Q-SUBALGEBRAS, MILNOR BASIS, AND COHOMOLOGY OF EILENBERG–MAC LANE SPACES

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Abstract. We describe mod $p$ cohomology rings of Eilenberg–Mac Lane spaces in terms of the Milnor basis rather than in terms of admissible monomials of the Steenrod algebra. We give a formula of excess for the Milnor basis elements, correcting Kraines’ formula in odd prime case. Using the Milnor basis description, we study and characterize certain polynomial subalgebras generated by elements obtained by applying maximum number of Milnor primitives on mod $p$ fundamental classes of Eilenberg–Mac Lane spaces. A simple and interesting unstable pattern emerges. These subalgebras are exact images of the BP-Thom map into mod $p$ cohomology rings.

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§1. Introduction and Summary of Results

The structure of mod $p$ cohomology rings of Eilenberg–Mac Lane spaces was determined in the early 1950s (see [1] and [8]) in terms of admissible monomials of Steenrod squares for even prime case, and of Steenrod reduced powers and Bockstein operators for odd prime case. In the late 1950s, Milnor [4] determined the Hopf algebra structure of the mod $p$ Steenrod algebra $\mathcal{A}(p)^*$, and he gave a new basis (Milnor basis) of the Steenrod algebra for all primes. Milnor basis elements are rather complicated in terms of admissible monomials, but the Milnor basis is, in a sense, a more natural basis of the Steenrod algebra because it comes from the structure theory of the Steenrod algebra as a Hopf algebra.

The first purpose of this paper is to reformulate mod $p$ cohomology rings of various Eilenberg–Mac Lane spaces in terms of the Milnor basis for both even and odd prime cases [see Theorem 1 below]. Although a long time has passed since the publication of [4], an account of an explicit description of the above result with all details is missing from literature, except that a formula for a certain version of excess of Milnor basis elements was discussed by Kraines [3, Definition 2, p. 363]. The author was informed of [3] after finishing the original version of this paper. The notion of excess is usually defined for admissible monomials in the Steenrod algebra. Kraines defined a notion of excess for any element in the Steenrod algebra in terms of (non)triviality of its action on mod $p$ fundamental classes of mod $p$ Eilenberg–Mac Lane spaces. However, his definition is not equivalent to the usual one in odd prime case, contrary to his claim. There is a very subtle but crucial difference between these two notions of excess. We calculate the correct formula of excess for Milnor basis elements in Lemma 5-4 for odd prime case. The excess formula for even prime case is given in Lemma 6-1. Kraines’ notion of excess does not detect free algebra generators of cohomology rings of Eilenberg–Mac Lane spaces in terms of Milnor basis elements: his notion of excess only detects nontriviality of certain elements which may not be algebra generators. Indeed, [3] does not go any further to discuss the algebra structure of these rings. But our correct excess formula does provide a simple description of free algebra generators of these cohomology rings in terms of Milnor basis elements, and this is the first purpose of this paper. Our paper is, in a sense, a continuation of his paper. But our approach is more systematic, detailed, and comprehensive.

Our main ingredients are certain decomposition formulae of Milnor’s Steenrod reduced powers and of Milnor’s Steenrod squares. We note that [3] also discusses similar formulae [cf. Lemma 4-1], but our formulae are a lot more precise [Proposition 4-6, Corollary 4-7], and directly give us what we want. As an element of the Steenrod algebra, any Milnor basis element is a stable cohomology operation. But usual Steenrod reduced powers and Steenrod squares also exhibit unstable property with respect to dimension of cohomology elements being acted on. Our decomposition formulae serve two purposes. The first purpose is to exhibit unstable nature of Milnor basis elements. The second purpose is to relate Milnor basis elements to certain closely associated admissible monomials.

Our second purpose of this paper is to describe the action of Milnor primitives on mod $p$ fundamental classes of Eilenberg–Mac Lane spaces [see Corollary 2]. We show that this action exhibits a surprisingly simple and regular pattern. For example, on the mod $p$ fundamental class $x_{n+1}$ of the mod $p$ Eilenberg–Mac Lane space $K(\mathbb{Z}/p, n+1)$, any product of $k$ distinct Milnor primitives can act nontrivially
as long as \( k \leq n + 1 \). However, as soon as \( k > n + 1 \), any product of \( k \) Milnor primitives acts trivially on \( \ell_{n+1} \). The case \( k = n + 1 \) is the borderline case, and this is the case we are most interested in. The element of this type of the smallest positive degree is \( Q_n \cdots Q_0 \ell_{n+1} \) of degree \( 2(p + \cdots + p^n) \). The \( \mathbb{Q} \)-subalgebra in the title is a polynomial subalgebra generated by elements obtained by the action of maximum number of Milnor primitives, in this case \( n + 1 \), on the fundamental class \( \ell_{n+1} \). We show that this algebra can be characterized as the smallest \( \mathcal{A}(p)^\ast \)-invariant subalgebra of the cohomology algebra \( H^\ast (K(\mathbb{Z}/p, n + 1); \mathbb{Z}_p) \) containing the element \( Q_n \cdots Q_0 \ell_{n+1} \) [see Theorem 3]. Similar subalgebras exist in mod \( p \) cohomology rings of all Eilenberg–Mac Lane spaces. In [10], these \( \mathbb{Q} \)-subalgebras are also characterized as the images of the Thom map from BP-cohomology to mod \( p \) cohomology of Eilenberg–Mac Lane spaces.

We note that in [2] actions of products of at most two Milnor primitives on mod \( p \) fundamental classes of integral Eilenberg–Mac Lane spaces are discussed.

We summarize our results. For any prime \( p \), let \( \mathbb{Z}/p = \mathbb{Z}_p \) be the ring of mod \( p \) integers. The Hopf algebra structure of the mod \( p \) Steenrod algebra \( \mathcal{A}^\ast = \mathcal{A}(p)^\ast \) was determined by Milnor for both even and odd prime \( p \) [4]. See §2 for a more detailed summary including all relevant facts needed for this paper. When \( p \) is odd, its dual algebra \( \mathcal{A}_s = \mathcal{A}(p)_s \) is a tensor product of a polynomial algebra and an exterior algebra of the following form:

\[
(1-1) \quad \mathcal{A}_s = \Lambda \mathbb{Z}_p (\tau_0, \tau_1, \ldots, \tau_r, \ldots) \otimes \mathbb{Z}_p [\xi_1, \xi_2, \ldots, \xi_r, \ldots],
\]

where \( |\tau_r| = 2p^r - 1 \), \( |\xi_r| = 2(p^r - 1) \). For even prime case, see Theorem 2-1. Let \( E = (\varepsilon_0, \varepsilon_1, \ldots) \) range over all sequences of zeroes and ones which are almost all zero, and let \( R = (r_1, r_2, \ldots) \) range over all sequences of non-negative integers which are almost all zero. Then the set of elements \( \tau(E)\xi(R) = \tau_0^{\varepsilon_0} \tau_1^{\varepsilon_1} \cdots \xi_1^{r_1} \xi_2^{r_2} \cdots \) forms an additive basis of the dual Steenrod algebra \( \mathcal{A}_s \). The element dual to \( \tau_j \) is the \( j \)-th Milnor primitive \( Q_j \in \mathcal{A}^\ast \) for \( j \geq 0 \). These elements generate an exterior subalgebra of the Steenrod algebra:

\[
(1-2) \quad Q_i Q_j + Q_j Q_i = 0, \quad \text{for all } i, j \geq 0.
\]

For a sequence \( E \) as above, let \( Q^E = Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots \). Let \( \mathcal{P}^R \in \mathcal{A}^\ast \) be the element dual to \( \xi(R) \). Elements of the form \( \mathcal{P}^R \) close under multiplication. See Theorem 2-2 for the explicit rule of multiplication. It is known that the set of elements \( \{Q^E \mathcal{P}^R\}_{E,R} \) forms an additive basis of \( \mathcal{A}^\ast \) dual to \( \{\tau(E)\xi(R)\}_{E,R} \) up to sign. Elements of the form \( Q^E \mathcal{P}^R \in \mathcal{A}^\ast \) are called Milnor basis elements. Elements of the form \( \mathcal{P}^R \) are called Milnor’s Steenrod reduced powers. For the mod 2 Steenrod algebra, we have Milnor’s Steenrod squares denoted by \( \text{Sq}^R \) [Theorem 2-3].

Let \( \Delta_i = (0, \ldots, 0, 1, 0, \ldots) \) be a sequence with 1 at \( i \)-th place and zero everywhere else. Let \( \mathcal{R} \) be the set of sequences \( R \) as above. For any \( R \in \mathcal{R} \), we define an integer \( \ell(R) \) and a shifted sequence \( t(R) \) by

\[
(1-3) \quad \ell[R] = \sum_{j \geq 1} r_j, \quad t(R) = (r_2, r_3, \ldots, r_k, \ldots) = \sum_{j \geq 1} r_{j+1} \Delta_j.
\]

Two sequences are added or subtracted componentwise.

The following four cases are discussed in this paper:
1. Mod $p$ cohomology of mod $p^h$ Eilenberg–Mac Lane spaces at an odd prime $p$.
2. Mod $p$ cohomology of integral Eilenberg–Mac Lane spaces at an odd prime $p$.
3. Mod 2 cohomology of mod $2^h$ Eilenberg–Mac Lane spaces at prime 2.
4. Mod 2 cohomology of integral Eilenberg–Mac Lane spaces at prime 2.

Situations are rather different among the above four cases which are discussed in §5.1, §5.2, §6.1, and §6.2, respectively. To illustrate our results in this paper, we explicitly describe our results here for the case of the mod $p^h$ Eilenberg–Mac Lane space $K(\mathbb{Z}/p^h, n+1)$.

**Theorem 1 [Theorem 5-2].** Let $p$ be an odd prime, and let $n \geq 0$ and $h \geq 1$. Let $\tau_{n+1} \in H^{n+1}(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$ be the mod $p$ fundamental class. Let $E$ and $R$ be sequences as above. Then the following identities hold:

\[
\begin{align*}
(1-4) & \quad Q^E_{\tau_{n+1}} = 0 \quad \text{if } \ell(E) + 2\ell(R) \geq n + 2, \\
(1-5) & \quad Q^E_{\tau_{n+1}} = (Q^{\ell(E)}_{\tau_{n+1}})^p \quad \text{if } \ell(E) + 2\ell(R) = n + 1 \text{ and } \varepsilon_0 = 0.
\end{align*}
\]

The mod $p$ cohomology ring of the mod $p^h$ Eilenberg–Mac Lane space is a free algebra described in terms of the Milnor basis as follows:

\[
H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p) = F_{\mathbb{Z}_p}[Q^E_{\tau_{n+1}} | \ell(\varepsilon) + 2\ell(R) < n + 1].
\]

Here we adopt a convention that $Q_{\tau_{n+1}}$ means $\delta_h \tau_{n+1}$, where $\delta_h$ is the $h$-th Bockstein operator.

Please note that the correct formula of excess of the Milnor basis element $Q^E_{\tau_{n+1}}$ is $\ell(\varepsilon) + 2\ell(R) = \ell(\varepsilon) + 2\ell(R) - \varepsilon_0$, which is defined as the excess of the associated admissible monomial [Lemma 5-4]. In [3], Kraines defines his version of excess of $Q^E_{\tau_{n+1}}$ and proves that it is given by $\ell(E) + 2\ell(R)$. This difference between his version and ours is more significant than it may seem. Kraines’ excess detects nontriviality of elements of the form $Q^E_{\tau_{n+1}}$, whereas our excess detects indecomposability of elements of the above form.

We have similar descriptions of mod $p$ cohomology rings of integral Eilenberg–Mac Lane spaces in terms of the Milnor basis [Theorem 5-10]. Since the mod $p$ fundamental class $\tau_{n+1}$ of the integral Eilenberg–Mac Lane space $K(\mathbb{Z}, n+2)$ has the property $Q_{\tau_{n+1}} = 0$, we need an extra condition on pairs of sequences $(E, R)$ to describe the mod $p$ cohomology ring, and the proof is more involved.

We are also interested in the action of Milnor primitives on mod $p$ fundamental classes of Eilenberg–Mac Lane spaces. Theorem 1 implies the following corollary.

**Corollary 2 [Corollary 5-7].** The action of Milnor primitives on the mod $p$ fundamental class $\tau_{n+1}$ of the mod $p^h$ Eilenberg–Mac Lane space $K(\mathbb{Z}/p^h, n+1)$ is described as follows:

\[
\begin{align*}
(1-7) & \quad Q^E_{\theta_{n+1}} = 0 \quad \text{if } \ell(E) \geq n + 2, \\
(1-8) & \quad Q^E_{\tau_{n+1}} = (Q^{\ell(E)}_{\tau_{n+1}})^p \quad \text{if } \ell(E) = n + 1 \text{ and } \varepsilon_0 = 0, \\
(1-9) & \quad Q^E_{\delta_{n+1}} = 0 \quad \text{if } \ell(E) \leq n + 1.
\end{align*}
\]

When $\ell(E) = n + 1$ and $\varepsilon_0 = 1$, or $\ell(E) \leq n$, the element $Q^E_{\tau_{n+1}}$ is a polynomial or exterior algebra generator of the cohomology ring $H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$.

Namely, on the fundamental class $\tau_{n+1}$, all products of $n + 2$ or more Milnor primitives act trivially, and any product of $n + 1$ or fewer distinct Milnor primitives
always acts nontrivially. Furthermore, except for the case $\ell[E] = n + 1$ and $\varepsilon_0 = 0$, any element of the form $Q^E_{\tau_{n+1}}$ with $\ell[E] \leq n + 1$ is a free algebra generator of the mod $p$ cohomology algebra.

A similar statement holds concerning the action of Milnor primitives on the mod $p$ fundamental class $\tau_{n+2}$ of the integral Eilenberg–Mac Lane space $K(\mathbb{Z}, n+2)$. It turns out that all products of $n + 1$ or more Milnor primitives act trivially on $\tau_{n+2}$, and any product of $n$ or fewer distinct Milnor primitives not including $Q_0$ acts nontrivially on $\tau_{n+2}$. In the second case, all of the resulting elements are free algebra generators of the cohomology ring [Proposition 5-11, Proposition 5-12].

We have corresponding statements for even prime case, with one difference. When $p = 2$, after applying maximum number of Milnor primitives on mod 2 fundamental classes of mod 2 rings [Corollary 6-6, Proposition 6-9].

In Corollary 2 we are particularly interested in the borderline case: $\ell[E] = n + 1$ and $\varepsilon_0 = 1$. Let $S^+_n$ be the set of sequences of $n$ strictly increasing positive integers:

$$(1-10) \quad S^+_n = \{(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n \mid 0 < s_1 < s_2 < \cdots < s_n\}.$$ 

Then for any $S \in S^+_n$, the element $Q_S Q_0^{\tau_{n+1}} = Q_{s_1} \cdots Q_{s_n} Q_0^{\tau_{n+1}}$ has even degree $2(1 + p^{s_1} + \cdots + p^{s_n})$, and it is a polynomial generator of the cohomology ring by the last part of Corollary 2. Let $Q$ be a polynomial subalgebra of the cohomology ring generated by these elements. That is,

$$(1-11) \quad Q = Q(\mathbb{Z}/p^h, n+1) = \mathbb{Z}_p[Q_S Q_0^{\tau_{n+1}} \mid S \in S^+_n].$$

Note that the element of the lowest positive degree in this polynomial subalgebra is $Q_n \cdots Q_1 Q_0^{\tau_{n+1}}$ of degree $2(1 + p + \cdots + p^n)$.

**Theorem 3 [Theorem 5-9].** The polynomial subalgebra $Q(\mathbb{Z}/p^h, n+1)$ of the cohomology ring $H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$ is the smallest $A(p)^*$-invariant subalgebra containing the element $Q_n \cdots Q_1 Q_0^{\tau_{n+1}}$. Any Milnor primitive acts trivially on this subalgebra.

We can also characterize the polynomial subalgebra $Q$ as the image of the BP-Thom map from BP-cohomology of $K(\mathbb{Z}/p^h, n+1)$ to its mod $p$ cohomology [10].

Similar polynomial subalgebras exist in mod $p$ cohomology rings of integral Eilenberg–Mac Lane spaces. For $n \geq 0$, they are given as follows:

$$(1-12) \quad Q(\mathbb{Z}, n+2) = \mathbb{Z}_p[Q_S \tau_{n+2} \mid S \in S^+_n] \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p).$$

We can show that the polynomial subalgebra $Q(\mathbb{Z}, n+2)$ is again the smallest $A(p)^*$-invariant subalgebra containing the element $Q_n \cdots Q_1 \tau_{n+2}$ of the smallest positive degree $2(1 + p + \cdots + p^n)$, and that this subalgebra is annihilated by any Milnor primitive [Theorem 5-14].

The above polynomial $Q$-subalgebras (1-11) and (1-12) are isomorphic as algebras by the homomorphism induced by the Bockstein map $\delta_h : K(\mathbb{Z}/p^h, n+1) \to K(\mathbb{Z}, n+2)$ [Proposition 5-15].

We also describe results corresponding to Theorem 1, Corollary 2, and Theorem 3 above for even prime case in §6. The statements are basically similar but simpler,
although there are some subtle differences in the details. One difference between odd prime case and even prime case is that when $p = 2$, all elements in the $\mathbb{Q}$-subalgebras are always squares in mod 2 cohomology rings [Theorem 6-7, Theorem 6-11].

The organization of this paper is as follows. In §2, we review the Hopf algebra structure of the mod $p$ Steenrod algebra determined by Milnor. In §3, we recall the classical description of mod $p$ cohomology rings of various Eilenberg–MacLane spaces in terms of admissible monomials. Sections 2 and 3 are included here in order to be more self-contained and for use in later sections. In §4, we present our key decomposition formulae for Milnor’s Steenrod reduced powers and Steenrod squares. Although the proof is elementary, these decomposition results are essential for the rest of the paper. These formulae are designed to make unstable nature of Milnor basis elements explicit. In §§5 and §6, we describe mod $p$ cohomology rings of Eilenberg–Mac Lane spaces for any prime $p$ in terms of the Milnor basis elements rather than in terms of admissible monomials. We also prove certain vanishing and $p$-th power properties of actions of certain Milnor basis elements.

In the same sections, we also describe the action of products of Milnor primitives on mod $p$ fundamental cohomology classes of mod $p^h$ or integral Eilenberg–Mac Lane spaces and we prove the characterizing property of the $\mathbb{Q}$-subalgebras.

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§2. The structure of the mod $p$ Steenrod algebra

We review those basic properties of the mod $p$ Steenrod algebra which are relevant to the present paper. Basic references are [4, 5, 9]. Let $p$ be a prime, even or odd, and let $A(p)^*$ be the mod $p$ Steenrod algebra. It is well known that $A(p)^*$ has the structure of a Hopf algebra [6]. Let $A(p)_*$ denote the dual Hopf algebra. To describe the structure of $A_*$, let $\mathbb{Z}_p[\cdot]$ denote a polynomial algebra over $\mathbb{Z}_p$ generated by elements inside of $[\cdot]$, and let $\Lambda_{\mathbb{Z}_p}(\cdot)$ denote an exterior algebra over $\mathbb{Z}_p$ generated by elements inside of $(\cdot)$.

Theorem 2-1 (Milnor [4]). Let $\phi : A(p)_* \to A(p)_* \otimes A(p)_*$ be the coalgebra map for the dual of the Steenrod algebra $A(p)^*$.

(I) The dual Hopf algebra $A(2)_*$ is a polynomial algebra described as follows:

\[
A(2)_* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots, \zeta_r, \ldots], \quad |\zeta_i| = 2^i - 1,
\]

\[
\phi(\zeta_k) = \sum_{i=0}^{k} \zeta_{k-i} \otimes \zeta_i.
\]

(II) Let $p$ be an odd prime. Then the structure of the dual Hopf algebra $A(p)_*$ is
described as follows:

\[ A(p)_* = A_{\mathbb{Z}_p}(\tau_0, \tau_1, \ldots, \tau_r, \ldots) \otimes \mathbb{Z}_p[\xi_1, \xi_2, \ldots, \xi_r, \ldots], \]

where \(|\tau_r| = 2p^r - 1, \) and \(|\xi_r| = 2(p^r - 1), \)

\(\phi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^p \otimes \xi_i, \quad \phi(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^p \otimes \tau_i + \tau_k \otimes 1.\)

Let \(R = (r_1, r_2, \ldots)\) range over all sequences of non-negative integers which are almost all zero, and let \(E = (\varepsilon_0, \varepsilon_1, \ldots)\) range over all sequences of zeroes and ones which are almost all zero. Let \(\Delta_i = (0, \ldots, 0, 1, 0, \ldots)\) be a sequence with 1 at the \(i\)-th entry and 0 everywhere else. So we can write \(R = \sum_{i \geq 1} r_i \Delta_i\) and \(E = \sum_{i \geq 0} \varepsilon_i \Delta_{i+1}\).

