

$K(N)$ -LOCAL DUALITY FOR FINITE GROUPS AND GROUPOIDS

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1. INTRODUCTION

The starting point of the investigations described here was our discovery of a natural inner product on the ring $K(n)^*BG$, the n 'th Morava K -theory of the classifying space of a finite group G . If $n = 1$ and G is a p -group then $K(1)^*BG$ is essentially the same as $R(G)/p$ (where $R(G)$ is the complex representation ring of G) and our inner product is just $(V, W) = \dim_{\mathbb{C}}(V \otimes W)^G \pmod{p}$. This is closely related to the classical inner product on $R(G)$, given by

$$(V, W) = \dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = (V, W^*).$$

(For more general groups G , there is still a relationship with the classical product but it is not too close; see Section 11 for some pitfalls.)

It turns out to be useful to work with an inner product on the spectrum $LG := L_{K(n)}\Sigma^\infty BG_+$ and then deduce consequences in Morava K -theory (and other generalised cohomology theories) by functorality. As background to this, in Section 2 we recall some results about inner products on objects in arbitrary compact closed categories. Moreover, to elucidate the relationship between the inner product and the ring structure on $K(n)^*BG$, it is helpful to recall some facts about Frobenius algebras and their relationship with topological quantum field theories, which we do in Sections 3 and 4. In Section 5 we give a version of Poincaré-Atiyah duality for manifolds which illustrates these ideas nicely, and which has striking formal similarities with our later treatment of LG ; indeed, one could probably set up a unifying categorical framework. We have also found that many aspects of our theory (for example homotopy pullbacks and free loop spaces) can be discussed more cleanly in terms of groupoids rather than groups. This is also convenient for a number of applications and calculations. Because of this, we give a fairly detailed treatment of the homotopy theory of groupoids in Section 6. In Section 7 we discuss transfers for coverings-up-to-homotopy, as outlined in [30, Remark 3.1]. In Section 8 we turn to the spectra LG . In [17] we used the Greenlees-May theory of generalised Tate spectra to exhibit an equivalence $LG \simeq F(LG, L_{K(n)}S^0)$. After comparing some definitions and feeding this into our machinery, we find that LG has a natural structure as a Frobenius object in the $K(n)$ -local stable category, whenever G is a finite groupoid. As part of the construction we define $K(n)$ -local transfer maps for arbitrary homomorphisms of finite groups, or functors of finite groupoids; these reduce to classical transfers when the homomorphisms or functors are injective or faithful. In Section 9 we deduce various consequences for the generalised cohomology of BG ; in the case where G is a finite Abelian group, we can be quite explicit. In Section 10, we deduce some further consequences in terms of the Hopkins-Kuhn-Ravenel generalised character theory [14], which gives a complete description of $\mathbb{Q} \otimes E^0BG$ for suitable cohomology theories E . Finally, in Section 11 we alert the reader to some possible pitfalls that can arise from overoptimism about the analogy with classical representation theory.

2. INNER PRODUCTS

Let \mathcal{C} be an additive compact closed category, in other words an additive closed symmetric monoidal category in which every object is dualisable. We write $X \wedge Y$ for the symmetric monoidal product, and S for the unit object. We also write $F(Y, Z)$ for the function objects, so that $\mathcal{C}(X, F(Y, Z)) \simeq \mathcal{C}(X \wedge Y, Z)$. Finally, we write $DX = F(X, S)$, so that $D^2X = X$ and $F(X, Y) = DX \wedge Y$.

Definition 2.1. An *inner product* on an object $X \in \mathcal{C}$ is a map $b: X \wedge X \rightarrow S$ such that

1. b is symmetric in the sense that $b \circ \tau = b$, where $\tau: X \wedge X \rightarrow X \wedge X$ is the twist map; and
2. the adjoint map $b^\#: X \rightarrow DX$ is an isomorphism.

Example 2.2. We could take \mathcal{C} to be the category of finitely generated projective modules over a commutative ring R , with the usual closed symmetric monoidal structure so that $M \wedge N = M \otimes_R N$ and $F(M, N) = \text{Hom}_R(M, N)$ and $DM = M^* = \text{Hom}_R(M, R)$. An inner product on M is then a symmetric R -bilinear pairing $M \times M \rightarrow R$ that induces an isomorphism $M \simeq M^*$. If R is a field then this just says that the pairing is nondegenerate. Note that we have no positivity condition.

Remark 2.3. We see from [21, Theorem III.1.6] that a symmetric map $b: X \wedge X \rightarrow S$ is an inner product iff it is a duality of X with itself in the sense discussed there, iff there is a map $c: S \rightarrow X \wedge X$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{1 \wedge c} & X \wedge X \wedge X \\ & \searrow 1 & \downarrow b \wedge 1 \\ & & X \end{array} \qquad \begin{array}{ccc} X \wedge X \wedge X & \xleftarrow{c \wedge 1} & X \\ & \swarrow 1 & \downarrow 1 \wedge b \\ & & X \end{array}$$

Moreover, if b is an inner product then there is a unique map c as above, and it is symmetric; in fact it is also the unique symmetric map making the left hand diagram commute.

Remark 2.4. The commutativity of the above diagrams can be expressed in terms of Penrose diagrams [18] as follows:

$$\begin{array}{c} \text{---} \bullet \\ \diagdown \quad \diagup \\ c \bullet \quad b \end{array} = \text{---} \quad 1 \quad = \begin{array}{c} c \bullet \quad \text{---} \\ \diagdown \quad \diagup \\ \bullet \quad b \end{array}$$

Similarly, the symmetry of b and c gives the following equations:

$$\begin{array}{c} \diagdown \quad \diagup \\ \bullet \quad b \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \bullet \quad b \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \quad c \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \bullet \quad c \\ \diagdown \quad \diagup \end{array}$$

Definition 2.5. If X and Y are equipped with inner products and $f: X \rightarrow Y$ then we write $f^t: Y \rightarrow X$ for the unique map making the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{f^t} & X \\ b_Y^\# \downarrow & & \downarrow b_X^\# \\ DY & \xrightarrow{Df} & DX. \end{array}$$

This can also be characterised by the equation

$$b_Y \circ (f \wedge 1) = b_X \circ (1 \wedge f^t): X \wedge Y \rightarrow S$$

or equivalently, the following equality between Penrose diagrams.

$$\begin{array}{ccc} X & \xrightarrow{f} & \bullet \\ & \searrow & \diagup \\ Y & & \bullet \quad b_Y \end{array} = \begin{array}{ccc} X & & \bullet \\ & \searrow & \diagup \\ Y & \xrightarrow{f^t} & \bullet \quad b_X \end{array}$$

It is clear that $f^{tt} = f$ and that $1^t = 1$ and $(gf)^t = f^t g^t$ whenever this makes sense. We call f^t the *transpose* of f .

Remark 2.6. Suppose that X and Y have inner products b_X and b_Y . We then define

$$b_{X \wedge Y} = (X \wedge Y \wedge X \wedge Y \xrightarrow{1 \wedge \tau \wedge 1} X \wedge X \wedge Y \wedge Y \xrightarrow{b_X \wedge b_Y} S).$$

It is easy to check that this is an inner product on $X \wedge Y$. Similarly, if \mathcal{C} is an additive category (with direct sums written as $X \vee Y$) and \wedge is bilinear then there is an obvious way to put an inner product on $X \vee Y$. By abuse of language, we call these inner products $b_X \wedge b_Y$ and $b_X \vee b_Y$. If we use these inner products, we find that $(f \wedge g)^t = f^t \wedge g^t$ and $(f \vee g)^t = f^t \vee g^t$.

3. FROBENIUS OBJECTS

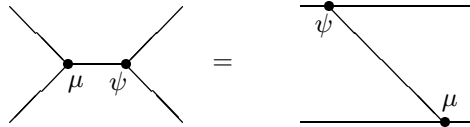
Definition 3.1. Let \mathcal{C} be a symmetric monoidal category. A *Frobenius object* in \mathcal{C} is an object $A \in \mathcal{C}$ equipped with maps $S \xrightarrow{\eta} A \xleftarrow{\mu} A \wedge A$ and $S \xleftarrow{\epsilon} A \xrightarrow{\psi} A \wedge A$ such that

- (a) (A, η, μ) is a commutative and associative ring object.
- (b) (A, ϵ, ψ) is a commutative and associative coring object.
- (c) (The ‘‘interchange axiom’’) The following diagram commutes:

$$\begin{array}{ccc} A \wedge A & \xrightarrow{\mu} & A \\ \psi \wedge 1 \downarrow & & \downarrow \psi \\ A \wedge A \wedge A & \xrightarrow{1 \wedge \mu} & A \wedge A \end{array}$$

The point for us will be that for any finite groupoid G , the spectrum $LG := L_{K(n)}\Sigma^\infty BG_+$ has a natural structure as a Frobenius object in the $K(n)$ -local stable category (Theorem 8.7).

The last axiom can be restated as the following equality of Penrose diagrams:



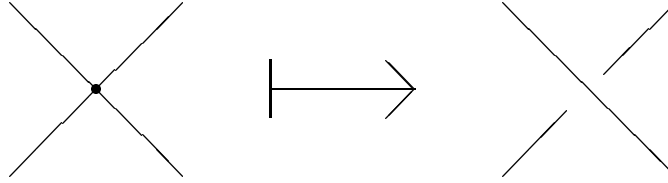
Remark 3.2. If $(A, \eta, \epsilon, \mu, \psi)$ is a Frobenius object in \mathcal{C} then $(A, \epsilon, \eta, \psi, \mu)$ is evidently a Frobenius object in \mathcal{C}^{op} .

Convention 3.3. For the rest of this paper, we use the following conventions for Penrose diagrams. Unless otherwise specified, each diagram will involve only a single object A , for which some subset of the maps $\mu, \psi, \eta, \epsilon$ will have been defined. We also automatically have a twist map $\tau: A \wedge A \rightarrow A \wedge A$.

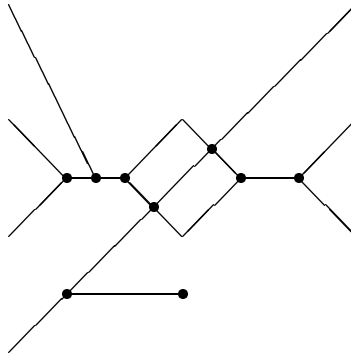
- Any unlabelled node with two lines in and one line out is implicitly labelled with μ .
- A node with one line in and no lines out is implicitly labelled ψ .
- A node with no lines in and one line out is implicitly labelled η .
- A node with one line in and no lines out is implicitly labelled ϵ .
- A node with two lines in and two lines out is implicitly labelled τ .

Another interesting point of view is that Frobenius objects are equivalent to topological quantum field theories (TQFT’s). In more detail, let \mathcal{S} be the $1 + 1$ -dimensional cobordism category, whose objects are closed 1-manifolds and whose morphisms are cobordisms. Some care is needed to set the details up properly: a good account is [1], although apparently the results involved were ‘‘folk theorems’’ long before this. The category \mathcal{S} has a symmetric monoidal structure given by disjoint unions. The circle S^1 is a Frobenius object in \mathcal{S} : the maps η and ϵ are the disc D^2 regarded as a morphism $\emptyset \rightarrow S^1$ and $S^1 \rightarrow \emptyset$ respectively, and the maps μ and ψ are the ‘‘pair of pants’’ regarded as a morphism $S^1 \amalg S^1 \rightarrow S^1$ and $S^1 \rightarrow S^1 \amalg S^1$ respectively. It follows easily from [1, Proposition 12] that this is a universal example of a symmetric monoidal category equipped with a Frobenius object. For further analysis of the category \mathcal{S} , see [7, 31].

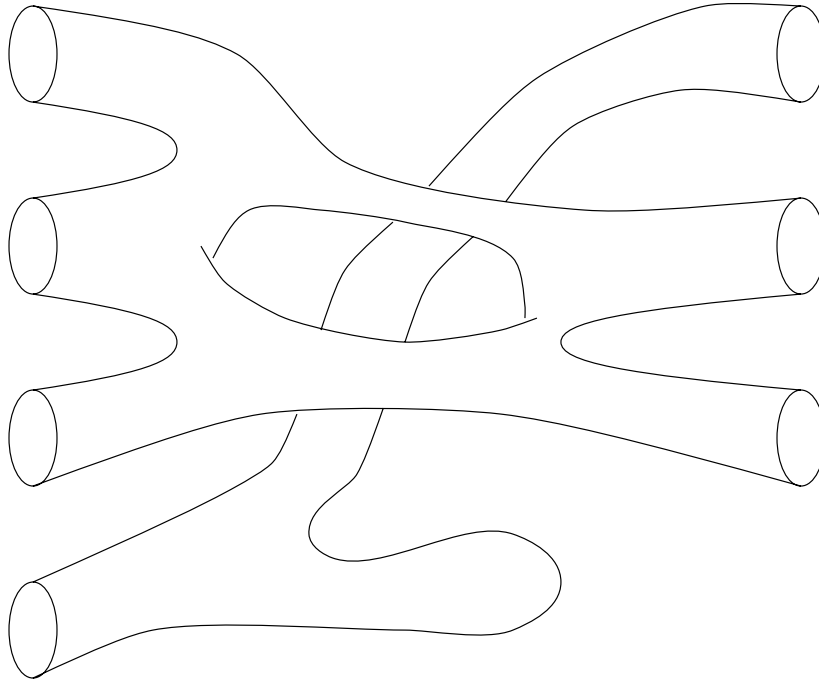
Remark 3.4. Using the Frobenius structure on S^1 , a Penrose diagram as in Convention 3.3 gives rise to a morphism in \mathcal{S} . This has the following appealing geometric interpretation. We first perform the replacement



(It makes no real difference whether we introduce an under crossing or an over crossing.) This converts the Penrose diagram to a graph embedded in $[0, 1] \times \mathbb{R}^2$. The boundary of a regular neighbourhood of this graph is a surface Σ which we can think of as a cobordism between $\Sigma \cap (\{0\} \times \mathbb{R}^2)$ and $\Sigma \cap (\{1\} \times \mathbb{R}^2)$ and thus as a morphism in \mathcal{S} . For example, the Penrose diagram



becomes the following cobordism:



Definition 3.5. Let \mathcal{C} be a compact closed category, and let A be an object of \mathcal{C} equipped with a commutative and associative product $\mu: A \wedge A \rightarrow A$. (We do not assume that there is a unit.) A *Frobenius form* on A is a map $\epsilon: A \rightarrow S$ such that the map $b = \epsilon\mu$ is an inner product.

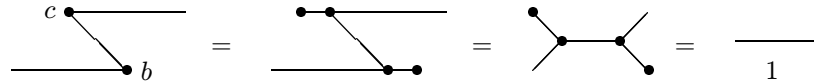
Example 3.6. The most familiar example in topology is that if M is a closed manifold with fundamental class $[M] \in H_*(M; \mathbb{F}_2)$ then the equation $\epsilon(u) = \langle u, [M] \rangle$ is a Frobenius form on $H^*(M; \mathbb{F}_2)$ (regarded as an ungraded module over \mathbb{F}_2). This can of course be generalised to other coefficients at the price of a few words about orientations and gradings. For a geometrised version of this, see Section 5.

