MULTIPLICATIVE INDECOMPOSABLE SPLITTINGS OF $\text{MSp}_{[2]}$

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Abstract. When 2 is inverted, the symplectic cobordism cohomology theory becomes complex oriented with respect to the Buhstaber orientation. We study multiplicative idempotents in this theory in detail. Such multiplicative idempotents can only annihilate polynomial generators in degree $-4n$, where $2n + 1$ is not a prime power. We split off a multiplicatively indecomposable smallest possible nontrivial cohomology theory from the above theory. The coefficient ring of this theory has generators in the same degrees as the odd primary Brown-Peterson theories for arbitrary odd primes.

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§1. Introduction and Summary of Results

Let $\text{MSp}$ be the symplectic cobordism Thom spectrum. It is a ring spectrum whose homotopy groups are isomorphic to the symplectic cobordism ring, $\text{MSp}_* = \Omega_*^{\text{Sp}}$. It is well-known that the symplectic cobordism ring has 2-torsion and when 2 is inverted, it becomes isomorphic to the oriented cobordism ring $\Omega_*^{\text{SO}}$ with 2 inverted, that is, in terms of spectra we have $\text{MSp}_{[2]} \simeq \text{MSO}_{[2]}$. The connection between these two spectra is that there are natural ring spectra maps $\text{MSp} \xrightarrow{\text{p}^*_{[2]}} \text{MU} \xrightarrow{\text{v}_*} \text{MSO}$ corresponding to the forgetful functors to the complex cobordism theory, and then to the oriented cobordism theory.

1991 Mathematics Subject Classification. 55.
Key words and phrases. Complex cobordism cohomology, formal group law, multiplicative idempotent, symplectic cobordism cohomology.

Typeset by AMS-TEX
It is well known that the $MU$ spectrum localized at a prime $p$ splits into a wedge sum of infinitely many suspension copies of the $BP$ spectrum for the prime $p$, and that the $BP$ spectrum is the smallest ring spectrum which $MU(p)$ can contain. In this paper, we decompose the symplectic cobordism Thom spectrum $MSp_{[2]}$ localized away from 2 in terms of the “smallest” ring spectrum in $MSp_{[2]}$, after describing multiplicative idempotents in $MSp_{[2]}$-theory.

Our method to split off the smallest ring spectrum from $MSp_{[2]}$ is rather different from the method used in [Q] to split off the $BP$ spectrum from the localized complex cobordism spectrum $MU(p)$. Quillen controlled logarithms of multiplicative idempotents on $MU(p)$, whereas we control maps on the cohomology group $MSp^*_{[2]}(\mathbb{H}P^\infty) = MSp^*_{[2]}([w])$ induced by multiplicative idempotents, where $w \in MSp^4_{[2]}(\mathbb{H}P^\infty)$ is the symplectic orientation given in (2-4) below. Our basic tool is the following result.

**Proposition 1** [Proposition 2-3]. Ring spectra maps $\tau : MSp_{[2]} \to MSp_{[2]}$ are in 1:1 correspondence with power series of the form

\[(1-1) \quad g(w) = w + \sum_{n \geq 1} (-1)^n 2^n w^{n+1} \in MSp^4_{[2]}([w]).\]

Furthermore, the ring spectra map $\tau$ is a multiplicative idempotent if and only if the induced map $\tau_* : MSp^4_{[2]} \to MSp^4_{[2]}$ annihilates all the higher coefficients of $g(w)$, that is, $\tau_*(\gamma_n) = 0$ for all $n \geq 1$.

This is completely analogous to the corresponding fact for the complex cobordism Thom spectrum $MU$ [Proposition 2-1]. The above proposition can be generalized to a slightly more general context [Proposition 2-4, Lemma 2-5].

Since all torsion elements in $MSp_*$ are 2-primary and the forgetful map $\rho^*_{U,*}$ is an injection modulo torsion, we can regard $MSp^*_{[2]} = MSp^* \otimes \mathbb{Z}[[\frac{1}{2}]]$ as a subring of $MU^*_{[2]}$. Buhstabber constructed a multiplicative idempotent $\kappa : MU_{[2]} \to MU_{[2]}$ such that for $z \in MU_{[2]}^*$, we have $\kappa_*(z) = z$ if and only if $z \in MSp^*_{[2]}[B1, B2]$. Thus the spectrum $MSp_{[2]}$ can be thought of as a subring spectrum of $MU_{[2]}$. Let $MU_{[2]} \xrightarrow{\pi} MSp_{[2]} \xrightarrow{i} MU_{[2]}$ be the projection and the inclusion map between these ring spectra. For more on the Buhstabber splitting, see Theorem 3-2.

To any ring spectra map $\tau : MSp_{[2]} \to MSp_{[2]}$, there corresponds a ring spectra map $\tilde{\tau} : MU_{[2]} \xrightarrow{\pi} MSp_{[2]} \xrightarrow{i} MU_{[2]}$ factoring through $MSp_{[2]}$. By Proposition 1, there corresponds a degree 4 element $g(w) \in MSp^4_{[2]}(\mathbb{H}P^\infty)$ to $\tau$. On the other hand, to $\tilde{\tau}$ there corresponds a degree 2 element $\tilde{\tau}_*(x)$ in $MU^2_{[2]}(\mathbb{C}P^\infty) = MU^2_{[2]}([x])$ by Proposition 2-1 in §2, where $x \in MU^2(\mathbb{C}P^\infty)$ is the standard $MU$-orientation. Since $MU_{Q_*} = \mathbb{Q}[m_1, m_2, \ldots, m_n, \ldots]$ where $m_n = [\mathbb{C}P^n]/(n+1)$, it is convenient to do calculations in $MU_{Q_*}$ to study properties of ring spectra maps $\tau$ between $MSp_{[2]}$. To describe the element $\tilde{\tau}_*(x)$, let $\overline{x} = \exp^{MU}(-\log^{MU}(x))$ be the inverse power series with respect to the $MU$-formal group law.