Let \(p\) be an odd prime. We let

\[ \tau(E)\xi(R) = \tau_0^\varepsilon_0 \tau_1^\varepsilon_1 \cdots \xi_1^{s_1} \xi_2^{s_2} \cdots \in A(p)_*. \]

For \(i \geq 0\), let \(Q_i \in A(p)_{2p^i-1}\) be the element dual to \(\tau_i\), and let \(\mathcal{P}^R \in A(p)^*\) be the element dual to \(\xi(R)\). Let \(Q^E \mathcal{P}^R = Q_0^E Q_1^E \cdots \mathcal{P}^R\) be the product of elements \(Q^E\) and \(\mathcal{P}^R\) in \(A(p)^*\). Elements \(\mathcal{P}^R\) are complicated expressions of Steenrod reduced powers \(\mathcal{P}^i = \mathcal{P}^{i \Delta_i}\). If any entry of the sequence \(R\) is negative, then we set \(\mathcal{P}^R = 0\) by convention.

**Theorem 2-2 (Milnor [4, 5])**. Let \(p\) be an odd prime. The set \(\{Q^E \mathcal{P}^R\}_{E, R}\) forms an additive basis of the Steenrod algebra \(A(p)^*\). This basis is dual to the basis \(\{\tau(E)\xi(R)\}_{E, R}\) of \(A(p)_*\) up to sign. Elements \(Q_i\) for \(i \geq 0\) are primitive and they generate an exterior subalgebra of \(A(p)_*\):

\[ Q_i Q_j + Q_j Q_i = 0, \quad i, j \geq 0. \]

For any sequence \(R\), the element \(Q_k\) commutes with \(\mathcal{P}^R\) by the following formula:

\[ \mathcal{P}^R Q_k = Q_k \mathcal{P}^R + Q_{k+1} \mathcal{P}^{R-p^k \Delta_1} + \cdots + Q_{k+j} \mathcal{P}^{R-p^k \Delta_j} + \cdots. \]

For two sequences \(R = (r_1, r_2, \ldots)\) and \(S = (s_1, s_2, \ldots)\) of non-negative integers, almost all zero, the product of \(\mathcal{P}^R\) and \(\mathcal{P}^S\) is given by

\[ \mathcal{P}^R \mathcal{P}^S = \sum_{R(X)=R, S(X)=S} b(X) \mathcal{P}^T(X), \]

where \(X\) range over all infinite matrices

\[ \begin{pmatrix}
* & x_{01} & x_{02} & \cdots \\
 x_{10} & x_{11} & x_{12} & \cdots \\
x_{20} & x_{21} & x_{22} & \cdots \\
 \vdots & \vdots & \vdots & \ddots 
\end{pmatrix} \]

of non-negative integers, almost all zero, with upper left corner omitted, such that

\[ r_i = \sum_{j \geq 0} p^j x_{ij} \quad \text{(weighted row sum)}, \quad s_j = \sum_{i \geq 0} x_{ij} \quad \text{(column sum)}. \]
These relations are denoted by $R(X) = R$ and $S(X) = S$, in short. From such a matrix $X$, the sequence $T(X) = (t_1, t_2, \ldots)$ and the coefficient $b(X)$ are defined by

\begin{equation}
(2-9) \quad t_n = \sum_{i+j=n} x_{ij} \quad \text{(diagonal sum)}, \quad b(X) = \prod_{n \geq 1} t_n^{1}, \quad \prod_{i+j \geq 1} x_{ij}^{1},
\end{equation}

The Cartan formula holds for $\mathcal{P}^R$: for any two cohomology elements $x, y$,

\begin{equation}
(2-10) \quad \mathcal{P}^R(xy) = \sum_{R_1 + R_2 = R} \langle \mathcal{P}^{R_1}x \rangle \langle \mathcal{P}^{R_2}y \rangle,
\end{equation}

where $\mathcal{P}^R_i = 0$ if any entry in $R_i$ is negative. The usual Steenrod reduced power $\mathcal{P}^m$ coincides with a Milnor basis element $\mathcal{P}^m \Delta^1 : H^k(X; \mathbb{Z}_p) \rightarrow H^{k+2(p-1)m} \mathbb{Z}_p$ for $m \geq 1$, and it has the following unstable property for any $x \in H^k(X; \mathbb{Z}_p)$:

\begin{equation}
(2-11) \quad \begin{cases}
\mathcal{P}^m \Delta^1 x = 0 & \text{if } 2m > k, \\
\mathcal{P}^m \Delta^1 x = x^p & \text{if } 2m = k.
\end{cases}
\end{equation}

The above unstable property in (2-11) will play a crucial role in our calculation of mod $p$ cohomology rings of Eilenberg–Mac Lane spaces in terms of the Milnor basis.

Next, we describe the mod 2 Steenrod algebra. Let $H = (h_1, h_2, \ldots)$ range over all sequences of non-negative integers which are almost all zero. In the dual Steenrod algebra, let $\zeta(H) = \zeta_1 h_1 \zeta_2 h_2 \cdots \in A(2)_*$. The set $\{\zeta(H)\}_H$ forms an additive basis of $A(2)_*$. Let $Sq^H \in A(2)^*$ be the element dual to $\zeta(H)$ with respect to this basis. For $i \geq 0$, let $Q_i = Sq^{\Delta_{i+1}}$ and let $\mathcal{P}^R = Sq^{2R}$ for any sequence $R$ of non-negative integers which are almost all zero.

**Theorem 2-3 (Milnor [4, 5]).** The set of elements $\{Sq^H\}_H$ forms an additive basis of the mod 2 Steenrod algebra $A(2)^*$ dual to the basis $\{\zeta(H)\}_H$ of $A(2)_*$. The elements $Q_i$ for $i \geq 0$ are primitive and they form an exterior subalgebra of $A(2)^*$:

\begin{equation}
(2-12) \quad Q_i^2 = 0, \quad Q_i Q_j = Q_j Q_i, \quad i, j \geq 0.
\end{equation}

For any sequence $E = (\varepsilon_0, \varepsilon_1, \ldots)$ of zeroes and ones which are almost all zero, and for any sequence $R$ of non-negative integers which are almost all zero, we have

\begin{equation}
(2-13) \quad Q^E \mathcal{P}^R = Q^E_0 Q^E_1 \cdots \mathcal{P}^R = Sq^{E+2R}.
\end{equation}

For any sequence $R$ of non-negative integers which are almost all zero, the element $Q_k$ commutes with $\mathcal{P}^R$ by the following formula:

\begin{equation}
(2-14) \quad \mathcal{P}^R Q_k = Q_k \mathcal{P}^R + \sum_{j \geq 1} Q_{k+j} \mathcal{P}^{R-2j} \Delta_j.
\end{equation}

For any two sequences $R, S$ of non-negative integers, almost all zero, the product of $Sq^R$ and $Sq^S$ is given by the same formula as in (2-6):

\begin{equation}
(2-15) \quad Sq^R Sq^S = \sum_{R(X)=R \atop S(X)=S} b(X) Sq^{T(X)}.
\end{equation}
The Cartan formula holds for $Sq^R$: for any cohomology elements $x, y$,

\begin{equation}
Sq^R(xy) = \sum_{R_1 + R_2 = R} Sq^{R_1}(x) \cdot Sq^{R_2}(y),
\end{equation}

where $Sq^{R_i} = 0$ if any entry of $R_i$ is negative. The usual Steenrod square $Sq^m$ coincides with a Milnor basis element $Sq^m_1$: $H^k(X; \mathbb{Z}_2) \to H^{k+m}(X; \mathbb{Z}_2)$, and it has the following unstable property for any element $x \in H^k(X; \mathbb{Z}_2)$:

\begin{equation}
\begin{cases}
Sq^m_1 x = 0 & \text{if } m > k, \\
Sq^m_1 x = x^2 & \text{if } m = k.
\end{cases}
\end{equation}

As we can see, we can treat Steenrod algebras for even or odd primes equally using the Milnor basis $\{Q^E, p^R\}_{E,R}$. However, we have to keep in mind that in $A(2)^*$, the elements $p^R$ do not close under multiplication unlike odd prime case, because $p^R = Sq^{2R}$ for even prime case.

\section{Description of mod $p$ cohomology rings of Eilenberg–MacLane spaces in terms of admissible monomials}

In this section, we recall the well-known structure of mod $p$ cohomology rings of Eilenberg–MacLane spaces in terms of admissible monomials [Theorems 3-4, 3-6]. We will need this description later.

First, we deal with odd prime case. Let

\begin{equation}
\theta = Q^0_p s_1^1 \cdot Q^0_{p^2} s_2^1 \cdot Q^0_{p^3} s_3^1 \cdot \ldots \cdot Q^0_{p^{j-1}} s_j^1 \cdot Q^0_{p^{j+1}} s_{j+1}^1 \cdot Q^0_{p^{j+2}} \cdot \ldots
\end{equation}

be a monomial in Steenrod reduced powers and the Bockstein operator $Q_0$, where $\varepsilon_j = 0, 1$ for $j \geq 0$ and $(s_1, s_2, \ldots)$ is a sequence of non-negative integers which are almost all zero. Put

\begin{equation}
i_j = |Q^0_{p^{j-1}} s_j^1| = \varepsilon_{j-1} + 2(p-1)s_j \quad \text{for } j \geq 1, \quad d(\theta) = \sum_{j \geq 1} i_j = \sum_{j \geq 0} \varepsilon_j + 2(p-1) \sum_{j \geq 1} s_j.
\end{equation}

Here $d(\theta)$ is the degree of the operation $\theta \in A(p)^*$. The next lemma is straightforward.

\begin{lemma}
Let $p$ be any prime. For a monomial $\theta$ as in (3-1), the following conditions on $\theta$ are equivalent:

1. $i_j \geq pi_{j+1}$ for all $j \geq 1$.
2. $s_j \geq ps_{j+1} + \varepsilon_j$ for all $j \geq 1$.

Any monomial $\theta$ of the form (3-1) satisfying one of the equivalent conditions in Lemma 3-1 is called an admissible monomial. Both of the above admissibility conditions are found in the literature. The condition (2) suits better for our purpose.

Next, we discuss the notion of excess of admissible monomials $\theta$ of the form (3-1). By the admissibility condition, we have $i_j \geq pi_{j+1}$ for all $j \geq 1$. We let

\begin{equation}
\tau_p(\theta) = (i_1 - pi_2) + (i_2 - pi_3) + \ldots + (i_j - pi_{j+1}) + \ldots
= pi_1 - (p-1)d(\theta).
\end{equation}
This is the usual definition of excess of admissible monomials. However, for odd prime case we use a slightly improved version of excess defined as follows. First note that $\tau_p(\theta) - \varepsilon_0$ is always divisible by $p - 1$. We then let

$$e_p(\theta) = \frac{\tau_p(\theta) - \varepsilon_0}{p - 1} \in \mathbb{Z}, \quad p \text{ : odd prime.} \tag{3-4}$$

We note the following relation between $\tau_p(\theta)$ and $e_p(\theta)$, which is immediate.

**Lemma 3-2.** Let $p$ be an odd prime and let $\theta$ be an admissible monomial in $A(p)^*$. Then for any positive integer $n$, we have $\tau_p(\theta) < n(p - 1)$ if and only if $e_p(\theta) < n$.

**Proof.** If $\tau_p(\theta) < n(p - 1)$, then $\tau_p(\theta) - \varepsilon_0 < n(p - 1)$. Dividing both sides by $p - 1$, we have $e_p(\theta) < n$. Conversely, suppose $e_p(\theta) < n$. Then, by definition, we have $\tau_p(\theta) - \varepsilon_0 < n(p - 1)$. Since $\tau_p(\theta) - \varepsilon_0$ is always divisible by $p - 1$ as we remarked right before (3-4), we have $\tau_p(\theta) - \varepsilon_0 = m(p - 1)$ for some $m < n$. Since $\varepsilon_0 < p - 1$ for any odd prime $p$, we have $\tau_p(\theta) = m(p - 1) + \varepsilon_0 < (m + 1)(p - 1) \leq n(p - 1)$. This completes the proof of Lemma 3-2. \qed

**Definition 3-3.** For any admissible monomial $\theta$ as in (3-1), the integer $e_p(\theta)$ defined in (3-4) is called (modified) excess of $\theta$.

Although $\tau_p(\theta)$ is the one we use for even prime case, we found it more convenient and simpler to use modified excess $e_p(\theta)$ for odd prime cases. Kraines gives a different and inequivalent definition of excess for any element in the Steenrod algebra [3].

To describe cohomology rings of Eilenberg–Mac Lane spaces, we use the following notation. For a non-negatively and integrally graded vector space $V$ over $\mathbb{Z}_p$, let $V^{\text{even}}$ and $V^{\text{odd}}$ be even and odd graded parts of $V$. The free algebra $F_{\mathbb{Z}_p}[V]$ generated by the graded vector space $V$ is the tensor product of the polynomial algebra on $V^{\text{even}}$ and the exterior algebra on $V^{\text{odd}}$.

$$F_{\mathbb{Z}_p}[V] = \mathbb{Z}_p[V^{\text{even}}] \otimes \Lambda_{\mathbb{Z}_p}(V^{\text{odd}}). \tag{3-5}$$

The well-known description of mod $p$ cohomology rings of Eilenberg–Mac Lane spaces in terms of admissible monomials goes as follows for any odd prime $p$:

**Theorem 3-4.** Let $p$ be an odd prime. Let $h \geq 1$ and $n \geq 0$. Let $\theta$ be a monomial in Steenrod reduced powers and the Bockstein operator $Q_0$ as in (3-1).

(I) Let $\tau_{n+1} \in H^{n+1}(K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p)$ be the fundamental class. The mod $p$ cohomology ring of the mod $p^h$ Eilenberg–Mac Lane space is a free algebra given by

$$H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p) = F_{\mathbb{Z}_p}[\theta_{n+1} \mid \theta \text{ is admissible and } e_p(\theta) < n + 1]. \tag{3-6}$$

Here if $h \geq 2$ and $\theta$ ends with a Bockstein operator $Q_0$, then this Bockstein should be regarded as the $h$-th Bockstein operator $\delta_h$.

(II) Let $\tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n + 2); \mathbb{Z}_p)$ be the fundamental class. The mod $p$ cohomology ring of the integral Eilenberg–Mac Lane space is a free algebra given by

$$H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p) = F_{\mathbb{Z}_p}[\theta_{n+2} \mid \theta \text{ is admissible, } e_p(\theta) < n + 2, \text{ and } \theta \text{ doesn't end with } Q_0]. \tag{3-7}$$

Next, we describe mod 2 cohomology rings of Eilenberg–Mac Lane spaces for even prime case $p = 2$. Let

$$\theta = \text{Sq}^{s_1} \Delta_1 \text{Sq}^{s_2} \Delta_1 \cdots \text{Sq}^{s_r} \Delta_1 \cdots \tag{3-8}$$

be a monomial in Steenrod squares for some sequence of non-negative integers $S = (s_1, s_2, \ldots, s_r, \ldots)$ which are almost all zero.
Definition 3-5. The monomial \( \theta \) in (3-8) is said to be admissible if \( s_j \geq 2s_{j+1} \) for all \( j \geq 1 \). The excess \( e_2(\theta) \) of an admissible monomial \( \theta \) is given by

\[
e_2(\theta) = \sum_{j \geq 1} (s_j - 2s_{j+1}) = 2s_1 - d(\theta),
\]

where \( d(\theta) = \sum_{j \geq 1} s_j \) is the degree of the monomial \( \theta \).

Note that the above excess \( e_2(\theta) \) is the \( p = 2 \) version of \( \bar{e}_p(\theta) \) in (3-3), rather than (3-4). With these definitions, the well-known description of mod 2 cohomology rings of Eilenberg–Mac Lane spaces is given as follows:

Theorem 3-6. Let \( \theta \) be a monomial in Steenrod squares as in (3-8). Let \( n \geq 0 \).

(I) Let \( n+1 \in H^{n+1}(K(\mathbb{Z}/2^n; n+1); \mathbb{Z}_2) \) be the mod 2 fundamental class, where \( n \geq 1 \) and \( h \geq 1 \), or \( n = 0 \) and \( h = 1 \). The mod 2 cohomology ring of the mod 2 Eilenberg–Mac Lane space \( K(\mathbb{Z}/2^n; n+1) \) is a polynomial algebra:

\[
H^*(K(\mathbb{Z}/2^n; n+1); \mathbb{Z}_2) = \mathbb{Z}_2[\theta_{n+1} | \theta \text{ is admissible and } e_2(\theta) < n + 1].
\]

Here, when \( h > 1 \) and \( \theta \) ends with \( Sq^{d_1} \), this last operator \( Sq^{d_1} \) should be regarded as the \( h \)-th Bockstein \( \delta_h \).

When \( n = 0 \) and \( h > 1 \), the mod \( p \) cohomology ring of \( K(\mathbb{Z}/2^n; 1) \) has an exterior factor and it is given by

\[
H^*(K(\mathbb{Z}/2^n; 1); \mathbb{Z}_2) = \mathbb{Z}_2[\delta_{h\tau_1}] \otimes \Lambda_{\mathbb{Z}_2}(\tau_1).
\]

(II) Let \( n+2 \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_2) \) be the mod 2 fundamental class. The mod 2 cohomology ring of the integral Eilenberg–Mac Lane space is a polynomial algebra

\[
H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2) = \mathbb{Z}_2[\theta_{n+2} | \theta \text{ is admissible, } e_2(\theta) < n + 2, \text{ and } \theta \text{ does not end with } Sq^{d_1}].
\]

§4. Decomposition formulae for Milnor’s Steenrod reduced powers and Steenrod squares

In this section, we prove decomposition formulae for Milnor’s Steenrod reduced powers and Steenrod squares. These decomposition formulae will play a crucial role in describing unstable action of Milnor basis elements. Although our formula [Proposition 4-6] is very precise, for the purpose of this paper, Corollary 4-7 and Proposition 4-8 are sufficient. We worked out a precise formula for future reference. Although the method of the proof is elementary, we need a very careful analysis to obtain an exact formula. The main point of these decomposition formulae is that they extract unstable nature of Milnor basis elements explicitly.

Let \( R \) be the set of all sequences \( R \) of non-negative integers which are almost all zero. We introduce a partial ordering in \( R \) as follows. Let \( R = (r_1, r_2, \ldots, r_k, \ldots) \) and \( S = (s_1, s_2, \ldots, s_k, \ldots) \) be two sequences in \( R \). Then we write \( S \leq R \) when \( s_k \leq r_k \) for all \( k \geq 1 \). We write \( S < R \) if \( S \leq R \) and \( r_k < r_k \) for some \( k \geq 1 \). For two such sequences \( R \) and \( S \), we define their multi-binomial coefficient \( (R, S) \) by

\[
(R, S) = \prod_{k \geq 1} (r_k, s_k) = \prod_{k \geq 1} \frac{(r_k + s_k)!}{r_k! s_k!}.
\]
For \( R = (r_1, r_2, \ldots, r_k, \ldots) \in \mathcal{R} \), let a weighted sum map \( \sigma : \mathcal{R} \to \mathbb{N} \cup \{0\} \), a translation operation \( t : \mathcal{R} \to \mathcal{R} \), and a length map \( \ell : \mathcal{R} \to \mathbb{N} \cup \{0\} \) be defined by

\[
\sigma[R] = r_1 + pr_2 + \cdots + p^{k-1}r_k + \cdots = \sum_{j \geq 1} p^{j-1}r_j, \\
t(R) = (r_2, r_3, \ldots, r_k, \ldots) = \sum_{j \geq 1} r_{j+1} \Delta_j, \\
\ell[R] = r_1 + r_2 + \cdots + r_k + \cdots = \sum_{j \geq 1} r_j.
\]

(4-2)

Here, \( \Delta_j = (0, \ldots, 0, 1, 0, \ldots) \) with 1 at the \( i \)-th place and 0 everywhere else. For convenience we let \( \Delta_0 \) be the zero sequence, that is, \( \Delta_0 = (0, 0, \ldots) \).

**Lemma 4-1.** Let \( p \) be an odd prime and let \( R \) be a sequence of non-negative integers which are almost all zero. Then

\[
\mathcal{P}^{\sigma[R] \Delta_1} \cdot \mathcal{P}^{t(R)} = \mathcal{P}^R + \sum_{S \in \mathcal{R}, s_i = 0} \left( R - S + \sigma[S] \Delta_1, t(S) \right) \mathcal{P}^{R - S + \sigma[S] \Delta_1 + t(S)}.
\]

(4-3)

Proof. We apply the product formula (2-6) to calculate \( \mathcal{P}^{\sigma[R] \Delta_1} \cdot \mathcal{P}^{t(R)} \). Since only the first entry of the sequence \( \sigma[R] \Delta_1 \) is nontrivial, matrices \( X \) of (2-7) which are relevant to the calculation of this product have trivial rows from the third rows on. Due to the column sum condition (2-8), any such matrix \( X \) must be of the form

\[
X = \begin{pmatrix}
* & s_2 & s_3 & \cdots & s_k & \cdots \\
r & r_2 - s_2 & r_3 - s_3 & \cdots & r_k - s_k & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots 
\end{pmatrix},
\]

where integers \( s_k \) are such that \( 0 \leq s_k \leq r_k \) for \( k \geq 2 \), and \( r \geq 0 \). The weighted row sum condition (2-8) requires that \( r + \sum_{k \geq 1} p^k(r_{k+1} - s_{k+1}) = \sigma[R] \), or \( r = r_1 + \sum_{k \geq 1} p^k s_{k+1} \). Thus we see that the set of matrices \( X \) relevant to the calculation of the product \( \mathcal{P}^{\sigma[R] \Delta_1} \cdot \mathcal{P}^{t(R)} \) are in 1:1 correspondence with the set of sequences \( S \in \mathcal{R} \) such that \( \Delta_0 \leq S \leq R \) and \( s_i = 0 \). In terms of \( S \), we have \( r = r_1 + \sigma[S] \).

