightarrow{\eta^*} \eta^* E_1) \in \text{EPR}(MP, \eta^* E_1) \]

is equal to \( \sigma \). Indeed, we certainly have \( \zeta_0(x_0) = x_0 \) and \( f_*(x_0) = b(x_1) \) by condition (d), so \( (f \zeta_0)_*(x_0) = b(x_1) \) as required. Next, we claim that the composite

\[ \gamma_0 = (MP(1) \xrightarrow{\xi} (\eta \land \eta)^* A \xrightarrow{\eta^* f'} \eta^* E_1) \in \text{EPR}(MP(1), \eta^* E_1) \]

is equal to \( \beta_0 \). Indeed, \( MP(1) \) is just the coproduct in \( \text{EPR} \) of two copies of \( MP \), so it will suffice to check that \( \gamma_0 \circ (\eta \land 1) = \beta_0 \circ (\eta \land 1) \) and \( \gamma_0 \circ (1 \land \eta) = \beta_0 \circ (1 \land \eta) \). By construction, we have \( \beta_0 \circ (1 \land \eta) = \sigma = f \zeta_0 \) and \( \beta_0 \circ (\eta \land 1) = \zeta_1 \). Now consider the following diagram, in which we have implicitly applied forgetful functors down to \( \text{EPR} \).

\[
\begin{array}{ccc}
MP & \xrightarrow{(1\land \eta)} & MP(1) & \xrightarrow{\eta^* f'} & MP \\
\downarrow{\zeta_0} & & \downarrow{\xi} & & \downarrow{\zeta_1} \\
E_0 & \xrightarrow{f'} & A & \xrightarrow{f''} & E_1
\end{array}
\]

As \( f' \) comes from a ring map \( E_0 \to (1 \land \eta)^* A \) in \( \text{EPA}_0 \), it is certainly a map of \( MP \)-algebras in the naive, homotopical sense. In particular, it preserves units, which means that the left hand square commutes. Similarly, so does the right hand square. From the right hand square, we see that \( \gamma_0 \circ (\eta \land 1) = \zeta_1 \). From the left hand square, we see that \( \gamma_0 \circ (1 \land \eta) = f \zeta_0 = \sigma \). It follows that \( \gamma_0 = \beta_0 \) as claimed, and so \( \pi_0 \gamma_0 = \beta = c[\phi_* F_0 \xrightarrow{b} F_1] \). We deduce that \( \pi_0 A \) is isomorphic as an algebra over \( \pi_0 MP(1) \) to \( (\pi_0 E_1, \phi_* F_0 \xrightarrow{b} F_1) \). As \( F_1 \) is a stably realisable formal group law, we see that this algebra is strongly realisable, so maps \( A \to A' \) in \( \text{EPA}_1 \) biject with maps \( \pi_0 A \to \pi_0 A' \) of \( \pi_0 MP(1) \)-algebras.

Now suppose we have another object \((B, g', g'')\) in \( CB \). We then have an isomorphism \( v = (g'')^{-1} f'' \): \((\eta \land 1)^* A \to (\eta \land 1)^* B \) in \( \text{EPA}_1 \). If we let \( \chi \) be the unit map for \( B \), the previous paragraph tells us that following diagram commutes (with both composites equal to \( \beta \)):

\[ \pi_0 MP(1) \xrightarrow{\pi_0 \xi} \pi_0 A \]

\[ \pi_0 \chi \downarrow \quad \pi_0 f'' \downarrow \quad \pi_0 \phi \downarrow \quad \pi_0 \eta \]

\[ \pi_0 B \quad \pi_0 E_1 \]

It follows easily that \( \pi_0 v \) is a map of \( \pi_0 MP(1) \)-algebras, so by strong realisability it agrees with \( \pi_0 u \) for a unique map \( u: A \to B \) in \( \text{EPA}_1 \). As \( \pi_0 u \) is an isomorphism,
the same is true of $u$. We also claim that $(\eta \land 1)^* u = v$ as a map from $(\eta \land 1)^* A$ to $(\eta \land 1)^* B$ in $\mathcal{E}_A$. Indeed, the $\pi_0 MP$-algebra $\pi_0((\eta \land 1)^* A)$ is isomorphic to $\pi_0 E_1$ and so is strongly realisable, so maps out of $(\eta \land 1)^* A$ are determined by their effect on $\pi_0$, and $\pi_0((\eta \land 1)^* u) = \pi_0 v$ by construction. This shows that $g'' \circ (\eta \land 1)^* u = f''$.