Example 3.7. Another elementary example is to let k be a field and G a finite Abelian group. We can then define a map $\epsilon: k[G] \rightarrow k$ sending $[1]$ to 1 and $[g]$ to 0 for $g \neq 1$. This is easily seen to be a Frobenius form.

Example 3.8. Let \mathcal{C} be the category of finitely generated free Abelian groups. Let G be a finite group, let $R = R(G)$ be its complex representation ring, and define $\epsilon: R \rightarrow \mathbb{Z}$ by $\epsilon[W] = \dim_{\mathbb{C}} W^G$. It is easy to see that this is a Frobenius form, and that the associated inner product is $([U], [W]) = \dim_{\mathbb{C}}(U \otimes W)^G$ as considered in the introduction. To generalise this to finite groupoids, let V be the category of finite dimensional complex vector spaces. A *representation* of G means a functor $G \rightarrow V$. The set $R_+(G) = \pi_0[G, V]$ of isomorphism classes of representations has a natural structure as a semiring, and we let $R(G)$ denote its group completion. If W is a representation then we write $W^G := \lim_{\leftarrow G} W \in V$ and $t[W] = \dim_{\mathbb{C}} W^G$ as before. One can easily deduce from the classical case that this is a Frobenius form on $R(G)$.

Lemma 3.9. *If $(A, \eta, \epsilon, \mu, \psi)$ is a Frobenius object in a compact closed category \mathcal{C} , then ϵ is a Frobenius form.*

Proof. Put $b = \epsilon\mu: A \wedge A \rightarrow S$; we need to show that this is an inner product. Put $c = \psi\eta: S \rightarrow A \wedge A$; it will suffice to check the identities in Remark 2.3. The symmetry conditions are clear, so we just need the two compatibility conditions for b and c . One of them is proved as follows:

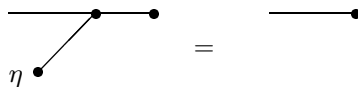


The first equation is just the definition of b and c , the second is the interchange axiom, and the third uses the (co)unit properties of η and ϵ . The other compatibility condition follows because b and c are symmetric. \square

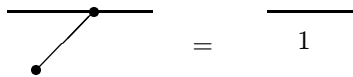
We now prove a converse to the above result.

Proposition 3.10. *Let \mathcal{C} be a compact closed category, and let A be an object equipped with a commutative and associative product μ and a Frobenius form ϵ . Then there are unique maps η, ψ making $(A, \eta, \epsilon, \mu, \psi)$ into a Frobenius object.*

Proof. By hypothesis $b = \epsilon\mu$ is an inner product on A , and trivially the canonical isomorphism $S \wedge S = S$ is an inner product on S . We can thus define $\eta := \epsilon^t: S \rightarrow A$, so η is the unique map such that $b \circ (1 \wedge \eta) = \epsilon$, or in other words the unique map giving the following equality of Penrose diagrams:



We claim that η is a unit for μ , or in other words that we have the following equality:



To prove this, we observe that for any two maps $f, g: B \rightarrow A$ we have $f = g$ if and only if

$$b \circ (1 \wedge f) = b \circ (1 \wedge g): A \wedge B \rightarrow S.$$

In view of this, the claim follows from the following diagram, in which the first equality comes from the associativity of μ and the second from the defining property of η .

We next equip $A \wedge A$ with the inner product $b \wedge b$ and define $\psi := \mu^t$. As (A, η, μ) is a commutative and associative monoid object, it is easy to deduce that $(A, \epsilon, \psi) = (A, \eta^t, \mu^t)$ is a commutative and associative comonoid object. Thus, to prove that A is a Frobenius object, we need only check the interchange axiom.

It follows directly from the definition that ψ is the unique map giving the following equality:

Using the perfectness of b , we see that two maps $f, g: B \rightarrow A \wedge A$ are equal if and only if we have

$$(b \wedge b)(1 \wedge f \wedge 1) = (b \wedge b)(1 \wedge g \wedge 1): A \wedge B \wedge A \rightarrow A \wedge A \wedge A \wedge A.$$

In view of this, the interchange axiom is equivalent to the following equation:

This equation can be proved as follows:

The first equality uses associativity of μ , the second uses the defining property of ψ , and the third uses the same two ideas backwards.

We still need to check that η and ψ are the *unique* maps giving a Frobenius structure. For η this is easy, because the unit for a commutative and associative product is always unique. For ψ , suppose that $\phi: A \rightarrow A \wedge A$ is another map giving a Frobenius structure. We then have the following equations:

The first equality is the interchange axiom, the second is the counit property of ϵ , and the third is the associativity of μ . This shows that ϕ has the defining property of ψ , so $\phi = \psi$ as required. \square

Scholium 3.11. Let $(A, \eta, \epsilon, \mu, \psi)$ be a Frobenius object. Give A the inner product $b = \epsilon\mu$ and give $A \wedge A$ the inner product $b \wedge b$. Then $\eta: S \rightarrow A$ is adjoint to $\epsilon: A \rightarrow S$ and $\psi: A \rightarrow A \wedge A$ is adjoint to $\mu: A \wedge A \rightarrow A$.

Proof. This is implicit in the proof of the proposition. \square

Scholium 3.12. The map $\epsilon: A \rightarrow S$ is the unique one such that $(\epsilon \wedge 1)\psi\eta = \eta: S \rightarrow A$.

Proof. We saw in the proof of the proposition that η is the unique map giving the following equality of Penrose diagrams.

$$\begin{array}{c} \mu \quad \epsilon \\ \bullet \quad \bullet \\ \hline \eta \bullet \end{array} = \begin{array}{c} \epsilon \\ \bullet \\ \hline \bullet \end{array}$$

The claim follows by working in the opposite category and using Remark 3.2. \square

Remark 3.13. Let A and B be Frobenius objects, and suppose that $f: A \rightarrow B$ is a ring map with respect to η and μ . We can use this to make B into an A -module. We claim that $f^t: B \rightarrow A$ is automatically a map of A -module objects. We will give the proof in the category of vector spaces over a field; it can easily be made diagrammatic. The claim is that $f^t(f(a)b) = a f^t(b)$. It suffices to prove that $b_A(a', f^t(f(a)b)) = b_A(a', a f^t(b))$. The left hand side is $b_B(f(a'), f(a)b) = \epsilon_B(f(a)f(a')b) = \epsilon_B(f(aa')b)$. The right hand side is

$$\epsilon_A(aa' f^t(b)) = b_A(aa', f^t(b)) = b_B(f(aa'), b) = \epsilon_B(f(aa')b),$$

as required.

4. THE TRACE FORM

We now construct an interesting map $A \rightarrow S$ which may or may not be a Frobenius form.

Definition 4.1. Let A be an arbitrary commutative ring object in an additive compact closed category \mathcal{C} . We can then transpose the multiplication map $\mu: A \wedge A \rightarrow A$ to get $\rho: A \rightarrow DA \wedge A$ and compose with the evaluation map $DA \wedge A = F(A, S) \wedge A \rightarrow S$ to get a map $\theta: A \rightarrow S$. This is called the *trace form*.

Remark 4.2. If K is a ring and \mathcal{C} is the category of finitely generated free modules over K then $\theta(a)$ is just the trace of the multiplication-by- a map. Now suppose that K is a perfect field. One can check that θ is a Frobenius form if and only if A has no nilpotents, if and only if A is a finite product of finite extension fields of K (this is well-known and can mostly be extracted from [11, Section I.1], for example).

Proposition 4.3. Let A be a Frobenius object in an additive compact closed category \mathcal{C} . Then the trace form θ is given by $\theta = b\psi = \epsilon\mu\psi: A \rightarrow S$. Moreover, if $\alpha = \mu c = \mu\psi\eta: S \rightarrow A$ then $\theta = b(\alpha \wedge 1)$.

Proof. The adjunction between the functors $A \wedge (-)$ and $F(A, -) = DA \wedge (-)$ is given by two maps $\text{unit}: S \rightarrow DA \wedge A$ and $\text{eval}: DA \wedge A \rightarrow S$. It follows from the basic theory of pairings and duality [21, Chapter III] that the following diagrams commute:

$$\begin{array}{ccc} A \wedge A & & S \xrightarrow{c} A \wedge A \\ \downarrow b^\# \wedge 1 & \searrow b & \downarrow b^\# \wedge 1 \\ DA \wedge A & \xrightarrow{\text{eval}} & S \end{array} \quad \begin{array}{ccc} S & \xrightarrow{c} & A \wedge A \\ \downarrow \text{unit} & & \downarrow b^\# \wedge 1 \\ DA \wedge A & & DA \wedge A \end{array}$$

It follows that the following diagram commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{c \wedge 1} & A \wedge A \wedge A & \xrightarrow{1 \wedge \mu} & A \wedge A & \xrightarrow{b} & S \\ \parallel & & \downarrow b^\# \wedge 1 \wedge 1 & & \downarrow b^\# \wedge 1 & & \parallel \\ A & \xrightarrow{\text{unit} \wedge 1} & DA \wedge A \wedge A & \xrightarrow{1 \wedge \mu} & DA \wedge A & \xrightarrow{\text{eval}} & S \end{array}$$

On the bottom row, the composite of the first two maps is ρ so the whole composite is just θ . Thus, $\theta = b(1 \wedge \mu)(c \wedge 1)$. To complete the proof, it is easiest to think in terms of TQFT's. Let M be a torus with a small open disc removed. We leave it to the reader to check that $\alpha = \mu c$ is represented by M , considered as a cobordism from \emptyset to S^1 . Moreover, the maps $b(1 \wedge \mu)(c \wedge 1)$, $b\psi$

and $b(\alpha \wedge 1)$ are all represented by M considered as a cobordism from S^1 to \emptyset . The proposition follows. \square

5. MANIFOLDS

We next show how to use manifolds to construct Frobenius objects in suitable categories of module spectra. This is of course just a reformulation of Atiyah-Poincaré duality, but it is a nice illustration of the theory of Frobenius objects. It is also strikingly formally similar to the constructions in the $K(n)$ -local stable category which we discuss later.

Let \mathcal{M} be the category of even-dimensional closed manifolds M equipped with a complex structure on the stable normal bundle, or equivalently a complex orientation on the map from M to the one-point manifold; we refer to Quillen's work [25] for a careful discussion of what this means.

Next, let MP denote the Thom spectrum of the tautological virtual complex bundle over $\mathbb{Z} \times BU$, so that $MP = \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU$ and $\Sigma^2 MP \simeq MP$. More generally, if V is a complex bundle over a space X then there is a canonical Thom class $u_V: X^V \rightarrow MP$ which combines with the usual diagonal map $X^V \rightarrow X_+ \wedge X^V$ to give a canonical equivalence $MP \wedge X^V \simeq MP \wedge \Sigma^\infty X_+$. With a little care, this also goes through for virtual bundles.

The spectrum MP can be constructed as an E_∞ ring spectrum, and thus as a strictly commutative ring spectrum (or “ S -algebra”) in the EKMM category [9]. We can thus define a category of MP -modules in the strict sense, and the associated derived category $\mathcal{D} = \mathcal{D}_{MP}$. (There are also other approaches to our results using less technology.) The category \mathcal{D} is a unital algebraic stable homotopy category in the sense of [16]; in particular it is a closed symmetric monoidal category. We write \mathcal{F} for the thick subcategory of \mathcal{D} generated by MP , which is the same as the category of small or strongly dualisable objects [16, Theorem 2.1.3(d)]. This is clearly a compact closed category.

Define $T: \mathcal{M} \rightarrow \mathcal{F}$ by $T(M) = MP \wedge \Sigma^\infty M_+$. This is clearly a covariant functor that converts products to smash products and disjoint unions to wedges.

Now suppose we have a smooth map $f: M \rightarrow N$ of closed manifolds. Let $j: M \rightarrow \mathbb{R}^k$ be a smooth map such that $(j, f): M \rightarrow \mathbb{R}^k \times N$ is a closed embedding, with normal bundle $\nu_{(j,f)}$ say. This is stably equivalent to $k + \nu_M - f^* \nu_N$. The Pontrjagin-Thom construction applied to the embedding (j, f) gives a map $\Sigma^k N_+ \rightarrow M^{\nu_{(j,f)}}$ and thus a stable map $f^!: \Sigma^\infty N_+ \rightarrow M^{\nu_M - f^* \nu_N}$.

Now suppose that M and N have specified complex orientations, so they are objects of \mathcal{M} . Then the virtual bundle $\nu_f = \nu_M - f^* \nu_N$ has a canonical complex structure, so there is a canonical equivalence $MP \wedge M^{\nu_M - f^* \nu_N} \simeq T(M)$. Thus, by smashing $f^!$ with MP we get a map $Uf: T(N) \rightarrow T(M)$. One can check that this construction gives a contravariant functor $U: \mathcal{M} \rightarrow \mathcal{D}$, which again converts products to smash products and disjoint unions to wedges. If f is a diffeomorphism, one checks easily that $U(f) = T(f)^{-1}$. We also have the following “Mackey property”. Suppose we have a commutative square in \mathcal{M} :

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ g \downarrow & & \downarrow h \\ M & \xrightarrow{k} & N. \end{array}$$

Suppose also that the square is a pullback and the maps h and k are transverse to each other, so that when $x \in K$ with $hf(x) = kg(x) = y$ say, the map of tangent spaces

$$(Dh, Dk): T_{f(x)}L \oplus T_{g(x)}M \rightarrow T_yN$$

is surjective. We then have $U(h)T(k) = T(f)U(g)$, as one sees directly from the geometry.

For any manifold $M \in \mathcal{M}$, there is a unique map $\epsilon: M \rightarrow 1$, where 1 is the one-point manifold. We also have a diagonal map $\psi: M \rightarrow M \times M$. We allow ourselves to write ϵ and ψ for $T(\epsilon)$ and $T(\psi)$, and we also write $\eta = U(\epsilon)$ and $\mu = U(\psi)$.

Proposition 5.1. *The above maps make $T(M)$ into a Frobenius object in \mathcal{M} . If we use the resulting inner product, then for any map $f: M \rightarrow N$ in \mathcal{M} we have $T(f)^t = U(f)$.*

Proof. If we make \mathcal{M} into a symmetric monoidal category using the cartesian product, it is clear that ϵ and ψ make M into a comonoid object. As T and U are monoidal functors, the first covariant and the second contravariant, we see that $\Sigma^\infty M_+ = T(M) = U(M)$ is a monoid object under μ and η , and a comonoid object under ψ and ϵ . For the interchange axiom $\psi\mu = (1 \wedge \mu)(\psi \wedge 1)$, we note that the following diagram is a transverse pullback and apply the Mackey property.