**Proposition 2** [Propositions 3-4, 3-5]. Let $\tau : MSp_{[2]} \to MSp_{[2]}$ be a ring spectra map, and let $\tilde{\tau} : MU_{[2]} \to MU_{[2]}$ be the associated ring map factoring through $MSp_{[2]}$. Then $\tilde{\tau}_*(x)$ is an odd power series given by

\[(1-2) \quad \tilde{\tau}_*(x) = x \sqrt{-\frac{g(x\overline{x})}{\overline{x}^2}} \in MU^2_{[2]}([x]),\]
where $g(w) = \tau_*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in \text{MSp}_{[2]}[[w]]$ is the power series associated to $\tau$. Furthermore, in $\text{MU}_{\mathbb{Q}_*}$, we have

\begin{equation}
\begin{aligned}
\hat{\tau}_*(m_{2k-1}) &= 0, &\text{for all } k \geq 1, \\
\hat{\tau}_*(m_{2k}) &\equiv m_{2k} - \gamma_k \pmod{\text{decomposables}}, &\text{for all } k \geq 1.
\end{aligned}
\end{equation}

We are primarily interested in multiplicative idempotents on $\text{MSp}_{[2]}$ among general ring spectra maps. We construct such multiplicative idempotents as successive compositions of certain multiplicative idempotents. Let

\begin{equation}
\begin{aligned}
\hat{L} &= \{ \ell \in \mathbb{N} \mid 2\ell + 1 \text{ is not a prime power} \}, \\
\hat{P} &= \{ k \in \mathbb{N} \mid 2k + 1 \text{ is a prime power} \}.
\end{aligned}
\end{equation}

Obviously, $\hat{L} \cup \hat{P} = \mathbb{N}$. Note that the smallest integer in the set $\hat{L}$ is 7. By Milnor’s criterion, for any $\ell \in \hat{L}$ there exists an indecomposable element $\lambda_\ell \in \text{MSp}_{[2]}^{-4\ell}$ such that $s_\ell(\lambda_\ell) = 1$, where $s_\ell(\ : \ )$ is a Pontryagin number which vanishes on decomposable elements. For $k \in \hat{P}$, there exists an indecomposable element $\lambda_k \in \text{MSp}_{[2]}^{-4k}$ such that $s_\ell(\lambda_k) = p$ if $2k + 1 = p^j$ for some odd prime $p$ and $j \geq 1$. The choice of these elements $\lambda_\ell$ and $\lambda_k$ is not unique.

First we describe the simplest nontrivial multiplicative idempotents in $\text{MSp}_{[2]}$-theory. For $\ell \in \hat{L}$, let $\lambda_\ell \in \text{MSp}_{[2]}^{-4\ell}$ be an indecomposable element such that $s_\ell(\lambda_\ell) = 1$. Let $\tau = \tau(\lambda_\ell) : \text{MSp}_{[2]} \to \text{MSp}_{[2]}$ be a ring spectra map whose associated power series is of the form

\begin{equation}
g_\ell(w) \overset{\text{def}}{=} \tau_*(w) \equiv w + (-1)^{\ell} 2\lambda_\ell w^{\ell+1} \pmod{\lambda_\ell w^{\ell+2}},
\end{equation}

that is, the first nontrivial higher coefficient is $(-1)^\ell 2\lambda_\ell$ and the rest of the higher coefficients are in the ideal $(\lambda_\ell) \subset \text{MSp}_{[2]}$. We know that such a ring map exists by Proposition 1.

**Proposition 3 [Proposition 4-1].** For $\ell \in \hat{L}$, let $\lambda_\ell \in \text{MSp}_{[2]}^{-4\ell}$ be any indecomposable element such that $s_\ell(\lambda_\ell) = 1$. Then any ring spectra map $\tau : \text{MSp}_{[2]} \to \text{MSp}_{[2]}$ whose associated power series is of the form (1-5) is a multiplicative idempotent such that $\text{Ker} \tau_* = (\lambda_\ell) \subset \text{MSp}_{[2]}$.

Let us call idempotents described in Proposition 3 basic idempotents. Two basic idempotents do not commute in general under compositions. However, by successively composing such multiplicative idempotents on the right, we can construct a multiplicative idempotent with specified kernel. When the kernel is finitely generated, such multiplicative idempotents are constructed in Theorem 4-2. For the general case, a limiting argument in Proposition 4-4 proves the following theorem.

**Theorem 4 [Theorem 4-5].** For any subset $\mathbb{L} \subset \hat{L}$ and for any choice of indecomposable elements $\lambda_\ell \in \text{MSp}_{[2]}^{-4\ell}$ with $s_\ell(\lambda_\ell) = 1$ for each $\ell \in \mathbb{L}$, there exists a multiplicative idempotent $\tau : \text{MSp}_{[2]} \to \text{MSp}_{[2]}$ such that the kernel of the induced map $\tau_* : \text{MSp}_{[2]} \to \text{MSp}_{[2]}$ is the ideal generated by the elements $\lambda_\ell$ for $\ell \in \mathbb{L}$. Furthermore, the associated power series $\tau_*(w) \in \text{MSp}_{[2]}[[w]]$ is of the form

\begin{equation}
\tau_*(w) \equiv w + \sum_{\ell \in \mathbb{L}} (-1)^\ell 2\lambda_\ell w^{\ell+1} \pmod{(\lambda_\ell \mid \ell \in \mathbb{L}) \cap \text{(decomposables)}}.
\end{equation}
On the other hand, if we compose basic idempotents successively on the left, the resulting ring map is not an idempotent in general. However, we can modify idempotents inductively so that the composition gives rise to a multiplicative idempotent [Theorem 4-6]. By considering the limiting case, we obtain an alternate proof of Theorem 4 above.

The kernels of the multiplicative idempotents in Theorem 4 are generated by indecomposable elements in degree $-4\ell$, where $\ell$ is in the set $\hat{\mathbb{L}}$. It turns out that no indecomposable elements in degree $-4k$ for $k \in \mathbb{P}$ can be annihilated by multiplicative idempotents on $MSp_{[2]}^*$ in which only 2 is inverted. To be more precise, for a given multiplicative idempotent $\tau : MSp_{[2]} \to MSp_{[2]}$, let $\tau^*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}^*[\{w\}]$ be the associated power series. Not all coefficients in $\tau^*(w)$ are indecomposable. But those which are indecomposable play an important role in the description of $\tau^*(w)$. We let

$$(1-7) \quad \mathbb{L}_\tau = \{n \in \mathbb{N} \mid s_n(\gamma_n) \neq 0\},$$

where $\gamma_n$’s are higher coefficients of $\tau^*(w)$. We show that this subset of $\mathbb{N}$ cannot be arbitrarily large. In fact, the set $\mathbb{L}_\tau$ is always contained in $\hat{\mathbb{L}}$.

**Theorem 5 [Theorem 5-3].** Let $\tau : MSp_{[2]} \to MSp_{[2]}$ be a multiplicative idempotent, and let the associated power series $\tau^*(w)$ and the set of integers $\mathbb{L}_\tau \subset \mathbb{N}$ be as above. Then the following statements hold.

(i) We have $\mathbb{L}_\tau \subset \hat{\mathbb{L}}$. Furthermore, for each $\ell \in \mathbb{L}_\tau$, we have in fact $s_\ell(\gamma_\ell) = 1$.