Finally, we claim that $(1 \land \eta)^* u \circ f' = g'$ as maps from $E_0$ to $(1 \land \eta)^* B$ in $\mathcal{E}_A$. As $\pi_0 E_0$ is strongly realisable, it again suffices to check the equation on $\pi_0$. From the definition of $B$ we have

$$\pi_0(f'') \circ \pi_0(f') = \phi = \pi_0(g'') \circ \pi_0(g') : \pi_0 E_0 \to \pi_0 E_1,$$

and by construction we have $\pi_0 u = \pi_0(g'')^{-1} \pi_0(f'')$; it follows that $\pi_0(u) \circ \pi_0(f') = \pi_0(g')$ as required.

This shows that there is actually a unique isomorphism between any two objects of $\mathcal{B}$, so in particular the function $\theta$ is constant on $\mathcal{B}$, so there is a unique good map inducing $p$, as claimed. \qed

**Lemma 4.3.** For any $E \in \mathcal{E}$, the identity map $1: \eta^* E \to \eta^* E$ is good.

**Proof.** Note that the multiplication map $MP(1) = MP \land MP \to MP$ is a map of ring spectra (in the strict sense) and so induces a functor $\mu^* : \mathcal{E}_A \to \mathcal{E}_A$ with $(1 \land \eta)^* \mu^* E = (\eta \land 1)^* \mu^* E = E$ on the nose. We can thus take $A = \mu^* E$ and $f' = f'' = 1_E$ to show that $1_E$ is good. \qed

**Proposition 4.4.** Suppose we have objects $E_0, E_1, E_2 \in \mathcal{E}$ and good morphisms $\eta^* E_0 \xrightarrow{f} \eta^* E_1 \xrightarrow{g} \eta^* E_2$. Then the composite $gf$ is also good.

**Proof.** Let $F_i$ be the formal group law for $E_i$, and let $(\phi, b)$ and $(\psi, c)$ be the morphisms of formal group laws coming from $f$ and $g$ (so in particular $\phi = \pi_0 f$ and $\psi = \pi_0 g$). Choose objects $A, B \in \mathcal{E}_A$ and maps

$$f' : E_0 \to (1 \land \eta)^* A$$
$$f'' : (\eta \land 1)^* A \xrightarrow{\sim} E_1$$
$$g' : E_1 \to (1 \land \eta)^* B$$
$$g'' : (\eta \land 1)^* B \xrightarrow{\sim} E_2$$

exhibiting the goodness of $f$ and $g$. Next, observe that we have a chain

$$\psi_\ast \phi_\ast F_0 \xrightarrow{\phi_\ast b} \psi_\ast F_1 \xrightarrow{c} F_2$$

of formal group laws over $\pi_0 E_2$. This makes $\pi_0 E_2$ into an algebra over $\pi_0 MP(2)$, and it is strongly realisable as such, because $F_2$ is stably realisable by assumption. We can thus choose an object $P \in \mathcal{E}_A$ and an isomorphism $w : \pi_0 P \to \pi_0 E_2$ of $\pi_0 MP(2)$-algebras. We now include $\pi_0 MP(1)$ in $\pi_0 MP(2)$ by the map $\eta \land 1 : MP(1) \to MP(2)$. The resulting algebra structure on $\pi_0 E_2$ classifies the chain $\psi_\ast F_1 \xleftarrow{c} F_2$. From the proof of Proposition 4.2, we see that the isomorphism $\pi_0 B \xrightarrow{\pi_0 g''} \pi_0 E_2$ is also a $\pi_0 MP(1)$-algebra map with respect to this structure. This means that we have an isomorphism $(\pi_0 g'')^{-1} w : \pi_0((\eta \land 1)^* P) \to \pi_0 B$ of $\pi_0 MP(1)$-algebras, and by strong realisability, it comes from a unique isomorphism $v : (\eta \land 1)^* P \to B$ in $\mathcal{E}_A$. 


Next, we have maps of algebras over \( \pi_0MP(1) \) as follows
\[
\pi_0 A \xrightarrow{\pi_0 f''} (\pi_0 E_1, \phi A F_0 \xleftarrow{\psi} F_1) \\
\xrightarrow{\psi} (\pi_0 E_2, \psi \phi A F_0 \xleftarrow{\psi b} \psi_b F_1) \\
\xrightarrow{w^{-1}} \pi_0((1 \land 1 \land \eta)^*P).
\]
As the first map is an isomorphism, we know that \( \pi_0 A \) is strongly realisable, so there is a unique map \( u : A \to (1 \land 1 \land \eta)^*P \) in EPA₁ inducing the composite map on \( \pi_0 \).