For any matrix \( X \) as above, the sequence \( T = T(X) \) in (2-5) is given by

\[
T = (r + s_2, r_2 - s_2 + s_3, \ldots, r_k - s_k + s_{k+1}, \ldots) = R - S + \sigma[S] \Delta_1 + t(S).
\]

The coefficient \( b(X) \) in (2-9) is given by

\[
b(X) = \frac{(r + s_2)! \prod_{k \geq 2} (r_k - s_k + s_{k+1})!}{r! s_2! (r_2 - s_2)! (r_3 - s_3)! \cdots (r_k - s_k)!} = (r, s_2) \prod_{k \geq 2} (r_k - s_k, s_{k+1})
\]

\[
= (R + \sigma[R] \Delta_1 - S, t(S)).
\]

Thus, the multiplication formula (2-6) gives the following result:

\[
\mathcal{P}^{\sigma[R] \Delta_1} \cdot \mathcal{P}^{t(R)} = \sum_{S \in \mathcal{R}, s_i = 0} \left( R - S + \sigma[S] \Delta_1, t(S) \right) \mathcal{P}^{R - S + \sigma[S] \Delta_1 + t(S)}.
\]
In this expression, the term corresponding to \( S = 0 \) gives \( P^R \). From this, we obtain (4.3). This completes the proof. \( \Box \)

The purpose of this section is to elaborate on (4.3).

In the summation on the right hand side of (4.3), the exponent sequence \( R - S + \sigma[S] \Delta_1 + t(S) \) \( \in \mathcal{R} \) is of interest to us. The corresponding Steenrod reduced power is nontrivial when \( R - S + \sigma[S] \Delta_1 + t(S) \geq \Delta_0 \). Applying (4.1), we also have \( t(R) - t(S) + t^2(S) \geq \Delta_0 \). This inequality is automatically satisfied when \( S \leq R \), which is the case relevant in (4.3). But there are some other sequences \( S \) satisfying this inequality. We give a necessary condition on \( S \) so that \( S \) satisfies \( t(R) - t(S) + t^2(S) \geq \Delta_0 \).

Lemma 4.2. Let \( R \) and \( S \) be sequences of non-negative integers which are almost all zero. Then \( t(R) - t(S) + t^2(S) \geq \Delta_0 \) implies \( t(S) \leq \sum_{i \geq 1} t_i(R) \). Furthermore, \( t(R) - t(S) + t^2(S) = \Delta_0 \) if and only if \( t(S) = \sum_{i \geq 1} t_i(R) \).

Proof. Applying the operator \( t(\cdot) \) to \( (t(R) + t^2(S) \geq t(S) \) a number of times, we have \( t(R) + t^2(R) \geq t^k(S) \) for \( k \geq 1 \). Since \( t(R) = 0 \) and \( t^i(S) = 0 \) for sufficiently large \( j \), summation of these inequalities side by side makes sense, and after some obvious cancellations, we have \( t(R) + t^2(R) + \cdots + t^k(R) + \cdots \geq t(S) \). This proves the first part.

If we have an identity \( t(R) + t^2(S) = t(S) \), then we have identities throughout the above argument. Hence we have \( \sum_{j \geq 1} t^j(R) = t(S) \). Conversely, if \( t(S) = \sum_{j \geq 1} t^j(R) \), then \( t(R) + t^2(S) = t(R) + \sum_{j \geq 2} t^j(R) = \sum_{j \geq 1} t^j(R) = t(S) \). This proves the second part. \( \Box \)

Rewriting (4.3), we have

\[
(4.4) \quad p^R = p^{\sigma[R]\Delta_1}, p^{t(R)} - \sum_{S^{\sigma[R]\Delta_1+t(S)} = 0} (R - S + \sigma[S] \Delta_1, t(S)) p^{R - S + \sigma[S] \Delta_1 + t(S)}.
\]

We can apply formula (4.4) to each term in the summation of (4.4) with different \( R \) to decompose \( P^R \) further. Let \( R' = R - S + \sigma[S] \Delta_1 + t(S) \). We have \( \sigma[R'] = \sigma(R) + \sigma[t(S)] \) and \( t(R') = t(R) - t(S) + t^2(S) \). Applying (4.4) to \( P^{R'} \), we have

\[
(4.5) \quad p^R = p^{\sigma[R]\Delta_1}, p^{t(R)} - \sum_{S^{\sigma[R]\Delta_1+t(S)} = 0} (R - S + \sigma[S] \Delta_1, t(S)) p^{\sigma[R'] + \sigma[t(S)] \Delta_1, p^{t(R) - t(S) + t^2(S)} + \sum_{S^{\sigma[R]\Delta_1+t(S)} = 0} (R - S + \sigma[S] \Delta_1, t(S)) p^{R'}.
\]

where \( R' = R - (S + S') + \sigma[S + S'] \Delta_1 + t(S + S') \). To continue this calculation further in a systematic way, we make the following definition.

Definition 4.3. For a sequence \( R \) of non-negative integers which are almost all zero, an \( R \)-admissible chain of length \( k \) is a sequence \( S_1, S_2, \ldots, S_k \) in \( \mathcal{R} \) such that

1. \( S_i > \Delta_0 \) and the first entry of \( S_i \) is zero for \( 1 \leq i \leq k \).
2. \( \Delta_0 < S_{i+1} \leq R - (S_1 + S_2 + \cdots + S_i) + t(S_1 + S_2 + \cdots + S_i) \) for \( 0 \leq i < k \).
The set of all $R$-admissible chains of length $k$ is denoted by $C_k(R)$. For any $\overrightarrow{S} = (S_1, S_2, \ldots, S_k) \in C_k(R)$, we let $|\overrightarrow{S}| = S_1 + S_2 + \cdots + S_k \in R$.

For a given $k$ and $R = (r_1, r_2, \ldots, r_t, \ldots)$ such that $r_\ell \neq 0$ for some $\ell \geq k + 1$, an example of an $R$-admissible chain of length $k$ is given as follows. Let $S_j$ be a sequence obtained from $t^{j-1}(R)$ by replacing the first entry by 0, namely, $S_j = (0, r_{j+1}, \ldots, r_t, \ldots) > \Delta_0$ for $1 \leq j \leq k$. Then a sequence $\overrightarrow{S} = (S_1, S_2, \ldots, S_k)$ is an $R$-admissible chain of length $k$. We list several properties of the set $C_k(R)$.

**Lemma 4-5.** Let $R$ be a sequence of non-negative integers which are almost all zero. The set $C_k(R)$ of $R$-admissible chains of length $k$ has the following properties:

1. For any $k \geq 1$, the set $C_k(R)$ has finitely many elements.
2. For any length $k$ admissible chain $\overrightarrow{S} \in C_k(R)$, we have $t(|\overrightarrow{S}|) \leq \sum_{i \geq 1} t^i(R)$.
3. The set $C_k(R)$ is empty if $k > \ell(\sum_{i \geq 1} t^i(R))$.

**Proof.** Let $\overrightarrow{S} = (S_1, S_2, \ldots, S_k) \in C_k(R)$ be any $R$-admissible chain of length $k$. By definition, $S_k$ satisfies $\Delta_0 < S_k \leq R - (S_1 + \cdots + S_{k-1}) + t(S_1 + \cdots + S_{k-1})$. From this, we immediately have $R - |\overrightarrow{S}| + t(|\overrightarrow{S}|) \geq \Delta_0$. By Lemma 4-2, this implies that $t(|\overrightarrow{S}|) \leq \sum_{i \geq 1} t^i(R)$. This proves (ii).

Since there are finitely many possibilities of $\overrightarrow{S}$ such that $|\overrightarrow{S}| \leq \sum_{i \geq 1} t^i(R)$, there can be finitely many elements in $C_k(R)$. This proves (i).

Suppose $C_k(R)$ is not empty and let $\overrightarrow{S} = (S_1, S_2, \ldots, S_k) \in C_k(R)$. Since $S_i > \Delta_0$ for $1 \leq i \leq k$ by definition, we have $\ell(S_i) \geq 1$. So $\ell(|\overrightarrow{S}|) = \sum_{i=1}^k \ell(S_i) \geq k$. By applying $\ell(\cdot)$ to the inequality in (ii), we have $k \leq \ell(|\overrightarrow{S}|) = \ell(t(|\overrightarrow{S}|)) \leq \ell(\sum_{i \geq 1} t^i(R))$. Thus, if $k > \ell(\sum_{i \geq 1} t^i(R))$, then $C_k(R) = \emptyset$. \hfill $\square$

Let $\overrightarrow{S} = (S_1, S_2, \ldots, S_k) \in C_k(R)$ be an $R$-admissible chain of length $k$. We let

$$a_k(\overrightarrow{S}) = \prod_{i=1}^k (R - (S_1 + \cdots + S_i) + \sigma(S_1 + \cdots + S_i) |\Delta_1 + t(S_1 + \cdots + S_{i-1})).$$

Here, by convention, $S_0 = \Delta_0$. When $k$ is understood, we simply write $a(\overrightarrow{S})$ for $a_k(\overrightarrow{S})$. With these preparations, we can further continue our process of decomposing $\mathcal{P}^R$ which was started in (4-4) and (4-5).

**Lemma 4-5.** Let $p$ be an odd prime. Let $R$ be a sequence of non-negative integers which are almost all zero. Then for any $r \geq 1$, we have

$$\mathcal{P}^R = \mathcal{P}^{\sigma[R|\Delta_1]} \mathcal{P}^{|R| + t(|\overrightarrow{S}|)} \mathcal{P}^{|R| - t(|\overrightarrow{S}|)} \mathcal{P}^{\sigma(|\overrightarrow{S}| + t(|\overrightarrow{S}|) + \sigma(|\overrightarrow{S}|)|\Delta_1}$$

$$+ \sum_{k=1}^{r-1} \sum_{\overrightarrow{S} \in C_k(R)} (-1)^k a_k(\overrightarrow{S}) \mathcal{P}^{\sigma(|\overrightarrow{S}| + \sigma(|\overrightarrow{S}|))|\Delta_1} \mathcal{P}^{t(|\overrightarrow{S}|) + t^2(|\overrightarrow{S}|)}$$

$$+ \sum_{\overrightarrow{S} \in C_r(R)} (-1)^r a_r(\overrightarrow{S}) \mathcal{P}^{|\overrightarrow{S}| + \sigma(|\overrightarrow{S}|)} \mathcal{P}^{\sigma(|\overrightarrow{S}|)|\Delta_1}. $$

**Proof.** We prove (4-7) by induction on $r \geq 1$. When $r = 1$, (4-7) is the same as (4-4) and we are done. We assume (4-7) for some $r \geq 1$ and we prove the formula corresponding to $r + 1$. For this, we apply the partial factorization formula (4-4)
to each term in the second summation on the right hand side of (4-7). For $\mathcal{S} = (S_1, S_2, \ldots, S_r) \in C_r(R)$, if we have $t(R) - t(\mathcal{S}) + t^2(\mathcal{S}) = \Delta_0$, then Lemma 4-1 applied to $\mathfrak{p}^{R-|\mathcal{S}|+t(\mathcal{S})+\sigma(\mathcal{S})}|\mathcal{S}|\Delta_1$ does not give anything different, because only the first entry of the exponent sequence is nontrivial. If $t(R) - t(\mathcal{S}) + t^2(\mathcal{S}) > \Delta_0$, then (4-4) gives

$$\mathfrak{p}^{R-|\mathcal{S}|+t(\mathcal{S})+\sigma(\mathcal{S})}|\mathcal{S}|\Delta_1 = \mathfrak{p}^{\sigma(\mathcal{R})+\sigma([\mathcal{S}])}\Delta_1, \mathfrak{p}^{t(R)-t(\mathcal{S})+t^2(\mathcal{S})}$$

where $R' = R - |\mathcal{S}|+t(\mathcal{S})+\sigma(\mathcal{S})\Delta_1$ and the first entry of $S_{r+1}$ is zero. By the choice of $S_{r+1}$, we see that $\mathcal{S}^t = (S_1, \ldots, S_r, S_{r+1})$ is an $R$-admissible sequence of length $r+1$. We observe that

$$a_r(\mathcal{S}) \cdot (R' - S_{r+1} + \sigma[S_{r+1}]\Delta_1, t(S_{r+1})) = a_{r+1}(\mathcal{S}'), \quad \text{and}$$

$$R' - S_{r+1} + \sigma[S_{r+1}]\Delta_1 + t(S_{r+1}) = R - |\mathcal{S}'|+t(\mathcal{S}')+\sigma(\mathcal{S}')\Delta_1.$$ 

From these formulae, the second summation on the right hand side of (4-7) can be written as follows:

$$(-1)^r \sum_{\mathcal{S} \in C_r(R)} a_r(\mathcal{S}) \mathfrak{p}^{R-|\mathcal{S}|+t(\mathcal{S})+\sigma(\mathcal{S})}\mathcal{S}^r \Delta_1$$

$$= (-1)^r \sum_{\mathcal{S} \in C_r(R)} a_r(\mathcal{S}) \mathfrak{p}^{\sigma(\mathcal{R})+\sigma([\mathcal{S}])}\Delta_1, \mathfrak{p}^{t(R)-t(\mathcal{S})+t^2(\mathcal{S})}$$

$$+ (-1)^{r+1} \sum_{\mathcal{S}' \in C_{r+1}(R)} a_{r+1}(\mathcal{S}') \mathfrak{p}^{R-|\mathcal{S}'|+t(\mathcal{S}')+\sigma(\mathcal{S}')}\mathcal{S}^r \Delta_1.$$ 

This completes the inductive step and Lemma 4-5 is proved. \(\square\)

The above process ends at a finite stage because $C_k(R)$ is empty for large $k$ by (3) of Lemma 4-4. The final outcome of this decomposition process is the following proposition whose proof is straightforward from Lemma 4-5.

**Proposition 4-6.** Let $p$ be an odd prime and let $R$ be a sequence of non-negative integers which are almost all zero. Then a decomposition of $\mathfrak{p}^R$ of the following form exists:

$$\mathfrak{p}^R = \mathfrak{p}^{\sigma(\mathcal{R})}\mathfrak{p}^{t(\mathcal{R})} + \sum_{k \geq 1} (-1)^k \sum_{\mathcal{S} \in C_k(R)} a(\mathcal{S}) \mathfrak{p}^{\sigma(\mathcal{R})+\sigma([\mathcal{S}])}\Delta_1, \mathfrak{p}^{t(R)-t(\mathcal{S})+t^2(\mathcal{S})}.$$ 

Here, the summation is finite.

Since the union $\bigcup_k C_k(R)$ is a finite set, after collecting similar terms, we have the following corollary.

**Corollary 4-7.** Let $p$ be an odd prime and let $R$ be a sequence of non-negative integers which are almost all zero. Then there exists a decomposition of $\mathfrak{p}^R$ of the following form for some coefficients $c(\mathcal{S}) \in \mathbb{Z}_p$:

$$\mathfrak{p}^R = \mathfrak{p}^{\sigma(\mathcal{R})}\mathfrak{p}^{t(\mathcal{R})} + \sum_{\mathcal{S} \in C_k(R)} c(\mathcal{S}) \mathfrak{p}^{\sigma(\mathcal{R})+\sigma([\mathcal{S}])}\Delta_1, \mathfrak{p}^{t(R)-t(\mathcal{S})+t^2(\mathcal{S})}.$$ 

Here, the summation is finite.
Proof. We only have to note that $\overrightarrow{S} \in C_k(R)$ implies that $t(\overrightarrow{S}) \leq \sum_{i \geq 1} t^i(R)$ for any $k \geq 1$ by (2) of Lemma 4-4. 

The point of this decomposition formula is that the first factor in each term on the right hand side of (4-9) is a usual Steenrod reduced power operation with unstable property (2-11).

Now we look at similar factorizations of Milnor’s Steenrod squares at prime 2. Since the multiplication formula for $Sq^R.Sq^S$ is the same as the multiplication formula for $p^R.p^S$, exactly the same proof works replacing the odd prime $p$ by 2 in the above and we obtain the following result.

**Proposition 4-8.** Let $p = 2$. For any sequence $R$ of non-negative integers which are almost all zero, there exists a decomposition of $Sq^R$ of the following form:

\[
(4-10) \quad Sq^R = Sq^{|R| \Delta_1} \cdot Sq^{t(R)} + \sum_{S \in R, \gamma_1 = 0} c(S) Sq^{\sigma[|R| + \sigma(t(S))] \Delta_1} \cdot Sq^{t(R) - t(S) + t^2(S)},
\]

for some coefficients $c(S) \in \mathbb{Z}_2$.

Although we can write down a precise formula for even prime case analogous to (4-8), the above form is enough for our purpose in later applications.

For an odd prime $p$, the mod $p$ Steenrod algebra $A(p)^*$ is spanned by elements of the form $Q^E.p^R$. When we apply the decomposition formula (4-9) to the element $Q^E.p^R$, the beginning of each term is of the form $Q^E.p^m \Delta_1$. We examine elements of this form.

Recall that for a sequence $E = (\varepsilon_1, \varepsilon_2, \ldots)$ of ones and zeroes which are almost all zero, we have let $Q^E = Q_0^E Q_1^E \cdots$ and $\ell[E] = \sum_{i > 0} \varepsilon_i$. For two such sequences $E_1$ and $E_2$, we define $\text{sgn}(E_1, E_2) \in \{\pm 1\}$ by the following rule:

\[
(4-11) \quad Q^{E_1}Q^{E_2} = \text{sgn}(E_1, E_2)Q^{E_1 + E_2}.
\]

For a sequence $E$ as above, we have defined $t(E) = (\varepsilon_1, \varepsilon_2, \ldots)$ and $\sigma[t(E)] = \varepsilon_1 + \varepsilon_2 p + \cdots + \varepsilon_r p^{r-1} + \cdots$ in (4-2).

**Proposition 4-9.** Let $p$ be an odd prime. Let $E$ be a sequence of zeroes and ones which are almost all zero, and let $m \geq 0$. Then

\[
(4-12) \quad Q^{E.p^m \Delta_1} = Q_0^m \sum_{E_1 + E_2 = \text{sgn}(E_1, E_2)(-1)^{\ell[E_1]} Q^{E_1} p^{(m + \sigma[t(E)]) \Delta_1} Q^{E_2}.
\]

**Proof.** By induction on $\ell[t(E)]$. When $\ell[t(E)] = 0$, the sequence $E$ is of the form $E = (\varepsilon_0, 0, \ldots)$ and the above identity is obviously true.

Assume that (4-12) holds for any $E$ such that $\ell[t(E)] = r$ for some $r \geq 0$. Suppose $\ell[t(E)] = r + 1 \geq 1$. In $t(E) = (\varepsilon_1, \varepsilon_2, \ldots)$, let $i$ be the position of the first nonzero entry. We let $Q^E = Q_0^E Q_i Q^{E'}$, where the first $i + 1$ entries of $E'$ are 0. Since $\ell[t(E')] = r$, by inductive hypothesis for $Q^{E'.p^m \Delta_1}$ we have

\[
Q^{E'.p^m \Delta_1} = Q_0^m \sum_{E_1' + E_2' = \text{sgn}(E_1', E_2')(t(E'))} (-1)^{\ell[E_1']} Q^{E_1'} p^{(m + \sigma[t(E')]) \Delta_1} Q^{E_2'}
\]

\[
= Q_0^m \sum_{E_1' + E_2' = \text{sgn}(E_1', E_2')} Q^{E_1'} Q_i Q^{E_2'} p^{(m + \sigma[t(E')]) \Delta_1} Q^{E_2'}.
\]
By formula (2-5), we have
\[ Q_1 \mathfrak{p}^{(m+\sigma[t(E')])\Delta_1} = \mathfrak{p}^{(m+\sigma[t(E')]+p^{i-1})\Delta_1} Q_{i-1} - Q_{i-1} \mathfrak{p}^{(m+\sigma[t(E')]+p^{i-1})\Delta_1}. \]
Since \( t(E) = t(E') + \Delta_i \), we have \( \sigma[t(E')] + p^{i-1} = \sigma[t(E)] \). Continuing our calculation, we have
\[ Q^E p^m \Delta_1 = Q_0^E \sum_{E_1' + E_2' = t(E')} \text{sgn}(E_1', E_2') Q_1 \mathfrak{p}^{(m+\sigma[t(E')])\Delta_1} Q_{i-1} Q^{E_2'} \]
\[ + Q_0^E \sum_{E_1' + E_2' = t(E')} \text{sgn}(E_1', E_2')(-1)^{\ell[E_1'] + 1} Q_{i-1} Q^{E_1'} \mathfrak{p}^{(m+\sigma[t(E')])\Delta_1} Q^{E_2'}. \]
Here we used \(-Q^{E_1'} Q_{i-1} = (-1)^{\ell[E_1']} Q_{i-1} Q^{E_1'} \). Since the first entry of \( E_1' + E_2' = t(E') \) comes after the \( i \)-th entry by the choice of \( i \geq 1 \), we have
\[ \text{sgn}(E_1', E_2') = \text{sgn}(E_1', \Delta_i + E_2') \cdot (-1)^{\ell[E_1']}, \]
\[ \text{sgn}(E_1', E_2')(-1)^{\ell[E_1'] + 1} = \text{sgn}(\Delta_i + E_1', E_2') \cdot (-1)^{\ell[E_1'] + 1}. \]
Rewriting the above, we have
\[ Q^E p^m \Delta_1 = Q_0^E \sum_{\substack{\Delta_i \leq E_2' \\ E_1' + E_2' = t(E')}} \text{sgn}(E_1', E_2')(-1)^{\ell[E_1']} Q^{E_1'} \mathfrak{p}^{(m+\sigma[t(E')])\Delta_1} Q^{E_2'} \]
\[ + Q_0^E \sum_{\substack{\Delta_i \leq E_1' \\ E_1' + E_2' = t(E')}} \text{sgn}(E_1', E_2')(-1)^{\ell[E_1'] + 1} Q^{E_1'} \mathfrak{p}^{(m+\sigma[t(E')])\Delta_1} Q^{E_2'}. \]
This completes the inductive step and Proposition 4-9 is proved. 

