NOTES ON $K(B\Sigma_4)$ AT $Q = 4$

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In this document we describe the Morava $K$ theory (with $n = p = 2$) of $\Sigma_4$ and its subgroups in excruciating detail. We use Chern classes and their transfers as generators, and describe the ring structure and all transfer and restriction maps. Much of the calculation was done using Mathematica. The Mathematica program includes very extensive internal consistency checks, so I have quite a high degree of confidence in its correctness.

1. Group Theory

We think of $\{0,1\}^2$ as $\{0,1,2,3\}$ via $(k,l) \mapsto k + 2l$, or as the corners of the unit square in $\mathbb{R}^2$:

\[
\begin{array}{ccc}
2 & 3 \\
0 & 1
\end{array}
\]

We take $\Sigma_4$ to be the permutation group on this set.

\[
\mu = (12) \quad \nu = (03) \quad \lambda = (01)(23) \\
\theta = (012)
\]

$P = \Sigma_2^2 = \{1, \nu, \mu, \nu\mu\}$

$W = \Sigma_2 \wr \Sigma_2 = D_4 = P \sqcup \lambda P$

$G = \Sigma_4 = W \sqcup \theta W \sqcup \theta^{-1} W$

$V = \{1, \lambda, \mu \nu, \lambda \mu \nu\} = W \cap W^\theta < W \simeq C_2^2$

$L = \{1, \lambda\} = V \cap W$

$C = \{1, \lambda \mu, \lambda \nu, \mu \nu\} \simeq C_4$

Note that the wreath product $W$ acts on the square as the dihedral group, equivalently, it respects the partition of the set $\{0,1,2,3\}$ of corners into two opposite pairs $\{\{0,3\}, \{1,2\}\}$.

2. Character Tables

In this section we give the character tables of the groups mentioned above. Later, we will describe the Morava $K$-theory of these groups in terms of Chern classes of these representations. We are really only interested in the 2-completion of the representation ring, so in the case of $G$ we can ignore the conjugacy class of order 3 and discard one character.

2.1. $P$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma = \alpha \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$1$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\mu \nu$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

\[\alpha^2 = \beta^2 = 1\]
2.2. $V$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& 1 & \zeta & \xi & \zeta \xi \\
\hline
1 & 1 & 1 & 1 & 1 \\
\lambda & 1 & 1 & -1 & -1 \\
\mu \nu & 1 & -1 & -1 & 1 \\
\lambda \mu \nu & 1 & -1 & 1 & -1 \\
\hline
\end{array}
\]

\[\zeta^2 = \xi^2 = 1\]

2.3. $W$.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& 1 & \delta & \epsilon & \delta \epsilon & \sigma \\
\hline
1 & 1 & 1 & 1 & 1 & 2 \\
\mu, \nu & 1 & -1 & -1 & 1 & 0 \\
\mu \nu & 1 & 1 & 1 & 1 & -2 \\
\lambda, \lambda \mu \nu & 1 & -1 & 1 & -1 & 0 \\
\lambda \mu, \lambda \nu & 1 & 1 & -1 & -1 & 0 \\
\hline
\end{array}
\]

\[\epsilon^2 = \delta^2 = 1\]

\[\epsilon \sigma = \delta \sigma = \sigma\]

\[\sigma^2 = 1 + \epsilon + \delta + \epsilon \delta\]

\[\psi^2 \sigma = 1 - \delta + \epsilon + \epsilon \delta\]

$\sigma$ is the obvious representation of $D_4$ on $\mathbb{R}^2$, and $\delta$ is its determinant. The kernel of the linear character $\delta \epsilon$ is $P$.