We now put
\[
C = (1 \land \eta \land 1)^*P \in \text{EPA}_1 \\
h' = (E_0 \xrightarrow{f'} (1 \land \eta)^*A \xrightarrow{(1 \land \eta)^* u} (1 \land \eta \land \eta)^*P = (1 \land \eta)^*C) \\
h'' = ((\eta \land 1)^*C = (\eta \land \eta \land 1)^*P \xrightarrow{(\eta \land 1)^* v} (1 \land \eta)^*B \xrightarrow{f''} E_2).
\]
As \( v \) and \( g'' \) are isomorphisms, the same is true of \( h'' \). We claim that after forgetting down to EPR, we have \( h'' h' = g f \); this will prove that \( g f \) is good as claimed. We certainly have \( h'' h' = g'' v u f' \) and \( g f = g'' g f'' f' \) so it will suffice to show that \( v u = g f'' : A \to B \) in EPR. For this, it will be enough to prove that the following diagram in EPA₀ commutes,
\[
\begin{array}{ccc}
(\eta \land 1)^*A & \xrightarrow{(\eta \land 1)^* u} & (\eta \land 1 \land \eta)^*P \\
\downarrow f'' & \simeq & \downarrow (1 \land \eta)^* v \\
E_1 & \xrightarrow{g'} & (1 \land \eta)^*B.
\end{array}
\]
As this is a diagram in EPA₀ and \( \pi_0((\eta \land 1)^*A) \simeq \pi_0 E_1 \) is strongly realisable, it will be enough to check that the diagram commutes after applying \( \pi_0 \). By construction we have \( \pi_0(u) = w^{-1} \circ \psi \circ \pi_0(f'') \) and \( \psi = \pi_0(g) = \pi_0(g'') \circ \pi_0(g') \) and \( \pi_0(v) = \pi_0(g'')^{-1} \circ w \). It follows directly that the above diagram commutes on homotopy groups, so it commutes in EPA₀, so it commutes in EPR, so \( g f = h'' h' \) in EPR as explained previously. Thus, the map \( g f \) is good, as claimed. \( \square \)

Proof of Theorem 1.1. We merely need to collect results together and explain the argument in the introduction in more detail. Lemma 4.3 and Proposition 4.4 show that we can make \( E \) into a category by taking the good maps from \( \eta^* E_0 \) to \( \eta^* E_1 \) as the morphisms from \( E_0 \) to \( E_1 \). Tautologically, we can define a faithful functor \( U : E \to \text{EPR} \) by \( U(E) = \eta^* E \) and \( U(f) = f \). We then define \( V = \Gamma U : E \to \text{FG} \); by the definition of \( E \), this actually lands in \( \text{GFG} \). Proposition 4.1 says that \( V \) is essentially surjective, and Proposition 4.2 says that \( V \) is full and faithful. This means that \( V \) is an equivalence, so we can invert it and define \( R = V U^{-1} : \text{GFG} \to \text{EPR} \). As \( V = \Gamma U \) we have \( \Gamma R = 1 \), so \( R \) is the required realisation functor. \( \square \)

Appendix A. The Product on \( MP \)

In this appendix we verify that \( MP \) can be constructed as an \( E_\infty \) ring spectrum.
Let \( \mathcal{U} \) be a complex universe. For any finite-dimensional subspace \( U \) of \( \mathcal{U} \), we write \( U_L = U \oplus 0 < U \oplus \mathcal{U} \) and \( U_R = 0 \oplus U < U \oplus \mathcal{U} \). We let \( \text{Grass}(U \oplus U) \) denote the Grassmannian of all subspaces of \( U \oplus U \) (of all possible dimensions),
and we write $\gamma_U$ for the tautological bundle over this space, and $\text{Thom}(U \oplus U)$ for the associated Thom space. If $U \leq U' < \mathcal{U}$ then we define $i : \text{Grass}(U^2) \to \text{Grass}((U')^2)$ by $i(A) = A \oplus (U' \oplus U)_R$. On passing to Thom spaces we get a map $\sigma : \Sigma^{U' \oplus U} \text{Thom}(U^2) \to \text{Thom}((U')^2)$. These maps can be used to assemble the spaces $\text{Thom}(U^2)$ into a $\Sigma$-inclusion prespectrum indexed by the complex subspaces of $\mathcal{U}$. We write $T_\mathcal{U}$ for this prespectrum, and $MP_\mathcal{U}$ for its specification.