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M \times M \\ \psi \downarrow & & \downarrow 1 \times \psi \\ M \times M & \xrightarrow{\psi \times 1} & M \times M \times M. \end{array}$$

Similarly, to prove that $T(f)^t = U(f)$, we note that the following square is a transverse pullback:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ (1, f) \downarrow & & \downarrow \psi \\ M \times N & \xrightarrow{f \times 1} & N \times N. \end{array}$$

We then apply the Mackey property, noting that $(1, f) = (1 \times f)\psi_M$; this gives the following commutative diagram:

$$\begin{array}{ccccc} T(M) \wedge T(N) & \xrightarrow{T(f) \wedge 1} & & T(N) \wedge T(N) & \\ \downarrow 1 \wedge U(f) & & & \downarrow \mu_N & \\ T(M) \wedge T(M) & \xrightarrow{\mu_M} & T(M) & \xrightarrow{T(f)} & T(N). \end{array}$$

We then compose with ϵ_N , noting that $\epsilon_N T(f) = \epsilon_M$ and $\epsilon_M \mu_M = b_M$ and $\epsilon_N \mu_N = b_N$. We conclude that $b_N(T(f) \wedge 1) = b_M(1 \wedge U(f))$, so $T(f)^t = U(f)$ as claimed. \square

6. GROUPOIDS

Let \mathcal{G} denote the category of groupoids and functors between them, and let $\overline{\mathcal{G}}$ be the quotient category in which two functors are identified if there is a natural isomorphism between them. We say that a groupoid G is *finite* if there are only finitely many isomorphism classes of objects, and $G(a, b)$ is finite for any $a, b \in G$. We write \mathcal{G}_f for the category of finite groupoids, and $\overline{\mathcal{G}}_f$ for the obvious quotient category.

We next exhibit an equivalence between $\overline{\mathcal{G}}$ and a certain homotopy category of spaces. As usual in homotopy theory, it will be convenient to work with compactly generated weakly Hausdorff spaces (so we have Cartesian closure). Let \mathcal{B} be the category of such spaces X for which $\pi_k(X, x) = 0$ for all $k > 1$ and all $x \in X$. We also write $\overline{\mathcal{B}}$ for the associated homotopy category (in which weak equivalences are inverted), and we let \mathcal{B}_f and $\overline{\mathcal{B}}_f$ be the subcategories whose objects are those $X \in \mathcal{B}$ for which $\pi_0 X$ is finite and $\pi_1(X, x)$ is finite for any basepoint x .

Milgram's classifying space construction gives a functor $\mathcal{B}: \mathcal{G} \rightarrow \mathcal{B}$. One can also define a functor $\Pi_1: \mathcal{B} \rightarrow \mathcal{G}$: the set of objects of $\Pi_1(X)$ is X , and the set of morphisms from x to y is the set of paths from x to y modulo homotopy relative to the endpoints. Both \mathcal{G} and \mathcal{B} have finite products and coproducts, and both our functors preserve them.

It is easy to check that these constructions give equivalences $\overline{\mathcal{G}} \simeq \overline{\mathcal{B}}$ and $\overline{\mathcal{G}}_f \simeq \overline{\mathcal{B}}_f$.

Any (finite) group G can be regarded as a (finite) groupoid with one object. If G and H are groups then $\mathcal{G}(G, H)$ is the set of homomorphisms from G to H , and $\overline{\mathcal{G}}(G, H)$ is the set of conjugacy classes of such homomorphisms.

Conversely, if G is a finite groupoid then we can choose a family $\{a_i\}_{i \in I}$ containing precisely one object of G from each isomorphism class and then let H_i be the group $G(a_i, a_i)$. We find that $G \simeq \coprod_i H_i$ in $\overline{\mathcal{G}}$. Thus, all our questions about groupoids can be reduced to questions about groups by some unnatural choices. Our next lemma sharpens this slightly.

Definition 6.1. A groupoid G is *discrete* if all its maps are identity maps, and *indiscrete* if there is precisely one map from a to a' for all $a, a' \in G$.

Remark 6.2. The category of discrete groupoids is equivalent to that of sets, as is the category of indiscrete groupoids. The classifying space of a discrete groupoid is discrete, and that of a nonempty indiscrete groupoid is contractible.

Lemma 6.3. *Any nonempty connected groupoid is isomorphic to $A \times H$ for some nonempty indiscrete groupoid A and some group H . Thus, any groupoid is isomorphic to $\coprod_I A_i \times H_i$ for some family of nonempty indiscrete groupoids A_i and groups H_i .*

Proof. Let G be a connected groupoid. Choose an object $x \in G$ and let H be the group $G(x, x)$. Let A be the indiscrete groupoid with $\text{obj}(A) = \text{obj}(G)$, and for each $a \in A$ choose a map $k_a: x \rightarrow a$ in G . Put $B = A \times H$, so $\text{obj}(B) = \text{obj}(G)$ and $B(a, a') = H$ for all a, a' . Composition is given by multiplication in H . Define $u: B \rightarrow G$ by $u(a) = a$ on objects, and

$$u_{a,a'}(h) = (a \xrightarrow{k_a^{-1}} x \xrightarrow{h} x \xrightarrow{k_{a'}} a')$$

on morphisms. This is easily seen to be functorial and to be an isomorphism.

The generalisation to the disconnected case is immediate. \square

6.1. Model category structure. We now complete an exercise assigned by Anderson [2] to his readers, by verifying that his definitions (reproduced below) do indeed make the category \mathcal{G} into a closed model category in the sense of Quillen [23] (see also [8] for an exposition and survey of more recent literature). As well as being useful for our applications, this seems pedagogically valuable, as the verification of the axioms is simpler than in most other examples. The homotopy theory of the category of all small categories has been extensively studied (see [22] for example), but the case of groupoids is easier so it makes sense to treat it independently.

Definition 6.4. We say that a functor $u: G \rightarrow H$ of groupoids is

- (a) a *weak equivalence* if it is full, faithful and essentially surjective (in other words, an equivalence of categories);
- (b) a *cofibration* if it is injective on objects; and
- (c) a *fibration* if for all $a \in G$, $b \in H$ and $h: u(a) \rightarrow b$ there exists $g: a \rightarrow a'$ in G such that $u(a') = b$ and $u(g) = h$.

As usual, an *acyclic fibration* means a fibration that is also an equivalence, and similarly for acyclic cofibrations.

Remark 6.5. Let $u: G \rightarrow H$ be a homomorphism of groups. Then u is automatically a cofibration of groupoids, and it is a fibration iff it is surjective. It is an equivalence of groupoids iff it is an isomorphism.

Remark 6.6. Let $v: X \rightarrow Y$ be a map of sets. If we regard X and Y as discrete categories then v is automatically a fibration. It is a cofibration iff it is injective, and an equivalence iff it is bijective. If we regard X and Y as indiscrete categories then v is automatically an equivalence (unless $\emptyset = X \neq Y$). It is a cofibration iff it is injective, and a fibration iff it is surjective.

Theorem 6.7. *The above definitions make \mathcal{G} into a closed model category.*

Proof. We need to verify the following axioms, numbered as in [8]:

MC1: \mathcal{G} has finite limits and colimits.

MC2: If we have functors $G \xrightarrow{u} H \xrightarrow{v} K$ and two of u , v and vu are weak equivalences then so is the third.

MC3: Every retract of a weak equivalence is a weak equivalence, and similarly for fibrations and cofibrations.

MC4: Cofibrations have the left lifting property for acyclic fibrations, and acyclic cofibrations have the left lifting property for all fibrations.

MC5: Any functor u has factorisations $u = pi = qj$ where i and j are cofibrations, p and q are fibrations, and i and q are equivalences.

MC1: This from the fact that \mathcal{G} is the category of models for a left-exact sketch [3, Section 4.4]. More concretely, for limits we just have $\text{obj}(\lim_{\leftarrow i} G_i) = \lim_{\leftarrow i} \text{obj}(G_i)$ and $\text{mor}(\lim_{\leftarrow i} G_i) = \lim_{\leftarrow i} \text{mor}(G_i)$. Similarly, for coproducts we have $\text{obj}(\coprod_i G_i) = \coprod_i \text{obj}(G_i)$ and $\text{mor}(\coprod_i G_i) = \coprod_i \text{mor}(G_i)$. Coequalisers are more complicated and best handled by the adjoint functor theorem.

MC2: This is easy.

MC3: Let v be an equivalence and let $u: G \rightarrow H$ be a retract of v . Then $\pi_0(u)$ is a retract of $\pi_0(v)$, so $\pi_0(u)$ is a bijection and so u is essentially surjective. If $a, b \in G$ then $u_{a,b}: G(a, b) \rightarrow H(ua, ub)$ is a retract of a map of the form $v_{c,d}$ and thus is a bijection, so u is full and faithful. Thus u is an equivalence as required.

It is clear that a retract of a cofibration is a cofibration.

For fibrations, let 1 be the terminal groupoid. Let I be the groupoid with objects $\{0, 1\}$ and two non-identity morphisms $u: 0 \rightarrow 1$ and $u^{-1}: 1 \rightarrow 0$. Let $i: 1 \rightarrow I$ be the inclusion of $\{0\}$. Then fibrations are precisely the maps with the right lifting property for i , and it follows that a retract of a fibration is a fibration.

MC4: Consider a commutative square as follows, in which i is a cofibration and p is a fibration.

$$\begin{array}{ccc} G & \xrightarrow{u} & K \\ i \downarrow & & \downarrow p \\ H & \xrightarrow{v} & L. \end{array}$$

Because p is a fibration, it is easy to see that the image of p is replete: if $d \in L$ is isomorphic to pc then d has the form pc' for some $c' \in K$.

Suppose that p is an equivalence; we must construct a functor $w: H \rightarrow K$ such that $pw = v$ and $wi = u$. As p is essentially surjective and the image is replete, we see that $\text{obj}(p)$ is surjective. By assumption i is a cofibration so $\text{obj}(i)$ is injective. Define a map $w: \text{obj}(H) \rightarrow \text{obj}(K)$ by putting $w(i(a)) = u(a)$ for $a \in \text{obj}(A)$ and choosing $w(b)$ to be any preimage under p of $v(b)$ if $b \notin \text{image}(i)$. Clearly $pw = v$ and $wi = u$ on objects. Given $b, b' \in H$ we define $w_{b,b'}$ to be the composite

$$H(b, b') \xrightarrow{v_{b,b'}} L(vb, vb') = L(pwb, pwb') \xrightarrow{p_{wb,wb'}^{-1}} K(wb, wb').$$

One can check that this makes w a functor with $pw = v$. Also $pw_i = vi = pu$ on morphisms and $wi = u$ on objects and p is faithful; it follows that $wi = u$ on morphisms, as required.

Now remove the assumption that p is an equivalence, and suppose instead that i is an equivalence. We must again define a functor $w: H \rightarrow K$ making everything commute. As i is injective on objects we can choose $r: \text{obj}(H) \rightarrow \text{obj}(G)$ with $ri = 1$. As $\pi_0(i)$ is a bijection we find that $\pi_0(r) = \pi_0(i)^{-1}$ so we can choose isomorphisms $\eta_b: b \rightarrow ir(b)$ for all $b \in H$. If $b = i(a)$ for some (necessarily unique) object a , we have $rb = a$ and $irb = b$, and we choose $\eta_b = 1_b$ in this case. There is a unique way to make r a functor $H \rightarrow G$ such that η is natural: explicitly, the map $r_{b,b'}$ is the composite

$$H(b, b') \xrightarrow{\eta_* \eta^*} H(irb, irb') \xrightarrow{i_{rb,rb'}^{-1}} G(rb, rb').$$

Next, if $b \in \text{image}(i)$ we define $wb = urb$ and $\zeta_b = 1: wb \rightarrow urb$. If $b \notin \text{image}(i)$ we instead apply the fibration axiom for p to the map $v\eta_b: vb \rightarrow virb = purb$ to get an object $wb \in K$ and a morphism $\zeta_b: wb \rightarrow urb$ such that $pwb = vb$ and $p\zeta_b = v\eta_b$. Note that these last two equations also hold in the case $b \in \text{image}(i)$. There is a unique way to make w into a functor such that

$\zeta: w \rightarrow ur$ is natural. Clearly $wi = u$ as functors, and $pw = v$ on objects. Given $h: b \rightarrow b'$ in H we can apply p to the naturality square for ζ and then use the naturality of η to deduce that $pw h = v h$; thus $pw = v$ on morphisms, as required.

MC5: Consider a functor $u: G \rightarrow H$. Let K be the category whose objects are triples (a, b, k) , with $a \in G$ and $b \in H$ and $k: u(a) \rightarrow b$. The morphisms from (a, b, k) to (a', b', k') are the pairs (g, h) where $g: a \rightarrow a'$ and $h: b \rightarrow b'$ and the following diagram commutes:

$$\begin{array}{ccc} u(a) & \xrightarrow{u(g)} & u(a') \\ k \downarrow & & \downarrow k' \\ b & \xrightarrow{h} & b'. \end{array}$$

We also consider the category L with the same objects as K , but with $L(a, b, k; a', b', k') = H(b, b')$, so there is an evident functor $v: L \rightarrow K$. There is also a functor $i: G \rightarrow K$ given by $i(a) = (a, ua, 1_{ua})$ and a functor $q: L \rightarrow H$ given by $q(a, b, k) = b$; we put $j = vi$ and $p = qv$. It is clear that $u = qvi = qj = pi$ and that i and j are cofibrations and that i is full and faithful. If $(a, b, k) \in K$ then $(1_a, k): i(a) \rightarrow (a, b, k)$ so i is essentially surjective and thus an equivalence. The functor q is clearly full and faithful, and its image is the repletion of the image of u .

We next claim that p and q are fibrations. Suppose that $(a, b, k) \in \text{obj}(K)$ and $h: b \rightarrow q(a, b, k) = b'$. Then $(a, b', hk) \in \text{obj}(K)$ and $(1_a, h): (a, b, k) \rightarrow (a, b', hk)$ and $q(1_a, h) = h$. This shows that q is a fibration, and the same construction also shows that p is a fibration.

We now have a factorisation $u = pi$ as required by axiom M3. If u is essentially surjective then the same is true of q and thus q is an equivalence and so the factorisation $u = qj$ is also as required. If u is not essentially surjective then we let L' be the full subcategory of H consisting of objects not in the repletion of the image of u and let $q': L' \rightarrow H$ be the inclusion. We then have an acyclic fibration $(q, q'): L \amalg L' \rightarrow H$ and a cofibration $G \xrightarrow{j} L \rightarrow L \amalg L'$ whose composite is u , as required. \square

Proposition 6.8. *The above model category structure is right proper (in other words, the pullback of a weak equivalence along a fibration is a weak equivalence.)*

Proof. Consider a pullback square as follows, in which v is a weak equivalence and q is a fibration.

$$\begin{array}{ccc} G & \xrightarrow{u} & K \\ p \downarrow & & \downarrow q \\ H & \xrightarrow{v} & L. \end{array}$$

Suppose that $a, a' \in G$ and put $d = qu(a) = vp(a)$ and $d' = qu(a') = vp(a')$. By the construction of pullbacks in \mathcal{G} , we see that the following square is a pullback square of sets:

$$\begin{array}{ccc} G(a, a') & \xrightarrow{u_{a, a'}} & K(u(a), u(a')) \\ p_{a, a'} \downarrow & & \downarrow q_{u(a), u(a')} \\ H(p(a), p(a')) & \xrightarrow{v_{p(a), p(a')}} & L(d, d'). \end{array}$$

As v is a weak equivalence, the map $v_{p(a), p(a')}$ is a bijection, and it follows that the same is true of $u_{a, a'}$. This means that u is full and faithful.