2.4. $G$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& 1 & \epsilon & \rho & \epsilon \rho \\
\hline
1 & 1 & 1 & 3 & 3 \\
2^1 & 1 & -1 & 1 & -1 \\
2^2 & 1 & 1 & -1 & -1 \\
4^1 & 1 & -1 & -1 & 1 \\
\hline
\end{array}
\]

\[\epsilon^2 = 1\]

\[\rho^2 = 2 + \epsilon + \rho + \epsilon \rho\]

$\epsilon$ is the signature and $\rho$ is the reduced standard representation.
3. Maps of Representation Rings

Restriction $W$ to $P$:
\[
\begin{align*}
\delta, \epsilon &\mapsto \gamma \\
\sigma &\mapsto \alpha + \beta
\end{align*}
\]

Restriction $W$ to $V$:
\[
\begin{align*}
\delta &\mapsto \zeta \xi \\
\epsilon &\mapsto 1 \\
\sigma &\mapsto \zeta + \xi
\end{align*}
\]

Restriction $W$ to $C$:
\[
\begin{align*}
\delta &\mapsto 1 \\
\epsilon &\mapsto \kappa^2 \\
\sigma &\mapsto \kappa + \kappa^3
\end{align*}
\]

Restriction $G$ to $W$:
\[
\rho \mapsto \sigma + \epsilon \delta
\]

Transfer $P$ to $W$:
\[
\begin{align*}
1 &\mapsto 1 + \delta \epsilon \\
\alpha, \beta &\mapsto \sigma \\
\gamma &\mapsto \delta + \epsilon
\end{align*}
\]

Transfer $V$ to $W$:
\[
\begin{align*}
1 &\mapsto 1 + \epsilon \\
\zeta, \xi &\mapsto \sigma \\
\zeta \xi &\mapsto \delta + \epsilon \delta
\end{align*}
\]

Transfer $W$ to $G$:
\[
\begin{align*}
1 &\mapsto 2 + \epsilon \\
\epsilon &\mapsto 2 \epsilon + 1 \\
\delta &\mapsto \epsilon \rho \\
\epsilon \delta &\mapsto \rho \\
\sigma &\mapsto \rho + \epsilon \rho
\end{align*}
\]

4. Morava $K$ Theory

Let $K$ be the extended version of $K(2)$ with period 2, so we can choose a generator of $K^* \mathbb{C}P^\infty$ in degree zero. This gives Chern classes in degree zero for complex bundles. We write $K(BG) = K^0(BG)$.

4.1. $P$.
\[
\begin{align*}
a = c_1(\alpha) &\quad a^4 = 0 \\
b = c_1(\beta) &\quad b^4 = 0 \\
c = c_1(\gamma) &\quad a + b + a^2 b^2 &\quad c^4 = 0 \\
K(BP) &\quad = K[a,b]/(a^4,b^4)
\end{align*}
\]

4.2. $V$.
\[
\begin{align*}
z = c_1(\zeta) &\quad z^4 = 0 \\
x = c_1(\xi) &\quad x^4 = 0 \\
K(BV) &\quad = K[z,x]/(z^4,x^4)
\end{align*}
\]
4.3. $W$.

\[
d = c_1(\delta) \quad d^4 = 0
\]
\[
e = c_1(\epsilon) \quad e^4 = 0
\]
\[
u = c_1(\delta \epsilon) = d + e + d^2 e^2 \quad u^4 = 0
\]
\[
\text{res}(d) = \text{res}(e) = c \quad \text{res}(u) = 0
\]
\[
t = c_1(\sigma) \quad \text{res}(t) = a + b
\]
\[
s = c_2(\sigma) \quad \text{res}(s) = ab
\]
\[
b_k = \text{tr}_P^W(b^k) \quad \text{res}_P^W(b_k) = a^k + b^k
\]
\[
b_0 = \text{tr}_P^W(1) = u^3
\]
\[
v = b_1
\]
\[
x_k = \text{tr}_V^W(x^k)
\]
\[
x_0 = e^3
\]

$K(BW)$ is generated by $v, s$ and $u$ subject to

\[
u v = 0
\]
\[
u^4 = 0
\]
\[
v^4 = s^2 u^3
\]
\[
s^4 = s^2 u + s u^3
\]
\[
s^3 v = 0
\]
\[
s^2 v^2 = s^3 u^3
\]
\[
sv^3 = s^2 v
\]

A basis is as follows:

\[
v^3 \quad v^2 \quad v \quad 1 \quad u \quad u^2 \quad u^3
\]
\[
su^2 \quad sv \quad s \quad su \quad su^2 \quad su^3
\]
\[
s^2 v \quad s^2 \quad s^2 u \quad s^2 u^2 \quad s^2 u^3
\]
\[
s^3 \quad s^3 u \quad s^3 u^2 \quad s^3 u^3
\]