§5. Mod \( p \) cohomology rings of Eilenberg–Mac Lane spaces in terms of the Milnor basis, and \( Q \)-subalgebras: odd prime case

Throughout this section, let \( p \) be an odd prime. We describe the action of Milnor basis elements \( Q^E p^R \) on mod \( p \) fundamental classes of Eilenberg–Mac Lane spaces, and we give a simple description of mod \( p \) cohomology rings of Eilenberg–Mac Lane spaces in terms of the Milnor basis. We pay particular attention to the action of Milnor primitives on the mod \( p \) fundamental classes. We then characterize \( Q \)-subalgebras generated by elements obtained by nontrivial actions of maximum number of Milnor primitives, in terms of \( \mathcal{A}(p)^2 \)-invariance and generation from a single element of the lowest positive degree in the \( Q \)-subalgebras. Even prime case is discussed in the next section.

When we apply cohomology operations to cohomology classes of spaces, unstable properties (2-11) and (2-17) of cohomology operations play an important role. To facilitate the application of the unstable property in our context, the following degree formulae will be useful.
Lemma 5-1. Let $p$ be a prime, even or odd. Let $R = (r_1, r_2, \ldots)$ be a sequence of non-negative integers which are almost all zero. Then
\begin{equation}
|p^{\ell(R)}| = 2\sigma[R] - 2\ell[R], \quad |p^{\sigma[R]\Delta_1}| = 2(p - 1)\sigma[R], \quad |p^R| = |p^{\ell(R)}| + |p^{\sigma[R]\Delta_1}| = 2p \cdot \sigma[R] - 2\ell[R].
\end{equation}

Let $E = (\varepsilon_0, \varepsilon_1, \ldots)$ be a sequence of zeroes and ones, almost all zero. Then
\begin{equation}
|Q^t(E)| = 2\sigma[t(E)] - \ell[t(E)] = 2\sigma[t(E)] - \ell[E] + \varepsilon_0.
\end{equation}

Proof. Straightforward calculations. For example,
\begin{equation}
|p^{\ell(R)}| = \sum_{k \geq 1} 2(p^k - 1)r_{k+1} = 2(pr_2 + p^2r_3 + \cdots) - 2(r_2 + r_3 + \cdots)
\end{equation}

\begin{equation}
= 2(r_1 + pr_2 + p^2r_3 + \cdots) - 2(r_1 + r_2 + \cdots) = 2\sigma[R] - 2\ell[R].
\end{equation}

Other calculations are similar.

§5.1. Mod $p$ cohomology rings of mod $p^h$ Eilenberg–MacLane spaces in terms of the Milnor basis, and $Q$-subalgebras. We now give a description of mod $p$ cohomology rings of mod $p^h$ Eilenberg–MacLane spaces in terms of the Milnor basis. To keep notations simple, we adopt the following convention:

Convention. From now on, whenever $Q_0$ is applied directly to the mod $p$ fundamental class $t_{n+1} \in H^{n+1}(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$, the element $Q_0t_{n+1}$ should be understood to be $\delta_ht_{n+1}$, where $\delta_h$ is the $h$-th Bockstein operator.

Formula (5-3) below appears in [3]. But our proof is more direct and straightforward using decomposition formulae in §4.

Theorem 5-2. Let $E = (\varepsilon_0, \varepsilon_1, \ldots)$ be a sequence of zeroes and ones which are almost all zero, and let $R = (r_1, r_2, \ldots)$ be a sequence of non-negative integers which are almost all zero. Let $h \geq 1$ and $n \geq 0$. Then the action of Milnor basis elements $Q^{E^0p^R}$ on the fundamental class $t_{n+1} \in H^{n+1}(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$ has the following properties:
\begin{equation}
Q^{E^0p^R}t_{n+1} = 0 \quad \text{if} \quad \ell[E] + 2\ell[R] \geq n + 2,
\end{equation}

\begin{equation}
Q^{E^0p^R}t_{n+1} = (Q^{t(E)}p^{\ell(R)}t_{n+1})^p \quad \text{if} \quad \ell[E] + 2\ell[R] = n + 1 \text{ and } \varepsilon_0 = 0.
\end{equation}

The mod $p$ cohomology of the mod $p^h$ Eilenberg–MacLane space is a free algebra described in terms of the Milnor basis as follows:
\begin{equation}
H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p) = F_{\mathbb{Z}_p}\{Q^{E^0p^R}t_{n+1} \mid \ell[t(E)] + 2\ell[R] < n + 1\}.
\end{equation}

Proof. Here, we prove (5-3) and (5-4). After some preparations, we prove (5-5). (See the paragraph after Proposition 5-6.) We use decomposition formulae (4-9) and (4-12). Applying $Q^{E^0p^R}$ to the fundamental class $t_{n+1}$, we obtain
\begin{equation}
Q^{E^0p^R}t_{n+1} = \sum_{S \in \mathcal{R}, s_1 = 0} c(S)Q^{E^0p(\sigma[R] + \sigma[t(S)])\Delta_1 \cdot p^{\ell(R)} - t(S) + t^2(S)}t_{n+1}
\end{equation}

\begin{equation}
= \sum_{S \in \mathcal{R}, s_1 = 0} c(S) \cdot Q^{0_0} \sum_{E_1 + E_2 = t(E)} \operatorname{sgn}(E_1, E_2)(-1)^{\ell[E_1]} Q^{E_1 \cdot p(\sigma[R] + \sigma[t(S)] + \sigma[t(E)])\Delta_1}
\end{equation}

\begin{equation}
\cdot Q^{E_2 \cdot p^{\ell(R)} - t(S) + t^2(S)}t_{n+1}.
\end{equation}
Using Lemma 5-1, we observe that
\[
|Q^{E_2, pt(R) - t(S) + t^2(S)}_{t_{n+1}}| \leq |Q^{t(E), pt(R) - t(S) + t^2(S)}_{t_{n+1}}| \\
= 2(\sigma[R] + \sigma[t(S)] + \sigma[t(E)]) + (n + 1 + \varepsilon_0 - \ell[E] - 2\ell[R]) - 2\sigma[t(S)].
\]
Hence if \(\ell[E] + 2\ell[R] > n + 1 + \varepsilon_0\), then for any \(E_2 \leq t(E)\) and \(S \geq \Delta_0\) as above we have
\[
|Q^{E_2, pt(R) - t(S) + t^2(S)}_{t_{n+1}}| < 2(\sigma[R] + \sigma[t(S)] + \sigma[t(E)]).
\]

Thus by dimensional reason (2-11), we have
\[
\sigma[R] + \sigma[t(S)] + \sigma[t(E)]|_{t_{n+1}} = 0,
\]
for any \(E_2\) and \(S\). Hence all the terms in the above summation vanish, and we have \(Q^{E, pt(R)}_{t_{n+1}} = 0\).

When \(\ell[E] + 2\ell[R] = n + 1 + \varepsilon_0\), the element
\[
\sigma[R] + \sigma[t(S)] + \sigma[t(E)]|_{t_{n+1}} = 0,
\]
which is nontrivial only when \(S = \Delta_0\) and \(E_2 = t(E)\) by dimensional reason as above. There is only one such term in the double summation above, and we have
\[
Q^{E, pt(R)}_{t_{n+1}} = Q_0^0 (Q^{t(E), pt(R)}_{t_{n+1}}) = 0,
\]
since \(|Q^{t(E), pt(R)}_{t_{n+1}}| = 2(\sigma[R] + \sigma[t(E)])\). This element vanishes if \(\varepsilon_0 = 0\) by the derivation property of \(Q_0\). This proves (5-3). When \(\varepsilon_0 = 0\), we get (5-4). □

The statement (5-5), which will be proved shortly, implicitly says that when \(\ell[E] + 2\ell[R] < n + 1 + \varepsilon_0\) (note that this includes the case \(\ell[E] + 2\ell[R] = n + 1\) and \(\varepsilon_0 = 0\)), the element \(Q^{E, pt(R)}_{t_{n+1}}\) is nontrivial because it is an algebra generator. For the case \(\ell[E] + 2\ell[R] = n + 1\) with \(\varepsilon_0 = 0\), formula (5-4) does not explicitly state nontriviality of the element. But repeated use of (5-4) reveals that it is nontrivial, as shown in the next corollary, assuming (5-5) for a moment. This corollary can also be found in [3].

**Corollary 5-3.** Let sequences \(E\) and \(R\) be such that \(\ell[E] + 2\ell[R] = n + 1\) with \(\varepsilon_0 = 0\). Let \(k = \min\{j \mid \varepsilon_j \neq 0 \text{ or } r_j \neq 0\}\). Then
\[
(5-6) \quad Q^{E, pt(R)}_{t_{n+1}} = (Q^{t^k(E), pt^k(R)}_{t_{n+1}})^p \neq 0,
\]
where \(\ell[t^{k+1}(E)] + 2\ell[t^k(R)] < n + 1\), and consequently \(Q^{t^k(E), pt^k(R)}_{t_{n+1}}\) is an indecomposable free algebra generator of the cohomology \(H^*(K(Z/p^n, n + 1); Z_p)\).

**Proof.** Since \(\ell[t^i(E)] + 2\ell[t^i(R)] = n + 1\) and \(\varepsilon_i = 0\) for \(0 \leq i \leq k - 1\), repeated use of (5-4) gives \(Q^{E, pt(R)}_{t_{n+1}} = (Q^{t^k(E), pt^k(R)}_{t_{n+1}})^p\). Since \(t^k(E) = (\varepsilon_k, \varepsilon_{k+1}, \ldots)\) and \(t^k(R) = (r_{k+1}, r_{k+2}, \ldots)\), we have \(\ell[t^{k+1}(E)] + 2\ell[t^k(R)] = \ell[E] + 2\ell[R] - 2r_k = n + 1 - 2r_k\). By our choice of \(k\), we have either \(r_k \neq 0\) or \(\varepsilon_k \neq 0\). If \(r_k \neq 0\), then \(n + 1 - 2r_k < n + 1 \leq n + 1 + \varepsilon_k\). If \(\varepsilon_k \neq 0\), then \(n + 1 - 2r_k \leq n + 1 < n + 1 + \varepsilon_k\). Hence in either case we have \(\ell[t^{k+1}(E)] + 2\ell[t^k(R)] < n + 1\), and \(Q^{t^k(E), pt^k(R)}_{t_{n+1}}\)
is an indecomposable free algebra generator of the cohomology ring by (5-5). This completes the proof. □

The statement (5-5) is proved by reformulating the classical description of the cohomology algebra described in §3 in terms of admissible monomials. We compare Milnor basis elements and admissible monomials in the Steenrod algebra. To do this, we apply (4-9) and (4-12) repeatedly to $Q^E_pR_{t_{n+1}}$. We have

$$Q^E_pR_{t_{n+1}} = Q^E_p\sigma[R]_{t_{n+1}} + \text{(other terms)}$$

where

$$Q^E_p\sigma[R]_{t_{n+1}} = Q_0^E\sigma[R] + \sigma[t(E)]\Delta_1Q_t^E + \text{(other terms)}.$$ 

Repeating this procedure on $Q^E_p\sigma[R]_{t_{n+1}}$ again and again, we get $Q^E_pR_{t_{n+1}} = \theta(E, R) + \text{(other terms)}$, where

$$\theta(E, R) = Q_0^{\epsilon_1}p_{s_1}\Delta_1Q_0^{\epsilon_2}p_{s_2}\Delta_1Q_0^{\epsilon_3}p_{s_3}\Delta_1 \ldots Q_0^{\epsilon_r}p_{s_r}\Delta_1Q_0^{\epsilon_{r+1}} \ldots,$$

with $s_r = \sigma[t^{-1}(R)] + \sigma[t^r(E)]$ for $r \geq 1$.

If we let $S = (s_1, s_2, \ldots)$, then after an easy calculation we find that

$$S = \sum_{j \geq 0} p^j \cdot t^j(R) + \sum_{j \geq 0} p^j \cdot t^{j+1}(E) = \frac{R + t(E)}{1 - pt},$$

where the last formal operator expression makes sense since the sequences $E$ and $R$ have entries which are almost all zero.

**Lemma 5-4.** Let $E$ be a sequence of zeroes and ones which are almost all zero, and let $R$ be a sequence of non-negative integers which are almost all zero. Let $\theta(E, R)$ be a monomial in Steenrod reduced powers and the Bockstein operator $Q_0$ as in (5-7). Then the following statements hold:

1. The monomial $\theta(E, R)$ is admissible.
2. The excess of the admissible monomial $\theta(E, R)$ is given by

$$(5-9) \quad e_p(\theta(E, R)) = \ell(t(E)) + 2\ell[R] = \ell[E] + 2\ell[R] - \epsilon_0.$$ 

This correspondence $(E, R) \rightarrow \theta(E, R)$ gives rise to the following bijection:

$$(5-10) \quad \{(E, R)\} \xrightarrow{1:1} \{\text{admissible monomials in } A(p)^*\},$$

where $E$ ranges over all sequences of zeroes and ones which are almost all zero, and $R$ ranges over all sequences of non-negative integers which are almost all zero.

**Proof.** To check the admissibility of the monomial $\theta(E, R)$, we can check either (1) or (2) in Lemma 3-1. For our purpose, condition (2) suits better. From (5-8), we have

$$S - p \cdot t(S) - t(E) = R,$$

or $s_j - (ps_{j+1} + \epsilon_j) = r_j \geq 0$ for all $j \geq 1$ in terms of components of the sequences. Hence condition (2) in Lemma 3-1 is satisfied, and $\theta(E, R)$ is an admissible monomial.

Next, we calculate excess of the monomial $\theta(E, R)$. First we calculate $\tau_p(\theta)$ given in (3-3). For the monomial $\theta(E, R)$, the integer $i_j$ in (3-2) is given by

$$i_j = \epsilon_{j-1} + 2(p - 1)s_j = \epsilon_{j-1} + 2(p - 1)\sigma[t^{j-1}(R)] + 2(p - 1)\sigma[t^j(E)].$$
A simple calculation shows that \( i_j - pi_{j+1} = \varepsilon_{j-1} + (p-2)\varepsilon_j + 2(p-1)r_j \) for all \( j \geq 1 \). Thus, \( \tau_p(\theta(E, R)) = \sum_{j \geq 1} (i_j - pi_{j+1}) = \varepsilon_0 + (p-1)(\ell(E) + 2p-1)(\ell[R].

Hence our modified excess given in (3-4) is equal to \( e_p(\theta(E, R)) = \ell[E] + 2\ell[R]. \) Since \( \ell[E] = \ell[E] - \varepsilon_0 \), we get (5-9).

To check the bijection between the set of pairs of sequences \((E, R)\) and the set of admissible monomials, let

\[
\theta = Q_0^{\varepsilon_0} Q_0^{\varepsilon_1} Q_0^{\varepsilon_2} \ldots \]

be any admissible monomial. Let \( E_\theta = (\varepsilon_0, \varepsilon_1, \ldots) \) and \( S_\theta = (s_1, s_2, \ldots) \) be the exponent sequences associated with the monomial \( \theta \). Then \( E_\theta \) is a sequence of zeroes and ones which are almost all zero. By the admissibility condition, these integers must satisfy \( s_j - (px_{j+1} + \varepsilon_j) \geq 0 \) for all \( j \geq 1 \). Let this integer be \( r_j \) and let \( R_\theta = (r_1, r_2, \ldots) \). Then \( R_\theta \) is a sequence of non-negative integers which are almost all zero. From this calculation, we have

\[
R_\theta = S_\theta - (p-t(S_\theta) + t(E_\theta)) = (1-p-t)(S_\theta) - t(E_\theta).
\]

Thus, given an admissible monomial \( \theta \) in \( A(p)^* \), we have obtained a pair of sequences \((E_\theta, R_\theta)\) with the property stated in Lemma 5-4. For this \((E_\theta, R_\theta)\), the \( S \) sequence for the corresponding \( \theta(E_\theta, R_\theta) \) given by the formula (5-8) is \( (R_\theta + t(E_\theta))/(1-p-t) = S_\theta \), which is the original \( S \) sequence for \( \theta \). Thus the correspondence \( \theta \rightarrow (E_\theta, R_\theta) \rightarrow \theta(E_\theta, R_\theta) \) is the identity map. It is also immediate to check that the correspondence \( (E, R) \rightarrow \theta(E, R) \rightarrow (E_\theta, R_\theta) \) is also the identity map. This proves that the correspondence (5-10) between the set of pairs of sequences \((E, R)\) and the set of admissible monomials in \( A(p)^* \) is a bijection. This completes the proof. \( \square \)

We remark that \( \ell[E] + 2\ell[R] \) is called excess in [3]. His excess differs from ours by 0 or 1. This difference is essential when we describe free algebra generators of the cohomology rings of Eilenberg–MacLane spaces.

Since sequences \( E \) and \( R \) terminate eventually, the associated admissible monomial \( \theta(E, R) \) also terminates eventually. We examine how \( \theta \) terminates depending on the pair of sequences \((E, R)\).

**Lemma 5-5.** Let \( E = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\ell, 0, \ldots) \) be a sequence of zeroes and ones such that \( \varepsilon_\ell = 1 \), and \( R = (r_1, r_2, \ldots, r_k, 0, \ldots) \) be a sequence of non-negative integers such that \( r_k > 0 \).

(I) If \( \ell \geq k \), then the associated admissible monomial \( \theta(E, R) \) ends with \( Q_0 \) and is of the form

\[
\theta(E, R) = Q_0^{\varepsilon_0} Q_0^{\varepsilon_1} \ldots Q_0^{\varepsilon_{\ell-k}} Q_0^{r_k} \Delta_1 \ldots Q_0^{r_k-1} \Delta_1 \ldots.
\]

(II) If \( \ell < k \), then the associated admissible monomial \( \theta(E, R) \) ends with a Steenrod reduced power \( \mathfrak{p}^{r_k} \Delta_1 \), and it is of the form

\[
\theta(E, R) = Q_0^{\varepsilon_0} Q_0^{\varepsilon_1} \ldots Q_0^{r_k} \mathfrak{p}^{r_k} \Delta_1.
\]

**Proof.** This follows immediately from the formula for \( S \) given in (5-8). \( \square \)
We want to examine the relationship between $Q^E \mathcal{P}^R$ and $\theta(E, R)$ more closely. Given sequences $E = (\varepsilon_0, \varepsilon_1, \ldots)$ and $R = (r_1, r_2, \ldots)$ as before, let

\begin{equation}
I(E, R) = (\varepsilon_0, r_1, \varepsilon_1, r_2, \ldots, r_k, \varepsilon_k, r_{k+1}, \ldots).
\end{equation}

Note that if $I(E, R)$ ends with an entry from $E$, then we are in (I) of Lemma 5-5 and $\theta(E, R)$ ends with $Q_0$. If $I(E, R)$ ends with an entry from $R$, then we are in (II) of Lemma 5-5 and $\theta(E, R)$ ends with a Steenrod reduced power.

Following Milnor, we introduce a lexicographic total ordering from the right on the set of all such sequences $I(E, R)$. Note that for a given pair $(E, R)$, there are finitely many pairs $(E', R')$ of the same degree such that $I(E', R') < I(E, R)$.

**Proposition 5-6** [cf. 4, Lemma 8]. For any pair of sequences $(E, R)$ as above,

\begin{equation}
\theta(E, R) = \pm Q^E \mathcal{P}^R + \sum_{\ell(E') = \ell(E)} c(E', R') Q^E \mathcal{P}^R,
\end{equation}

for some coefficients $c(E', R') \in \mathbb{Z}_p$. The above summation is finite.

**Proof.** The above statement is in [4] except the statement that $c(E', R') = 0$ unless $\ell[E] = \ell[E']$. To see this, it is enough to show that that the nontriviality of the dual pairing $(\theta(E, R), \tau^E \xi^R) = c(E', R')$ implies that $\ell[E] = \ell[E']$.