Next suppose we have $c \in K$, so $q(c) \in L$. As v is essentially surjective there exists $b \in H$ and $l: q(c) \rightarrow v(b)$ in L . As q is a fibration there is a map $k: c \rightarrow c'$ in K with $q(c') = v(b)$ and $q(k) = l$. By the pullback property there is a unique $a \in G$ with $u(a) = c'$ and $p(a) = b$. Thus $u(a) \simeq c$, proving that u is essentially surjective and thus an equivalence. \square

6.2. Classifying spaces. Let N be the nerve functor from groupoids to simplicial sets, and put $BG = |NG|$; this is called the *classifying space* of G . It is easy to see that N converts groupoids to Kan complexes and fibrations to Kan fibrations, and that it preserves coproducts and finite limits. The geometric realisation functor preserves coproducts (easy) and finite limits [10, Theorem 4.3.16] and it converts Kan fibrations to fibrations [24] (see also [10, Theorem 4.5.25]). Thus, the composite functor $B: \mathcal{G} \rightarrow \mathcal{B}$ preserves coproducts, finite limits and fibrations.

6.3. Homotopy pullbacks.

Definition 6.9. Suppose we have functors $G \xrightarrow{u} H \xleftarrow{v} K$ of groupoids. We define a new groupoid L whose objects are triples (a, c, h) with $a \in G$ and $c \in K$ and $h: u(a) \rightarrow v(c)$. The morphisms from (a, c, h) to (a', c', h') are pairs (r, s) where $r: a \rightarrow a'$ and $s: c \rightarrow c'$ and the following diagram commutes:

$$\begin{array}{ccc} u(a) & \xrightarrow{u(r)} & u(a') \\ \downarrow h & & \downarrow h' \\ v(c) & \xrightarrow{v(s)} & v(c'). \end{array}$$

We also define functors $K \xleftarrow{u'} L \xrightarrow{v'} G$ by $u'(a, c, h) = a$ and $v'(a, c, h) = c$, and a natural transformation $\phi: uv' \rightarrow vu'$ by $\phi_{(a,c,h)} = h$. This gives a square as follows, which commutes in $\overline{\mathcal{G}}$:

$$\begin{array}{ccc} L & \xrightarrow{v'} & G \\ u' \downarrow & & \downarrow u \\ K & \xrightarrow{v} & H. \end{array}$$

We call L the *homotopy pullback* of u and v . We say that an arbitrary commutative square in $\overline{\mathcal{G}}$ is *homotopy-cartesian* if it is isomorphic to one of the above form.

Remark 6.10. We can also consider the actual pullback rather than the homotopy pullback, which can be identified with the full subcategory $M \subseteq L$ consisting of pairs $(a, c, 1)$ where $u(a) = v(c)$. One checks that the inclusion $M \rightarrow L$ is an equivalence if u or v is a fibration.

Remark 6.11. Suppose that H is a group and u and v are inclusions of subgroups. Then M is the group $G \cap K$. Let $T \subseteq H$ be a set containing one element of each double coset in $G \setminus H/K$; we may as well assume that $1 \in T$. We find that L is equivalent to the groupoid $\coprod_T G^t \cap K$, and the term indexed by $t = 1$ is just M . It follows that the map $M \rightarrow L$ is an equivalence if and only if $H = GK$. Note that this is only predicted by the previous remark when $G = H$ or $K = H$.

Remark 6.12. By standard methods of abstract homotopy theory, we see that a square S in $\overline{\mathcal{G}}$ is homotopy-cartesian iff there is a pullback square S' in \mathcal{G} whose maps are fibrations, which becomes isomorphic to S in $\overline{\mathcal{G}}$.

Remark 6.13. It is easy to see that if G, H and K are finite then so is their homotopy pullback.

Definition 6.14. Suppose we have functors u, v, s, t such that the following square is commutative in $\overline{\mathcal{G}}$.

$$\begin{array}{ccc} F & \xrightarrow{t} & G \\ s \downarrow & & \downarrow u \\ K & \xrightarrow{v} & H \end{array}$$

Let L be the homotopy pullback of u and v , and let u', v' be as above. Choose an isomorphism $\sigma: ut \rightarrow vs$. We can then define a functor $\hat{\sigma}: F \rightarrow L$ by $\hat{\sigma}(d) = (t(d), s(d), \sigma_d)$; this has $u'\hat{\sigma} = t$ and $v'\hat{\sigma} = s$. If $\zeta: s \rightarrow s'$ and $\xi: t' \rightarrow t$ and $\sigma' = v(\zeta) \circ s \circ u(\xi)$ then it is easy to see that $\sigma \simeq \hat{\sigma}'$.

Lemma 6.15. *A square as in the above definition is homotopy Cartesian if and only if there exists $\sigma: ut \rightarrow vs$ such that $\hat{\sigma}: F \rightarrow L$ is an equivalence.*

Proof. If there exists such a map σ then the square is visibly equivalent in $\overline{\mathcal{G}}$ to a homotopy pullback square, and thus is homotopy cartesian. For the converse, suppose that the square is homotopy Cartesian. We can then find a diagram as follows which commutes in $\overline{\mathcal{G}}$, such that the outer square is a homotopy pullback, and the diagonal functors are equivalences.

$$\begin{array}{ccccc}
 L_1 & \xrightarrow{v'_1} & G_1 & & \\
 \delta \swarrow & & \nearrow \alpha & & \\
 & F \xrightarrow{t} G & & & \\
 s \downarrow & & \downarrow u & & \\
 & K \xrightarrow{v} H & & & \\
 \gamma \swarrow & & \searrow \beta & & \\
 K_1 & \xrightarrow{v_1} & H_1 & & \\
 u'_1 \downarrow & & \downarrow u_1 & &
 \end{array}$$

There is a “tautological” natural isomorphism $\phi_1: u_1v'_1 \rightarrow v_1u'_1$, and we write $\rho = \phi_1\delta: u_1v'_1\delta \rightarrow v_1u'_1\delta$ so that $\delta = \hat{\rho}$. As the top and left-hand regions of the diagram commute in $\overline{\mathcal{G}}$, we have natural maps $\alpha t \rightarrow v'_1\delta$ and $u'_1\delta \rightarrow \gamma s$, which we can use to form a natural map

$$\kappa = (u_1\alpha t \rightarrow u_1v'_1\delta \xrightarrow{\rho} v_1u'_1\delta \rightarrow v_1\gamma s).$$

Using the remark in the preceding definition, we see that $\hat{\kappa} \simeq \hat{\rho} = \delta: F \rightarrow L_1$. As δ is an equivalence, we see that the same is true of $\hat{\kappa}$. Next, we note that the functors $u_1\alpha v', v_1\gamma u': L \rightarrow L'$ are joined by the natural map

$$\tau = (u_1\alpha v' \rightarrow \beta u v' \xrightarrow{\beta(\phi)} \beta v u' \rightarrow v_1\gamma u'),$$

where the first and third maps come from the commutativity of the right-hand and bottom regions of the diagram. This gives a functor $\hat{\tau}: L \rightarrow L'$; we leave it to the reader to check directly that this is an equivalence.

Next, consider the composite

$$\beta u t \rightarrow u_1\alpha t \xrightarrow{\hat{\kappa}} v_1\gamma s \rightarrow \beta v s.$$

As β is full and faithful, this composite has the form $\beta(\sigma)$ for a unique natural map $\sigma: ut \rightarrow vs$, which gives rise to $\hat{\sigma}: F \rightarrow L$. One checks directly that $\hat{\tau}\hat{\sigma} = \hat{\kappa}$, and both $\hat{\tau}$ and $\hat{\kappa}$ are equivalences, so $\hat{\sigma}$ is an equivalence, as required. \square

6.4. Coverings and quasi-coverings.

Definition 6.16. A functor $u: G \rightarrow H$ is a *covering* if for each $a \in G$ and each $h: u(a) \rightarrow b$ in H there is a *unique* pair (a', g) such that $a' \in G$ and $g: a \rightarrow a'$ and $u(a') = b$ and $u(g) = h$. More generally, we say that u is a *quasi-covering* if it can be factored as an equivalence followed by a covering.

Remark 6.17. It is easy to check that pullbacks, products and composites of coverings are coverings.

Remark 6.18. A group homomorphism is only a covering if it is an isomorphism. We will see later that it is a quasi-covering iff it is injective.

Definition 6.19. A functor $u: G \rightarrow H$ *reflects identities* if whenever $g: a \rightarrow a'$ and $u(g) = 1_b$ for some b , we have $a = a'$ and $g = 1_a$. Such a functor is easily seen to be faithful.

We leave the following easy lemma to the reader.

Lemma 6.20. *A functor $u: G \rightarrow H$ is a covering iff it reflects identities and is a fibration.* \square

Proposition 6.21. *If $u: G \rightarrow H$ is a covering, then $Bu: BG \rightarrow BH$ is a covering map of topological spaces.*

Proof. Suppose for the moment that H is indiscrete and G is connected. Then for $a, a' \in G$ we have $G(a, a') \neq \emptyset$ and $u: G(a, a') \rightarrow H(ua, ua')$ is injective but the codomain has only one element so the same is true of $G(a, a')$. Thus u is full and faithful. It is also a fibration and H is connected so it is surjective on objects. If $ua = ua'$ then the unique map $a \rightarrow a'$ in G must become an identity map in H but u reflects identities so $a = a'$. We now see that u is an isomorphism so Bu is a homeomorphism and thus certainly a covering.

If H is indiscrete and G is disconnected, we can still show that Bu is a covering by looking at one component at a time.

Now suppose merely that H is connected. We can then split H as $A \times K$, where A is indiscrete and K is a group, as in Lemma 6.3. Let K' be the indiscrete category with object set K , and define $q: K' \rightarrow K$ by sending the unique morphism $k \rightarrow k'$ to $k'k^{-1} \in \text{mor}(K)$. One checks that $BK' = EK$ and that $Bq: EK \rightarrow BK = EK/K$ is the usual covering map. Thus, $H' = A \times K'$ is indiscrete and $r = 1 \times q: H' \rightarrow H$ is a covering with the property that Br is also a covering. Now form a pullback square as follows:

$$\begin{array}{ccc} G' & \xrightarrow{u'} & H' \\ r' \downarrow & & \downarrow r \\ G & \xrightarrow{u} & H. \end{array}$$

Note that u' is a covering. As H' is discrete we know that Bu' is a covering by the first paragraph. Thus, the pullback of Bu along the surjective covering map Br is a covering, and it follows easily that Bu is a covering.

Finally, if H is disconnected we just look at one component at a time. \square

Proposition 6.22. *Fix a groupoid H . Then the category of coverings $q: G \rightarrow H$ is equivalent to the category of functors $X: H \rightarrow \text{Sets}$, and thus (by [22, Section 1]) to the category of covering spaces of BH .*

Proof. This is a simple translation of Quillen's analysis of coverings of BG .

Suppose we start with a functor $X: H \rightarrow \text{Sets}$. We then define a category G whose objects are pairs (b, x) with $b \in H$ and $x \in X_b$; the morphisms $(b, x) \rightarrow (b', x')$ are the maps $h: b \rightarrow b'$ in H such that $X_h: X_b \rightarrow X_{b'}$ sends x to x' . There is an evident forgetful functor $q: G \rightarrow H$ sending (b, x) to b ; one checks that this is a covering.

Conversely, suppose we start with a covering $q: G \rightarrow H$. For each $b \in H$, we define $X_b = q^{-1}\{b\} \subseteq \text{obj}(G)$. Given a morphism $h: b \rightarrow b'$ in H and an element $a \in X_b$, the definition of a covering gives a unique morphism $g: a \rightarrow a'$ in G with $q(g) = h$; we define a map $X_h: X_b \rightarrow X_{b'}$ by $X_h(a) = a'$.

We leave it to the reader to check that these constructions give the claimed equivalence. \square

We next let C be the class of all coverings, and let E be the class of functors that are full and essentially surjective.

Proposition 6.23. *The pair (C, E) is a factorisation system in \mathcal{G} ; in other words*

- (a) *Both C and E contain all identity functors and are closed under composition by isomorphisms on either side.*
- (b) *Every functor $u: G \rightarrow H$ can be factored as $u = pr$ with $p \in C$ and $r \in E$.*

- (c) Every functor in E has the unique left lifting property relative to every functor in C . In other words, given functors $u, w, r \in E$ and $p \in C$ making the diagram below commute, there is a unique functor v such that $pv = w$ and $vr = u$.

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ \downarrow r & \nearrow v & \downarrow p \\ K & \xrightarrow{w} & H. \end{array}$$

See [3, Exercises 5.5] (for example) for generalities about factorisation systems.

Proof. (a): This is clear.

(b): Let $u: G \rightarrow H$ be a functor. We define a new groupoid K as follows. The objects are equivalence classes of triples (a, b, h) , where $a \in G$ and $b \in H$ and $h: u(a) \rightarrow b$; the equivalence relation identifies (a, b, h) with (a', b', h') if and only if $b = b'$ and there is a map $g: a \rightarrow a'$ such that $h = h' \circ u(g)$. The maps from $[a, b, h]$ to $[a', b', h']$ are the maps $k: b \rightarrow b'$ in H such that there exists a map $j: a \rightarrow a'$ in G with $k \circ h = h' \circ u(j)$. Equivalently, k gives a map $[a, b, h] \rightarrow [a', b', h']$ if and only if $[a', b', h'] = [a, b', kh]$.

There is an evident functor $r: G \rightarrow K$ defined by $r(a) = [a, u(a), 1_{u(a)}]$. Given $c = [a, b, h] \in K$ we find that h can be thought of as a map $r(a) \rightarrow b$ in K , so r is essentially surjective. Moreover, we find that $K(r(a), r(a'))$ is just the image of $G(a, a')$ in $H(u(a), u(a'))$, and thus that r is full. Thus we have $r \in E$.

There is also an evident functor $p: K \rightarrow H$ defined by $p[a, b, h] = b$. It is easy to check that p is a covering and $u = pr$ as required. In terms of Proposition 6.22, the covering p corresponds to the functor $X: H \rightarrow \text{Sets}$ defined by $X_b = \pi_0(u \downarrow b)$.

