HIGHER ORDER PHANTOM MAPS

JEFFREY STROM

Abstract For each ordinal number α, we define phantom maps of order α. We construct universal phantom maps out of X with order α, and show that under easily verifiable conditions, every one of these universal phantom maps is essential.

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1. The Main Result

A map $f : X \longrightarrow Y$ is a phantom map if it can be factored

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \nearrow{\scriptstyle f_n} & \\ & X \cup CX_n & \end{array}$$

for each n. The Gray index of a phantom map, $G(f)$, is the least integer n such that the map f_n cannot be chosen to be a phantom map. The Gray index was introduced by Brayton Gray in his thesis [1]. There he claimed that every essential phantom map has finite index. However, there is a flaw in his argument; recently McGibbon and Strom [4] have shown that there are essential phantom maps with infinite Gray index.

This might seem like the end of the story, but that is far from the case. In this paper, we show that the phantom maps with infinite Gray index are just the second step in a filtration on $[X, Y]$ whose length can take on any ordinal number value.

To begin, we define the order of a phantom map.

Definition *A phantom map of order at least* 0 *is simply a map. Let* α *be an ordinal number. If* $\alpha = \beta + 1$*, then* $f : X \longrightarrow Y$ *is a phantom of order at least* α *if, for each* n*, it can be factored*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \nearrow{\scriptstyle f_n} & \\ & X \cup CX_n & \end{array}$$

in such a way that f_n is a phantom map of order at least β. If α is a limit ordinal, then a phantom map is of order at least α if it has order at least β for every $\beta < \alpha$.

Thus, a phantom map of order 1 is simply an ordinary phantom map with finite Gray index, and a phantom map has order at least 2 if and only if it is a phantom map with infinite Gray index. There is a generalized Gray index that is defined for all phantom maps of order α; it is the least integer n such that f does not factor through a phantom of order α out of $X \cup CX_n$.

In the spirit of Gray and McGibbon [2], we inductively construct maps

$$\Theta^\alpha : X \longrightarrow W^\alpha(X)$$

out of X which are universal with the property of having order at least α. Define Θ^0 to be the identity map. Now let α be an ordinal number. If $\alpha = \beta + 1$, then Θ^α is the natural map from X to the colimit of the diagram of maps

$$X \longrightarrow X \cup CX_n \xrightarrow{\Theta^\beta} W^\beta(X \cup CX_n).$$

If α is a limit ordinal, then Θ^α is the colimit of the diagram of the $\Theta^\beta : X \longrightarrow W^\beta(X)$ for $\beta < \alpha$. It is not hard to see that Θ^1 is the universal phantom map of Gray and McGibbon [2].

Proposition 1. *The map $\Theta^\alpha : X \longrightarrow W^\alpha(X)$ has order at least α. Furthermore, any essential phantom $f : X \longrightarrow Y$ with order at least α must factor through Θ^α.*

Proof
The order of Θ^α is at least α by construction. The factorization property follows by induction and the definition of the colimit. ∎

It follows that a space X is the domain for an essential phantom with order at least α if and only if the map $\Theta^\alpha : X \longrightarrow W^\alpha(X)$ is essential. Now we can state our main result.

Theorem 2. *Let X be of finite type, and suppose there is a class $u \in H^*(\Sigma X; G)$ and cohomology operations $\theta_1, \theta_2, \ldots$ of positive degree such that $\theta_n \theta_{n-1} \cdots \theta_1(u) \neq 0$ for each n. Then the map*

$$\Theta^\alpha : X \longrightarrow W^\alpha(X)$$

is essential for each ordinal number α.

It is a consequence of Miller's solution of the Sullivan conjecture [5] that a Postnikov section of a nilpotent space of finite type satisfies the condition in the theorem. But the condition is easily verified for many familiar spaces X without such heavy machinery.

Example 3. *For example, take* $X = \mathbb{CP}^\infty$, $u \neq 0 \in H^3(\Sigma X; \mathbb{Z}/2)$ *and* $\theta_n = Sq^{2^n}$.

There is a dual version of the order of a phantom map, and a dual construction of a universal map into Y with order at least α. Unfortunately, the dual to Theorem 2 does not seem to be true; at least, our proof does not dualize. The problem lies in dualizing Lemma B below–there is no reasonable concept dual to compactness.

We end this section with a nagging question.

Question Does every phantom map have an order? In other words, is it possible that an essential phantom map has order at least α for every ordinal number α? If we could form colimits indexed on the class of all ordinal numbers, then our proof would show that the answer is yes!

2. Proof of the Main Result

Our proof depends on two Lemmas. The proofs of the Lemmas are postponed to the end of the section.

Lemma A *Let* $f : X \longrightarrow W$ *be a map, and construct the cofiber sequence*

$$X \xrightarrow{f} W \xrightarrow{\partial} W/X \xrightarrow{j} \Sigma X.$$

Then $f \simeq *$ *if and only if there is a map* $s : \Sigma X \longrightarrow C$ *such that* $j \circ s \simeq 1_{\Sigma X}$.

Lemma B *Let* X *be of finite type, and let* $u \in H^n(X; G)$. *Suppose*

$$f : X \longrightarrow \bigvee K_\alpha$$

and $u = f^*(v)$ *for some* $v \in H^n(\bigvee K_\alpha; G)$. *Then there is a finite subwedge* $\bigvee_{\alpha \in I} K_\alpha$ *such that* $(j \circ f)^*(i^*(v)) = u$; *here* $i : \bigvee_{\alpha \in I} K_\alpha \longrightarrow \bigvee K_\alpha$ *is the inclusion and* j *is its left inverse.*

In view of Lemma A, it suffices to show that if X is as in the Theorem, then u cannot be in the image of a map induced by

$$\Sigma X \longrightarrow W^\alpha(X)/X.$$

To prove this, we actually prove the following stronger statement.