To show this, following Milnor, for $k \geq 1$, let $M_k = \mathcal{P}^{p^{k-1} \Delta_1} \ldots \mathcal{P}^{p \Delta_1} \mathcal{P} \Delta_1$. Then it is well-known that $(\theta, \xi_k)$ is nontrivial only when $\theta = M_k$, and that $(\theta, \tau_k) \neq 0$ only when $\theta = M_k Q_0$. (See Lemma 7 in [4].)

Now suppose $(\theta(E, R), \tau^E \xi^R) \neq 0$. We apply the diagonal map in the Steenrod algebra $\ell[E'] + \ell[R']$ times to the element $\theta(E, R)$. Each term in the resulting expression consists of $\ell[E'] + \ell[R']$ tensor products. For the nontriviality of the above pairing, in this iterated diagonal expression there must be a term such that $\ell[E']$ tensor factors among $\ell[E'] + \ell[R']$ factors contain exactly one Bockstein each, due to the above property of dual pairings for $\xi_k$ and $\tau_k$. Since $Q_0$ is primitive, the number of Bocksteins does not change under the diagonal map, and hence it is equal to $\ell[E]$. So we must have $\ell[E] = \ell[E']$. This completes the proof. \qed

We can now complete the proof of (5-5) in Theorem 5-2.

**Completion of the proof of Theorem 5-2.** From Theorem 3-4 (I) and Lemma 5-4, the mod $p$ cohomology ring of the mod $p^h$ Eilenberg–Mac Lane space $K(\mathbb{Z}/p^h, n+1)$ can be described as follows in terms of admissible monomials $\theta(E, R)$:

\begin{equation}
H^* (K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p) = F_{\mathbb{Z}_p} [\theta(E, R)_{t_{n+1}} | \ell[t(E)] + 2\ell[R] < n+1].
\end{equation}

Since $Q^E \mathcal{P}^R_{t_{n+1}} \in H^* (K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p)$ for any pair of sequences $(E, R)$, we consider a subalgebra generated by some of these elements:

\begin{equation}
\mathbb{Z}_p \{ Q^E \mathcal{P}^R_{t_{n+1}} | \ell[t(E)] + 2\ell[R] < n+1 \} \subset H^* (K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p).
\end{equation}

Here, $\mathbb{Z}_p \{ \cdot \}$ denotes a subalgebra generated by elements in $\{ \cdot \}$. There may be some algebraic relations among these generators. Our aim is to show that there are no extra relations among the elements $Q^E \mathcal{P}^R_{t_{n+1}}$ other than obvious relations that
squares of odd degree elements are 0, and the inclusion relation in (5-16) is actually an identity. To see this, we use Milnor’s result (5-14). For any pair of sequences \((E, R)\) such that \(\ell(t(E)) + 2\ell(R) < n + 1\), we have 

\[
\theta(E, R)_{t_{n+1}} = \pm Q^E PR_{t_{n+1}} + \sum_{\ell(E', R') \leq \ell(E, R)} c(E', R')Q^E PR_{t_{n+1}}.
\]

In the above summation, if \(\ell(E') + 2\ell(R') \geq n + 2\), then we have \(Q^E PR_{t_{n+1}} = 0\) by (5-3). If \(\ell(E') + 2\ell(R') = n + 1\) and \(\varepsilon_0 = 0\), then the element \(Q^E PR_{t_{n+1}}\) is a \(p\)-th power of some element \(Q^E PR_{t_{n+1}}\) for which \(\ell(t(E')) + 2\ell(R') < n + 1\) for some \(k\) by (5-6) in Corollary 5-3. This shows that for any pair of sequences \((E, R)\) such that \(\ell(t(E)) + 2\ell(R) < n + 1\), we have

\[
(5-17) \quad \theta(E, R)_{t_{n+1}} \in \mathbb{Z}_p \{ Q^E PR_{t_{n+1}} \mid \ell(t(E')) + 2\ell(R') < n + 1 \}.
\]

This means that any algebra generator \(\theta(E, R)_{t_{n+1}}\) of the cohomology ring (5-15) is contained in the subalgebra (5-16). Thus this subalgebra must be the entire cohomology algebra and the inclusion relation in (5-16) is actually an identity. Since the number of elements of the form \(\theta(E, R)_{t_{n+1}}\) and that of the form \(Q^E PR_{t_{n+1}}\) in a given degree are equal, we see that the algebra generators in the left hand side of (5-16) are in fact free algebra generators, because elements \(\theta(E, R)_{t_{n+1}}\) are free generators by (5-15). This completes the proof of (5-5), and also completes the proof of Theorem 5-2. □

We examine the action of Milnor primitives \(Q_i\) on the fundamental cohomology class \(t_{n+1}\). By letting \(R = 0\) in Theorem 5-2, we have the following corollary.

**Corollary 5-7.** In the cohomology ring \(H^*(K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p)\), the following statements hold:

1. \(Q^E t_{n+1} = 0\) if \(\ell[E] \geq n + 2\).
2. \(Q^E t_{n+1} = (Q^t(E)_{t_{n+1}})^p\) if \(\ell[E] = n + 1\) and \(\varepsilon_0 = 0\).
3. \(Q^E t_{n+1} \neq 0\) if \(\ell[E] \leq n + 1\).
4. When \(\ell[E] = n + 1\) and \(\varepsilon_0 = 1\), or \(\ell[E] \leq n\), the element \(Q^E t_{n+1}\) is a free generator of the cohomology ring \(H^*(K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p)\).

We note that (3) in Corollary 5-7 shows that any product of \(n + 1\) distinct Milnor primitives can act nontrivially on the fundamental cohomology class \(t_{n+1}\). The element of the smallest positive degree obtained in this way is

\[
(5-18) \quad (Q_0 t_{n+1})_{\mathbb{Z}_p} H_2(1 + p + p^2 + \cdots + p^n) (K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p).
\]

Here when \(h > 1\), \(Q_0 t_{n+1}\) really means \(\delta_h t_{n+1}\) in terms of the \(h\)-th Bockstein. But we keep using the notation \(Q_0\) in accordance with the convention stated right before Theorem 5-2. By (1) in Corollary 5-7, no more Milnor primitives can act nontrivially. However, some Steenrod reduced powers can still act nontrivially. Nontrivial actions are described as follows. Recall that \(\Delta_0 = (0, 0, \ldots, 0, \ldots)\).

**Lemma 5-8.** Let sequences \(E\) and \(R\) be as usual. Then

\[
Q^E PR_{Q_i Q_n Q_{n-1} \cdots Q_0 t_{n+1} \neq 0} \iff \begin{cases} E = \Delta_0 \text{ and } R = \sum_{j=0}^{n} p^j \Delta_{s_j - j} \text{ for some} \\ \text{set of mutually distinct non-negative} \\ \text{integers } \{s_0, s_1, \ldots, s_n\} \text{ such that } s_j \geq j. \end{cases}
\]
When $E = \Delta_0$ and $R = \sum_{j=0}^{n} p^j \Delta_{s_j-j}$ for a set $\{s_0, s_1, \ldots, s_n\}$ as above,
\begin{equation}
\mathcal{P}^R Q_n \cdots Q_{0^t n+1} = Q_{s_0} Q_{s_0-1} \cdots Q_{s_0 t n + 1} \\
= (Q_{s_0-k} Q_{s_0-1-k} \cdots Q_{s_0-k Q_{s_0-k t n + 1}})^{p^k} \neq 0,
\end{equation}
where $k$ is the smallest integer among $s_j$'s, and the last element inside the parenthesis is a polynomial generator of the cohomology ring $H^* (K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p)$.

**Proof.** Suppose $Q^E \mathcal{P}^R Q_n Q_{n-1} \cdots Q_{0^t n+1} \neq 0$. By (2-5), we have
\[Q^E \mathcal{P}^R Q_n Q_{n-1} \cdots Q_{0^t n+1} = \sum_{(i_0, \ldots, i_n) \geq \Delta_0} Q^E Q_{n+i_0} \cdots Q_{1+i_1} Q_{i_0} \mathcal{P}^R - \sum_{k=0}^{n} p^k \Delta_{i_k t n + 1}.\]
Since there are $n + 1 + \ell[E]$ Milnor primitives, an element of the above form can be nonzero only when $E = \Delta_0$ and $R - \sum_{k=0}^{n} p^k \Delta_{i_k} = 0$ by (5-3). In this case,
\[Q^E \mathcal{P}^R Q_n \cdots Q_{0^t n+1} = Q_{n+i_0} \cdots Q_{1+i_1} Q_{i_0} \mathcal{P}^R - \sum_{(i_0, \ldots, i_n) \geq \Delta_0} Q^E \mathcal{P}^R Q_n \cdots Q_{0^t n+1} = 0.
\]
This element is nonzero if and only if the integers $n + i_n, 1 + i_1, i_0$ are mutually distinct. Letting $s_j = j + i_j$ for $0 \leq j \leq n$, we get the first part of the statement. The second part follows from (2), (3), and (4) of Corollary 5-7. \qed

From Theorem 5-2, an element of the form $Q_{s_0} \cdots Q_{s_1} Q_{s_0 t n + 1}$ for some integers $0 \leq s_0 < s_1 < \cdots < s_n$ is an indecomposable polynomial generator of the cohomology ring if and only if $s_0 = 0$. To deal with all such sequences, we let
\begin{equation}
S_n^+ = \{(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n \mid 0 < s_1 < s_2 < \cdots < s_n\}.
\end{equation}
To each sequence $S = (s_1, s_2, \ldots, s_n) \in S_n^+$, we associate an element
\begin{equation}
Q_S Q_{0^t n+1} = Q_{s_1} \cdots Q_{s_1} Q_{0^t n+1} \in H^* (K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p).
\end{equation}
Note that this generator has even degree $2(1 + p^{s_1} + p^{s_2} + \cdots + p^{s_n})$, and hence it is a polynomial generator in the cohomology ring for any $S \in S_n^+$. We consider a subring generated by these elements.

**Theorem 5-9.** The polynomial subalgebra
\begin{equation}
\mathcal{Q} = \mathcal{Q}(\mathbb{Z}/p^h, n + 1) = \mathbb{Z}_p[Q_S Q_{0^t n+1} \mid S \in S_n^+] \subset H^* (K(\mathbb{Z}/p^h, n + 1); \mathbb{Z}_p)
\end{equation}
is the smallest $A(p)^*$-invariant subalgebra containing the element $Q_n \cdots Q_{0^t n+1}$ of degree $2(1 + p + \cdots + p^n)$. Any Milnor primitive acts trivially on the subalgebra $\mathcal{Q}$.

**Proof.** Since there are already $n + 1$ Milnor primitives in the element $Q_S Q_{0^t n+1}$, no more Milnor primitives can act nontrivially on it. By the derivation property of Milnor primitives, they act trivially on the entire polynomial subalgebra $\mathcal{Q}$. By Lemma 5-8 and the Cartan formula of the Steenrod reduced powers (2-10), we also see that the above subalgebra is preserved under the action of Steenrod reduced powers. Thus $\mathcal{Q}$ is invariant under the action of the entire Steenrod algebra $A(p)^*$. The algebra $\mathcal{Q}$ contains the element $Q_n \cdots Q_{0^t n+1}$, and all the other algebra generator of $\mathcal{Q}$ can be obtained by the action of Steenrod reduced powers as
\( \mathcal{A}(p)^*-\)invariant subalgebra of the cohomology ring \( H^*(K(\mathbb{Z}/p^h, n+1); \mathbb{Z}_p) \) containing the element \( Q_n \cdots Q_1Q_0 \mathcal{I}_{n+1} \) should also contain the subalgebra \( Q \). Thus, \( Q \) is the smallest \( \mathcal{A}(p)^*-\)invariant subalgebra containing \( Q_n \cdots Q_1 Q_0 \mathcal{I}_{n+1} \). This completes the proof. \( \square \)

**Remark.** In [10], it is shown that the \( Q \)-subalgebra (5-22) is precisely the image of the BP-Thom map for the Eilenberg–Mac Lane space:

\[
\rho_* : \text{BP}^*(K(\mathbb{Z}/p^h, n+1)) \to HZ_p^*(K(\mathbb{Z}/p^h, n+1)).
\]

The mod \( p \) cohomology ring of the BP-spectrum is the following cyclic module over \( \mathcal{A}(p)^* \) generated by the BP-Thom map \( \rho \):

\[
HZ_p^*(\text{BP}) = \mathcal{A}(p)^*/(Q_0, Q_1, \ldots, Q_n, \ldots).p.
\]

It follows that the image of the BP-Thom map \( \rho_* \) is always annihilated by Milnor primitives, whatever the space is. This “explains” the fact that the subalgebra \( Q \) in (5-22) is annihilated by all Milnor primitives.

### §5.2. Mod \( p \) cohomology rings of integral Eilenberg–Mac Lane spaces in terms of the Milnor basis, and \( Q \)-subalgebras.

We consider the mod \( p \) cohomology of the integral Eilenberg–Mac Lane space \( K(\mathbb{Z}, n+2) \) which is related to the mod \( p^h \) Eilenberg–Mac Lane space \( K(\mathbb{Z}/p^h, n+1) \) by the Bockstein map \( \delta_h : K(\mathbb{Z}/p^h, n+1) \to K(\mathbb{Z}, n+2) \) for \( n \geq 0 \). The mod \( p \) cohomology rings for these spaces are closely related (see Proposition 5-15 below). Since \( K(\mathbb{Z}, 1) \cong S^1 \) is homotopically rather trivial, we do not deal with it.

Recall that for sequences \( E = (\varepsilon_0, \varepsilon_1, \ldots) \) of zeroes and ones which are almost all zero, and \( R = (r_1, r_2, \ldots) \) of non-negative integers which are almost all zero, we defined another sequence \( I(E, R) \) in (5-13).

Proof of the next theorem is basically the same as the proof of Theorem 5-2. However, the property \( Q_0 \mathcal{I}_{n+2} = 0 \) requires extra care for the proof of (5-25).

**Theorem 5-10.** Let \( n \geq 0 \) and let \( p \) be odd. Let \( \mathcal{I}_{n+2} \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_p) \) be the mod \( p \) fundamental class. Let \( E, R, \) and \( I(E, R) \) be as above. Then the following statements hold:

\[
\begin{align*}
(5-23) \quad & Q^E P^R \mathcal{I}_{n+2} = 0 \quad \text{if } \ell[E] + 2\ell[R] \geq n + 3, \\
(5-24) \quad & Q^E P^R \mathcal{I}_{n+2} = (Q^{I(E)} P^{I(R)} \mathcal{I}_{n+2})^p \quad \text{if } \ell[E] + 2\ell[R] = n + 2 \text{ and } \varepsilon_0 = 0.
\end{align*}
\]

The mod \( p \) cohomology ring of the integral Eilenberg–Mac Lane space \( K(\mathbb{Z}, n+2) \) is a free algebra given by

\[
H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p) = F_{\mathbb{Z}_p} \left[ Q^E P^R \mathcal{I}_{n+2} \mid \ell[I(E)] + 2\ell[R] < n + 2 \text{ and } I(E, R) \text{ ends with an entry from } R \right].
\]

**Proof.** For the proof of (5-23) and (5-24), we can apply the same argument used to prove (5-3) and (5-4), since only dimension of fundamental classes is relevant for the argument.
In (3-7), the mod $p$ cohomology ring of an integral Eilenberg–MacLane space was described in terms of admissible monomials. First we rewrite this description in terms of $\theta(E, R)$’s. By (5-9), Lemma 5-5, and (5-13), we have

\[(5-26)\]

\[H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p) = F_{Z_p}[\theta(E, R)\tau_{n+2} \mid \ell[\ell(t(E))] + 2\ell[R] < n + 2 \text{ and } I(E, R) \text{ ends with an entry from } R].\]

Now we consider the following subalgebra of this cohomology algebra:

\[(5-27)\]

\[\mathbb{Z}_p \left\{ Q^{E\cdot p^R \tau_{n+2}} \mid \ell[\ell(t(E))] + 2\ell[R] < n + 2 \text{ and } I(E, R) \text{ ends with an entry from } R \right\} \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p).\]

Here, as before, we are considering a subalgebra generated by elements inside of \(\{ \cdot \}\), which might satisfy some nontrivial algebraic relations. Our aim is to show that there are no nontrivial algebraic relations other than obvious ones coming from dimensional reason, and that the inclusion relation in (5-27) is actually an identity.

Let \((E, R)\) be such that \(\ell[t(E)] + 2\ell[R] < n + 2\) and \(I(E, R)\) ends with an entry from \(R\). By Proposition 5-6, we have

\[(5-28)\]

\[\theta(E, R)\tau_{n+2} = \pm Q^{E\cdot p^R \tau_{n+2}} + \sum_{I(E', R') < I(E, R)} c(E', R')Q^{E\cdot p^R' \tau_{n+2}},\]

for some constants \(c(E', R') \in \mathbb{Z}_p\). In (5-28) if \(\ell[t(E')] + 2\ell[R'] > n + 2\), then \(Q^{E\cdot p^R' \tau_{n+2}}\) is zero by (5-23). So we may restrict the summation in (5-28) to those \((E', R')\) such that \(\ell[t(E')] + 2\ell[R'] \leq n + 2\).

If \(I(E', R')\) ends with an entry from \(E'\), then \(\theta(E', R')\tau_{n+2} = 0\) since \(\theta(E', R')\) ends with \(Q_0\) by (1) in Lemma 5-5 and \(Q_0\tau_{n+2} = 0\). Hence Proposition 5-6 implies

\[Q^{E\cdot p^R' \tau_{n+2}} = \sum_{I(E', R') < I(E, R)} c(E', R', R'')Q^{E''\cdot p^{R''} \tau_{n+2}},\]

where the summation can be restricted to those with \(\ell[t(E'')] + 2\ell[R''] \leq n + 2\) by the same reason as above. Repeating this procedure, we see that an element \(Q^{E\cdot p^R' \tau_{n+2}}\) for which \(I(E', R')\) ends with an entry from \(E'\) can be replaced by a linear combination of elements \(Q^{E''\cdot p^{R''} \tau_{n+2}}\) for which \(I(E'', R'')\)’s end with entries from \(R''\), \(I(E', R'') < I(E', R')\), and \(\ell[t(E'')] + 2\ell[R''] \leq n + 2\).

Since there are finitely many \((E', R')\) such that \(I(E', R') < I(E, R)\), after repeating the above process finitely many times for each term in the summation of (5-28), we see that in formula (5-28) we only have to consider a summation over \((E', R')\) for which \(I(E', R')\) ends with an entry from \(R'\), and for which \(\ell[t(E')] + 2\ell[R'] \leq n + 2\). Since any element \(Q^{E\cdot p^R \tau_{n+2}}\) with \(\ell[t(E')] + 2\ell[R'] = n + 2\) is either 0 or a \(p^k\)-th power of some element \(Q^{E\cdot p^R' \tau_{n+2}}\) for which \(\ell[t(E'')] + 2\ell[R''] < n + 2\) for some \(k\) by (5-23) and (5-24), it follows that any algebra generator \(\theta(E, R)\tau_{n+2}\) of \(H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p)\) is in the subalgebra (5-27). Hence this subalgebra is in fact the entire cohomology algebra. Since the algebra generators in (5-26) and (5-27) are in 1:1 correspondence, we see that the algebra generators in (5-27) are free algebra generators, since generators in (5-26) are free generators. This completes the proof of (5-25). □

Just as in the case of mod \(p^h\) Eilenberg–MacLane spaces, we apply products of Milnor primitives on the fundamental class \(\tau_{n+2}\), and we check whether the resulting elements are trivial or not. It turns out that products of at most \(n\) Milnor primitives can act nontrivially on \(\tau_{n+2}\), as shown in the next proposition.
Proposition 5-11. Let \( \tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_p) \) be the mod \( p \) fundamental class and let sequences \( E, R \) as before. Then

\[
(5-29) \quad Q^E p^R \tau_{n+2} = 0 \quad \text{if } \ell[E] \geq n + 1, \text{ or } \ell[E] = n \text{ and } \ell[R] \geq 2.
\]

Suppose \( \ell[E] = n \) and \( \ell[R] = 1 \). Let \( R = \Delta_j \) for some \( j \geq 1 \), and \( E = (\varepsilon_0, \varepsilon_1, \ldots) \).