(c): Suppose we have a square as in the statement of the proposition. We first define a map $v: \text{obj}(K) \rightarrow \text{obj}(G)$ as follows. Suppose that $c \in \text{obj}(K)$. As r is essentially surjective, we can choose $d \in \text{obj}(L)$ and $k: r(d) \rightarrow c$ in K . We apply w to get $w(k): pu(d) = wr(d) \rightarrow w(c)$. As p is a covering, there is a unique pair (a, g) with $a \in \text{obj}(G)$ and $g: u(c) \rightarrow a$ such that $p(a) = w(c)$ and $p(g) = w(k)$. We would like to define $v(c) = a$. To check that this is well-defined, consider another $d' \in \text{obj}(L)$ and another $k': r(d') \rightarrow c$, giving rise to a unique pair (a', g') . As r is full there exists $l: d' \rightarrow d$ such that $k^{-1}k' = r(l)$ and one checks that $(a, g \circ u(l))$ has the defining property of (a', g') . Thus $a = a'$ as required. This means that we have a well-defined map $v: \text{obj}(K) \rightarrow \text{obj}(G)$ with $pv = w$ on objects. It is easy to check that $vr = u$ on objects as well.

Now suppose we have a map $m: c \rightarrow c'$ in K . We can choose maps $k: r(d) \rightarrow c$ and $k': r(d') \rightarrow c'$ with $d, d' \in L$. By the definition of v on objects we have maps $g: u(d) \rightarrow v(c)$ and $g': u(d') \rightarrow v(c')$ such that $p(g) = w(k)$ and $p(g') = w(k')$. As r is full we can choose $n: d \rightarrow d'$ such that $r(n) = (k')^{-1}mk$. One then checks that the map $g'' = g' \circ u(n) \circ g^{-1}: v(c) \rightarrow v(c')$ has $p(g'') = w(m)$. As p is faithful, there is at most one map $v(c) \rightarrow v(c')$ with this property, so g'' is independent of the choices made. We can thus define v on morphisms by $v(m) = g''$, so that $pv = w$. Using the faithfulness of p , we check easily that v is a functor and that $vr = u$. Thus v fills in the diagram as required.

Finally suppose that $v': K \rightarrow G$ is another functor making the diagram commute. We must check that $v' = v$. As p is faithful it is enough to check this on objects. Given $c \in \text{obj}(K)$ we choose $k: r(d) \rightarrow c$ as before and write $a = v'(c)$ and $g = v'(k): u(d) \rightarrow a$. We then have $p(g) = pv'(k) = w(k)$, so the definition of v gives $v(c) = a = v'(c)$ as required. \square

Corollary 6.24. (i) The factorisation in (b) is unique up to isomorphism.

(ii) $C \cap E$ is precisely the class of isomorphisms in \mathcal{G} .

(iii) C and E are closed under compositions and retracts.

(iv) C is closed under pullbacks, and E is closed under pushouts.

Proof. See [3, Exercises 5.5]. Of course, in our case, many of these things are immediate from the definitions. \square

Proposition 6.25. A functor $u: G \rightarrow H$ is a quasi-covering if and only if it is faithful.

Proof. As equivalences and coverings are faithful, we see that quasi-coverings are faithful.

For the converse, let $u: G \rightarrow H$ be faithful. We can factor u as pr where p is a covering and r is full and essentially surjective, as in Proposition 6.23. As $u = pr$ is faithful we see that r is faithful and thus an equivalence, as required. \square

Lemma 6.26. *Suppose we have functors $L \xrightarrow{p} K \xrightarrow{v} H$ such that p is a covering and v is an equivalence. Then there is a pullback square as follows, in which q is a covering and u is an equivalence.*

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ p \downarrow & & \downarrow q \\ K & \xrightarrow{v} & H. \end{array}$$

Proof. We can factor vp as qu with q a covering and u full and essentially surjective. Now consider the following diagram:

$$\begin{array}{ccccc} L & \xrightarrow{w} & M & \xrightarrow{\tilde{v}} & G \\ & \searrow p & \downarrow \tilde{q} & & \downarrow q \\ & & K & \xrightarrow{v} & H. \end{array}$$

The square is defined to be the pullback of v and q , and w is the unique functor such that $\tilde{q}w = p$ and $\tilde{v}w = u$. By Proposition 6.8 we know that \tilde{v} is an equivalence. It will thus be enough to show that w is an isomorphism in \mathcal{G} .

As $u = \tilde{v}w$ is full and essentially surjective, and \tilde{v} is an equivalence, we see that w is full and essentially surjective.

We next show that w is surjective on objects. Suppose $e \in M$, and put $a = \tilde{v}(e) \in G$ and $c = \tilde{q}(e) \in K$ so that $q(a) = v(c) = b$ say. As u is essentially surjective, we can choose $d \in L$ and $g: u(d) \rightarrow a$ in G . Thus $q(g): vp(d) = qu(d) \rightarrow q(a) = v(c)$ in H . As v is an equivalence, there is a unique $k: p(d) \rightarrow c$ such that $v(k) = q(g)$. As p is a covering, there is a unique $d' \in L$ and $l: d \rightarrow d'$ such that $p(d') = c$ and $p(l) = k$. Thus $u(l)g^{-1}: a \rightarrow u(d')$ satisfies $q(u(l)g^{-1}) = vp(l)v(k)^{-1} = 1$. As q is a covering, it reflects identity maps, so $u(l)g^{-1} = 1_a$ and $a = u(d')$. Thus u is surjective on objects, as claimed.

Now consider the following diagram:

$$\begin{array}{ccc} L & \xrightarrow{1} & L \\ w \downarrow & \nearrow z & \downarrow p \\ M & \xrightarrow{\tilde{q}} & K. \end{array}$$

It follows from Proposition 6.23 that there is a unique map z making everything commute. In particular, we have $zw = 1$. It follows that $(wz)w = w$ and w is full and surjective on objects so $wz = 1$. Thus w is an isomorphism, as required. \square

Lemma 6.27. *Suppose that we have a homotopy cartesian square as follows, in which p is a quasicovering.*

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ q \downarrow & & \downarrow p \\ K & \xrightarrow{v} & H \end{array}$$

Then there is a diagram as follows, in which p' and q' are coverings, r and s are equivalences, the bottom square is cartesian, the top square commutes up to homotopy, and $p = p'r$, $q = q's$.

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ s \downarrow & & \downarrow r \\ L' & \xrightarrow{u'} & G' \\ q' \downarrow & & \downarrow p' \\ K & \xrightarrow{v} & H \end{array}$$

Proof. Using Lemma 6.15, it is not hard to reduce to the case in which L is the standard homotopy pullback of p and v . As p is a quasicovering we can factor it as $p = p'r$ where p' is a covering and r is an equivalence. We can then define L' , u' and q' so that the bottom square is cartesian, which implies that q' is a covering.

Our next task is to define the functor s . An object $d \in L$ is a triple $(a, c, h: p(a) \rightarrow v(c))$. As p' is a covering and $h: p'r(a) \rightarrow v(c)$, we see that there is a unique morphism $g': r(a) \rightarrow a'$ in G' such that $p'(a') = v(c)$ and $p'(g') = h$. Thus $(a', c) \in L'$ and we can define s on objects by $s(d) = (a', c)$. Note that $u(d) = a$ and $u's(d) = u'(a', c) = a'$ so we can define

$$\alpha_d := g': ru(d) \rightarrow u's(d).$$

Next, consider a morphism $(g, k): d_0 \rightarrow d_1$ in L , where $d_i = (a_i, c_i, h_i: p(a_i) \rightarrow v(c_i))$ for $i = 0, 1$. We define a'_i and g'_i as above, and define

$$\bar{g} = g'_1 \circ r(g) \circ (g'_0)^{-1}: a'_0 \rightarrow a'_1,$$

so that the following diagram commutes.

$$\begin{array}{ccccc} p'r(a_0) & \xrightarrow{p'r(g)} & & p'r(a_1) & \\ & \searrow h_0 & & \swarrow h_1 & \\ p'(g'_0) & & v(c_0) \xrightarrow{v(k)} v(c_1) & & p'(g'_1) \\ & \swarrow \parallel & & \searrow \parallel & \\ p'(a'_0) & \xrightarrow{p'(\bar{g})} & & p'(a'_1) & \end{array}$$

We now define s on morphisms by putting $s(g, k) = (\bar{g}, k)$. It is easy to check that this makes s into a functor, and that $\alpha: ru \rightarrow u's$ is a natural map. Thus, the top square in our diagram commutes up to homotopy. It is also clear that $q's = q$.

Thus, all that is left is to check that s is an equivalence. Let d_0 and d_1 be as above, and suppose given $k: c_0 \rightarrow c_1$. As p' is faithful and r is an equivalence, we see that there is at most one map g making the upper trapezium of the above diagram commute, and at most one map \bar{g} making the lower trapezium commute. Moreover, g exists if and only if \bar{g} does, and they determine each other by $\bar{g} = g'_1 \circ r(g) \circ (g'_0)^{-1}$ and $g = r^{-1}((g'_1)^{-1} \circ g \circ g'_0)$. Note also that $L(d_0, d_1)$ is the set of pairs (g, k) such that the top trapezium commutes, and $L'(s(d_0), s(d_1))$ is the set of pairs (\bar{g}, k) making the bottom trapezium commute. It follows easily that s is full and faithful.

Now consider an object $d' = (a', c) \in L'$, so $v(c) = p'(a')$. As r is essentially surjective we can choose an object $a \in G$ and a map $g': r(a) \rightarrow a'$ in G' . We thus have an object $d = (a, c, p'(g'): p(a) = p'r(a) \rightarrow v(c))$ of L . Clearly $s(d) = d'$ so s is surjective on objects, and thus an equivalence as claimed. \square

6.5. Cartesian closure. Let G and H be groupoids, and let $[G, H]$ denote the category of functors from G to H . It is easy to see that this is a groupoid and that functors $K \rightarrow [G, H]$ biject naturally with functors $K \times G \rightarrow H$. It follows that \mathcal{G} is cartesian-closed. One can also check that this descends to $\bar{\mathcal{G}}$ in the obvious way.

We next want to check how this works out in the equivalent category $\bar{\mathcal{B}}$.

Lemma 6.28. *Suppose that X and Y are objects of \mathcal{B} and they have the homotopy type of CW complexes. Then the space $C(X, Y)$ of maps from X to Y also lies in \mathcal{B} .*

Proof. By well-known results of Milnor, the space $C(S^k, Y)$ also has the homotopy type of a CW complex. Evaluation at the basepoint of S^2 gives a surjective Hurewicz fibration $\epsilon_Y : C(S^2, Y) \rightarrow Y$ whose fibres have the form $\epsilon_Y^{-1}\{y\} = \Omega^2(Y, y)$. As $Y \in \mathcal{B}$ we know that these fibres are acyclic and so ϵ_Y is a weak equivalence, and thus a homotopy equivalence. By a standard result (the dual of [4, Corollary II.1.12], for example) we deduce that ϵ_Y is fibre-homotopy equivalent to 1_Y . One can also see that for any $f : X \rightarrow Y$ we have

$$\Omega^2(C(X, Y), f) \simeq \{g : X \rightarrow C(S^2, Y) \mid \epsilon_Y \circ g = f\}$$

and our fibre-homotopy equivalence shows that this is contractible. The result follows. \square

Proposition 6.29. *If $G, H \in \mathcal{G}$ then $B[G, H] \simeq C(BG, BH)$ in $\overline{\mathcal{B}}$.*

Proof. It follows from the lemma that $C(BG, BH) \in \mathcal{B}$. Recall that $B : \overline{\mathcal{G}} \rightarrow \overline{\mathcal{B}}$ is an equivalence. Thus, for any K we have

$$\begin{aligned} \overline{\mathcal{B}}(BK, C(BG, BH)) &= \overline{\mathcal{B}}(BK \times BG, BH) \\ &= \overline{\mathcal{B}}(B(K \times G), BH) \\ &= \overline{\mathcal{G}}(K \times G, H) \\ &= \overline{\mathcal{G}}(K, [G, H]) \\ &= \overline{\mathcal{B}}(BK, B[G, H]). \end{aligned}$$

As B is an equivalence we conclude that $\overline{\mathcal{B}}(X, C(BG, BH)) = \overline{\mathcal{B}}(X, B[G, H])$ for all $X \in \overline{\mathcal{B}}$, and it follows by Yoneda's lemma that $C(BG, BH) \simeq B[G, H]$ as claimed. \square

7. TRANSFERS

Let $u : G \rightarrow H$ be a covering with finite fibres. Then $Bu : BG \rightarrow BH$ is a finite covering map of spaces, so it is well-known how to define an associated transfer map $Tu : \Sigma^\infty BH_+ \rightarrow \Sigma^\infty BG_+$ of spectra. This construction is contravariantly functorial and it converts disjoint unions to wedges and cartesian products to smash products. If p is a homeomorphism then $Tp = \Sigma^\infty p_+^{-1}$. The well-known Mackey property of transfers says that if we have a pullback square as shown on the left, in which p is a covering, then q is also a covering and the square on the right commutes up to homotopy.

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ q \downarrow & & \downarrow p \\ K & \xrightarrow{v} & H \end{array} \qquad \begin{array}{ccc} \Sigma^\infty BL_+ & \xrightarrow{\Sigma^\infty Bu_+} & \Sigma^\infty BG_+ \\ Tq \uparrow & & \uparrow Tp \\ \Sigma^\infty BK_+ & \xrightarrow{\Sigma^\infty Bv_+} & \Sigma^\infty BH_+ \end{array}$$

It will be convenient to extend this to quasicoverings rather than just coverings. If $u : G \rightarrow H$ is a quasicovering then we can factor u as $G \xrightarrow{v} K \xrightarrow{p} H$ where v is an equivalence and p is a covering. We then define

$$Tu = (\Sigma^\infty Bv_+^{-1}) \circ Tp.$$

To see that this is well-defined, note (using Proposition 6.23) that any other such factorisation has the form $G \xrightarrow{wv} L \xrightarrow{pw^{-1}} H$ for some isomorphism $w : K \rightarrow L$. Using this and the equation $Tw = \Sigma^\infty Bw_+^{-1}$ we see that $(\Sigma^\infty B(wv)_+^{-1}) \circ T(pw^{-1})$ as required.

Now suppose we have quasicoverings $G \xrightarrow{u} H \xrightarrow{v} K$; we want to check that $T(vu) = T(u)T(v)$. It is easy to reduce to the case where we have functors $L \xrightarrow{p} K \xrightarrow{v} H$ such that p is a covering and v is an equivalence; we need to check that $T(p)T(v) = T(p)v^{-1} = T(vp)$, where we allow ourselves

to write v instead of $\Sigma^\infty Bv_+$. Lemma 6.26 gives us a pullback diagram as follows, in which q is a covering and u is an equivalence.

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ p \downarrow & & \downarrow q \\ K & \xrightarrow{v} & H. \end{array}$$

By definition we have $T(vp) = u^{-1}T(q)$. The Mackey property gives $uT(p) = T(q)v$ so

$$T(p)T(v) = T(p)v^{-1} = u^{-1}T(q) = T(vp)$$

as required.

