The Lusternik-Schnirelmann Category of $S^1_q \times S^1_q$ and $S^1_q \times S^1_q$

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Abstract

We answer a question of Rudyak by showing that $\text{cat}(S^1_q \times S^1) = \text{cat}(S^1_q \times \tilde{S}^1_q) = 3$. The second formula shows that $X = \tilde{S}^1_q$ is an example of a space for which $\text{cat}(X \times X) < 2 \text{cat}(X)$. These calculations are derived from a general formula for the category weight of elements of $H^*(BG; \pi)$ that is of independent interest.

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The Lusternik-Schnirelmann category of a map $f : X \to Y$ between CW complexes is the least integer $n$ for which $X$ has a cover $\{A_0, \ldots, A_n\}$ by subcomplexes with the property that $f|A_i \simeq *$ for each $i$ [1]. The category of a space $X$, $\text{cat}(X)$, is the category of the identity map $\text{id}_X$. A classical result due to Bassi [4, Thm. 9] shows that, for two CW complexes $X$ and $Y$, $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$. Until recently, the only known cases in which this formula was not an equality involved torsion phenomena in homology. In [5], Ganea asked whether $\text{cat}(X \times S^k) = \text{cat}(X) + 1$ for every space $X$. That this is true has come to be known as Ganea’s conjecture. The rational version of the conjecture, which can be stated $\text{cat}(X_q \times S^k_q) = \text{cat}(X_q) + 1$ when $X_q \times S^k$ is simply connected, was proved for simply connected spaces by Jessup and Hess in [10, 7] (we denote by $X_q$ the rationalization of the space $X$). More recently, counterexamples to the conjecture have been constructed by Iwase [8], by Stanley [17], and others. In the wake of these counterexamples, there
has been some interest in the related problem of finding space \( X \) such that \( \text{cat}(X \times X) < 2 \text{cat}(X) \) [12].

In this note we use category weight techniques to compute the Lusternik-Schnirelmann category of the spaces \( S^1_Q \times S^1 \) and \( S^1_Q \times S^1 \), thereby answering a question asked by Y. Rudyak [15]. The calculation shows that \( S^1_Q \) satisfies the Ganea conjecture, and also has the property that \( \text{cat}(X \times X) < 2 \text{cat}(X) \).

Recall that the **category weight** of a map \( f : X \to Y \) is the least integer \( n \) such that \( f \circ g \simeq * \) for every \( g : Z \to X \) with \( \text{cat}(g) < n \). We write \( \text{wgt}(f) = n \) if the category weight of \( f \) is \( n \), and \( \text{wgt}(f) = \infty \) if there is no such integer. The category weight of a cohomology class is obtained from the isomorphism \( H^n(X; \pi) \cong [X, K(\pi, n)] \). Clearly \( \text{wgt}(f) \) is a lower bound for \( \text{cat}(X) \), provided \( f \) is nontrivial. This concept has also been called strict category weight (see Rudyak [13, 14, 11]) or essential category weight [18, 19].

We will make use of a well-known alternative characterization of the category of a map \( f : X \to Y \). For each space \( Y \) and \( n \geq 0 \) there is a fibration \( p_n : G_n(Y) \to Y \) with the property that \( \text{cat}(f) \leq n \) if and only if \( f \) has a lift into \( G_n(Y) \) [6]; these are known as the Ganea fibrations. If \( X \simeq BG \) for some discrete group \( G \) then the map \( p_n \) is homotopically equivalent to the inclusion \( B_n G \to BG \simeq X \) [20], see also [18]. The following result is a basic property of category weight [13, 14, 18, 19].

**Theorem 1** Let \( f : X \to Y \), and let \( p_n : G_n(X) \to X \) be the \( n \)th Ganea fibration. Then \( \text{wgt}(f) \geq n \) if and only if \( f \circ p_n \simeq * \).

This follows immediately from the definitions. The following corollary, which can be found in [18, Cor. 79] and implicitly in the proof of [11, Thm. 4.1], has proved useful in differential geometry; in fact, it is an essential ingredient in the proof of the Arnold conjecture for certain special symplectic manifolds [11, 16].

**Corollary 2** If \( G \) is a discrete group and \( u \in H^n(BG; \pi) \), then \( \text{wgt}(u) = n \).

**Proof** Since \( G \) is a discrete group, \( G_n(BG) \simeq B_n G \) which is \( (n-1) \)-dimensional, and hence has trivial cohomology in dimensions \( \geq n \). \( \Box \)

This corollary generalizes to finite-dimensional groups: if \( G \) is a \( d \)-dimensional topological group and \( u \in H^n(BG; \pi) \), then \( \text{wgt}(u) \geq \frac{d + 1}{d + 1} \). This shows, for example, that every nonzero class \( u \in H^n(\mathbb{H}P^m; \pi) \) has \( \text{wgt}(u) = n \), even without a cup product structure.

Corollary 2 immediately implies a result of Eilenberg and Ganea: if \( G \) is a discrete group then the category of \( BG \) is bounded below by its cohomological
dimension [2]. Since $S^1_Q$ is a $K(Q, 1)$ and $H^2(S^1_Q; Z) \cong \text{Ext}(Q, Z)$ which is isomorphic to $\mathbb{R}$ as rational vector spaces, it follows that $\text{cat}(S^1_Q) \geq 2$; since $S^1_Q$ is homotopy equivalent to a 2-dimensional space we see that $\text{cat}(S^1_Q) = 2$. Therefore, Ganea’s conjecture predicts

$$\text{cat}(S^1_Q \times S^1) = \text{cat}(S^1_Q) + 1 = 3.$$  

Since $H^*(S^1_Q)$ is torsion free, the general product formula which motivated Ganea’s conjecture predicts that $\text{cat}(S^1_Q \times S^1_Q)$ is equal to 4.

With these preliminaries in place, we state and prove our main theorem.

**Theorem 3** \( \text{cat}(S^1_Q \times S^1) = \text{cat}(S^1_Q \times S^1_Q) = 3. \)

**Proof** Notice first that it follows from Bassi’s formula that $\text{cat}(S^1_Q \times S^1) \leq 3$, and since $S^1_S \times S^1_Q \cong (S^1 \times S^1)_Q$ can be constructed as a 3-dimensional CW complex, $\text{cat}(S^1_S \times S^1_Q) \leq 3$ as well. Now we have from the universal coefficient formula

$$H^3(S^1_Q \times S^1; Z) \cong \text{Ext}(Q, Z) \otimes Z \cong \text{Ext}(Q, Z) \cong \mathbb{R} \neq 0$$

and

$$H^3(S^1_S \times S^1_Q; Z) \cong \text{Ext}(Q, Z) \cong \mathbb{R} \neq 0.$$

Since $S^1_Q \times S^1 \cong K(Q \times Z, 1)$ and $S^1_S \times S^1_Q \cong K(Q \times Q, 1)$, the result follows immediately from Corollary 2. \[\square\]

**Remark** In fact, a similar argument shows that $\text{cat}((S^1_Q)^n) = n + 1 < 2n$.

**References**