(ii) For any integer $k \in \mathbb{N} \setminus \mathbb{L}_\tau$, there exists an indecomposable element $\lambda_k \in MSp_{[2]}^{4k}$ such that (a) $MSp_{[2]}^* = \mathbb{Z}[\frac{1}{2}][\gamma_\ell, \lambda_k]$, where $\ell \in \mathbb{L}_\tau, k \in \mathbb{N} \setminus \mathbb{L}_\tau$ as a polynomial algebra, and (b) the induced multiplicative idempotent $\tau^* : MSp_{[2]} \to MSp_{[2]}^*$ on the homotopy groups is given by

$$(1-8) \quad \left\{ \begin{array}{ll} \tau^*(\gamma_\ell) = 0, & \ell \in \mathbb{L}_\tau \\ \tau^*(\lambda_k) = \lambda_k, & k \in \mathbb{N} \setminus \mathbb{L}_\tau. \end{array} \right.$$ 

The kernel $\text{Ker} \tau^* \subset MSp_{[2]}^*$ is the ideal generated by the indecomposable elements $\gamma_\ell$ for $\ell \in \mathbb{L}_\tau$.

(iii) For any $k \in \mathbb{N} \setminus \mathbb{L}_\tau$, the corresponding coefficient $\gamma_k$ of $w^{k+1}$ in $\tau^*(w)$ is a decomposable element in the ideal $(\gamma_\ell | \ell \in \mathbb{L}_\tau)$.

As a simple consequence of Theorem 4, we have the following description of the kernel of $\tau^* : MSp_{[2]}^* \to MSp_{[2]}^*$.

**Corollary 6 [Corollary 5-4].** Let $\tau : MSp_{[2]} \to MSp_{[2]}$ be a multiplicative idempotent. Then the kernel of the induced map $\tau^* : MSp_{[2]}^* \to MSp_{[2]}^*$ is the ideal generated by the higher coefficients of the associated power series $\tau^*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}^*[\{w\}]$.

In Theorem 5 and Corollary 6, we assumed that a multiplicative idempotent $\tau : MSp_{[2]} \to MSp_{[2]}$ is given, and we studied its properties through its associated power series. Now suppose we are given a power series $g(w) = w + \sum_{k \geq 1} (-1)^k 2\gamma_k w^{k+1} \in MSp_{[2]}^*[\{w\}]$. We consider the problem of determining the
necessary and sufficient condition on the power series \( g(w) \) so that the corresponding ring map \( \tau_g : MSp_{[2]} \rightarrow MSp_{[2]} \) is actually a multiplicative idempotent. We give an inductive criterion on \( g(w) \). For \( n \geq 1 \), let

\[
g^{(n)}(w) = w + \sum_{k=1}^{n} (-1)^k 2\gamma_k w^{k+1}
\]

be a truncated polynomial of \( g(w) \), and let \( \tau^{(n)} : MSp_{[2]} \rightarrow MSp_{[2]} \) be the corresponding ring map. In Lemma 5-6, we observe that \( \tau \) is a multiplicative idempotent if and only if \( \tau^{(n)} \) is a multiplicative idempotent for all \( n \geq 1 \). By examining the transition from \( \tau^{(n)} \) to \( \tau^{(n+1)} \) [Lemma 5-7, Lemma 5-8], we obtain the following inductive characterization of those power series corresponding to idempotents.

**Theorem 7 [Theorem 5-9].** The ring spectra map \( \tau_g : MSp_{[2]} \rightarrow MSp_{[2]} \) corresponding to the power series \( g(w) = w + \sum_{k \geq 1} (-1)^k 2\gamma_k w^{k+1} \) is an idempotent if and only if \( \tau^{(n-1)} \gamma_n = s_n(\gamma_n) \gamma_n \) for all \( n \geq 1 \), where \( \tau^{(n)} \) is the ring map corresponding to the degree \( n+1 \) truncated polynomial of \( g(w) \). Equivalently and more concretely, \( \tau_g \) is a multiplicative idempotent if and only if for all \( n \geq 1 \), we have

1. \( \gamma_n \in \text{Ker} \tau^{(n-1)} \) and \( \gamma_n \) is decomposable, if \( 2n + 1 \) is a prime power,
2. \( \gamma_n \in \text{Ker} \tau^{(n-1)} \), if \( 2n + 1 \) is not a prime power and \( \gamma_n \) is decomposable,
3. \( \gamma_n \in \text{Im} \tau^{(n-1)} \) and \( s_n(\gamma_n) = 1 \), if \( 2n + 1 \) is not a prime power and \( \gamma_n \) is indecomposable.

From (i) in Theorem 5 above, the set \( L_\tau \) is always contained in \( \hat{L} \), and from Theorem 4 there are multiplicative idempotents \( \tau \) on \( MSp_{[2]} \) for which \( L_\tau = \hat{L} \).

Let \( LSp_\tau \) be the ring spectrum split off from \( MSp_{[2]} \) as the image of a multiplicative idempotent \( \tau \) for which \( L_\tau = \hat{L} \). From Theorem 5 (ii), its homotopy group is a polynomial algebra given by

\[
LSp_{\tau_*} = \mathbb{Z}[[1/2]][\lambda_k \mid k \in \hat{P}],
\]

where \( \lambda_k \in MSp_{[2]}^{-4k} \) is an indecomposable element such that \( s_k(\lambda_k) = p \) if \( 2k + 1 \) is a power of the odd prime \( p \). Although the way \( LSp_\tau \) sits inside of \( MSp_{[2]} \) depends on the multiplicative idempotent \( \tau \), it turns out that the homotopy type of the spectrum \( LSp_\tau \) is independent of \( \tau \) satisfying the property \( L_\tau = \hat{L} \).

**Theorem 8 [Theorem 6-2, Proposition 6-3].** Let \( \tau : MSp_{[2]} \rightarrow MSp_{[2]} \) be a multiplicative idempotent such that \( L_\tau = \hat{L} \). Then the ring spectrum \( LSp_\tau \) split off from \( MSp_{[2]} \) as the image of \( \tau \) is multiplicatively indecomposable in the sense that any multiplicative idempotent on \( LSp_\tau \) is the identity map. For any multiplicative idempotents \( \tau_1 \) and \( \tau_2 \) on \( MSp_{[2]} \) such that \( L_{\tau_i} = \hat{L} \) for \( i = 1, 2 \), the associated ring spectra \( LSp_{\tau_1} \) and \( LSp_{\tau_2} \) are homotopy equivalent.