It is easy to check that in this greater generality we still have $T(p \amalg q) = T(p) \vee T(q)$ and $T(p \times q) = T(p) \wedge T(q)$. We also have an extended Mackey property: if the square on the left is homotopy-cartesian and p is a quasicovering then q is also a quasicovering and the right hand square commutes up to homotopy (this follows easily from Lemma 6.27).

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ q \downarrow & & \downarrow p \\ K & \xrightarrow{v} & H \end{array} \qquad \begin{array}{ccc} \Sigma^\infty BL_+ & \xrightarrow{\Sigma^\infty Bu_+} & \Sigma^\infty BG_+ \\ Tq \uparrow & & \uparrow Tp \\ \Sigma^\infty BK_+ & \xrightarrow{\Sigma^\infty Bv_+} & \Sigma^\infty BH_+ \end{array}$$

8. THE $K(n)$ -LOCAL CATEGORY

Fix a prime p and an integer $n > 0$, and let $K = K(n)$ denote the associated Morava K -theory spectrum. Let \mathcal{K} denote the category of spectra that are local with respect to $K(n)$ in the sense of Bousfield [6, 26], and let \mathcal{D} be the full subcategory of strongly dualisable objects in \mathcal{K} . These categories are studied in detail in [17]. We write $X \wedge Y$ for the $K(n)$ -localised smash product, which makes \mathcal{K} into a symmetric monoidal category. The unit object is $S := L_K \Sigma^\infty S^0$.

Definition 8.1. We define a functor $L: \overline{\mathcal{G}} \rightarrow \mathcal{D}$ by $LG := L_K \Sigma^\infty BG_+$. (We know from [17, Corollary 8.7] that LG is always dualisable, so this lands in \mathcal{D} as indicated.) It is clear that $L(G \times H) = LG \wedge LH$ and $L(G \amalg H) = LG \vee LH$.

Definition 8.2. Let 1 denote the terminal groupoid (with one object and one morphism), and write ϵ for the unique functor $G \rightarrow 1$. Let $\delta: G \rightarrow G \times G$ be the diagonal functor. Define

$$b_G = (LG \wedge LG \xrightarrow{L_K T \delta} LG \xrightarrow{L \epsilon} S).$$

It is not hard to see that $b_{G \times H} = b_G \wedge b_H$ and $b_{G \amalg H} = b_G \oplus b_H$.

The following result is the key to the whole paper.

Proposition 8.3. *The map b_G is an inner product on LG .*

Proof. We can easily reduce to the case where G is a group rather than a groupoid. It was observed in the proof of [17, Corollary 8.7] that a certain map $c_G: LG \rightarrow DLG$ (arising from the Greenlees-May theory of generalised Tate spectra) is an isomorphism. It is thus enough to show that $c_G = b_G^\#$.

We will need some notation. Firstly, we will need to consider various unlocalised spectra, so in this proof only we write S for the ordinary, unlocalised sphere spectrum, and \widehat{S} for $L_{K(n)}S$. Similarly, we write $X \wedge Y$ for the unlocalised smash product and $X \widehat{\wedge} Y = L_{K(n)}(X \wedge Y)$. Next, we will work partially in the equivariant categories of G -spectra and G^2 -spectra, indexed over complete universes [21]. We write S' and S'' for the corresponding 0-sphere objects. Also, we can regard \widehat{S} as a naive G -spectrum with trivial action and then extend the universe to obtain a genuine G -spectrum, which we denote by \widehat{S}' . We define a genuine G^2 -spectrum \widehat{S}'' in the analogous way.

We next recall the definition of c_G . It is obtained from a certain map $d_G: \widehat{S} \wedge BG_+ \rightarrow F(BG_+, \widehat{S}) \simeq DLG$ by observing that DLG is $K(n)$ -local and that any map from $\widehat{S} \wedge BG_+$ to a $K(n)$ -local spectrum factors uniquely through $L_{K(n)}(\widehat{S} \wedge BG_+) \simeq LG$. It will be enough to check that d_G is adjoint to $1 \wedge b'_G: \widehat{S} \wedge BG_+^2 \rightarrow \widehat{S}$, where b'_G is the composite

$$\Sigma^\infty BG_+^2 \xrightarrow{\psi^!} \Sigma^\infty BG_+ \xrightarrow{B\epsilon} S.$$

We thus need to show that two elements of the group $[\widehat{S} \wedge BG_+^2, \widehat{S}]$ are equal. Theorem II.4.5 of [21] (applied to $G^2/G^2 \simeq 1$) gives a natural isomorphism

$$[\widehat{S} \wedge BG_+^2, \widehat{S}] \simeq [\widehat{S}'' \wedge EG_+^2, \widehat{S}'']_{G^2}.$$

Let $\zeta: EG_+ \rightarrow S^0$ and $\xi: G^2/\Delta_+ \rightarrow S^0$ be the collapse maps. Desuspending Construction II.5.1 of [21] gives a pretransfer map $t: S'' \rightarrow \Sigma^\infty G^2/\Delta_+$ of genuine G^2 -spectra. By smashing this with EG_+^2 and passing to orbits we get the transfer map $\psi^!: \Sigma^\infty BG_+^2 \rightarrow \Sigma^\infty B\Delta_+$. Using this and the proof of [21, Theorem II.4.5] we find that $1 \wedge b'_G$ corresponds to the composite

$$\widehat{S}'' \wedge EG_+^2 \xrightarrow{1 \wedge \zeta \wedge \zeta \wedge t} \widehat{S}'' \wedge G^2/\Delta_+ \xrightarrow{1 \wedge \xi} \widehat{S}''$$

in $[\widehat{S}'' \wedge EG_+^2, \widehat{S}'']_{G^2}$.

We now return to the definition of d_G . We have a map

$$\widehat{S}' \wedge EG_+ \xrightarrow{1 \wedge \zeta} \widehat{S}' \xrightarrow{\zeta^*} F(EG_+, \widehat{S}')$$

of G -spectra. We next apply the fixed point functor, noting that $F(EG_+, \widehat{S}')^G = F(BG_+, \widehat{S})$ and that [21, Theorem II.7.1] gives an equivalence $\tilde{\tau}: \widehat{S} \wedge BG_+ \rightarrow (\widehat{S}' \wedge EG_+)^G$. The resulting map $\widehat{S} \wedge BG_+ \rightarrow F(BG_+, \widehat{S})$ is d_G (see [12, Section 5]). To understand this better, we need to follow through the construction of $\tilde{\tau}$. We use the notation of [21, Section II.7], noting that in our case we have $N = G$. The construction uses the group $\Gamma = G \times_c N$, the semidirect product of G with N using the action by conjugation. There are two natural maps $\epsilon, \theta: \Gamma \rightarrow G$ given by $\epsilon(g, n) = g$ and $\theta(g, n) = gn$. In our case we find that the resulting map $\Gamma \rightarrow G^2$ is an isomorphism, so we can replace Γ by G^2 everywhere. The subgroup Π becomes $1 \times G$, the standard embedded copy $G \times_c 1$ of G becomes Δ , and the maps ϵ and θ become the projections $\pi_0, \pi_1: G^2 \rightarrow G$. The relevant spectrum D is $\widehat{S} \wedge EG_+$, so $i_*\theta^*D = \widehat{S}' \wedge \pi_1^*EG_+$ and $j_*i_*\theta^*D = \widehat{S}'' \wedge \pi_1^*EG_+$. The map $\tilde{\tau}$ is obtained from

$$1 \wedge t: \widehat{S}'' \wedge \pi_1^*EG_+ \rightarrow \widehat{S}'' \wedge \pi_1^*EG_+ \wedge G^2/\Delta_+$$

by shrinking the universe, passing to orbits and adjoining as described in [21, Construction II.7.5]. It follows that d_G is obtained from the composite

$$\widehat{S}'' \wedge \pi_1^*EG_+ \xrightarrow{1 \wedge t} \widehat{S}'' \wedge \pi_1^*EG_+ \wedge G^2/\Delta_+ \xrightarrow{1 \wedge \zeta \wedge 1 \wedge \zeta^*} \widehat{S}'' \wedge G^2/\Delta_+ \wedge F(\pi_0^*EG_+, S)$$

by a similar procedure. We can identify EG_+^2 with $\pi_0^*EG_+ \wedge \pi_1^*EG_+$, and we find that the adjoint of d_G is obtained by applying another similar procedure to the map

$$\widehat{S}'' \wedge EG_+^2 \xrightarrow{1 \wedge \zeta \wedge \zeta \wedge t} \widehat{S}'' \wedge G^2/\Delta_+.$$

This procedure amounts to just composing with $\xi: G^2/\Delta_+ \rightarrow S^0$ and using our isomorphism $[\widehat{S} \wedge BG_+^2, \widehat{S}] \simeq [\widehat{S}'' \wedge EG_+^2, \widehat{S}'']_{G^2}$. It follows that the adjoint of d_G is b'_G , as required. \square

Definition 8.4. For any functor $u: G \rightarrow H$ we put $Ru = (Lu)^t: LH \rightarrow LG$.

Proposition 8.5. *If $u: G \rightarrow H$ is faithful then $Ru = L_K T u$.*

Proof. We first claim that the following square is homotopy-cartesian:

$$\begin{array}{ccc} G & \xrightarrow{u} & H \\ (1, u) \downarrow & & \downarrow \psi_H \\ G \times H & \xrightarrow{u \times 1} & H \times H. \end{array}$$

To see this, let K be the homotopy pullback of the functors ψ_H and $u \times 1$. The square is clearly cartesian, which means that G embeds as a full subcategory of K ; we need only check that the inclusion is essentially surjective. The objects of K are 5-tuples (a, b, c, h, k) where $a \in G$ and $b, c \in H$ and $h: u(a) \rightarrow c$ and $k: b \rightarrow c$. The maps from (a, b, c, h, k) to (a', b', c', h', k') are triples (r, s, t) making the following diagram commute:

$$\begin{array}{ccccc} u(a) & \xrightarrow{h} & b & \xleftarrow{k} & c \\ u(r) \downarrow & & \downarrow s & & \downarrow t \\ u(a') & \xrightarrow{h'} & b' & \xleftarrow{k'} & c' \end{array}$$

The canonical functor $v: G \rightarrow K$ is given by $v(a) = (a, u(a), u(a), 1, 1)$. We define $w: K \rightarrow G$ by $w(a, b, c, h, k) = a$. Then $wv = 1$, and we have a natural map $vw(a, b, c, h, k) \rightarrow (a, b, c, h, k)$ given by $(1, k, k^{-1}h)$. This proves that v is an equivalence, and if we compose it with the projections $K \rightarrow G \times H$ and $K \rightarrow H$ we get the functors $(1, u)$ and u . This proves that our original square is homotopy-cartesian, so the Mackey property tells us that

$$T(\psi_H) \circ (Bu \times 1) = Bu \circ T(1, u): \Sigma^\infty B(G \times H)_+ \rightarrow \Sigma^\infty BH_+.$$

We now use the fact that $(1, u) = (1 \times u) \circ \psi_G$ and compose with the projection $\Sigma^\infty BH_+ \rightarrow S^0$ to get

$$b_H \circ (Bu \wedge 1) = \epsilon_H \circ Bu \circ (T\psi_G) \circ (1 \wedge Tu).$$

We next note that $\epsilon_H \circ Bu = \epsilon_G$ and $K(n)$ -localise to conclude that $b_H \circ (Lu \wedge 1) = b_G \circ (1 \wedge L_K Tu)$, as required. \square

We can thus think of the maps Ru as generalised transfers. It turns out that we also have a generalised Mackey property.

Proposition 8.6. *If we have a homotopy-cartesian square as shown on the left, then the diagram on the right commutes.*

$$\begin{array}{ccc} M & \xrightarrow{u} & G \\ \downarrow t & & \downarrow s \\ K & \xrightarrow{v} & H \end{array} \qquad \begin{array}{ccc} LM & \xrightarrow{Lu} & LG \\ \uparrow Rt & & \uparrow Rs \\ LK & \xrightarrow{Lv} & LH \end{array}$$

Proof. We may assume that the square is actually a pullback square of fibrations (see Remark 6.12), so in particular it commutes on the nose. As b_H is a perfect pairing, it suffices to check that $b_G \circ (1 \wedge (Rs)(Lv)) = b_G \circ (1 \wedge (Lu)(Rt))$. By transposition, this is equivalent to

$$b_H \circ (Ls \wedge Lv) = b_M \circ (Ru \wedge Rt): LG \wedge LK \rightarrow S.$$

To verify this, we consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{vt=su} & H \\ (u, t) \downarrow & & \downarrow \psi_H \\ G \times K & \xrightarrow{s \times v} & H \times H. \end{array}$$

We claim that this is homotopy-cartesian. It is clearly cartesian, so it suffices (as in the previous proof) to show that the obvious functor from M to the homotopy pullback is essentially surjective. Suppose we are given an object of the homotopy pullback, in other words a 5-tuple $d = (a, b, c, k, l)$ where $a \in G$, $b \in H$, $c \in K$ and $s(a) \xrightarrow{k} b \xleftarrow{l} v(c)$. As s is a fibration we can choose $a' \in G$ and

$g: a \rightarrow a'$ such that $s(a') = v(c)$ and $s(g) = l^{-1}k$. Thus $d' := (a', s(a') = v(c), c, 1, 1)$ is an object of M and the following diagram gives an isomorphism $d \rightarrow d'$:

$$\begin{array}{ccccc} s(a) & \xrightarrow{k} & b & \xleftarrow{l} & v(c) \\ s(g) \downarrow & & \downarrow l^{-1} & & \downarrow v(1) \\ s(a') & \xlongequal{\quad} & v(c) & \xlongequal{\quad} & v(c) \end{array}$$

This shows that our square is homotopy-cartesian. The vertical functors are faithful and thus are quasicoverings, so the Mackey property tells us that

$$(R\psi_H) \circ (Ls \wedge Lv) = L(su) \circ R(u, t): LG \wedge LK \rightarrow LH.$$

We next compose with the map $L\epsilon_H: LH \rightarrow S$, noting that $\epsilon_H s u = \epsilon_M: M \rightarrow 1$ and that $(u, t) = (u \times t)\psi_M: M \rightarrow G \times K$. We conclude that $b_H \circ (Ls \wedge Lv) = b_M \circ (Ru \wedge Rt)$, as required. \square

Theorem 8.7. *For any finite groupoid G , the maps $(R\epsilon, R\psi, L\epsilon, L\psi)$ make LG into a Frobenius object.*

Proof. This is formally identical to the proof of Proposition 5.1; we need only check that the following square is homotopy-cartesian, and that is easy.

$$\begin{array}{ccc} G & \xrightarrow{\psi} & G \times G \\ \psi \downarrow & & \downarrow 1 \times \psi \\ G \times G & \xrightarrow{\psi \times 1} & G \times G \times G. \end{array}$$

\square

Definition 8.8. Given a finite groupoid G , define $\Lambda G = [\mathbb{Z}, G]$; Proposition 6.29 tells us that $B\Lambda G$ is homotopy-equivalent to the free loop space on BG . The objects of ΛG are pairs (a, u) where $u \in G(a, a)$, and the maps from (a, u) to (b, v) are maps $g: a \rightarrow b$ such that $v = gug^{-1}$. It is thus easy to see that $\pi(a, u) = a$ gives a functor $\Lambda G \rightarrow G$, and that this is actually a covering. If G is a group then ΛG is equivalent the disjoint union of the groups $Z_G(g)$ as g runs over the conjugacy classes in G , so the free loop space on BG is $\coprod BZ_G(g)$; this is actually well-known, and a more elementary account appears in [5, Section 2.12], for example.