Then the following identities hold:

\[
(5-30) \quad Q^E p^\Delta_j \tau_{n+2} = \begin{cases} 
(Q^E p^{\Delta_{j-1}} \tau_{n+2})^p & \text{if } \varepsilon_0 = 0, \\
(-1)^n Q^E - \Delta_1 Q_j \tau_{n+2} & \text{if } \varepsilon_0 = 1 \text{ and } \varepsilon_j = 0, \\
0 & \text{if } \varepsilon_0 = 1 \text{ and } \varepsilon_j = 1.
\end{cases}
\]

Proof. Suppose \( \ell[E] \geq n + 1 \). If \( \ell[R] \geq 1 \), then \( \ell[E] + 2\ell[R] \geq n + 3 \) and it follows that \( Q^E p^R \tau_{n+2} = 0 \) by (5-23). If \( \ell[R] = 0 \), then applying Proposition 5-6, we have

\[
(\ast) \quad Q^E \tau_{n+2} = \pm \theta(E, 0) \tau_{n+2} + \sum_{\substack{I(E', R') < I(E, R) \\
\ell[E'] = \ell[E]}} Q^{E'} p^{R'} \tau_{n+2},
\]

for some \( c(E', R') \in \mathbb{Z}_p \). Since \( \theta(E, 0) \) ends with \( Q_0 \) by (5-11) and \( Q_0 \tau_{n+2} = 0 \), we have \( \theta(E, 0) \tau_{n+2} = 0 \). By the reason described in the proof of Theorem 5-10, we may assume that the second summation is over those pairs \( (E', R') \) such that \( I(E', R') \) ends with an entry from \( R' \), in addition to the above listed conditions. But then, since \( \ell[E'] = \ell[E] \geq n + 1 \) and \( \ell[R'] \geq 1 \), we have \( Q^E p^{R'} \tau_{n+2} = 0 \) by (5-23). Thus all the terms on the right hand side of \( (\ast) \) vanish and we must have \( Q^E \tau_{n+2} = 0 \).

When \( \ell[E] \geq n \) and \( \ell[R] \geq 2 \), then \( \ell[E] + 2\ell[R] \geq n + 4 \). Thus, by (5-23), \( Q^E p^R \tau_{n+2} = 0 \) in this case also. This proves (5-29).

For (5-30), the case for \( \varepsilon_0 = 0 \) follows from (5-24). When \( \varepsilon_0 = 1 \), we have

\[
Q^E = Q_0 Q^E - \Delta_1 = (-1)^{n-1} Q^E - \Delta_1 Q_0 \quad \text{since } \ell[E] = n.
\]

Using \( Q_0 p^\Delta_j = p^\Delta_j Q_0 - Q_j \) which follows from (2-5), we have

\[
Q^E p^\Delta_j \tau_{n+2} = (-1)^{n-1} Q^E - \Delta_1 Q_0 p^\Delta_j \tau_{n+2} = (-1)^{n-1} Q^E - \Delta_1 (p^\Delta_j Q_0 - Q_j) \tau_{n+2} = (-1)^n Q^E - \Delta_1 Q_j \tau_{n+2},
\]

since \( Q_0 \tau_{n+2} = 0 \). When \( \varepsilon_j = 1 \), this element is trivial because \( Q^E - \Delta_1 \) also contains \( Q_j \) and \( Q_j^2 = 0 \). This completes the proof. \( \square \)

Formulae (5-29) and (5-30) provide extra (vanishing) information for those \( (E, R) \) not dealt with in Theorem 5-10.

Suppose \( \ell[E] = n, \ell[R] = 1 \) and \( I(E, R) \) ends with an entry from \( R \). If \( \varepsilon_0 = 0 \), then \( Q^E p^R \tau_{n+2} \) is a \( p \)-th power of some other element by (5-24). If \( \varepsilon_0 = 1 \), then \( \ell[I(E)] + 2\ell[R] = n + 1 \), and by (5-25) the element \( Q^E p^R \tau_{n+2} \) is an algebra generator of the cohomology algebra. Let \( R = \Delta_j \) for some \( j \geq 1 \). Since \( I(E, R) \) must end with an entry from \( R \), the sequence \( E \) must be of the form \( E = \sum_{i=0}^{n-1} \Delta_{s_i+1} \) for some integers \( 0 = s_0 < s_1 < \cdots < s_{n-1} < j \). Let \( j = s_n \). Recall that \( Q^{\Delta_{s_{n+1}}} = Q_{s_n} \).

Then by (5-30), this free algebra generator can be written as

\[
(5-31) \quad Q^{\sum_{i=0}^{n-1} \Delta_{s_{i+1}}} p^\Delta_n \tau_{n+2} = \pm Q_{s_n} Q_{s_{n-1}} \cdots Q_{s_1} \tau_{n+2}.
\]

Since the degree of this element is \( 2(1 + p^{s_1} + p^{s_2} + \cdots + p^{s_n}) \) which is even, the above element is actually a polynomial generator of the cohomology algebra. Recall that we defined a set of sequences \( S^+_n \) in (5-20). We have observed the following.
Proposition 5-12. For any sequence \( S = (s_1, s_2, \ldots, s_n) \in \mathcal{S}_n^+ \), the corresponding even dimensional element \( Q_S \tau_{n+2} = Q_{s_1} \cdots Q_{s_n} \tau_{n+2} \in H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p) \) is an indecomposable polynomial generator of the cohomology ring.

Note that (5-25) only gives one description of a set of free algebra generators. Other descriptions of the same set of generators are certainly possible. Proposition 5-12 says that we can replace the element \( Q_E \mathcal{P} R \tau_{n+2} \) corresponding to \((E, R) = (\sum_{i=1}^{n-1} \Delta_{s_{i+1}}, \Delta_{s_n})\) by the element corresponding to \((E, R) = (\sum_{i=1}^{n} \Delta_{s_{i+1}}, \Delta_0)\), because the only possible difference is the sign by (5-31).

Using these elements, we define the following polynomial subalgebra of the mod \( p \) cohomology ring of the integral Eilenberg–MacLane space \( K(\mathbb{Z}, n+2) \):

\[
Q = Q(\mathbb{Z}, n+2) = \mathbb{Z}_p[Q_{S} \tau_{n+2} \mid S \in \mathcal{S}_n^+] \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p).
\]

The element of the lowest positive degree in the subalgebra \( Q \) is \( Q_n Q_{n-1} \cdots Q_1 \tau_{n+2} \) of degree \( 2(1+p+p^2+\cdots+p^n) \). We calculate the effect of the Steenrod algebra on this element.

Lemma 5-13. We have \( Q_E \mathcal{P} R Q_n \cdots Q_1 \tau_{n+2} \neq 0 \) only when \( E = \Delta_0 \) and \( R \) is of the form \( R = \sum_{j=1}^{n} p^j \Delta_{\ell_j} + \Delta_k \) for some non-negative integers \( \ell_j \) and \( k \). In this case

\[
(5-32) \quad Q = Q(\mathbb{Z}, n+2) = \mathbb{Z}_p[Q_{S} \tau_{n+2} \mid S \in \mathcal{S}_n^+] \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_p).
\]

If \( k > 0 \), then this element is either trivial or a \( p \)-primary power of \( Q_S \tau_{n+2} \) for some \( S' \in \mathcal{S}_n^+ \), and hence it is not a free algebra generator of the cohomology ring. If \( k = 0 \), then the element is trivial unless the set of integers \( \{j + \ell_j\}_{j=1}^{n} \) is a set of mutually distinct integers, in which case the above element is an indecomposable polynomial generator of the cohomology ring.

Similarly, for any \( S = (s_1, \ldots, s_n) \in \mathcal{S}_n^+ \), \( Q_E \mathcal{P} R Q_S \tau_{n+2} \neq 0 \) only when \( E = \Delta_0 \) and \( R \) is of the form \( R = \sum_{j=1}^{n} p^j \Delta_{\ell_j} + \Delta_k \) for some non-negative integers \( \ell_j \) and \( k \). If \( k > 0 \), then this element is either trivial or a \( p \)-primary power of \( Q_S \tau_{n+2} \) for some \( S' \in \mathcal{S}_n^+ \), and hence it is not a free generator. If \( k = 0 \), then the element is trivial unless the set of integers \( \{s_j + \ell_j\}_{j=1}^{n} \) is a set of mutually distinct integers, in which case the element \( \mathcal{P} R Q_S \tau_{n+2} = Q_{s_1+\ell_1} \cdots Q_{s_n+\ell_n} \tau_{n+2} \) is an indecomposable polynomial generator of the cohomology ring.

Proof. Since there are already \( n \) Milnor primitives acting on \( \tau_{n+2} \) in the element \( Q_n \cdots Q_1 \tau_{n+2} \), no more Milnor primitives can act nontrivially on this element by (5-29). Thus, for nontriviality of \( Q_E \mathcal{P} R Q_n \cdots Q_1 \tau_{n+2} \), we must have \( E = \Delta_0 \). Now repeatedly applying (2-5), we have

\[
(5-33) \quad \mathcal{P} R Q_n \cdots Q_1 \tau_{n+2} = \sum_{\ell_1, \ldots, \ell_n \geq 0} Q_{\ell_1+\ell_2} \cdots Q_{\ell_1+\ell_n} \mathcal{P} R - p^\ell \Delta_{\ell_1} - \cdots - p^\ell_{\ell_n} \tau_{n+2}.
\]

Here, \( \mathcal{P} R - p^\ell \Delta_{\ell_1} - \cdots - p^\ell_{\ell_n} = 0 \) unless \( R - p^\ell \Delta_{\ell_1} - \cdots - p^\ell_{\ell_n} \geq \Delta_0 \). Since the term corresponding to the indices \( \ell_1, \ell_2, \ldots, \ell_n \geq 0 \) has \( n \) Milnor primitives, it can be nontrivial only when \( \ell | R - p^\ell \Delta_{\ell_1} - \cdots - p^\ell_{\ell_n} | \leq 1 \) by Proposition 5-11. This implies that the sequence \( R \) must be of the form \( R = \Delta_k + p \Delta_{\ell_1} + \cdots + p^\ell \Delta_{\ell_n} \) for some \( k \geq 0 \). In this case, only one term in (5) is nonzero and we have (5-33). When \( k \geq 1 \),
this is a $p$-th power of another element by the first case in (5-30). Thus it can be
indecomposable only when $k = 0$, and in this case the resulting element is nonzero
and a polynomial generator when the set of integers $\{1 + \ell_1, 2 + \ell_2, \ldots, n + \ell_n\}$ is
a set of $n$ distinct integers by Proposition 5-12.

The second part can be proved in a similar way. □

Now we can characterize the subalgebra $Q$ in (5-32) in terms of the action of the
Steenrod algebra $A(p)^*$. 

**Theorem 5-14.** Let $n \geq 0$. The polynomial subalgebra

$$Q = Q(\mathbb{Z}, n + 2) = \mathbb{Z}_p[Q_S \tau_{n+2} \mid S \in \mathcal{S}^+_n] \subset H^*(K(\mathbb{Z}, n + 2); \mathbb{Z}_p)$$

is the smallest $A(p)^*$-invariant subalgebra containing the element $Q_n \cdots Q_1 \tau_{n+2}$. On
this subalgebra $Q$, all Milnor primitives act trivially.

**Proof.** Since any Milnor primitive acts trivially on any polynomial generator of $Q$
by (5-29) in Proposition 5-11, Milnor primitives act trivially on the entire algebra $Q$
by their derivation property. The action of a Steenrod reduced power $P^R$ on
a polynomial generator $Q_S \tau_{n+2}$ for $S \in \mathcal{S}^+_n$ is described in Lemma 5-13, which
shows that we have $P^R Q_S \tau_{n+2} \in Q$ for any $S \in \mathcal{S}^+_n$ and for any sequence $R$
of non-negative integers which are almost all zero. From this and the Cartan formula
(2-10), it is clear that $P^R$ preserves the algebra $Q$ for any $R$. Hence the polynomial
subalgebra $Q$ is invariant under the action of the Steenrod algebra $A(p)^*$.

Since all the polynomial generator of $Q$ can be obtained by applying certain
Steenrod reduced powers to the lowest positive degree element $Q_n \cdots Q_1 \tau_{n+2}$ by
Lemma 5-13, the algebra $Q$ is contained in any $A(p)^*$-invariant subalgebra of the
mod $p$ cohomology ring $H^*(K(\mathbb{Z}, n + 2); \mathbb{Z}_p)$ containing the element $Q_n \cdots Q_1 \tau_{n+2}$. Hence $Q$ is the smallest $A(p)^*$-invariant subalgebra containing $Q_n \cdots Q_1 \tau_{n+2}$. This
completes the proof. □

**Remark.** In [10], we have shown that the subalgebra $Q$ is precisely the image of the
BP-Thom map for the integral Eilenberg–MacLane space:

$$\rho_n : BP^*(K(\mathbb{Z}, n + 2)) \to HZ^p_p(K(\mathbb{Z}, n + 2)).$$

Since any Milnor primitives annihilate the class of the Thom map $[\rho] \in HZ^p_p(BP)$,
it is no wonder that the image algebra $Q$ is annihilated by Milnor primitives.

We compare $Q$-subalgebras $Q(\mathbb{Z}/p^h, n + 1)$ in (5-22) for various $h \geq 1$ and
$Q(\mathbb{Z}, n + 2)$ in (5-32). The relationship among these spaces is supplied by the
following homotopy commutative diagram:

$$
\begin{array}{ccc}
K(\mathbb{Z}, n + 2) & \xrightarrow{\delta_h} & K(\mathbb{Z}, n + 2) \\
\uparrow & & \uparrow \\
K(\mathbb{Z}/p^h, n + 1) & \xrightarrow{\alpha} & K(\mathbb{Z}/p^{h+1}, n + 1),
\end{array}
$$

where $\delta_h$ is the $h$-th Bockstein map and $\alpha$ is the map induced by an injective
homomorphism $\mathbb{Z}/p^h \to \mathbb{Z}/p^{h+1}$, which is multiplication by $p$. The above diagram
(5-34) commutes because $\delta_h$ is essentially division by $p^h$. The above diagram
induces a commutative diagram on mod $p$ cohomology rings containing the relevant
$Q$-subalgebras.
Proposition 5-15. We have the following isomorphisms of $Q$-subalgebras:
\[
\begin{align*}
Q(\mathbb{Z}, n + 2) & \cong \delta^r_{n+1} \\
Q(\mathbb{Z}/p^h, n + 1) & \cong Q(\mathbb{Z}/p^{h+1}, n + 1).
\end{align*}
\]
(5-35)

Proof. To avoid possible ambiguity, we denote the fundamental class for the mod $p^h$ Eilenberg–Mac Lane space $K(\mathbb{Z}/p^h, n + 1)$ by $\tau^{(h)}_{n+1}$. Since $\delta^r_{n+1} \tau^{(h)}_{n+1} = \delta^r_{n+1}$, where $\delta^r$ on the right-hand side is the $r$-th Bockstein, the commutativity of (5-35) implies that $\alpha^*(\delta^r_{n+1}) = \delta^r_{n+1}$. Since the cohomology operations commute with induced maps, for any $S \in S^+_n$ we have $\delta^r_{n+1}(Q_S \tau^{(h)}_{n+2}) = Q_S \delta^r_{n+1}(\tau^{(h)}_{n+2})$. Similarly, we have $\alpha^*(Q_S \delta^r_{n+1}(\tau^{(h)}_{n+1})) = Q_S \delta^r_{n+1}(\tau^{(h)}_{n+1})$. Thus the maps $\alpha^*$ induce isomorphisms of $Q$-subalgebras. Hence they induce isomorphisms of $Q$-subalgebras. \(\square\)

In Theorems 5-9 and 5-14, we considered subalgebras of cohomology algebras invariant under the entire Steenrod algebra $A(p)$. The Steenrod algebra has various interesting subalgebras. For example, for each positive integer $m$, Milnor considered a subalgebra $A[m]^*$ generated by elements $Q_0, \mathcal{P}^{\Delta^m}, \mathcal{P}^{p^{\Delta^m}}, \ldots, \mathcal{P}^{p^{m-1}\Delta^m}$.

Proposition 5-16 [4, §8 Proposition 2]. For each $m \geq 1$, the subalgebra $A[m]^*$ of the Steenrod algebra is finite dimensional, and its vector space basis over $\mathbb{Z}_p$ is given by the collection of elements of the form
\[
Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots Q_m^{\varepsilon_m} \mathcal{P}^{\sum_{j=1}^m r_j \Delta_j},
\]
where $0 \leq r_1 < p^{m}, 0 \leq r_2 < p^{m-1}, \ldots, 0 \leq r_m < p, \text{ and } \varepsilon_j = 0, 1 \text{ for } 0 \leq j \leq m.$

The above form of Steenrod reduced power is very interesting. For example, we have the following result.

Lemma 5-17. Suppose $0 \leq r_j < p^{m-j+1}$ for $1 \leq j \leq m$. Then
\[
[\mathcal{P}^{\sum_{j=1}^m r_j \Delta_j}, Q_0 \cdots Q_m] = 0.
\]
(5-37)

Proof. By repeatedly applying (2-5), we have
\[
\mathcal{P}^{\sum_{j=1}^m r_j \Delta_j} Q_0 \cdots Q_m = \sum_{i_j \geq j \geq 0} Q_{i_0} Q_{i_1} \cdots Q_{i_m} \mathcal{P}^{\sum_{j=1}^m r_j \Delta_j - \sum_{j=0}^m p^j \Delta_{i_j - j}}.
\]
The term corresponding to $(i_0, \ldots, i_m)$ is nontrivial if the corresponding exponent sequence of $\mathcal{P}$ is non-negative, that is, when $\sum_{j=0}^m p^j \Delta_{i_j - j} \leq \sum_{j=1}^m r_j \Delta_j$. Since $r_j < p^{m+1-j}$, the only way to have $p^m \Delta_{i_m - m} \leq \sum_{j=1}^m r_j \Delta_j$ is $i_m = m$. Similarly, $p^{m-1} \Delta_{i_{m-1} - (m-1)} \leq \sum_{j=1}^m r_j \Delta_j$ implies that $i_{m-1} = m - 1$ or $m$. Continuing this process, we see that we must have $j \leq i_j \leq m$ for all $0 \leq j \leq m$. Since Milnor primitives generate an exterior algebra, $Q_{i_0} \cdots Q_{i_m} \neq 0$ only when integers $i_0, \ldots, i_m$ are distinct. This can happen only when $i_j = j$ for all $0 \leq j \leq m$. In the above summation, there is only one term corresponding to this case, and we have
\[
\mathcal{P}^{\sum_{j=1}^m r_j \Delta_j} Q_0 \cdots Q_m = Q_0 \cdots Q_m \mathcal{P}^{\sum_{j=1}^m r_j \Delta_j}.
\]
This shows that the Steenrod reduced power $\mathcal{P}^{\sum_{j=1}^m r_j \Delta_j}$, where $0 \leq r_j < p^{m+1-j}$ for $1 \leq j \leq m$, commutes with the product $Q_0 \cdots Q_m$ of Milnor primitives. This completes the proof. \(\square\)

We can give examples of $A[m]^*$-invariant subalgebras of the cohomology algebras.
Proposition 5-18. Let $X$ be a topological space. Then

\[(5-38) \quad Q_0Q_1 \cdots Q_m: H^*(X; \mathbb{Z}_p) \subset H^*(X; \mathbb{Z}_p) \]

is an $A[m]^*$-invariant subalgebra on which Milnor primitives $Q_0, \ldots, Q_m$ in $A[m]$ act trivially.

Proof. It is obvious that Milnor primitives $Q_0, \ldots, Q_m$ act trivially on this subspace of the cohomology ring over $\mathbb{Z}_p$, because they generate an exterior algebra. To see that the subspace $(5-38)$ is actually a subalgebra, let $Q_0 \cdots Q_m x$ and $Q_0 \cdots Q_m y$ be any two elements in this subspace. Then by the derivation property of Milnor primitives, we have

\[(Q_0 \cdots Q_m x) \cdot (Q_0 \cdots Q_m y) = Q_0 \cdots Q_m [x \cdot (Q_0 \cdots Q_m y)].\]

This shows that the subspace is closed under the cup product.

It remains to be shown that the above subspace is invariant under the action of the subalgebra $A[m]^*$ of the Steenrod algebra. By Proposition 5-16, we only have to show that it is invariant under the action of the Steenrod reduced power of the form $\mathcal{P}^{\sum_{i=1}^m r_i} \Delta_j$, where $0 \leq r_j < p^{m-j}+1$. But by Lemma 5-17, this form of Steenrod reduced power commutes with the product $Q_0 \cdots Q_m$. Since obviously the cohomology ring $H^*(X; \mathbb{Z}_p)$ is invariant under the action of Steenrod reduced powers, we see that the above subalgebra in $(5-38)$ is invariant under the action of $A[m]^*$. This completes the proof. □

Remark. Let $BP(m)$ be the Wilson spectrum for $m \geq 0$ and let $\rho^{(m)} : BP(m) \to \mathbb{H}_p$ be the Thom map [11]. Using Sullivan exact sequences, we see that

\[(5-39) \quad Q_0Q_1 \cdots Q_m: H^*(X; \mathbb{Z}_p) \subset \text{Im} [\rho^{(m)}_* : BP(m)^*(X) \to H^*(X; \mathbb{Z}_p)],\]

for any space $X$. Since the mod $p$ cohomology of the spectrum $BP(m)$ is given by $\mathbb{H}_p^*(BP(m)) = A(p)^*/(Q_0, \ldots, Q_m)p$, it is clear that the image of the Thom map $\rho^{(m)}$ is annihilated by the first $m+1$ Milnor primitives $Q_0, Q_1, \ldots, Q_m$ for any space or even for any spectrum $X$.

§6 Mod 2 cohomology rings of Eilenberg–Mac Lane spaces in terms of the Milnor basis, and $Q$-subalgebras

In this section, we describe mod 2 cohomology rings of Eilenberg–Mac Lane spaces in terms of Milnor’s Steenrod squares $Sq^R$ rather than in terms of admissible monomials as in Theorem 3-6.