Since the homotopy type of \( LSp_\tau \) is independent of \( \tau \), we simply denote this spectrum \( LSp \) without any reference to the multiplicative idempotent used. It follows that \( MSp_{[2]} \) can be decomposed as a wedge sum of many suspension copies of \( LSp \) as follows [Corollary 6-4]:

\[
MSp_{[2]} \simeq \bigvee_i \Sigma^{2|I|} LSp,
\]
where $I$ ranges over all finite (possibly empty) sequences of integers from $2\mathbb{L}$:

$$I = (i_1 \leq i_2 \leq \cdots \leq i_r | i_k \in 2\mathbb{L} \text{ for } 1 \leq k \leq r, r \geq 0),$$

and $|I| = \sum_j i_j$.

The organization of this paper is as follows. In §2, we describe how complex orientations and symplectic orientations control ring spectra maps for complex or symplectic cobordism Thom spectra. Here, we describe the symplectic case in detail. We also characterize multiplicative idempotents among ring spectra maps. In §3, we first describe the relationship between the standard symplectic orientation and the standard complex orientation. Using the Buhštaber splitting, we describe the ring spectra map on the symplectic cobordism Thom spectra $MSp_{[2]}$ localized away from 2 in terms of the complex orientation and we derive various useful formulae. In §4, we construct various multiplicative idempotents on $MSp_{[2]}$ as successive compositions of basic multiplicative idempotents on the right or on the left. In §5, we discuss general properties of multiplicative idempotents in $MSp_{[2]}$, and in particular we show a close relationship between kernel ideals of idempotents and coefficients of the associated power series. We also give an inductive characterization of those power series corresponding to multiplicative idempotents on $MSp_{[2]}$. In §6, we split off multiplicatively indecomposable ring spectra from $MSp_{[2]}$ and we decompose $MSp_{[2]}$ into a wedge sum in terms of these indecomposable spectra.

Acknowledgement. The author thanks the referee for various comments which lead to improvement of the exposition.

§2. Complex and Symplectic Orientations: Ring Maps for the Thom Spectra $MU$ and $MSp$

In this section, we describe how orientations in $MU$-theory and $MSp$-theory can be used to control ring spectra maps. Since the literature is readily available for relevant facts on $MU$-theory (see [A2], for example), we only recall the basic facts for $MU$-theory. Here we describe the $MSp$-theory case in some detail.

**Notation.** Let $P$ be a set of primes. For a spectrum $E$, let $E_{[P]}$ denote the spectrum obtained from $E$ by inverting primes in $P$. When $P$ is the set of all primes, then $E_{[P]}$ is also denoted by $E_\mathbb{Q}$ because its homotopy group is a $\mathbb{Q}$-algebra. If $P$ contains all primes except one prime $p$, then $E_{[P]}$ is the spectrum $E_{(p)}$ localized at $p$. In particular, $E_{[2]}$ denotes the spectrum obtained by inverting only 2.

Let $\eta \to \mathbb{C}P^\infty = BU(1)$ be the universal (tautological) complex line bundle on $\mathbb{C}P^\infty$. The inclusion map of $\mathbb{C}P^\infty$ as a zero section into $\eta$ induces a homotopy equivalence with the associated Thom complex: $\mathbb{C}P^\infty \xrightarrow{\eta} \mathbb{C}P^\infty \xrightarrow{\eta} M\eta = MU(1)$. The standard complex orientation is defined by

$$(2-1) \quad x = x^{MU} : \mathbb{C}P^\infty \xrightarrow{\eta} M\eta = MU(1) \to \Sigma^2 MU.$$ 

This defines a nontrivial element $x \in MU^2(\mathbb{C}P^\infty)$. If we apply the Thom map $\rho : MU \to H\mathbb{Z}$ from $MU$ to the integral Eilenberg-Mac Lane spectrum, the resulting element $\rho(x) \in H\mathbb{Z}^2(\mathbb{C}P^\infty)$ is the Euler class of the universal bundle $\eta \to \mathbb{C}P^\infty$ generating the second integral cohomology group of $\mathbb{C}P^\infty$. 


A spectral sequence argument implies that $MU^*(\mathbb{C}P^n) = MU^*[x]/(x^{n+1})$ for all $n \geq 0$, where $x$ is the restriction of the standard complex orientation $x^M$ to $\mathbb{C}P^n$. The relation $x^{n+1} = 0$ comes from the fact that $x^{n+1} : \mathbb{C}P^n \to MU(n+1)$ is null homotopic, since $MU(n+1)$ is $2n+1$-connected and $\mathbb{C}P^n$ is $2n$ dimensional. From this calculation, it follows that the restriction map $MU^*(\mathbb{C}P^{n+1}) \to MU^*(\mathbb{C}P^n)$ is surjective for all $n \geq 0$. Then from the Milnor exact sequence [M1], we have $MU^*(\mathbb{C}P^\infty) = MU^*[x^M]$, a formal power series ring with the usual filtration topology. There are no nonzero elements of infinite filtration because the surjectivity of each map in the inverse system implies the vanishing of the $\lim^1$ term in the Milnor exact sequence. As usual, let $m_k = \lfloor \mathbb{C}P^k \rfloor/(k+1) \in MU_{Q2k} = MU_{Q}^{2k}$ for $k \geq 1$. We let $m_0 = 1$. The logarithm series $\log MU(T)$ is defined by

$$\log MU(T) = T + \sum_{k \geq 1} m_k T^{k+1} \in MU^*[\lfloor T \rfloor],$$

where $T$ is a formal variable. We remark that the set of all the primitive elements in $MU^*_Q(\mathbb{C}P^\infty)$ is $MU^*_Q$-free generated by $\log MU(x)$. The power series inverse to $\log MU(T)$ is denoted by $\exp MU(T)$, that is, $\exp MU(\log MU(T)) = T = \log MU(\exp MU(T))$.

Now let $\tau : MU_P \to MU_P$ be a ring spectra map for some set of primes $P$. Let $\tau : MU^*_P(\mathbb{C}P^\infty) \to MU^*_P(\mathbb{C}P^\infty)$ be the induced algebra map. Then $\tau(x) = g(x) \in MU^*_P([x])$ is a formal power series in $x$. The following fact is well-known. For example see Lemma 4.6 and Proposition 15.3 in [A2], or Lemma 1.53 and the formula (3.8) in [W].

**Proposition 2-1.** (I) The above correspondence induces a bijection between the set of ring spectra maps $\tau : MU_P \to MU_P$ and the set of formal power series $g(x) \in MU^*_P([x])$ of homogeneous degree 2 such that $g(x) \equiv x \mod (x^2)$.