Remark 8.9. It is important to distinguish between $[\mathbb{Z}, G]$ and $[\mathbb{Z}_p, G]$; see Section 11 for more discussion of this.

We can now identify the maps $\theta = \epsilon\mu\psi: LG \rightarrow S$ and $\alpha = \mu\psi\eta: S \rightarrow LG$ discussed in Proposition 4.3.

Proposition 8.10. *We have $\theta = (L\epsilon_{\Lambda G})(R\pi)$ and $\alpha = (L\pi)(R\epsilon_{\Lambda G})$.*

Proof. The key point is that the following square is homotopy-cartesian:

$$\begin{array}{ccc} \Lambda G & \xrightarrow{\pi} & G \\ \pi \downarrow & & \downarrow \psi \\ G & \xrightarrow{\psi} & G^2. \end{array}$$

To see this, let H be the homotopy pullback of ψ and ψ . The objects of H are tuples (a, b, u, v) where $a, b \in G$ and $u, v: a \rightarrow b$. The morphisms from (a, b, u, v) to (a', b', u', v') are pairs (g, h) where $g: a \rightarrow a'$ and $h: b \rightarrow b'$ and $hu = u'g$ and $hv = v'g$. We can define a functor $\phi: \Lambda G \rightarrow H$ by $(a, u) \mapsto (a, a, u, 1)$ and a functor ξ in the opposite direction by $(a, b, u, v) \mapsto (a, v^{-1}u)$. We find that these are equivalences and that either projection $H \rightarrow G$ composed with ϕ is just π ; it follows

that the square is homotopy-cartesian, as claimed. We conclude that $\mu\psi = (R\psi)(L\psi) = (L\pi)(R\pi)$. We also know from Proposition 4.3 that $\theta = \epsilon\mu\psi = (L\epsilon)(R\psi)(L\psi)$ and $\alpha = \mu\psi\eta = (R\psi)(L\psi)(R\epsilon)$. Everything now follows from the evident fact that $\epsilon_G\pi = \epsilon_{\Lambda G} : \Lambda G \rightarrow 1$. \square

We conclude this section by discussing the case of a finite abelian group A , considered as a groupoid with one object. There is then a unique functor $\zeta : 1 \rightarrow A$, and also a division homomorphism $\nu : A \times A \rightarrow A$ given by $\nu(g, h) = gh^{-1}$.

Proposition 8.11. *We have $b = (R\zeta)(L\nu) : LA \wedge LA \rightarrow S$. We also have $\alpha = |A|\eta$ and $\theta = |A|\epsilon$.*

Proof. We have a commutative diagram as follows, which is easily seen to be both cartesian and homotopy-cartesian:

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & 1 \\ \psi \downarrow & & \downarrow \zeta \\ A \times A & \xrightarrow{\nu} & A. \end{array}$$

The vertical functors are faithful and thus are quasicoverings. The Mackey property now tells us that $b = (L\epsilon)(R\psi) = (R\zeta)(L\nu)$ as claimed.

Next, consider the groupoid $\Lambda A = [\mathbb{Z}, A]$. It is easy to see that this is just a disjoint union of $|A|$ copies of A , and that the functor $\pi : \Lambda A \rightarrow A$ just sends each copy isomorphically to A . The remaining claims now follow easily from Proposition 8.10. \square

9. INNER PRODUCTS IN COHOMOLOGY

We next study E^*BG for suitable cohomology theories E .

If p is an odd prime, let E be a p -local commutative ring spectrum such that

- (a) E^0 is a complete local Noetherian ring
- (b) $E^1 = 0$
- (c) E^2 contains a unit
- (d) The associated formal group over $\text{spec}(E^0/\mathfrak{m})$ has height n .

In the language of [13, Section 2], these are precisely the $K(n)$ -local admissible ring spectra. In the case $p = 2$ we would like to allow E to be a two-periodic version of $K(n)$, but this is not commutative. We therefore relax the requirement that E be commutative and assume instead that there is a derivation $Q : E \rightarrow \Sigma E$ and an element $v \in \pi_2 E$ such that $2v = 0$ and

$$ab - ba = vQ(a)Q(b),$$

so that E is quasicommutative in the sense of [29, Definition 8.1.1]. This is of course satisfied if E is commutative, with $Q = 0$ and $v = 0$. Other examples, including the two-periodic version of $K(n)$, can most easily be produced by the methods of [27], which also contains detailed references to previous work in this direction.

We consider LG as a Frobenius object just as in the previous section. As usual we use the maps $S \xleftarrow{L\epsilon} LG \xrightarrow{L\psi} LG \wedge LG$ to make $E^0LG = E^0BG$ into a ring and $E^*LG = E^*BG$ into a graded ring. We also use $(R\epsilon) : S \rightarrow LG$ to give a map $\epsilon := (R\epsilon)^* : E^0BG \rightarrow E^0$, which in turn gives a bilinear form $b(x, y) = \epsilon(xy)$ on E^0BG .

Remark 9.1. If G is a group then the inclusion of the trivial group gives a map $\zeta : 1 \rightarrow G$ and thus an augmentation map $(L\zeta)^* : E^0BG \rightarrow E^0$. In other contexts this is often denoted by ϵ , but it is *not* the same as the map ϵ defined above.

We say that G is E -good if E^0LG is free of finite rank over E^0 and $E^1LG = 0$. If so then we have a Künneth isomorphism $E^0(LG \wedge LG) = E^0LG \otimes_{E^0} E^0LG$. Using this and Theorem 8.7 we find that the above maps make E^0LG into a Frobenius object in the compact-closed category of finitely generated free modules over E^0 . In particular, we deduce that our bilinear form is an inner product. A functor $u : G \rightarrow H$ gives a ring map $u^* : E^0BH \rightarrow E^0BG$ induced by

$Lu: LG \rightarrow LH$, and also a map $u_l := (Ru)^*: E^0BG \rightarrow E^0BH$ that is adjoint to u^* . If u is the inclusion of a subgroup in a group then u_l is the corresponding transfer map (by Proposition 8.5). The adjointness of u_l and u^* is thus a version of Frobenius reciprocity.

As usual we have a trace map $\theta: E^0BG \rightarrow E^0$ (which can be computed using only the ring structure) and an element $\alpha = \mu\psi(1) \in E^0BG$. Proposition 8.10 tells us how to compute α in terms of ordinary transfers, and Proposition 4.3 tells us that $\epsilon(x\alpha) = \theta(x)$. We will see later that α becomes invertible in $\mathbb{Q} \otimes E^0BG$, so the previous equation characterises ϵ up to torsion terms.

Now let A be a finite Abelian group. It is known that such groups are E -good for all E ; see [13, Proposition 2.9] for a proof in the present generality, although the basic idea of the proof is much older [20, 14]. We know from Proposition 8.11 that $\alpha = |A|$ in this context so that $|A|\epsilon(x) = \theta(x)$. We next give another formula for ϵ that is more useful when $p = 0$ in E^0 . It is easy to see that $\epsilon_{A \times B} = \epsilon_A \otimes \epsilon_B$, and if $|A|$ is coprime to p then $E^0BA = E^0$ with $\epsilon_A = |A|.1: E^0 \rightarrow E^0$. It is thus enough to treat the case where $A = C_{p^m}$ for some $m > 0$.

Choose a complex orientation $x \in \tilde{E}^0\mathbb{C}P^\infty$, or equivalently a coordinate on the associated formal group \mathbb{G} . This gives a formal group law F with associated p^m -series $[p^m](x)$, and we have

$$E^0B(C_{p^m}) = \mathcal{O}_{\mathbb{G}(m)} = E^0[[x]]/[p^m](x).$$

There is a unique invariant differential form ω on G that agrees with dx at zero: if E^0 is torsion-free this is most easily expressed as $\omega = d(\log_F(x)) = \log'_F(x)dx$. Given a function $f \in \mathcal{O}_{\mathbb{G}} = E^0[[x]]$ we get a meromorphic differential form $f\omega/[p^m](x) \in \mathcal{M}\Omega_{\mathbb{G}}^1$, and the residue of this form clearly only depends on f modulo $[p^m](x)$. (See [29, Sections 5.3 and 5.4] for an exposition of meromorphic forms and their residues.)

Proposition 9.2. *The canonical Frobenius form on $E^0BC_{p^m}$ is given by $\epsilon(f) = \text{res}(f\omega/[p^m](x))$.*

Proof. For any E -good group G , we can define

$$c := \psi\eta(1) = \text{tr}_{\Delta}^{G^2}(1) \in E^0(BG^2) = E^0BG \otimes_{E^0} E^0BG.$$

We see from Scholium 3.12 that $\epsilon: E^0BG \rightarrow E^0$ is the unique map such that $(\epsilon \otimes 1)(c) = 1 \in E^0BG$.

Now take $G = C_{p^m}$, so $E^0BG = E^0[[x]]/[p^m](x)$ and $E^0BG^2 = E^0[[x, y]]/([p^m](x), [p^m](y))$. Write $\langle p^m \rangle(t) = [p^m](t)/t \in E^0[[t]]$; we know from [25, Section 4] that $\text{tr}_1^G(1) = \langle p^m \rangle(x)$ (a simpler proof appears in [32]). Put $z = x -_F y$; it follows from Proposition 8.11 that $c = \langle p^m \rangle(z)$.

Now consider the form

$$\gamma = c\omega/[p^m](x) \in \frac{E^0[[y]]}{[p^m](y)} \otimes_{E^0} \mathcal{M}\Omega_{\mathbb{G}}^1,$$

so that $\text{res}(\gamma) \in E^0[[y]]/[p^m](y)$. In view of the above, it will suffice to check that $\text{res}(\gamma) = 1$.

For this, we note that $[p^m](y) = 0$ so $zc = [p^m](z) = [p^m](x)$ so $z\gamma = \omega$ so $\gamma = \omega/z$. Now, $\omega = g(x)dx$ for some power series g with $g(0) = 1$ and this differential is invariant under translation, which implies that $\omega = g(z)dz$ also. Thus $\text{res}(\gamma) = \text{res}(g(z)dz/z) = g(0) = 1$ as required. \square

Corollary 9.3. *Let E be the usual two-periodic version of $K(n)$ (with $n > 1$), and let x be the usual p -typical orientation. Then the Frobenius form on the ring*

$$E^0BC_{p^m} = E^0[[x]]/x^{p^{nm}} = E^0\{x^k \mid 0 \leq k < p^{nm}\}$$

is given by $\epsilon(x^k) = 0$ for $k < p^{nm-1}$ and $t(x^{p^{nm}-1}) = 1$. In the case $n = 1$ we have $\epsilon(x^{p^m-p^j}) = 1$ for $0 \leq j \leq m$ and $\epsilon(x^k) = 0$ for all other k .

Proof. For the integral two-periodic version of $K(n)$ we have $\log_F(x) = \sum_{k \geq 0} x^{p^{nk}}/p^k$. When $n > 1$ it follows easily that $\omega = \log'_F(x)dx = dx \pmod{p}$. We also have $[p^m](x) = x^{p^{mn}}$ so $\epsilon(x^k) = \text{res}(x^{k-p^{mn}}dx)$ and the claim follows easily. In the case $n = 1$ we have $\omega = \sum_{k \geq 0} x^{p^k-1}dx$ and the stated formula follows in the same way. \square

10. CHARACTER THEORY

Let G be a finite groupoid. Write $C(G) := \mathbb{Q}\{\pi_0 G\}$ for the rational vector space freely generated by the set of isomorphism classes of objects of G . Given $a \in G$ we write $[a]$ for the corresponding basis element in $C(G)$. We define a bilinear form on $C(G)$ by

$$([a], [b]) := |G(a, b)|.$$

It is convenient to write $G(a) := G(a, a)$ and to introduce the elements $[a]' := [a]/|G(a)|$, so that $([a], [a]') = 1$. We also write $C(G)^* = \text{Hom}_{\mathbb{Q}}(C(G), \mathbb{Q}) = F(\pi_0 G, \mathbb{Q})$ for the dual of $C(G)$. Given a functor $u: G \rightarrow H$ we define $Lu: C(G) \rightarrow C(H)$ by $(Lu)[a] = [u(a)]$, and we let $Ru: C(H) \rightarrow C(G)$ be the adjoint of this, so that

$$(Ru)[b]' = \sum_{[a] \mid u(a) \simeq b} [a]'.$$

The sum here is indexed by isomorphism classes of objects $a \in G$ such that $u(a)$ is isomorphic to b in H .

We next show that these constructions have the expected Mackey property.