To describe the relationship between Milnor basis elements and admissible monomials, we repeatedly apply the factorization formula (4-10). Let $R = (r_1, r_2, r_3, \ldots)$ is a sequence of non-negative integers which are almost all zero. We have

\[\begin{align*}
    Sq^R &= Sq^{|R|} \Delta_1 \cdot Sq^{\ell(R)} + \text{(other terms)} \\
    &= Sq^{|R|} \Delta_1 \cdot Sq^{\ell(R)} + \text{(other terms)} \\
    &= \cdots \\
    &= \theta(R) + \text{(other terms)},
\end{align*}\]
where \( \theta(R) \) is a monomial in Steenrod squares given by

\[
\theta(R) = \operatorname{Sq}^{\sigma[R] \Delta_1} \operatorname{Sq}^{\sigma[t(R)] \Delta_1} \ldots \operatorname{Sq}^{\sigma[\tau(R)] \Delta_1} \\
= \operatorname{Sq}^{s_1 \Delta_1} \operatorname{Sq}^{s_2 \Delta_1} \ldots \operatorname{Sq}^{s_p \Delta_1} ,
\]

where \( S = (s_1, s_2, \ldots) \) is a sequence of non-negative integers \( s_r = \sigma[t^{r-1}(R)] \) for \( r \geq 1 \) which are almost all zero. Recall from (4-2) that for a sequence of non-negative integers \( R = (r_1, r_2, r_3, \ldots) \) which are almost all zero, we defined a weighted sum \( \sigma[R] \in \mathbb{Z} \) and a shifted sequence \( t(R) \) by

\[
\sigma[R] = r_1 + 2r_2 + 2^2 r_3 + \cdots + \sum_{j \geq 1} 2^{j-1} r_j , \quad t(R) = (r_2, r_3, \ldots).
\]

Recall that a monomial \( \theta \) as in (6-1) is called admissible if \( s_j \geq 2s_{j+1} \) for \( j \geq 1 \), and its excess \( e_2(\theta) \) is defined by \( e_2(\theta) = \sum_{j \geq 1} (s_j - 2s_{j+1}) \). Note that \( e_2(\theta) \) is the \( p = 2 \) version of \( \tau_p(\theta) \) given in (3-3), rather than \( e_p(\theta) \) given in (3-4).

**Lemma 6-1.** For any sequence \( R = (r_1, r_2, r_3, \ldots) \) of non-negative integers which are almost all zero, the monomial \( \theta(R) \) given in (6-1) is admissible and the excess of \( \theta(R) \) is given by \( e_2(\theta(R)) = \ell[R] = \sum_{j \geq 1} r_j \).

The sequences \( R \) and \( S \) of non-negative integers in (6-1) correspond to each other in a 1:1 manner, and they are related by the following formulae:

\[
\begin{aligned}
S &= R + 2t(R) + 2^2 t^2(R) + \cdots = \frac{R}{1 - 2t} , \\
R &= S - 2t(S) = (1 - 2t)(S).
\end{aligned}
\]

Thus there is a 1:1 correspondence between the set of sequences \( R \) of non-negative integers which are almost all zero and the set of admissible monomials in Steenrod squares.

**Proof.** To check the admissibility, we calculate \( s_j - 2s_{j+1} \). Since \( s_j = \sigma[t^{j-1}(R)] \) for \( r \geq 1 \), we have \( s_j - 2s_{j+1} = \sigma[t^{j-1}(R)] - 2\sigma[t^j(R)] = r_j \), which is non-negative because \( R \) is a non-negative sequence. Hence the monomial \( \theta(R) \) in (6-1) is admissible. Its excess is then given by \( e_2(\theta(R)) = \sum_{j \geq 1} (s_j - 2s_{j+1}) = \sum_{j \geq 1} r_j = \ell[R] \). The relationship between the two sequences \( R \) and \( S \) is straightforward to prove, so is the 1:1 correspondence of non-negative finite sequences \( R \) and admissible monomials in \( \mathcal{A}(2)^* \). \( \square \)

On the set of all sequences \( R \) of non-negative integers which are almost all zero, we introduce a lexicographic total ordering from the right. We can then be more precise about the relationship between admissible monomials \( \theta(R) \) and Milnor basis elements \( \operatorname{Sq}^R \).

**Proposition 6-2.** For any sequence \( R \) as above, we have

\[
\theta(R) = \operatorname{Sq}^R + \sum_{R' < R} c(R') \operatorname{Sq}^{R'} ,
\]

This is the \( p = 2 \) version of Proposition 5-6, and its proof is essentially the same. Recall that we can also use elements \( Q^{E,R} = \operatorname{Sq}^{E+2R} \) for any sequence \( E \) of
zeros and ones which are almost all zero and for any sequence $R$ of non-negative integers which are almost all zero as in odd prime case. See (2-13). Note however that elements of the form $p^R = Sq^R$ are not preserved under the coproduct map, unlike odd prime case.

As a preparation for the description of mod 2 cohomology rings of Eilenberg–Mac Lane spaces in terms of Milnor’s Steenrod squares, we calculate degrees of operators of the form $Sq^t(R)$.

Lemma 6-3. For a sequence $R$ as above, we have

\[(6-3) \quad |Sq^t(R)| = \sigma[R] - \ell[R] = \sigma[t(R)] + |Sq^2(R)|.\]

Proof. Let $R = (r_1, r_2, r_3, \ldots)$. Then $t(R) = (r_2, r_3, \ldots)$ and we have

\[|Sq^t(R)| = \sum_{i \geq 1} (2^i - 1)r_{i+1} = \sum_{i \geq 1} 2^i r_{i+1} - \sum_{i \geq 1} r_{i+1}\]

\[= (r_1 + \sum_{i \geq 1} 2^i r_{i+1}) - (r_1 + \sum_{i \geq 1} r_{i+1}) = \sigma[R] - \ell[R].\]

For the second identity, we proceed as follows.

\[|Sq^t(R)| - |Sq^2(R)| = \sum_{i \geq 1} (2^i - 1)r_{i+1} - \sum_{i \geq 1} (2^i - 1)r_{i+2}\]

\[= r_2 - \sum_{i \geq 1} ((2^{i+1} - 1) - (2^i - 1))r_{i+2} = r_2 + \sum_{i \geq 1} 2^i r_{i+2} = \sigma[t(R)].\]

This completes the proof. \(\square\)

\$6.1. \quad$ Mod 2 cohomology rings of mod 2$^h$ Eilenberg–Mac Lane spaces in terms of the Milnor basis, and $Q$-subalgebras. We can now give a description of mod 2 cohomology rings of mod $2^h$ Eilenberg–Mac Lane spaces in terms of the Milnor basis. When $n = 0$ and $h \geq 2$, the mod 2 cohomology ring of the Eilenberg–Mac Lane space $K(\mathbb{Z}/2^h, 1)$ is given in (3-10'). This cohomology ring is rather different from the other cases in that it contains an exterior subalgebra. Since there is no need to use Milnor basis elements to describe this cohomology ring, from now on we assume that $n \geq 1$ and $h \geq 1$, or $n = 0$ and $h = 1$.

**Theorem 6-4.** Let $n \geq 1$, or $n = 0$ and $h = 1$. Let $R$ be a sequence of non-negative integers which are almost all zero. Let $i_{n+1} \in H^{n+1}(K(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2)$ be the mod 2 fundamental class. Then

\[(6-4) \quad Sq^{i_{n+1}} = 0 \quad \text{if} \quad \ell[R] > n + 1,\]

\[(6-5) \quad Sq^{i_{n+1}} = (Sq^{i(R)} i_{n+1})^2 \quad \text{if} \quad \ell[R] = n + 1.\]

In terms of Milnor primitives and Steenrod reduced powers, equivalent statements are given as follows:

\[(6-4') \quad Q^{E} p^{R} i_{n+1} = 0 \quad \text{if} \quad \ell[E] + 2\ell[R] > n + 1,\]

\[(6-5') \quad Q^{E} p^{R} i_{n+1} = (Q^{(E)} p^{i(R)} i_{n+1})^2 \quad \text{if} \quad \ell[E] + 2\ell[R] = n + 1.\]
The mod 2 cohomology ring of \( K(\mathbb{Z}/2^h, n + 1) \) is a polynomial algebra described in terms of the Milnor basis as follows:

\[
H^*(K(\mathbb{Z}/2^h, n + 1); \mathbb{Z}_2) = \mathbb{Z}_2[\Sigma^R t_{n+1} | \ell[R] < n + 1] \\
= \mathbb{Z}_2[Q^E \Sigma^R t_{n+1} | \ell[E] + 2\ell[R] < n + 1].
\]

**Proof.** To prove (6-4) and (6-5), we apply the factorization formula (4-10) of \( \Sigma^R \) to the fundamental class \( t_{n+1} \). We have

\[
\Sigma^R t_{n+1} = \sum_{S \in \mathcal{R}, s_1 = 0} c(S)\Sigma(\sigma[R] + \sigma[t(S)])\Delta_1, \Sigma^R t^{(R) - t(S) + t^2(S)} t_{n+1}.
\]

We compare the degree of \( \Sigma^R t^{(R) - t(S) + t^2(S)} t_{n+1} \) and \( \sigma[R] + \sigma[t(S)] \) and use the unstable property of Steenrod squares (2-17). By Lemma 6-3, we have

\[
|\Sigma^R t^{(R) - t(S) + t^2(S)} t_{n+1}| = \sigma[R] - \sigma[t(S)] + (n + 1 - \ell[R]).
\]

When \( \ell[R] > n + 1 \), the above is strictly less than \( \sigma[R] + \sigma[t(S)] \) for any \( S \). Hence all terms in the summation (**) vanish and we have \( \Sigma^R t_{n+1} = 0 \). This proves (6-4).

If \( \ell[R] = n + 1 \), then we have (**) \( \leq \sigma[R] + \sigma[t(S)] \) except for the case \( S = \Delta_0 \). Thus only one term in (***) remains, and by degree reason this term gives

\[
\Sigma^R t_{n+1} = \Sigma^R \sigma[R] \Delta_1 (\Sigma^R t^{(R)} t_{n+1}) = (\Sigma^R t^{(R)} t_{n+1})^2,
\]

since \( |\Sigma^R t^{(R)} t_{n+1}| = \sigma[R] \). This proves (6-5). The equivalent statements (6-4') and (6-5') can be derived using the identity (2-13).

To show (6-6), we first rewrite the description of mod 2 cohomology rings of mod \( 2^h \) Eilenberg–Mac Lane spaces in terms of sequences \( R \). Since we are assuming that \( n \geq 1 \), or \( n = 0 \) and \( h = 1 \), the cohomology ring for \( K(\mathbb{Z}/2^h, n + 1) \) is given by (3-10), and together with Lemma 6-1 we have

\[
H^*(K(\mathbb{Z}/2^h, n + 1); \mathbb{Z}_2) = \mathbb{Z}_2[\theta(R) t_{n+1} | \ell[R] < n + 1].
\]

We consider the following subalgebra generated by some of the elements defined by the Milnor basis:

\[
\mathbb{Z}_2\{ \Sigma^R t_{n+1} | \ell[R] < n + 1 \} \subset H^*(K(\mathbb{Z}/2^h, n + 1); \mathbb{Z}_2).
\]

Our aim is to show that there are no algebraic relations among these generators, and that the subalgebra (6-8) coincides with the entire cohomology ring. Applying Proposition 6-2 for \( R \) with \( \ell[R] < n + 1 \) to the fundamental class \( t_{n+1} \), we have

\[
\theta(R) t_{n+1} = \Sigma^R t_{n+1} + \sum_{R' < R} c(R') \Sigma^R t_{n+1}.
\]

By (6-4) and (6-5), the summation above is actually over all sequences \( R' \) such that \( R' < R \) and \( \ell[R'] \leq n + 1 \). When \( \ell[R'] = n + 1 \), the corresponding element
In particular, if \( R \) with \( \ell[R] < n + 1 \) for some \( k \) by (6-5). Hence for those sequences \( R \) with \( \ell[R] < n + 1 \), the algebra generator \( \theta(R)\tau_{n+1} \) in (6-7) lies in the subalgebra (6-8). Hence the subalgebra (6-8) must be the entire subalgebra. Since there is a 1:1 degree preserving correspondence between the set \( \{ \theta(R)\tau_{n+1} \mid \ell[R] < n + 1 \} \) and the set \( \{ \tau_{n+1} \mid \ell[R] < n + 1 \} \), and since both sets generate the same algebra, we conclude that the latter set is a set of algebraically independent elements because the former set is. This proves (6-6). □

We are particularly interested in the action of Milnor primitives \( Q_i \) on the fundamental class \( \tau_{n+1} \). Recall that \( Q_i = \text{Sq}^{A_{n+1}} \) and that for any sequence \( E = (e_0, e_1, \ldots) \) of zeroes and ones which are almost all zero, we have \( Q^E = \text{Sq}^E \).

**Corollary 6-5.** Let \( n \geq 1 \), or \( n = 0 \) and \( h = 1 \). The following relations hold with respect to the action of Milnor primitives on the mod 2 fundamental class \( \tau_{n+1} \in H^{n+1}(K(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2) \):

\[
Q^E\tau_{n+1} = 0 \quad \text{if} \quad \ell[E] > n + 1,
\]

\[
Q^E\tau_{n+1} = (Q^E\tau_{n+1})^2 \neq 0 \quad \text{if} \quad \ell[E] = n + 1.
\]

In particular, if \( E = (0, \ldots, 0, e_j-1, e_j, \ldots) \) with \( e_j = 1 \) and \( \ell[E] = n + 1 \), then

\[
Q^E\tau_{n+1} = (Q^E\tau_{n+1})^{2j} \neq 0,
\]

where the element \( Q^E\tau_{n+1} \) in \( H^*(K(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2) \) with \( \ell[t^j(E)] = n \) is an indecomposable polynomial generator.

**Proof.** Straightforward from (6-4') and (6-5'). Since \( Q^E\tau_{n+1} \) with \( \ell[E] = n \) is a polynomial generator by (6-6), all of its \( 2^j \)-th powers are nontrivial, which implies non triviality of the element in (6-10). □

In particular, for any sequence \( S = (s_1, s_2, \ldots, s_{n+1}) \) of \( n + 1 \) strictly increasing non-negative integers, the corresponding element \( Q_0Q_{s_1}\cdot\cdot\cdot Q_{s_{n+1}}\tau_{n+1} \) is non-trivial by (6-11). The element of this type with the smallest degree is \( Q_n\cdot\cdot\cdot Q_{0}\tau_{n+1} \) of even degree \( 2(1 + 2^{s_1} + \cdot\cdot\cdot + 2^n) \). One can obtain any element of the above type \( Q_0Q_{n+1} \) from the element \( Q_n\cdot\cdot\cdot Q_{0}\tau_{n+1} \) by the action of Steenrod squares as shown in the next lemma.

**Lemma 6-6.** Let \( n \geq 1 \), or \( n = 0 \) and \( h = 1 \). Let \( R \) be a sequence of non-negative integers which are almost all zero. Then \( \varphi^nQ_0Q_{n-1}\cdot\cdot\cdot Q_{0}\tau_{n+1} \neq 0 \) only if the sequence \( R \) is of the form \( R = \sum_{j=0}^{n} 2^j\Delta_{\ell_j} \) for some non-negative integers \( \ell_j \). In this case, we have

\[
\varphi^nQ_0Q_{n-1}\cdot\cdot\cdot Q_{0}\tau_{n+1} = Q_n\ell_{n+1}\cdot\cdot\cdot Q_1+\ell_0Q_0\tau_{n+1}.
\]

This element is nontrivial if the set of integers \( \{\ell_0, 1 + \ell_1, \ldots, n + \ell_n\} \) is a set of distinct non-negative integers.

Similarly, for any sequence of non-negative integers \( 0 \leq s_1 < \cdots < s_{n+1} \), we have \( \varphi^nQ_{s_{n+1}}\cdots Q_{s_{1}}\tau_{n+1} \neq 0 \) only if \( R \) is of the form \( R = \sum_{j=1}^{n+1} 2^j\Delta_{\ell_j} \) for some non-negative integers \( \ell_j \). In this case,

\[
\varphi^nQ_{s_{n+1}}\cdots Q_{s_{1}}\tau_{n+1} = Q_{s_{n+1}}(s_{n+1}+\ell_{n+1})\cdots Q_{s_1+\ell_1}\tau_{n+1}.
\]
Proof. From the general formula (2.14), for any sequence \( R \) we have
\[
(\ast) \quad \mathcal{P}^R Q_n \cdots Q_1 Q_0 t_{n+1} = \sum_{\ell_0, \ldots, \ell_n \geq 0} Q_{n+\ell_n} \cdots Q_{1+\ell_1} Q_0 \mathcal{P}^R - \sum_{j=0}^{n} 2^{j} \Delta_{ij} t_{n+1}.
\]
By (6-4'), \( E_j^R t_{n+1} = 0 \) when \( \ell[E] = n + 1 \) and \( R' \neq \Delta_0 \). Thus a nontrivial term can result in the above summation (\( \ast \)) only when the exponent sequence of \( \mathcal{P} \) on the right hand side vanishes, that is, only when \( R = \sum_{j=0}^{n} 2^j \Delta_{ij} \) for some \( \ell_0, \ldots, \ell_n \geq 0 \). In this case, all other terms in the summation on the right hand side of (\( \ast \)) vanish and we have
\[
\mathcal{P}^R Q_n \cdots Q_1 Q_0 t_{n+1} = Q_{n+\ell_n} \cdots Q_{1+\ell_1} Q_0 t_{n+1}.
\]
By (6-10), this element is nontrivial when the integers \( j + \ell_j \) are all distinct. This proves the first half of the lemma. The second part can be proved in a similar way. \( \square \)

Now we consider a subalgebra of \( H^* (\mathbb{K}(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2) \) generated by elements obtained by applying maximum number of Milnor primitives on the fundamental class. For any sequence \( S = (s_1, s_2, \ldots, s_n) \in S_n^+ \) of strictly increasing positive integers, let \( Q_S Q_0 t_{n+1} = Q_{s_n} \cdots Q_{s_1} Q_0 t_{n+1} \) as before. Its degree is given by \( |Q_S Q_0 t_{n+1}| = 2(1 + 2^{s_1} + 2^{s_2} + \cdots + 2^{s_n}) \). For \( S \) as above, let \( t(S) = (s_1 - 1, \ldots, s_n - 1) \) be a sequence of strictly increasing non-negative integers.

**Theorem 6-7.** Let \( n \geq 1 \), or \( n = 0 \) and \( h = 1 \). The subalgebra \( \mathcal{Q} \) of the cohomology ring \( H^* (\mathbb{K}(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2) \) given by
\[
(6.13) \quad \mathcal{Q} = \mathbb{Q}(\mathbb{Z}/2^h, n + 1) = \mathbb{Z}_2 [Q_S Q_0 t_{n+1} \mid S \in S_n^+] \quad (6-13)
\]
is the smallest \( \mathcal{A}(2)^- \)-invariant polynomial subalgebra of \( [H^* (\mathbb{K}(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2)]^2 \) containing \( Q_n \cdots Q_1 Q_0 t_{n+1} \). Any element in this algebra is a square and we have
\[
Q_S Q_0 t_{n+1} = (Q_{t(S)}^{t(S)+1})^2, \quad \text{for } S \in S_n^+.
\]
This subalgebra is annihilated by any Milnor primitive \( Q_j \) for \( j \geq 0 \).

**Proof.** Any generator \( Q_S Q_0 t_{n+1} \) for \( S \in S_n^+ \) of the subalgebra \( \mathcal{Q} \) has already \( n + 1 \) Milnor primitives in it, so no more Milnor primitives can act nontrivially. By the derivation property of Milnor primitives, all Milnor primitives annihilate the entire subalgebra \( \mathcal{Q} \).

By (6-11) and (6-12'), any Steenrod reduced power operation acting on an element \( Q_S Q_0 t_{n+1} \) produces either a trivial element or a \( 2^k \)-th power of another element \( Q^{2^k} Q_0 t_{n+1} \) for some \( k \geq 0 \) and for some \( S' \in S_n^+ \). Since \( Q_j \)'s act trivially on the subalgebra \( \mathcal{Q} \), the Cartan formula (2.16) for \( \text{Sq}^R \) reduces to a Cartan formula for the Steenrod reduced powers \( \mathcal{P}^R = \text{Sq}^{2R} \) when these operators act on \( \mathcal{Q} \). Hence \( \mathcal{Q} \) is preserved under the action of the entire mod 2 Steenrod algebra \( \mathcal{A}(2)^- \).

By (6-10), all generators in the algebra \( \mathcal{Q} \) are squares. Since the characteristic is 2, the algebra \( \mathcal{Q} \) is a subalgebra of \( [H^* (\mathbb{K}(\mathbb{Z}/2^h, n+1); \mathbb{Z}_2)]^2 \).