Other interesting elements are

\[
e = v + s^2 + u^2 s
\]
\[
d = u + v + s^2 + u^2 s + s^2 u^3
\]
\[
b_0 = u^3
\]
\[
b_1 = v
\]
\[
b_2 = v^2 + su^3
\]
\[
b_3 = v^3 + sv
\]
\[
x_0 = e^3 = v^3 + s^2 u^2 + s^3 u^3
\]
\[
x_1 = u + s^2 + su^2 + s^3 u + s^2 u^3
\]
\[
x_2 = s^2 u + u^2 + s^3 u^2 + s^3 u + s^2 v
\]
\[
x_3 = s^3 + su + u^3 + s^3 u^3
\]
\[
t = v + u + su^2
\]
4.4. $G$.

\[ c_k = c_k(\rho) \]
\[ w_0 = \text{tr}^G_P(1) \]
\[ w_1 = \text{tr}^G_P(a) \]

$K(BG)$ is generated by $c_2, w_1$ and $w_0$ subject to

\[ c_2^5 = c_2 w_0 \]
\[ w_1^5 = w_0^3 = w_1 w_0 = 0 \]
\[ c_2^3 w_0 = w_0^3 \]
\[ c_2^2 w_1 = c_2 w_1^2 \]
\[ c_2^2 w_0 = w_1^4 \]
\[ c_2 w_1^4 = w_0^2 \]
\[ c_2 w_0^2 = 0 \]

A basis is as follows:

\[
\begin{array}{cccccccccc}
1 & c_2 & c_2^2 & c_2^3 & c_2^5 & c_2^7 & c_2^9 & c_2^{11} & c_2^{13} & c_2^{15} \\
& w_1 & c_2 w_1 & c_2^2 w_1 & c_2^3 w_1 & c_2^5 w_1 & c_2^7 w_1 & c_2^9 w_1 & c_2^{11} w_1 & c_2^{13} w_1 \\
& w_1^2 & c_2 w_1^2 & c_2^2 w_1^2 & c_2^3 w_1^2 & c_2^5 w_1^2 & c_2^7 w_1^2 & c_2^9 w_1^2 & c_2^{11} w_1^2 & c_2^{13} w_1^2 \\
& w_1^4 & c_2 w_1^4 & c_2^2 w_1^4 & c_2^3 w_1^4 & c_2^5 w_1^4 & c_2^7 w_1^4 & c_2^9 w_1^4 & c_2^{11} w_1^4 & c_2^{13} w_1^4 \\
& w_0 & c_2 w_0 & c_2^2 w_0 & c_2^3 w_0 & c_2^5 w_0 & c_2^7 w_0 & c_2^9 w_0 & c_2^{11} w_0 & c_2^{13} w_0 \\
& w_0^2 & c_2 w_0^2 & c_2^2 w_0^2 & c_2^3 w_0^2 & c_2^5 w_0^2 & c_2^7 w_0^2 & c_2^9 w_0^2 & c_2^{11} w_0^2 & c_2^{13} w_0^2 \\
\end{array}
\]

The ideal of transfers from $P$ is generated by $w_0$ and $w_1$, and is spanned by those basis elements which are visibly divisible by $w_0$ or $w_1$.

Other interesting elements are

\[ c_1(\rho) = c_2^5 + w_1 \]
\[ c_3(\rho) = c_2^3 + c_2^6 + c_2 w_1^4 + w_0 \]
\[ c_1(\epsilon) = c_2^7 + c_2^5 + w_1 \]

The image of $K(BG)$ in $K(BW)$ is spanned by the following elements:

\[
\begin{array}{cccccccccc}
1 & s + s^2 u + u^2 & su & su^2 + v \\
& s^2 & s^2 u + su^3 & s^2 u^2 & s^2 u^3 \\
& s^3 & s^3 u & s^3 u^2 & s^3 u^3 \\
& v^2 & v^3 & sv & sv^2 \\
& s^2 v
\end{array}
\]