(II) Given a power series $g(x)$ as above, the effect on $MU^*_P$ of the corresponding ring map $\tau = \tau_g : MU_P \to MU_P$ is given by the following formula:

$$T + \sum_{k \geq 1} \tau_s(m_k) T^{k+1} = \log MU(g^{-1}(T)),$$

where $T$ is a formal variable.

(III) Let $g(x) = x + \sum_{k \geq 1} a_k x^{k+1} \in MU^*[\lfloor x \rfloor]$ be the power series corresponding to a ring map $\tau : MU_P \to MU_P$. Then $\tau$ is a multiplicative idempotent if and only if $\tau_a(a_k) = 0$ for all $k \geq 1$.

We call power series $g(x)$ as in (I) general complex orientations. The part (III) of Proposition 2-1 is not usually found in the literature, although the proof is straightforward. See the proof of Proposition 2-3 below.

The identity (2-3) follows from the fact that $\log MU$ is primitive. That is, when $\tau$ is a ring map, the induced algebra map $\tau : MU^*_P(\mathbb{C}P^\infty) \to MU^*_P(\mathbb{C}P^\infty)$ is also a coalgebra map preserving primitives. By checking the leading coefficient, we have $\log MU x = \tau_s(\log MU) x = g(x) + \sum_{k \geq 1} \tau_s(m_k) g(x)^{k+1}$. If we let $g(x) = T$ or $x = g^{-1}(T)$ in this formula, we obtain (2-3).

We show that there is a corresponding statement for the symplectic cobordism Thom spectrum $MSp$. First we define the standard $MSp$-orientation. Let $\zeta \to
be the universal (tautological) symplectic line bundle. Its Thom complex is \( M\zeta = MSp(1) \). The inclusion map of \( \mathbb{HP}^\infty \) as a zero section into \( \zeta \) induces a homotopy equivalence with the Thom complex \( \mathbb{HP}^\infty \xrightarrow{\sim} MSp(1) \). The symplectic orientation \( w \in MSp_{[P]}(\mathbb{HP}^\infty) \) is defined by

\[
(2-4) \quad w : \mathbb{HP}^\infty \xrightarrow{\sim} MSp(1) \to \Sigma^4 MSp.
\]

By an argument using spectral sequences and the Milnor exact sequences similar to the one used for \( MU_{[P]}^*([\mathbb{C}P^\infty]) \) for any set of primes \( P \), we have

\[
(2-5) \quad MSp_{[P]}^*(\mathbb{HP}^\infty) = \lim_{n \to \infty} MSp_{[P]}^*(\mathbb{HP}^n) = MSp_{[P]}^*[w].
\]

We note that the spectral sequence \( H_*([\mathbb{HP}^\infty]; MSp_{[P]}^*) \Rightarrow MSp_{[P]}^*(\mathbb{HP}^\infty) \) collapses since the image of the element \([\mathbb{HP}^n] \in MSp_{4n}(\mathbb{HP}^\infty) \) in \( H_{4n}(\mathbb{HP}^\infty; \mathbb{Z}) \) under the Thom map \( \rho : MSp \to H\mathbb{Z} \) is the generator of that group. Thus, by the Universal Coefficient Theorem [A1,A2], we have

\[
(2-6) \quad MSp_{[P]}^*(\mathbb{HP}^\infty) \cong \text{Hom}_{MSp_{[P]}^*}(MSp_{[P]}^*(\mathbb{HP}^\infty), MSp_{[P]}^*)
\]

For any \( i \geq 0 \), let \( q_i \in MSp_{[P]}^*([\mathbb{HP}^\infty]) \) be defined by \( \langle w^i, q_j \rangle = \delta_{ij} \) for \( i, j \geq 0 \). We have \( q_0 = 1 \). Then \( MSp_{[P]}^*([\mathbb{HP}^\infty]) \) is a \( MSp_{[P]}^* \)-module generated by \( q_i \)'s. Let \( Q_i \in MSp_{[P]}^*([\mathbb{HP}^\infty]) \) be the image of \( q_i+1 \) under the induced map \( w_* : MSp_{[P]}^*([\mathbb{HP}^\infty]) \to MSp_{[P]}^*(MSp_{[P]}^*) \). Note that \( MSp_{[P]}^*(MSp_{[P]}^*) \) is a \( MSp_{[P]}^* \)-algebra. Using Thom isomorphisms and spectral sequences, we have

\[
(2-7) \quad MSp_{[P]}^*(MSp_{[P]}^*) = MSp_{[P]}^*[Q_1, Q_2, \ldots, Q_n, \ldots].
\]

Since the spectral sequence for (2-7) collapses, again by the Universal Coefficient Theorem, the Kronecker pairing induces the following isomorphism:

\[
(2-8) \quad MSp_{[P]}^*(MSp_{[P]}^*) \cong \text{Hom}_{MSp_{[P]}^*}(MSp_{[P]}^*(MSp_{[P]}^*), MSp_{[P]}^*).
\]

The above isomorphism is as \( MSp_{[P]}^* \)-modules. Given a spectra map \( \tau : MSp_{[P]} \to MSp_{[P]} \), the corresponding \( MSp_{[P]}^* \)-module map \( \theta \) under (2-8) is given by

\[
\theta_* : MSp_{[P]}^*(MSp_{[P]}^*) \xrightarrow{(1 \wedge \tau)_*} MSp_{[P]}^*(MSp_{[P]}^*) \xrightarrow{\mu_*} MSp_{[P]}^*
\]

where \( \mu : MSp_{[P]} \wedge MSp_{[P]} \to MSp_{[P]} \) is the product map in the Thom spectrum \( MSp \). The next lemma is a straightforward generalization of the \( MU \) case and the proof is essentially the same. But we give its proof for the sake of completeness.

**Lemma 2.2.** Under the isomorphism (2-8), there is a bijective correspondence between the set of ring spectra maps \( \tau : MSp_{[P]} \to MSp_{[P]} \) and the set of algebra maps \( \theta : MSp_{[P]}^*(MSp_{[P]}^*) \to MSp_{[P]}^* \).

**Proof.** Suppose a spectra map \( \tau : MSp_{[P]} \to MSp_{[P]} \) is a ring map making the following diagram commutative:

\[
(2-9) \quad MSp_{[P]} \贴水 MSp_{[P]} \xrightarrow{\tau \wedge \tau} MSp_{[P]} \贴水 MSp_{[P]}
\]

\[
\mu \downarrow \quad \mu \downarrow
\]

\[
MSp_{[P]} \xrightarrow{\tau} MSp_{[P]},
\]
Proposition 2.3. The above correspondence gives rise to a bijection between the set of ring spectra maps of \( Q \) algebra map corresponding to and only if power series is an element in \( MSp \) series in \( w \). Here, for simplicity, we have let \( \Lambda = MSp[1] \). Using the definition of the map \( \theta_{\tau} \), we see that the commutativity of (2-10) shows that \( \theta_{\tau} \) is an algebra map.