Since all algebra generators of \( \mathcal{Q} \) are obtained by applying Steenrod reduced powers to the element \( Q_n \cdots Q_1 Q_0 t_{n+1} \), by Lemma 6-6, the subalgebra \( \mathcal{Q} \) is contained in any \( \mathcal{A}(2)^- \)-invariant subalgebra containing the element \( Q_n \cdots Q_1 Q_0 t_{n+1} \). Hence \( \mathcal{Q} \) is the smallest \( \mathcal{A}(2)^- \)-invariant subalgebra containing \( Q_n \cdots Q_1 Q_0 t_{n+1} \).
It remains to be shown that elements $Q^j Q_{0^{t_{n+1}}}$ are polynomial generators of the algebra $Q$, that is, there are no algebraic relations among these elements in the subalgebra $Q$. To see this, suppose we have an algebraic relation of the form

$$P(Q^j Q_{0^{t_{n+1}}}, Q^j Q_{0^{t_{n+1}}}, \ldots, Q^j Q_{0^{t_{n+1}}}) = 0$$

for some nontrivial polynomial $P(x_1, x_2, \ldots, x_\ell)$ over $Z_2$ in $\ell$ variables, and for some sequences $S_1, S_2, \ldots, S_\ell \in S_n^+$. Since $Q^j Q_{0^{t_{n+1}}} = (Q^j t_{n+1})^2$ for $1 \leq j \leq \ell$, and since the ring $H^*(K(Z/2^n, n+1); Z_2)$ is a polynomial algebra over the field of characteristic 2, we can take the unique square root of the above relation:

$$P(Q^j t_{n+1}, Q^j t_{n+1}, \ldots, Q^j t_{n+1}) = 0.$$ 

This means that there is a nontrivial algebraic relation among polynomial generators $Q^j t_{n+1}$ of the cohomology ring $H^*(K(Z/2^n, n+1); Z_2)$. This is a contradiction to our description of the cohomology ring given in (6-6) which says that elements of the form $Q^j t_{n+1}$ where $\ell[S_j] = n$ must be polynomial generators for any sequence $S_j$ of strictly increasing $n$ non-negative integers, and hence there cannot be any algebraic relation. Thus elements $Q^j Q_{0^{t_{n+1}}}$ must be algebraically independent within the ring $Q$. Consequently they are polynomial generators of $Q$. ∎

We emphasize that although the elements $Q^j Q_{0^{t_{n+1}}}$ for $S \in S_n^+$ are algebraically independent in the ring $Q$, all of these elements are squares in the cohomology ring $H^*(K(Z/2^n, n+1); Z_2)$ and hence they are decomposable in this larger ring.

§6.2. Mod 2 cohomology rings of integral Eilenberg–Mac Lane spaces in terms of the Milnor basis, and $Q$-subalgebras. Next, we describe the mod 2 cohomology ring of the integral Eilenberg–Mac Lane space $K(Z, n+2)$ for $n \geq 0$ in terms of Milnor basis elements, and we describe its various properties. Recall that $I(E, R)$ was defined in (5-13).

**Theorem 6-8.** Let $n \geq 0$ and let $\tau_{n+2} \in H^{n+2}(K(Z, n+2); Z_2)$ be the mod 2 fundamental class. Then the following identities hold:

\begin{align}
(6-14) & \quad Sq^R \tau_{n+2} = 0 & \text{if } \ell[R] > n + 2, \\
(6-15) & \quad Sq^R \tau_{n+2} = (Sq^{(R)} \tau_{n+2})^2 & \text{if } \ell[R] = n + 2.
\end{align}

These identities can be restated in terms of Milnor primitives and Steenrod reduced powers as follows:

\begin{align}
(6-14') & \quad Q^E \mathcal{P}^R \tau_{n+2} = 0 & \text{if } \ell[E] + 2\ell[R] > n + 2, \\
(6-15') & \quad Q^E \mathcal{P}^R \tau_{n+2} = (Q^{(E)} \mathcal{P}^{(R)} \tau_{n+2})^2 & \text{if } \ell[E] + 2\ell[R] = n + 2.
\end{align}

The mod 2 cohomology ring of $K(Z, n+2)$ is a polynomial algebra given by

\begin{align}
H^*(K(Z, n+2); Z_2) = Z_2 \left[ Sq^R \tau_{n+2} \bigg| \begin{array}{l}
\ell[R] < n + 2 \text{ and } r_k > 1 \text{ if } \\
R = (r_1, \ldots, r_k, 0, \ldots) \text{ with } r_k \neq 0
\end{array} \right] \\
= Z_2 \left[ Q^E \mathcal{P}^R \tau_{n+2} \bigg| \begin{array}{l}
\ell[E] + 2\ell[R] < n + 2 \text{ and } I(E, R) \text{ ends with an entry from } R
\end{array} \right].
\end{align}
Proof. Identities (6-14) and (6-15) can be proved in a way similar to the proof of (6-4) and (6-5), since only dimension of cohomology classes are relevant for the argument.

In (3-11), a description of the cohomology ring $H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2)$ is given in terms of admissible monomials in the mod 2 Steenrod algebra. We first rewrite this description in terms of $\theta(R)$ given in (6-1). Let $R = (r_1, r_2, \ldots, r_k, 0, \ldots)$ be any sequence of non-negative integers with $r_k \neq 0$. Then $\theta(R)$ is of the form

$$\theta(R) = Sq^{\sigma[R]\Delta_1} \cdot Sq^{\sigma[R]\Delta_2} \cdots Sq^{\sigma[R]\Delta_k},$$

where $Sq^{\sigma[R]\Delta_1} = Sq^1\Delta_1$. Since $\theta(R)$ is always admissible for any sequence $R$ and its excess $e_2(\theta(R))$ is given by $\ell[R]$ by Lemma 6-1, the description of (3-11) in terms of admissible monomials can be rewritten as follows:

$$(6-17) \quad H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2) = \mathbb{Z}_2 \left[ \theta(R)\tau_{n+2} \right] \quad \text{if } \ell[R] < n + 2 \text{ and } r_k > 1 \text{ if } R = (r_1, \ldots, r_k, 0, \ldots) \text{ with } r_k \neq 0.$$ 

We consider the following subalgebra of the cohomology ring:

$$(*) \quad \mathbb{Z}_2 \{ Sq^R\tau_{n+2} \mid \ell[R] < n + 2, R \text{ does not end with } 1 \} \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2).$$

Our objective is to show that this subalgebra in fact coincides with the entire cohomology ring, and that the generators are algebraically independent. Let $R$ be a sequence with $\ell[R] < n + 2$ and ending with an integer greater than 1. Applying the identity in Proposition 6-2 to the fundamental class $\tau_{n+2}$ and using (6-14), we can write

$$\theta(R)\tau_{n+2} = Sq^R\tau_{n+2} + \sum_{\ell[R'] \leq n+2} c(R') Sq^{R'}\tau_{n+2}.$$

Furthermore, if $R' = (r_1, r_2, \ldots, r_k, 0, \ldots)$ ends with $r_k = 1$, then $\theta(R')$ ends with $Sq^{\Delta_1} = Q_0$ and thus $\theta(R')\tau_{n+2} = 0$. Hence the above identity applied to $\theta(R')$ implies that $Sq^{R'}\tau_{n+2}$ can be expressed in terms of $Sq^{R''}\tau_{n+2}$ for which $R'' < R'$ and $R''$ does not end with 1. This procedure allows us to eliminate those sequences $R'$ ending with 1 from the above summation. Since $Sq^{R'}\tau_{n+2}$ with $\ell[R'] = n + 2$ is either trivial or a $2^k$-th power of some element $Sq^{R''}\tau_{n+2}$ where $R''$ is such that $\ell[R''] < n + 2$ and the last integer in $R''$ is greater than 1, it follows that when $\ell[R] < n + 2$ and $R$ doesn’t end with 1, $\theta(R)\tau_{n+2}$ is in the subalgebra $(*)$. Since elements $\theta(R)\tau_{n+2}$ generate the entire cohomology algebra, it follows that the subalgebra $(*)$ is in fact the entire cohomology algebra. By comparing the number of algebra generators in a given dimension in (6-17) and $(*)$, we find that they are equal and hence we see that generators given in $(*)$ must be polynomial generators since those given in (6-17) are polynomial generators. This completes the proof of (6-16). $\square$

Next, we study the action of Milnor primitives on the fundamental class in detail.

**Proposition 6-9.** Let $\tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_2)$ be the mod 2 fundamental class. Let $E$ be a sequence of zeroes and ones which are almost all zero, and let $R$ be a sequence of non-negative integers which are almost all zero.
(I) Suppose $\ell[E] = n$ and let $E = \Delta_{s_1+1} + \cdots + \Delta_{s_n+1}$ for some strictly increasing sequence of positive integers $0 < s_1 < s_2 < \cdots < s_n$. Then

$$Q^E \tau_{n+2} = Q_{s_1} \cdots Q_{s_n} \tau_{n+2} = (Q_{s_1-1} \cdots Q_{s_{n-1}-1} \rho^{\Delta_{s_{n-1}}+1} \tau_{n+2})^2 \neq 0.$$

The element inside the square on the right hand side is a polynomial generator of the cohomology ring. For the same $E$ as above, suppose $\ell[R] = 1$, say $R = \Delta_j$ for some $j \geq 1$. Then

$$Q^E \rho^j \tau_{n+2} = \begin{cases} (Q_{s_2-s_1} \cdots Q_{s_n-s_1} Q_{j-s_1} \tau_{n+2})^{2^s} & \text{if } j > s_1, \\ 0 & \text{if } j = s_1, \\ (Q_{s_1-j} Q_{s_2-j} \cdots Q_{s_{n-1}-j} \tau_{n+2})^{2^s} & \text{if } j < s_1. \end{cases}$$

For the same $E$ as above, suppose $\ell[R] \geq 2$. Then $Q^E \rho^R \tau_{n+2} = 0$.

(II) Suppose $\ell[E] \geq n + 1$. Then for any sequence $R$, we have $Q^E \rho^R \tau_{n+2} = 0$.

Proof. By (2-12), we have $Q_{s_n} = \rho^{\Delta_{s_n}} Q_0 - Q_0 \rho^{\Delta_{s_n}}$. For $E$ as in (I), let $E' = \sum_{j=1}^{n-1} \Delta_{s_j+1}$. Since $Q_{s_n} \tau_{n+2} = 0$, we have $Q^E \tau_{n+2} = Q^{E'} Q_0 \rho^{\Delta_{s_n}} \tau_{n+2}$. Since the excess of this element is given by $\ell[E'] + 1 + 2\ell[R] = n + 2$, by (6-15) this is further equal to $(Q^{E'} \rho^{\Delta_{s_n-1}} \tau_{n+2})^2$. Since the element inside of the parenthesis is a polynomial generator of the cohomology ring by (6-16), its square is nonzero. This proves (6-18). For (6-19), we can apply (6-15) as long as the excess of the cohomology operation is $n + 2$. Suppose $s_1 < j$. Then we have

$$Q^E \rho^j \tau_{n+2} = \left(0 Q_{s_2-s_1} \cdots Q_{s_n-s_1} \rho^{\Delta_{s_1}} \tau_{n+2}\right)^{2^s} = \left(Q_{s_2-s_1} \cdots Q_{s_{n-1}-s_1} Q_{j-s_1} \tau_{n+2}\right)^{2^s}.$$ 

Here we used $Q_0 \rho^{\Delta_{s_1}} \tau_{n+2} = Q_{j-s_1} \tau_{n+2}$. When $j = s_1$, we do not have the reduced power term in the above calculation and due to the identity $Q_0 \rho^{\Delta_{s_n}} \tau_{n+2} = 0$, the above vanishes. When $s_1 > j$, applying (6-15) $j$ times, we have

$$Q^E \rho^j \tau_{n+2} = (Q_{s_1-j} \cdots Q_{s_{n-j} \tau_{n+2}})^{2^s}.$$ 

since $\rho^{\Delta_0} = 1$. This proves (6-19). When $\ell[E] = n$ and $\ell[R] \geq 2$, it follows that $\ell[E] + 2\ell[R] \geq n + 4$. Hence by (6-14') we have $Q^E \rho^R \tau_{n+2} = 0$.

For (II), when $\ell[E] = n + 1$ and $R = \Delta_0$, we let $E = \Delta_{s_0+1} + \sum_{j=1}^{n} \Delta_{s_j+1}$ for $0 \leq s_0 < \cdots < s_n$. Then using (6-18), we have

$$Q^E \tau_{n+2} = Q_{s_0} \left(Q_{s_1-1} \cdots Q_{s_{n-1}-1} \rho^{\Delta_{s_{n-1}}+1} \tau_{n+2}\right)^2 = 0,$$

because $Q_{s_0}$ acts as a derivation. When $\ell[E] = n + 1$ and $\ell[R] \geq 1$, we have $\ell[E] + 2\ell[R] \geq n + 3$. Hence by (6-14') we have $Q^E \rho^R \tau_{n+2} = 0$. This completes the proof of Proposition 6-9. □

Note that the elements inside of the parenthesis on the right hand side of (6-19) is a square by (6-18), if it is nontrivial.

From (II) of Proposition 6-9, no $n + 1$ products of Milnor primitives can act non-trivially on the fundamental class $\tau_{n+2}$, whereas any product of $n$ distinct Milnor
primitives can act nontrivially on \( \tau_{n+2} \) by (6-18) of Proposition 6-9. The collection of such elements \( \{ Q_S \tau_{n+2} \mid S \in S_n \} \) is a set of algebraically independent elements, which can be seen in a similar way as in the proof of Theorem 6-7, although such elements are always squares of some other elements in the cohomology ring \( H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2) \) by (6-18). So these elements generate a polynomial subalgebra of the cohomology ring. The lowest degree element among such elements is \( Q_n \cdots Q_2 Q_1 \tau_{n+2} \) of degree \( 2(1 + 2 + 2^2 + \cdots + 2^n) \). We examine the action of the Steenrod algebra \( A(2)^* \) on this element.

**Lemma 6-10.** Let \( n \geq 0 \) and let \( \tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_2) \) be the fundamental class. Let \( R \) be a sequence of non-negative integers which are almost all zero. Then \( P^R Q_n \cdots Q_2 Q_1 \tau_{n+2} \neq 0 \) only when the sequence \( R \) is of the form \( R = \sum_{j=0}^n 2^j \Delta_{\ell_j} \) for some non-negative integers \( \ell_j \) for \( 0 \leq j \leq n \). In this case,

\[
P^{\sum_{j=0}^n 2^j \Delta_{\ell_j}} Q_n \cdots Q_2 Q_1 \tau_{n+2} = Q_{n+\ell_n} \cdots Q_{2+\ell_2} Q_{1+\ell_1} P^{\Delta_{\ell_0}} \tau_{n+2},
\]

whose actual value in terms only of Milnor primitives is determined by (6-19). Consequently, for any sequence \( R \),

\[
(6-20) \quad P^R Q_n \cdots Q_2 Q_1 \tau_{n+2} \in Q = \mathbb{Z}_2[Q_S \tau_{n+2} \mid S \in S_n].
\]

In particular, for any sequence \( S = (s_1, s_2, \ldots, s_n) \) in \( S_n^+ \) of strictly increasing \( n \) positive integers, we have

\[
(6-21) \quad Q_S \tau_{n+2} = P^{\sum_{j=1}^n 2^j \Delta_{s_j}} Q_n \cdots Q_1 \tau_{n+2}.
\]

**Proof.** Applying (2-14) repeatedly, we have

\[
(*) \quad P^R Q_n \cdots Q_2 Q_1 \tau_{n+2} = \sum_{\ell_1, \ldots, \ell_n \geq 0} Q_{n+\ell_n} \cdots Q_{2+\ell_2} Q_{1+\ell_1} P^{R - \sum_{j=1}^n 2^j \Delta_{\ell_j}} \tau_{n+2}.
\]

Since there are \( n \) Milnor primitives in each term of the right hand side, for nontriviality of the term corresponding to \( \ell_1, \ldots, \ell_n \) we must have \( |R - \sum_{j=1}^n 2^j \Delta_{\ell_j}| \leq 1 \) by (6-14'). Thus the sequence \( R \) must be of the form \( R = \sum_{j=1}^n 2^j \Delta_{\ell_j} + \Delta_{\ell_0} \) for some \( \ell_0 \geq 0 \). In this case, all other terms on the right hand side of (*) vanish and

\[
(**) \quad P^{\sum_{j=1}^n 2^j \Delta_{s_j}} Q_n \cdots Q_2 Q_1 \tau_{n+2} = Q_{n+\ell_n} \cdots Q_{2+\ell_2} Q_{1+\ell_1} P^{\Delta_{\ell_0}} \tau_{n+2}.
\]

Suppose \( \ell_0 = 0 \). The element (**) is trivial unless the set of integers \( \{ j + \ell_j \}_{j=1}^n \) is a set of mutually distinct integers, in which case this element is in the subalgebra \( Q \) in (6-20). If \( \ell_0 > 0 \), then by (6-19) this element is either trivial or a \( 2^{\ell_0} \)-th power of an element of the form \( Q_S \tau_{n+2} \) for some \( S \in S_n^+ \) and for some \( k \). In either case, elements of the form (**) are in the subalgebra \( \mathbb{Z}_2[Q_S \tau_{n+2} \mid S \in S_n^+] \). This proves (6-20). The formula (6-21) is a straightforward consequence of (**). \( \square \)

The subalgebra in (6-20) has a very special property and it is of particular interest.
Theorem 6-11. Let \( n \geq 0 \) and let \( \tau_{n+2} \in H^{n+2}(K(\mathbb{Z}, n+2); \mathbb{Z}_2) \) be the fundamental class. Then the subalgebra
\[
Q = Q(K(\mathbb{Z}, n+2)) = \mathbb{Z}_2[Q_S \tau_{n+2} \mid S \in S^n_+ \subset H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2)]
\]
is the smallest \( \mathcal{A}(2)^* \)-invariant polynomial subalgebra of \( [H^*(K(\mathbb{Z}, n+2); \mathbb{Z}_2)]^2 \) containing the element \( Q_n \cdot Q_2 \tau_{1,n+2} \). If \( S = (0 < s_1 < s_2 < \cdots < s_n) \), then
\[
Q_S \tau_{n+2} = (Q_{s_1-1} \cdots Q_{s_{n-1}-1}^n) \mathbb{Z}_2[Q_0, Q_1, Q_2, \ldots, Q_n, Q_{n+2}]^2.
\]
The element inside of the parenthesis on the right hand side is a polynomial generator of the cohomology ring. The subalgebra \( Q \) is annihilated by the Milnor primitive \( Q_j \) for any \( j \geq 1 \).

Proof. Since there are already \( n \) Milnor primitives in \( Q_S \tau_{n+2} \), no more Milnor primitives can act nontrivially on \( Q_S \tau_{n+2} \) for any \( S \in S^n_+ \) by (II) of Proposition 6-9. Since Milnor primitives act as derivations, they act trivially on the entire subalgebra (6-22). (This can also be concluded from the fact that every element of the form \( Q_S \tau_{n+2} \) for \( S \in S^n_+ \) is a square and Milnor primitives act as derivations.)

As for the action of Steenrod reduced powers, we recall that for each \( S \in S^n_+ \), there is a sequence \( R_S \) such that \( p^{R_S} Q_n \cdots Q_1 \tau_{n+2} = Q_S \tau_{n+2} \) by (6-21). Thus for any sequence \( R \), we have \( p^{R} Q_{S} \tau_{n+2} = p^{R} Q_{S} \tau_{n+2} = p^{R} p^{R_S} Q_n \cdots Q_1 \tau_{n+2} \). We can rewrite the element \( p^{R} p^{R_S} \in \mathcal{A}(2)^* \) in terms of a basis \( \{ Q_S, p^{R_S} \} \) of \( \mathcal{A}(2)^* \). Using (6-20), we then conclude that the element \( p^{R} Q_{S} \tau_{n+2} \) is in the algebra \( Q \). This shows that elements resulting from the action of \( \mathcal{A}(2)^* \) on any generator \( Q_S \tau_{n+2} \) are in the algebra \( Q \). Using the Cartan formula (2-16), we see that \( \mathcal{A}(2)^* \) preserves the entire subalgebra \( Q \). Since all algebra generators of \( Q \) are obtained from \( Q_n \cdots Q_1 \tau_{n+2} \) by applying certain Steenrod reduced powers in \( \mathcal{A}(2)^* \) as in (6-21), \( Q \) is contained in any \( \mathcal{A}(2)^* \)-invariant subalgebra containing the element \( Q_n \cdots Q_1 \tau_{n+2} \). Hence \( Q \) is the smallest \( \mathcal{A}(2)^* \)-invariant subalgebra containing \( Q_n \cdots Q_1 \tau_{n+2} \). The formula (6-23) is obtained by rewriting (6-18). This completes the proof. \( \square \)

As in odd prime case, the \( Q \)-subalgebras in (6-13) and (6-22) are isomorphic to each other.

Proposition 6-12. Let \( \delta_h : K(\mathbb{Z}/2^h, n+1) \rightarrow K(\mathbb{Z}, n+2) \) be the \( h \)-th Bockstein map for \( h \geq 1 \) and \( n \geq 0 \). Then \( \delta_h \) induces an isomorphism between \( Q \)-subalgebras:
\[
\delta_h^* : Q(Z, n+2) \rightarrow Q(\mathbb{Z}/2^h, n+1).
\]

Proof. The proof is the same as the proof of Proposition 5-15. \( \square \)

References


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