Proposition 2.1 *Let* X *and* u *be as in Theorem 2. Then* u *is not in the image of any map induced by a map*

$$\Sigma X \longrightarrow \bigvee_\kappa W^{\alpha(\kappa)}(Z_\kappa)/Z_\kappa$$

for any set of spaces $\{Z_\kappa\}$.

Proof
We work by transfinite induction. We know that the Proposition is true for all Z and $\alpha \leq 0$, because $W^0(Z)/Z = *$ for all Z. Assume that the result is known for all Z and all $\alpha < A$. By way of contradiction, assume that u is in the image of a map induced by

$$\Sigma X \longrightarrow \bigvee_{\kappa} W^{\alpha(\kappa)}(Z_\kappa)/Z_\kappa$$

in which each $\alpha(\kappa) \leq A$ (from now on, we will suppress the dependence of α on κ). We will show that this implies that u is in the image of a map induced by a map to a wedge of this kind with each $\alpha < A$, which will contradict the inductive hypothesis.

The case in which A is a limit ordinal is easiest, for in this case we have

$$W^A(Z)/Z \simeq \bigvee_{\alpha < A} W^\alpha(Z)/Z,$$

so the wedge with $\alpha \leq A$ is actually a wedge in which each $\alpha < A$.

Now assume that $A = B + 1$, and that u is in the image of a map induced by

$$f : \Sigma X \longrightarrow \overbrace{\bigvee_{\kappa} W^\alpha(Z_\kappa)/Z_\kappa}^{V} \vee \bigvee_{\lambda} W^A(Z_\lambda)/Z_\lambda$$

in which each $\alpha < A$. We'll write $V = \bigvee_\kappa W^\alpha(Z_\kappa)/Z_\kappa$. From our description of $W^A(Z)$ as a colimit we have

$$W^A(Z)/Z \simeq \bigvee_{n} \frac{W^B(Z \cup CZ_n)}{Z},$$

so this is really a map

$$f : \Sigma X \longrightarrow V \vee \bigvee_{\lambda} \left(\bigvee_{n} \frac{W^B(Z_\lambda \cup C(Z_\lambda)_n)}{Z_\lambda} \right).$$

Applying Lemma B, we see that we can restrict to a finite subwedge– that is, we may assume that λ and n run over finite index sets.

Now we need to understand the spaces $W^B(Z_\lambda \cup C(Z_\lambda)_n)/Z_\lambda$. For any space Z, there is a diagram of cofiber sequences

$$\begin{array}{ccccc}
Z & \longrightarrow & Z & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
Z \cup CZ_n & \longrightarrow & W^B(Z \cup CZ_n) & \longrightarrow & \bigvee_n \frac{W^B(Z\cup CZ_n)}{Z\cup CZ_n} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma Z_n & \longrightarrow & \frac{W^B(Z\cup CZ_n)}{Z} & \longrightarrow & \bigvee_n \frac{W^B(Z\cup CZ_n)}{Z\cup CZ_n}
\end{array}$$

From the cofiber sequence on the bottom row of this diagram, we see that we have a cofiber sequence

$$\bigvee_{\lambda,n} \Sigma(Z_\lambda)_n \longrightarrow V \vee \bigvee_{\lambda,n} \frac{W^B(Z_\lambda \cup C(Z_\lambda)_n)}{Z_\lambda} \longrightarrow V \vee \bigvee_{\lambda,n} \frac{W^B(Z_\lambda \cup C(Z_\lambda)_n)}{Z_\lambda \cup C(Z_\lambda)_n}.$$

Since the first term is a finite dimensional space, we can choose N larger than the dimension of $\bigvee_{\lambda,n} \Sigma(Z_\lambda)_n$, (a finite wedge of finite complexes) and let $v = \theta_N \cdots \theta_1(u)$. Then v must be in the image of the map induced by the composite

$$X \xrightarrow{f} V \vee \bigvee_{\lambda,n} \frac{W^B(Z_\lambda \cup C(Z_\lambda)_n)}{Z_\lambda} \longrightarrow V \vee \bigvee_{\lambda,n} \frac{W^B(Z_\lambda \cup C(Z_\lambda)_n)}{Z_\lambda \cup C(Z_\lambda)_n}.$$

Since v also satisfies the conditions of the Theorem, this contradicts the inductive hypothesis. ∎

Proof of Lemma A

The proof amounts to examining the following diagram.

$$\begin{array}{ccccccc}
X & \longrightarrow & * & \longrightarrow & \Sigma X & \xrightarrow{=} & \Sigma X \\
{\scriptstyle =}\downarrow & \textcircled{A} & \downarrow & \textcircled{B} & \downarrow{\scriptstyle s} & \textcircled{C} & \downarrow{\scriptstyle =} \\
X & \xrightarrow{f} & W & \xrightarrow{\partial} & W/X & \xrightarrow{j} & \Sigma X \\
 & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow \\
 & & W & \xrightarrow[=]{} & W & \longrightarrow & *
\end{array}$$

If $f \simeq *$, then A commutes; this makes both B and C commute, and s is the desired section. If C commutes, then so does B. There results a square of cofibrations, and so $f \simeq 1_W \circ f \simeq r \circ \partial \circ f \simeq *$. ∎

Proof of Lemma B

Since X is of finite type X_m is compact, so $f(X_m)$ must be contained in a finite subwedge $\bigvee_{\alpha \in I} K_\alpha$. It is a simple matter to verify the claims of the Lemma for this subwedge. ∎

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Dartmouth College, Hanover, NH 03755
jeffrey.strom@dartmouth.edu