Conversely, let \( \theta : MSp[1]^* \rightarrow MSp[1]^* \) be an \( MSp[1]^* \)-algebra map and let \( \tau \circ MSp[1]^* \rightarrow MSp[1]^* \) be the corresponding spectra map under (2-8). Since \( \theta = \mu_{\ast} \circ (1 \land \tau)_{\ast} \), the condition that \( \theta \) is an algebra map implies that we have a commutative diagram as in (2-10) with \( \tau \) replaced by \( \tau_{\theta} \). This implies that we have the following commutative diagram:

\[
\begin{array}{ccc}
\downarrow{\mu_{\ast}} & & \downarrow{\mu_{\ast}} \\
\end{array}
\]

The commutativity of this diagram means that under the isomorphism


the \( MSp[1]^* \)-module maps corresponding to the two maps \( \mu \circ (\tau_{\theta} \land \tau_{\theta}) \) and \( \tau_{\theta} \circ \mu \) from \( MSp[1]^* \land MSp[1]^* \) to \( MSp[1]^* \) are the same. Hence, these two spectra maps must be the same. But this means that the diagram (2-9) commutes, with \( \tau \) replaced by \( \tau_{\theta} \). Hence \( \tau_{\theta} \) is a ring spectra map. \( \square \)

Given a ring spectra map \( \tau : MSp[1]^* \rightarrow MSp[1]^* \), let \( \tau_{\ast} : MSp[1]^* (\mathbb{HP}^\infty) \rightarrow MSp[1]^* (\mathbb{HP}^\infty) \) be the induced map. By (2-5), \( \tau_{\ast} (w) \in MSp[1]^* (\mathbb{HP}^\infty) \) is a power series in \( w \) with coefficients in \( MSp[1]^* \). Note that the coefficient of \( w^0 = 1 \) of this power series is an element in \( MSp[1]^* \), hence it must be zero.

**Proposition 2.3.** The above correspondence gives rise to a bijection between the set of ring spectra maps \( \tau : MSp[1]^* \rightarrow MSp[1]^* \) and the set of formal power series \( g(w) \in MSp[1]^*[w] \) of homogeneous degree 4 such that \( g(w) \equiv w \mod (w^3) \).

Let \( g(w) = w + \sum_{k \geq 1} a_k w^{k+1} \) with \( a_k \in MSp[1]^*[k] \) be a power series. The corresponding ring map \( \tau \circ \tau : MSp[1]^* \rightarrow MSp[1]^* \) is an idempotent, that is \( \tau \circ \tau = \tau \), if and only if \( \tau_{\ast} (a_k) = 0 \) for all \( k \geq 1 \).

**Proof.** Given a ring map \( \tau : MSp[1]^* \rightarrow MSp[1]^* \), let \( \tau_{\ast} (w) = g_{\tau} (w) \in MSp[1]^*[w] \) be the induced power series, and let \( \theta_{\tau} : MSp[1]^* (MSp[1]^*) \rightarrow MSp[1]^* \) be the algebra map corresponding to \( \tau \) given in Lemma 2.2.

Let \( g_{\tau} (w) = \sum_{k \geq 0} a_k w^{k+1} \), and \( \theta_{\tau} (Q_k) = d_k \in MSp[1]^* \) for \( k \geq 0 \). Since \( Q_0 = 1 \) and \( \theta_{\tau} \) is an algebra map, we must have \( d_0 = 1 \). Now, from the definition of \( Q_k \)'s and the Kronecker pairings, we have \( d_k = \theta_{\tau} (Q_k) = \theta_{\tau} (w_{\ast} (Q_{k+1})) = \ldots \)
\[ \langle \tau, w_*(q_{k+1}) \rangle = \langle w^*(\tau), q_{k+1} \rangle. \] Here, \( w^*(\tau) = \tau \circ w = \tau_*(w) : H^\infty P \to \Sigma^4 MSp_{[P]} P \to \Sigma^4 MSp_{[P]}. \) Since \( \tau_*(w) = \sum_{k \geq 0} a_k w^{k+1} \) and \( (w^i, q_j) = \delta_{ij} \) for \( i, j \geq 0, \) continuing the above calculation, we have \( d_k = \langle \sum_{k \geq 0} a_k w^{k+1}, q_{k+1} \rangle \) for all \( k \geq 0. \) Since \( d_0 = 1, \) we must have \( a_0 = 1. \) Thus the leading term of \( g_*(w) \) is \( w. \)

Let \( \tau_1, \tau_2 : MSp_{[P]} \to MSp_{[P]} \) be two ring maps, and suppose that their induced power series are the same, that is, \( g_{\tau_1} \) (\( w \)) = \( g_{\tau_2} \) (\( w \)). From the above calculation, this implies that the corresponding algebra maps \( \theta_{\tau_1} \) and \( \theta_{\tau_2} \) must be the same. But then, by Lemma 2-2, we must have \( \tau_1 = \tau_2. \) This shows that the correspondence \( \tau \to g_*(w) \) is injective.

Now let \( g(w) \in MSp_{[P]} (w) \) be an arbitrary power series of the form \( g(w) = \sum_{k \geq 0} a_k w^{k+1} \) with \( a_0 = 1. \) Let \( \theta : MSp_{[P]} \to MSp_{[P]} \) be an algebra map given by \( \theta(q_k) = a_k \) for all \( k \geq 0. \) If \( \tau : MSp_{[P]} \to MSp_{[P]} \) is the unique ring map corresponding to the algebra map \( \theta \) by Lemma 2-2, then we must have \( g(w) = g_*(w) \) by the choice of \( \tau. \) Hence our correspondence \( \tau \to g_*(w) \) is surjective. This proves the first part of Proposition 2-3.