Proposition 10.1. *If we have a homotopy-cartesian square as shown on the left, then the diagram on the right commutes.*

$$\begin{array}{ccc} M & \xrightarrow{u} & G \\ \downarrow t & & \downarrow s \\ K & \xrightarrow{v} & H \end{array} \qquad \begin{array}{ccc} C(M) & \xrightarrow{Lu} & C(G) \\ \uparrow Rt & & \uparrow Rs \\ C(K) & \xrightarrow{Lv} & C(H) \end{array}$$

Proof. We may assume that the square is actually a pullback square of fibrations (see Remark 6.12), so in particular it commutes on the nose. Fix $c \in K$, so $(Lv)[c] = [vc] = |H(vc)||[vc]'$. We need to check that $(Rs)(Lv)[c] = (Lu)(Rt)[c]$. Because s is a fibration, any isomorphism class in G that maps to $[vc]$ in H has a representative $a \in G$ such that $sa = vc$. Using this, we find that

$$(Rs)(Lv)[c] = \sum_{[a] \mid sa=vc} |H(vc)||G(a)|^{-1}[a].$$

We also know that t is a fibration, so every isomorphism class in M that maps to c contains a representative d with $t(d) = c$, in other words d has the form (a, c) for some $a \in G$ with $sa = vc$. It follows that

$$(Lu)(Rt)[c] = \sum_{[a, c] \mid sa=vc} |M((a, c))|^{-1}|K(c)||[a].$$

Fix $a \in G$ with $sa = vc$. The coefficient of $[a]$ in $(Rs)(Lv)[c]$ is then $|H(vc)||G(a)|^{-1}$. For $(Lu)(Rt)[c]$ we need to be more careful, because there will typically be objects $a' \in G$ with $[a'] = [a] \in \pi_0 G$ but $[a', c] \neq [a, c] \in \pi_0 M$. Put

$$X = \{a' \in G \mid a' \simeq a \text{ and } sa' = vc\} / \sim,$$

where $a' \sim a''$ iff there exist $g: a' \rightarrow a''$ and $k: c \rightarrow c$ such that $sg = vk: vc \rightarrow vc$. It is easy to see that $a' \sim a''$ iff $(a', c) \simeq (a'', c)$ in M , and it follows that the coefficient of $[a]$ in $(Lu)(Rt)[c]$ is

$$\lambda := \sum_{[a'] \in X} |M((a', c))|^{-1}|K(c)|.$$

To analyse this further, we introduce the set

$$Y = \{(a', g') \mid a' \in G, sa' = vc \text{ and } g': a' \rightarrow a\} / \sim,$$

where $(a', g') \sim (a'', g'')$ iff $sg' = sg'': vc \rightarrow vc$. Using the fact that s is a fibration, one checks that the map $[a', g'] \mapsto sg'$ gives a bijection $Y \simeq H(vc)$, so that $|Y| = |H(vc)|$. On the other hand, there is an evident projection $\pi: Y \rightarrow X$ sending $[a', g']$ to $[a']$. If $\pi[a'', g''] = [a']$ then we

can choose $f: a' \rightarrow a''$ and $k: c \rightarrow c$ such that $sf = vk$, and then observe that there is a unique $g: a' \rightarrow a'$ such that $f = (g'')^{-1}g'g$. One checks that the coset $(g, k) \cdot M((a', c))$ depends only on the equivalence class $[a'', g'']$ and that this construction gives a bijection

$$\pi^{-1}\{[a']\} \simeq (G(a') \times K(c))/M((a', c)).$$

Note also that $|G(a')| = |G(a)|$ because $a' \simeq a$. It follows that

$$\begin{aligned} \lambda &= \sum_{[a', g'] \in Y} |G(a') \times K(c)/M((a', c))|^{-1} |M((a', c))|^{-1} |K(c)| \\ &= \sum_{[a', g'] \in Y} |G(a)|^{-1} \\ &= |H(vc)| |G(a)|^{-1}. \end{aligned}$$

This is the same as the coefficient of $[a]$ in $(Rs)(Lv)[c]$, as required. \square

If we let $1 \xleftarrow{\epsilon} G \xrightarrow{\psi} G^2$ be the obvious functors, then it follows easily that the maps $(L\epsilon)^*$, $(L\psi)^*$, $(R\epsilon)^*$, $(R\psi)^*$ make $C(G)^*$ into a Frobenius algebra over \mathbb{Q} . The Frobenius form is just

$$\epsilon(f) = (f, 1) = \sum_{[a] \in \pi_0 G} f(a)/|G(a)|.$$

Next, let Λ denote the group $(\mathbb{Q}_p/\mathbb{Z}_p)^n$, whose dual is $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathbb{Z}_p^n$. We regard Λ^* as a groupoid with one object in the usual way. We also consider $\Lambda(m) = \ker(\Lambda \xrightarrow{p^m} \Lambda)$, so $\Lambda(m)^* = \Lambda^*/p^m$.

Note that if G is a finite group and $\lambda: \Lambda^* \rightarrow G$ then the image of λ is Abelian and p -local (because it is a quotient of Λ^*) and finite (because it is a subgroup of G). It follows that $\lambda(p^m \Lambda^*) = 1$ for large m , so that λ is automatically continuous. It follows that $\text{Hom}(\Lambda^*, G)$ bijects with the set of n -tuples of commuting elements of G of p -power order. More generally, if G is a finite groupoid then a functor $\lambda: \Lambda^* \rightarrow G$ consists of an object $a \in G$ together with an n -tuple of commuting p -elements of the group $G(a, a)$.

The generalised character theory of Hopkins, Kuhn and Ravenel [14, 15] can be repackaged slightly to relate $\mathbb{Q} \otimes E^0 BG$ to $C([\Lambda^*, G])^*$ for admissible cohomology theories E , as we now explain. Given such a spectrum we need to define an associated ring D' . In the special case of Morava E -theory, this was defined in [14]; the details necessary for the general case are given in [28]. Associated to E we have a formal group \mathbb{G} over $\text{spf}(E^0)$ and thus a level-structure scheme $\text{Level}(\Lambda(m), \mathbb{G})$ with coordinate ring D_m say. These form a directed system in an obvious way and we define $D' = \mathbb{Q} \otimes \lim_{\rightarrow m} D_m$. (This was called L in [14] but we have renamed it to avoid clashes of notation.) If \mathbb{G} is the universal deformation of its restriction to the special fibre (as is the case with Morava E -theory) then D' is the integral domain obtained from $\mathbb{Q} \otimes E^0$ by adjoining a full set of roots of $[p^m](x)$ for all x . For any E one can show that D' is a free module of countable rank over $\mathbb{Q} \otimes E^0$.

As mentioned earlier, the following theorem is merely a repackaging of results of Hopkins, Kuhn and Ravenel [14].

Theorem 10.2. *For any admissible ring spectrum E , there is a natural isomorphism of Frobenius algebras over L*

$$D' \otimes_{E^0} E^0 BG = D' \otimes_{\mathbb{Q}} C([\Lambda^*, G])^*.$$

Moreover, this respects the constructions $u \mapsto (Lu)^*$ and $u \mapsto (Ru)^*$ for functors between groupoids.

Proof. We first construct a map

$$\tau: D' \otimes_{E^0} E^0 BG \rightarrow D' \otimes_{\mathbb{Q}} C([\Lambda^*, G])^*$$

of D' -algebras. By juggling various adjunctions we see that it suffices to construct, for each functor $\lambda: \Lambda^* \rightarrow G$, a map $\tau_\lambda: E^0 BG \rightarrow D'$ of E^0 -algebras, such that $\tau_\lambda = \tau_\mu$ when λ is isomorphic to μ . We know from our previous remarks that λ must factor through $\Lambda(m)^* = \Lambda^*/p^m$ for some

m . We thus get a map $E^0BG \rightarrow E^0B\Lambda(m)^*$, and we know from [13, Proposition 2.9] that $E^0B\Lambda(m)^* = \mathcal{O}_{\text{Hom}(\Lambda(m), \mathbb{G})}$, and D_m is a quotient of this ring, so we get the required map τ_λ as the composite

$$E^0BG \xrightarrow{B\lambda^*} E^0B\Lambda(m)^* \rightarrow D_m \rightarrow D'.$$

One checks easily that this is independent of the choice of m . Isomorphic functors λ, μ give homotopic maps $B\Lambda(m)^* \rightarrow BG$ and thus $\tau_\lambda = \tau_\mu$ as required. The resulting map τ is easily seen to be natural for functors of groupoids and to convert equivalences to isomorphisms. Both source and target of τ convert disjoint unions to products. Any finite groupoid is equivalent to a finite disjoint union of finite groups, so it suffices to check that τ is an isomorphism when G is a group. This is just [14, Theorem B].

To say that this isomorphism respects the construction $u \mapsto (Lu)^*$ is just to say that τ is a natural map, which is clear. We also need to check that for any functor $u: G \rightarrow H$, the following diagram commutes:

$$\begin{array}{ccc} D' \otimes_{E^0} E^0BG & \xrightarrow{\tau_G} & D' \otimes_{\mathbb{Q}} C([\Lambda^*, G])^* \\ (Ru)^* \downarrow & & \downarrow (Ru)^* \\ D' \otimes_{E^0} E^0BH & \xrightarrow{\tau_H} & D' \otimes_{\mathbb{Q}} C([\Lambda^*, H])^* \end{array}$$

We first make this more explicit. The functor u induces $Ru: LH \rightarrow LG$. By applying $E^0(-)$ and noting that $E^0LK = E^0BK$ we get a map $(Ru)^*: E^0BG \rightarrow E^0BH$. After tensoring with D' we obtain the left hand vertical map in the above diagram. On the other hand, u also induces a functor $u_*: [\Lambda^*, G] \rightarrow [\Lambda^*, H]$ and thus a map $R(u_*): C[\Lambda^*, H] \rightarrow C[\Lambda^*, G]$. By dualising and tensoring with D' we obtain the right hand vertical map.

We first prove that the diagram commutes when u is a quasi-covering. This reduces easily to the case where H is a group and G is connected. It is not hard to see that in this case u is equivalent to the inclusion of a subgroup $G \leq H$ and $Ru: LH \rightarrow LG$ is just the $K(n)$ -localisation of the transfer map $\Sigma^\infty BH_+ \rightarrow \Sigma^\infty BG_+$. It follows from [14, Proposition 3.6.1] that $\tau_\lambda((Ru)^*x) = \sum \tau_{\lambda^h}(x)$, where the sum runs over cosets hG such that $\lambda^h := h^{-1}\lambda h$ maps Λ^* into G . The right hand side can be rewritten as $|G|^{-1} \sum_h \tau_{\lambda^h}(x)$, where the sum now runs over elements rather than conjugacy classes. Fix a homomorphism $\mu: \Lambda^* \rightarrow G$ that becomes conjugate to λ in H . Then the number of h 's for which $\lambda^h = \mu$ is the order of the group $Z_H(\lambda) = \{h \in H \mid \lambda^h = \lambda\}$, so

$$\tau_\lambda((Ru)^*x) = |G|^{-1} \sum_{\mu} |Z_H(\lambda)| \tau_\mu(x).$$

If we want to index this sum using conjugacy classes of μ 's rather than the μ 's themselves, we need an extra factor of $|G|/|Z_G(\mu)|$, the number of conjugates of μ . This gives

$$\tau_\lambda((Ru)^*x) = \sum_{[\mu]} |Z_G(\mu)|^{-1} |Z_H(\lambda)| \tau_\mu(x).$$

On the other hand, $Z_G(\mu)$ is just the automorphism group of μ in the category $[\Lambda^*, G]$, so the map $R(u_*): C[\Lambda^*, H] \rightarrow C[\Lambda^*, G]$ is given by

$$R(u_*)[\lambda]/|Z_H(\lambda)| = \sum_{[\mu] \mid u\mu \simeq \lambda} [\mu]/|Z_G(\mu)|.$$

The claim follows easily by comparing these formulae.

We have an inner product on $D' \otimes_{E^0} E^0BG$ obtained from the inner product b_G on LG , and an inner product on $D' \otimes_{\mathbb{Q}} C([\Lambda^*, G])^*$ obtained from the standard inner product on $C(K)^*$ for any K . By taking u to be the diagonal functor $G \rightarrow G \times G$ in the above discussion, we see that our isomorphism τ converts the former inner product to the latter one. Thus τ is compatible with taking adjoints and with the construction $u \mapsto (Lu)^*$, so it is compatible with the construction $u \mapsto (Ru)^*$ as well. \square

We next reformulate Theorem 10.2 in the spirit of [13, Theorem 3.7].

Definition 10.3. Given a finite groupoid G , we define a new groupoid AG as follows. The objects are pairs (a, A) , where $a \in G$ and A is a finite Abelian p -subgroup of $G(a)$. The morphisms from (a, A) to (b, B) are maps $g: a \rightarrow b$ in G such that $B = gAg^{-1}$. For any finite Abelian p -group A we can define a ring $D'_A = \mathbb{Q} \otimes \mathcal{O}_{\text{Level}(A^*, G)}$ as in [28, Proposition 22]. There is an evident way to make the assignment $(a, A) \mapsto D'_A$ into a functor $AG^{\text{op}} \rightarrow \text{Rings}$, and we define $TG = \lim_{\leftarrow (a, A) \in AG} D'_A$.

If we write $\alpha'_{(a, A)} = |Z_{G(a)}(A)|$ then $\alpha' \in TG$.

Theorem 10.4. *There is a natural isomorphism $\mathbb{Q} \otimes E^0BG = TG$, and this is a finitely generated free module over E^0 . The element $\alpha = \mu\psi(1) \in E^0BG$ becomes $\alpha' \in TG$, so the resulting Frobenius form on TG is just $\epsilon(x) = \theta(x/\alpha')$, where θ is the trace form.*

Proof. The isomorphism $\mathbb{Q} \otimes E^0BG = TG$ can be proved either by reducing to the case of a group and quoting [13, Theorem 3.7], or by taking the fixed-points of both sides in Theorem 10.2 under the action of $\text{Aut}(\Lambda)$. From the latter point of view, the term in TG indexed by (a, A) corresponds to the terms in $C[\Lambda^*, G]^*$ coming from homomorphisms $\Lambda^* \rightarrow G(a)$ with image A , so α' becomes the function $\pi_0[\Lambda^*, G] \rightarrow L$ that sends $[\lambda]$ to $|\Lambda^*, G](\lambda)|$. Proposition 8.10 identifies this with α , as required. \square

11. WARNINGS

We started this paper by considering the representation ring $R(G)$, but unfortunately the analogy between our rings E^0LG and $R(G)$ fails in a number of respects, even in the height one case. In this section we point out some possible pitfalls.

Let E be the p -adic completion of the complex K -theory spectrum, so E is an admissible ring spectrum of height one. Then E^0BG is the completion of $R(G)$ at $I + (p)$, where I is the augmentation ideal. The ring $R(G)$ is a free Abelian group of rank equal to the number of conjugacy classes, generated by the irreducible characters. These are orthonormal, so the inner product on $R(G)$ is equivalent to the standard diagonal, positive definite inner product on \mathbb{Z}^h . It also follows that $R(G)$ is a permutation module for the outer automorphism group of G .

The ring E^0BG is a free module over \mathbb{Z}_p of rank equal to the number of conjugacy elements of elements of p -power order. The canonical map $R(G) \rightarrow E^0BG$ does not preserve inner products. There is no canonical set of generators for E^0BG , so there is no reason for it to be a permutation module for $\text{Out}(G)$. In fact, Igor Kriz has constructed examples of extensions $G \rightarrow G' \rightarrow C_p$ where G is good but $H^1(C_p; E^0BG) \neq 0$ and one can deduce that E^0BG is not a permutation module in this case [19]. There is also no reason to expect that E^0BG has an orthonormal basis.

A related set of issues involves the comparison between the free loop space of BG (which is $B[\mathbb{Z}, G]$) and the space of maps from the p -adically completed circle to BG (which is $B[\mathbb{Z}_p, G]$). The former enters into Proposition 8.10, and the latter into Theorem 10.2. The two spaces are not even p -adically equivalent: if G is a group and T is a set of representatives for the conjugacy classes of elements whose order is not a power of p then $B[\mathbb{Z}, G] \simeq B[\mathbb{Z}_p, G] \amalg \coprod_{g \in T} BZ_G(g)$, and each term in the coproduct contributes at least a factor of E^0 in $E^0B[\mathbb{Z}, G]$ even if $Z_G(g)$ is a p' -group. Note that [14, 5.3.10] is slightly inaccurate in this regard; the proof given there really shows that $\chi_n BG = \chi_{n-1}(B[\mathbb{Z}_p, G])$, rather than $\chi_{n-1}(B[\mathbb{Z}, G])$.

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