5. Maps of Morava $K$-Theory

Restriction $G$ to $W$:

\[
\begin{align*}
w_1 &\mapsto v + su^2 + s^3 u \\
w_0 &\mapsto s^3 + su + s^3 u^3 \\
c_1 &\mapsto su^2 + v \\
c_2 &\mapsto s + u^2 + su^3 \\
c_3 &\mapsto su
\end{align*}
\]

Restriction $W$ to $P$:

\[
\begin{align*}
u &\mapsto 0 \\
v &\mapsto a + b
\end{align*}
\]
\[
s \mapsto ab \\
t \mapsto a + b \\
d, e \mapsto c = a + b + a^2b^2 \\
b_k \mapsto a^k + b^k
\]

Restriction \(W\) to \(V\): 
\[
\begin{align*}
  u & \mapsto z + x + z^2x^2 \\
v & \mapsto z^3x + z^2x^2 + zx^3 \\
s & \mapsto zx \\
t & \mapsto z + x \\
d & \mapsto z + x + z^2x^2 \\
e & \mapsto 0 \\
b_k & \mapsto z^k(z + x)^3
\end{align*}
\]

Restriction \(W\) to \(C\): 
\[
K(BW) = K[r]/r^{16} \\
u \mapsto r^4 \\
v \mapsto r^{13} \\
s \mapsto r^2 + r^5 + r^{11} \\
t \mapsto r^4 + r^{10} \\
d \mapsto 0 \\
e \mapsto r^4 \\
b_k \mapsto r^{k+12}
\]

Transfer \(P\) to \(W\): 
\[
1 \mapsto u^3 \\
ab^l \mapsto s^kb_{l-k} \quad (k \leq l)
\]

Transfer \(V\) to \(W\): 
\[
1 \mapsto e^3 \\
x^kz^l \mapsto s^kx_{l-k} \quad (k \leq l)
\]

Transfer \(W\) to \(G\): 
\[
1 \mapsto 1 + c_2^6 + w_1^3 \\
u \mapsto c_2^2 + c_2^5 + c_2w_1^2 + w_1^4 \\
u^2 \mapsto c_2^4 + c_2w_1^3 + c_2w_0 \\
u^3 \mapsto w_0 \\
v \mapsto w_1
\]

This determines everything because \(K(BW)\) is generated by \(\{1, u, u^2\}\) as a module over \(K(BG)\).
6. Generalised Character Table

Let $F$ be the obvious spectrum with $\pi_k F = W_{\mathbb{F}_4}[u^{\pm 1}]$, where $|u| = 2$ and $u^3 = v_2$, and let $R$ be obtained from $W_{\mathbb{F}_4}$ by adjoining a full set of roots for $|4|(x)$. Write $\Lambda = (\mathbb{Z}/2^\infty)^2$, so $\Lambda^* = \mathbb{Z}_2^2$, and choose an identification of the points of order 4 in $\Lambda$ with the roots of the $[4]$-series.

If $\{a, b, c\}$ are the nonzero roots of $[2](x)$ in $R$ then

$$a + b + c = ab + bc + ca = 0$$

$$c = a + F b = -a - b \quad ab = c^2$$

There is an element $\tau \in \mathbb{Z}_2$ with $\tau = 2 \pmod{8}$ such that

$$abc = a^3 = b^3 = c^3 = -\tau$$

Conjugacy classes of maps $\Lambda^* \to W$ biject with isomorphism classes of $\Lambda^*$-sets $X$, equipped with an equivariant partition into two blocks of order two (the action is allowed to exchange the two blocks of the partition, but not to mix them up). We write $\overline{X}$ for the set of blocks, considered as a $\Lambda^*$-set. We will typically draw $\overline{X}$ as a square, with the blocks as diagonals. If the action of $\Lambda^*$ is transitive, then it is forced to be Euclidean.

Suppose $a \in \Lambda$. Then $a^* = (a)^* \simeq \Lambda^*/\text{ann}(a)$ is a $\Lambda^*$-set of size equal to the order of $a$. We also write $n$ for the trivial $\Lambda^*$-set of order $n$.