Next, given a power series \( g(w) = w + \sum_{k \geq 1} a_k w^{k+1} \in MSp_{[P]} (w) \) whose leading term is \( w, \) let \( g_0 : MSp_{[P]} \to MSp_{[P]} \) be the corresponding ring map. This map is idempotent, that is, \( g_0 \circ g_0 = g_0, \) and only if \( g_{g_0} \circ g_{g_0}(w) = g_{g_0}(w) \) because the induced power series uniquely determine ring spectra maps by the first part of Proposition 2-3. By construction of \( g_0, \) we have \( g_{g_0}(w) = g_*(w). \) Hence, the above identity means that \( g(w) = g_{g_0}(g(w)) = g(w) + \sum_{k \geq 1} g_{g_0}(a_k) g(w)^{k+1}. \) Since \( g(w) = w + \) (higher powers of \( w \)), by examining the coefficients of \( w^k \) inductively, this identity implies that \( g_{g_0}(a_k) = 0 \) for all \( k \geq 1. \) This completes the proof of Proposition 2-3. \( \square \)

We call power series \( g(w) \) as in Proposition 2-3 general symplectic orientations.

**Remark.** Since there is no \( H \)-space structure on \( H^\infty P = BSp(1) \), the algebra \( MSp_{[P]} (H^\infty P) \) does not have a structure of a coalgebra. Consequently, we cannot talk about primitives in \( MSp_{[P]} (H^\infty P) \) unlike the case for \( MU_{[P]} (CP^\infty) \). Hence there is no logarithm in this symplectic context. See the remark right after (2-2).

In fact, the same method as in the first half of Proposition 2-3 applies to classify ring maps from \( MSp \) to \( MU. \) Let \( \tau : MSp_{[P]} \to MU_{[P]} \) be a ring map, and let \( \tau_0 : MSp_{[P]} (H^\infty P) \to MU^*_{[P]} (H^\infty P) \) be the induced map. Let \( \tau_0 (w^{MSp}) = g(w^{MU}) \in MU^*_{[P]} (w^{MU}). \)

**Proposition 2-4.** There is a bijection between the set of ring spectra maps \( \tau : MSp_{[P]} \to MU_{[P]} \) and the set of formal power series \( g(w) \in MU^*_{[P]} (w) \) of homogeneous degree 4 such that \( g(w) \equiv w \mod (w^2). \)

**Proof.** The proof is similar to the proof of Proposition 2-3 using the fact that the Kronecker pairing gives rise to an isomorphism

\[ MU^*_{[P]} (MSp_{[P]}) \cong \text{Hom}_{MU_{[P]}} (MU_{[P]} (MSp_{[P]}), MU_{[P]}), \]

and the ring maps \( \tau : MSp_{[P]} \to MU_{[P]} \) corresponds to algebra maps under the above isomorphism. Here, \( MU_{[P]} (MSp_{[P]}) \cong MU_{[P]} [Q_1, Q_2, \ldots]. \) \( \square \)

The statement in the second half of Proposition 2-3 can be generalized in the following way.
Lemma 2-5. Let $\tau_1, \tau_2 : \text{MSp}[P] \to \text{MSp}[P]$ be two ring maps, and let $\tau_2(w) = w + \sum_{k \geq 1} b_k w^{k+1}$ be the power series associated to $\tau_2$. Then $\tau_1 \circ \tau_2 = \tau_1 \text{ and only if } \tau_1$ annihilates all the higher coefficients of $\tau_2 (w)$, that is, if and only if $\tau_1 (b_k) = 0$ for all $k \geq 1$.

Proof. Let $\tau_1 (w) = g_1(w)$ and $\tau_2 (w) = g_2(w)$. Then $(\tau_1 \circ \tau_2)_*(w) = \tau_1 (g_2(w)) = g_1(w) + \sum_{k \geq 1} \tau_1 (b_k) g_2 (w)^{k+1}$. This is equal to $\tau_1 (w) = g_1(w)$ if and only if $\tau_1 (b_k) = 0$ for all $k \geq 1$. □

§3. Buhštaber Splitting and Ring Maps of $MU_{[2]}$ factoring through $\text{MSp}[2]$

In §2, we saw that general complex orientations in $MU^*_P(\mathbb{C}P^\infty)$ control ring maps $MU_P \to MU_P$, and that general symplectic orientations in $\text{MSp}^*_P(\mathbb{H}P^\infty)$ control ring maps $\text{MSp}_P \to \text{MSp}_P$. In this section we study the relationship between these two objects.

First we examine the relationship between the standard $\text{MSp}$-orientation $w \in \text{MSp}^*(\mathbb{H}P^\infty)$ given in (2-4) and the standard $MU$-orientation $x \in MU^2(\mathbb{C}P^\infty)$ given in (2-1). Let $\mathbb{H}^\infty$ be an infinite dimensional left $\mathbb{H}$-vector space. The $\mathbb{H}P^\infty$ can be thought of as the set of all the left $\mathbb{H}$-lines $L$ in $\mathbb{H}^\infty$. The fibre of the tautological left $\mathbb{H}$ line bundle $\zeta \to \mathbb{H}P^\infty$ over $L \in \mathbb{H}P^\infty$ is the set of all vectors in $L$. Since $\mathbb{C} \subset \mathbb{H} = \mathbb{C} \oplus j \cdot \mathbb{C}$, $\mathbb{H}^\infty$ can be thought of as an infinite dimensional $\mathbb{C}$-vector space and the set of all $\mathbb{C}$-lines $\ell$ in $\mathbb{H}^\infty$ is $\mathbb{C}P^\infty$. These $\mathbb{C}$-lines form a tautological complex line bundle $\eta \to \mathbb{C}P^\infty$. We have a canonical map $\iota : \mathbb{C}P^\infty \to \mathbb{H}P^\infty$ by mapping a $\mathbb{C}$-line $\ell \in \mathbb{C}P^\infty$ into the $\mathbb{H}$-line $L \in \mathbb{H}P^\infty$ containing $\ell$, that is, $L = \mathbb{H} \cdot \ell = \ell \oplus j \cdot \ell$. As left $\mathbb{C}$-line bundles, $j \cdot \eta$ is isomorphic to the conjugate bundle $\overline{\eta}$. Thus it follows that $\iota^*(\zeta) = \eta \oplus \overline{\eta}$ as (left) complex bundles over $\mathbb{C}P^\infty$.

Let $\rho^S_P : \text{MSp} \to MU$ be the ring spectra map corresponding to the forgetful functor. We consider the following maps:

\[
\text{MSp}^*(\mathbb{H}P^\infty) \xrightarrow{\rho^S_P} MU^*(\mathbb{H}P^\infty) \xrightarrow{\iota^*} MU^*(\mathbb{C}P^\infty).
\]

Let $w^{\text{MSp}} = \rho^S_P(w^{\text{MSp}}) \in MU^*(\mathbb{H}P^\infty)$. By a spectral sequence argument, we have $MU^*(\mathbb{H}P^\infty) = MU^*([w^{\text{MSp}}])$. Since $MU^*(\mathbb{C}P^\infty) = MU^*([x^{MU}])$, the image of the symplectic orientation $w^{\text{MSp}} \in \text{MSp}^1(\mathbb{H}P^\infty)$ under the above map is a power series in $x^{MU}$. This power series is identified in the next lemma.