Now let $\mathcal{V}$ be another complex universe, so we have a prespectrum $T_\mathcal{V}$ over $\mathcal{V}$, and thus an external smash product $T_\mathcal{U} \wedge^{\text{ext}} T_\mathcal{V}$ indexed on the complex subspaces of $\mathcal{U} \oplus \mathcal{V}$ of the form $U \oplus V$. The direct sum gives a map $\text{Grass}(U^2) \times \text{Grass}(V^2) \to \text{Grass}((U \oplus V)^2)$ which induces a map $\text{Thom}(U^2) \wedge \text{Thom}(V^2) \to \text{Thom}((U \oplus V)^2)$. These maps fit together to give a map $T_\mathcal{U} \wedge^{\text{ext}} T_\mathcal{V} \to T_{\mathcal{U} \oplus \mathcal{V}}$, and thus a map $MP_\mathcal{U} \wedge^{\text{ext}} MP_\mathcal{V} \to MP_{\mathcal{U} \oplus \mathcal{V}}$ of spectra over $\mathcal{U} \oplus \mathcal{V}$. Essentially the same construction gives maps

$$MP_\mathcal{U} \wedge^{\text{ext}} \ldots \wedge^{\text{ext}} MP_\mathcal{U} \to MP_{\mathcal{U} \oplus \ldots \oplus \mathcal{U}}.$$ 

If $\mathcal{U}_1 = \ldots = \mathcal{U}_t = \mathcal{U}$, then this map is $\Sigma_t$-equivariant.

Now suppose instead that we have a complex linear isometry $f : \mathcal{U} \to \mathcal{V}$. This gives evident homeomorphisms $\text{Thom}(U^2) \to \text{Thom}((fU)^2)$, which fit together to induce a map $MP_\mathcal{U} \to f^* MP_\mathcal{V}$, which is adjoint to a map $f_* MP_\mathcal{U} \to MP_\mathcal{V}$. We next observe that this construction is continuous in all possible variables, including $f$. (This statement requires some interpretation, but there are no new issues beyond those that are well-understood for $MU$; the cleanest technical framework is provided by [1].) It follows that they fit together to give a map $L_\mathcal{C}(\mathcal{U}, \mathcal{V}) \times MP_\mathcal{U} \to MP_\mathcal{V}$ of spectra over $\mathcal{V}$.

We now combine this with the product structure mentioned earlier to get a map

$$L_\mathcal{C}(\mathcal{U}, \mathcal{V}) \times_{\Sigma_*} (MP_\mathcal{U} \wedge^{\text{ext}} \ldots \wedge^{\text{ext}} MP_\mathcal{U}) \to MP_\mathcal{U}.$$ 

This means that $MP_\mathcal{U}$ has an action of the $E_\infty$ operad of complex linear isometries, as required.

All that is left is to check that the spectrum $MP = MP_{E_\infty}$ constructed above has the right homotopy type. As $T$ is a $\Sigma$-inclusion prespectrum, we know that specification works in the simplest possible way and that $MP$ is the homotopy colimit of the spectra

$$\Sigma^{-2n} \text{Thom}(\mathbb{C}^n \oplus \mathbb{C}^n) = \bigvee_{k=-n}^n \Sigma^{-2n} \text{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma,$$

where $\text{Grass}_d(V)$ is the space of $d$-dimensional subspaces of $V$. It is not hard to see that the maps of the colimit system preserve this splitting, so that $MP$ is the wedge over all $k \in \mathbb{Z}$ of the spectra

$$X_k := \text{holim} \Sigma^{-2n} \text{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma.$$ 

This can be rewritten as

$$X_k = \Sigma^{2k} \text{holim} \Sigma^{-2(k+n)} \text{Grass}_{k+n}(\mathbb{C}^n \oplus \mathbb{C}^n)^\gamma.$$ 

We can reindex by putting $n = i - k$ and $m = j + k$, and then pass to the limit in $j$. We find that

$$X_k = \Sigma^{2k} \text{holim} \Sigma^{-2i} \text{Grass}_i(\mathbb{C}^\infty \oplus \mathbb{C}^i)^\gamma.$$
It is well-known that Grass$_i(\mathbb{C}^\infty \oplus \mathbb{C})$ is a model for $BU(i)$, and it follows that $X_k = \Sigma^{2k} MU$, so $MP = V_k \Sigma^{2k} MU$ as claimed. We leave it to the reader to check that the product structure is the obvious one,

References


Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, UK
E-mail address: N.P.Strickland@sheffield.ac.uk