These isomorphism classes can be classified as follows:

1. $X \simeq 4$, $\overline{X} \simeq 2$ (one class, denoted by $\overline{\square}$).
2. $X \simeq a^* \cup a^*$, $\overline{X} \simeq 2$ (three classes, denoted by $\overline{\otimes}$).
3. $X \simeq 2 \cup a^*$, $\overline{X} \simeq 2$ (three classes (one for each point $a$ of exact order 2), denoted by $\overline{\boxtimes}$).
4. $X \simeq 2 \times a^*$, $\overline{X} \simeq a^*$ (three classes, denoted by $\overline{\bigcirc}$).
5. $X \simeq a^* \cup b^*$, $\overline{X} \simeq 2$, $a \neq b$ (three classes, denoted by $\overline{\otimes}$).
6. $X \simeq a^* \times b^*$, $\overline{X} \simeq (a + F b)^*$ (three classes, denoted by $\overline{\boxtimes}$).
7. $X \simeq a^*$, $\overline{X} \simeq [2](a)^*$ where $a$ is a point of exact order 4, determined up to sign (six classes, denoted $\overline{\square}$). In this case we write $\overline{\tau} = [-1](a)$.

Generalised character theory assigns to each $X$ as above and each $y \in K(W)$ an element $\chi(X, y)$ as follows:

\[
\begin{array}{cccccc}
\text{ } & u & v & s & t & d & e \\
\overline{\square} & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{\otimes} & 0 & 2a & a^2 & 2a & 0 & 0 \\
\overline{\boxtimes} & 0 & a & 0 & a & a & a \\
\bigcirc & a & 0 & 0 & a & a & 0 \\
\overline{\boxtimes} & 0 & a + b & ab & a + b & a + F b & a + F b \\
\overline{\square} & [2](a) & 0 & ab & a + b & a + F b & 0 \\
\end{array}
\]
7. Formal Formulae

Here we work with the usual formal group law over $F_0 = W_{F_4}$, whose logarithm is $\sum_k x^4^k / 2^k$.

Mod 2, $[2](a), [2](b)$:

\[
\begin{align*}
  a^4 &= 0 \\
  a + F b &= a + b + a^2 b^2
\end{align*}
\]

Mod 8, $[2](a), [2](b)$:

\[
\begin{align*}
  a^4 + 2a &= 0 \\
  a + F b &= a + b - 2ab(a + b) - 3a^2 b^2
\end{align*}
\]

Mod 8, $[4]/[2](a)$:

\[
\begin{align*}
  a^{12} - 2a^9 + 4a^6 + 2 &= 0 \\
  \bar{a} = [1](a) &= 3a - 3a^4 + 2a^7 - a^{10} \\
  b &= a\bar{a} = 3a^2 - 3a^5 + 2a^8 - a^{11} \\
  [2](a) &= 2a + a^4 + 4a^{10} = b^2 + 3b^5 \\
  \langle 2 \rangle (a) &= 2 + a^3 + 4a^9 \\
  a + \bar{a} &= b^2 \\
  b^6 - 2b^3 + 2 &= 0
\end{align*}
\]

8. A Little Justification

By the last part of [1], the spectral sequence of the extension $P \to W \xrightarrow{\delta_5} C_2$ collapses. Moreover, $K(BW)$ is spanned by the transfers of elements $a^k b^l$ with $k < l$, together with a free module over $K(BC_2) = K[u]/u^4$ on elements which restrict to $a^k b^k$. We may take these elements to be $\{1, s, s^2, s^3\}$. Note that

\[
\text{tr}(a^k b^l) = \text{tr}(\text{res}(s^k) b^{l-k}) = s^k b_{l-k}
\]

We thus have a basis for $K(BW)$ as follows:

\[
\{s^k u^l \mid 0 \leq k, l < 4\} \cup \{s^k b_j \mid k \geq 0, l > 0, k + l < 4\}
\]

If $k \leq l$ then

\[
\begin{align*}
  b_k b_l &= \text{tr}((a^k + b^k)b^l) = s^k b_{l-k} + b_{k+l} \\
  b_0 b_1 &= 0 \\
  b_1^2 &= b_2 + su^3 \\
  b_1 b_3 &= b_3 + sb_1 \\
  b_2 b_3 &= sb_2 \\
  b_2^2 &= s^2 u^3 \\
  b_2 b_3 &= s^2 b_1 \\
  b_3^2 &= s^3 u^3 \\
  ub_k &= \text{tr}(\text{res}(u) b^k) = 0 \\
  v &= b_1 \\
  b_2 &= v^2 + su^3 \\
  b_3 &= v^3 + sv
\end{align*}
\]

Using the above, one can check that the following is also a basis for $K(BW)$, as claimed earlier:

\[
\{s^k u^l \mid k, l \geq 0, k + l < 4\} \cup \{s^k u^l \mid k \geq 0, l > 0, k, l < 4\}
\]
It is well-known that $\text{tr}^G_1(1) = [2](x)/x = x^3$, where $x$ is the usual generator of $K(BC_2)$. A naturality argument gives

$$b_0 = u^3$$

Using $s^k b_l = 0$ for $k + l > 3$, we find

$$s^3 v = 0$$
$$s^2 v^2 = s^3 v^3$$
$$sv^3 = s^2 v$$

For any bundle $V$ we write $c(V, x) = \sum c_k(V) x^k$. I claim that $c(\psi^2 \sigma, x^4) = c(\sigma, x)^4$. To see this, analyse the Chern classes of $\psi^2 V$ using the splitting principle and the fact that $[2](x) = x^4$.

Using $\psi^2 \sigma = 1 - \delta + \epsilon + \epsilon \delta$ we get

$$c(1 - \delta + \epsilon + \epsilon \delta, x^4) = (x^2 + tx + s)^4$$

Moving $c(-\delta, x^4)$ to the right hand side and putting $y = x^4$ we get

$$y(y + e)(y + d + e + d^2 e^2) = (y + d)(y^2 + t^4 y + s^4)$$

This means that $d$ is a root of the left hand side, so

$$d^2 e + de^2 + d^3 e^3 = 0$$

Multiplying by $d^2 e$ gives $d^3 e^3 = 0$, and thus $de(e + d) = 0$.

$$de(d + e) = 0 = d^3 e^3$$
$$d + e = u(1 + e^3) = u(1 + d^3)$$
$$u = (d + e)(1 + e^3) = (d + e)(1 + d^3)$$

$$s^4 = e(e + d) = eu$$
$$t^4 = d^2 e^2$$

Some delicate calculations in generalised character theory give

$$e = v + s^2 + u^2 s$$
$$t = v + u + su^2$$
$$x_1 = u + s^2 + su^2 + s^3 u + s^2 u^3$$

We can use the above to get the complete ring structure of $K(BW)$ just by bashing the relations in a simple-minded manner.

We have double-coset formulae as follows:

$$\text{res}^W_P \text{tr}^V_P(x) = \text{tr}^V_{\gamma \cap P} \text{res}^V_{\gamma \cap P}(x)$$
$$\text{res}^W_P \text{tr}^V_P(x) = \text{tr}^V_{\gamma \cap P} \text{res}^V_{\gamma \cap P}(x)$$

In both cases, the right hand side involves only transfers between Abelian groups. These are easy to calculate, because the corresponding restriction maps are epi. Using this and the naturality of Chern classes, we can calculate the restriction map $K(BW) \to K(BV)$. Combining this with our knowledge of $x_1$, we can compute the transfer map $K(BV) \to K(BBW)$.

It is well known that $\text{res}: K(BG) \to K(BW)$ is mono. To identify the image, we restrict further to $K(BV)$; the image is fixed under the action of $\theta$ by conjugation (note that $V$ is normal in $G$). Calculation shows that the set of elements of $K(BW)$ with this property has dimension 17, which we know by generalised character theory to be the rank of $K(BG)$. This shows that the set in question is precisely the image of $K(BG) \to K(BW)$.
Naturality of Chern classes gives
\[ \text{res}^G_W c_2(\rho) = c_2(\sigma + \epsilon \delta) = c_2(\sigma) + c_1(\sigma)c_1(\epsilon \delta) = s + tu \]

We also have the double-coset formula
\[ \text{res}^G_W \text{tr}^G_W (x) = x + \text{tr}^W_V c_0^W \text{res}^V_W (x) \]

This reduces everything to monomial-bashing.

References