Lemma 3-1. The image of the symplectic orientation $w^{\text{MSp}}$ under the map (3-1) is given by

\[
\iota^* \circ \rho^S_P(w^{\text{MSp}}) = x \cdot \overline{x} \in MU^4(\mathbb{C}P^\infty),
\]

where $\overline{x} = [-1]_{MU}(x) = \exp^{MU}(-\log^{MU}(x))$. In particular, the induced map $\iota^* : MU^*(\mathbb{H}P^\infty) \to MU^*(\mathbb{C}P^\infty)$ is an injection and $\iota^*(w^{MU}) = x \cdot \overline{x}$.

Proof. Elements $x$ and $\overline{x}$ are given by the following maps:

\[
x : \mathbb{C}P^\infty \hookrightarrow MU(1) \to \Sigma^2 MU, \quad \overline{x} : \mathbb{C}P^\infty \xrightarrow{(-1)} \mathbb{C}P^\infty \hookrightarrow MU(1) \to \Sigma^2 MU,
\]
Here \((-1)\cdot\) denotes the homotopy inverse map. The product \(x \cdot \mathcal{T} \in MU^4(\mathbb{C}P^\infty)\) is given by the composition of the upper arrows followed by the vertical map and the map into \(\Sigma^4MU\) in the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}P^\infty & \stackrel{(1,-1)\circ \Delta}{\longrightarrow} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\
\downarrow \eta \oplus \overline{\eta} & & \downarrow \eta \times \eta \\
BU(2) & \longrightarrow & BU(2) \longrightarrow MU(2) \longrightarrow \Sigma^4MU.
\end{array}
\]

In the above diagram, the vertical map labeled \(\eta \oplus \overline{\eta}\) is the classifying map for the Whitney sum bundle \(\eta \oplus \overline{\eta}\). Similarly, the vertical map labeled \(\eta \times \eta\) is the classifying map for the product bundle \(\eta \times \eta\). Since the composition \(\zeta \circ \iota : \mathbb{C}P^\infty \to \mathbb{H}P^\infty \to BU(2)\) is such that \((\zeta \circ \iota)(\eta \oplus \overline{\eta}) = \eta \oplus \overline{\eta}\), where \(\zeta\) is the classifying map for \(\zeta\) as a complex 2-dimensional vector bundle, the map \(\zeta \circ \iota\) is the classifying map for the sum \(\eta \oplus \overline{\eta}\). Hence we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}P^\infty & \stackrel{\iota}{\longrightarrow} & \mathbb{H}P^\infty \\
\downarrow \eta \oplus \overline{\eta} & & \downarrow \zeta \\
BU(2) & \longrightarrow & BU(2) \longrightarrow MU(2) \longrightarrow \Sigma^4MU.
\end{array}
\]

The element \(\iota^*\left(\rho^p_{\mu}\left(w^{MSp}\right)\right)\) is represented by the composition of upper arrows followed by the rightmost vertical arrow. By the commutativity of the above diagram, this is equal to the composition of the leftmost vertical arrow followed by the bottom arrows, which is equal to \(x \cdot \mathcal{T}\) by the commutative diagram \((\ast)\). This proves \((3-2)\). The injectivity of the map \(\iota^*\) is now obvious. \(\square\)

We consider Conner-Floyd Chern classes \(c_i = c_i^{MU}\) of the quaternionic line bundle \(\zeta \to \mathbb{H}P^\infty\) viewed as a 2-dimensional complex vector bundle. Recall that \(\iota^*(\zeta) = \eta \oplus \overline{\eta}\) for the canonical map \(\iota : \mathbb{C}P^\infty \to \mathbb{H}P^\infty\). Thus by the naturality of characteristic classes, the induced map \(\iota^* : MU^*\left(\mathbb{H}P^\infty\right) \to MU^*\left(\mathbb{C}P^\infty\right)\) is such that \(\iota^*(c_1(\zeta)) = c_1(\eta \oplus \overline{\eta}) = x + \mathcal{T}\) and \(\iota^*(c_2(\zeta)) = c_1(\eta)\eta(\overline{\eta}) = x\mathcal{T}\). Since \(MU^*\left(\mathbb{H}P^\infty\right) = MU^*\left(\mathbb{C}P^\infty\right)\), we have

\[
c_1(\zeta) = \sum_{i \geq 1} \alpha_i(w^{MU})^i \text{ for some uniquely determined } \alpha_i \in MU^{2-4i}.
\]

Note that the above summation starts with \(i = 1\) by dimensional reason. Since \(\iota^*\) is injective and \(\iota^*(w^{MU}) = x\mathcal{T}\) by Lemma 3-1, we have

\[
c_2(\zeta) = w^{MU}.\]

By comparing the formula for \(\iota^*(c_1(\zeta))\), we have

\[
(3-3) \quad x + \mathcal{T} = \sum_{i \geq 1} \alpha_i(\mathcal{A}\mathcal{T})^i \in MU^2(\mathbb{C}P^\infty),
\]

for some uniquely determined elements \(\alpha_i \in MU^{2-4i}\) for \(i \geq 1\).

The standard \(MU\)-symplectic orientation \(w^{MU}\) is such that \(w^{MU} = x\mathcal{T} = -x^2 + (\text{higher order terms}) \text{ in } MU^*(\mathbb{C}P^\infty)\). Let \(\kappa(x)\) be a power series in \(x\) given by

\[
(3-4) \quad \kappa(x) = x\sqrt{1 + (\text{higher order terms in } x)} \in MU^2_{[2]}[[x]].
\]

Here by convention, square roots of the form \(\sqrt{1 + (\text{higher order terms in } x)}\) are always taken as those with leading term 1. Note that \(\kappa(x)^2 = -x\mathcal{T}\). Let \(\kappa : MU_{[2]} \to MU_{[2]}\) be the ring map whose associated power series is \(\kappa_*(x) = \kappa(x)\). This ring map was first considered by Buhştaber.
Theorem 3.2 (Buhšteber Splitting) [B1, B2]. The Buhšteber map \( \kappa : MU_{[2]} \to MU_{[2]} \) is a multiplicative idempotent and the spectrum split off as the image of \( \kappa \) is the isomorphic image of \( MS\mathbb{P}[2] \) under the forgetful ring spectra map \( \rho^S_{U} \).

More precisely, in the following natural cohomology transformations,

\[
MS\mathbb{P}_{[2]}(\cdot) \xrightarrow{\rho^S_{U} \ast} MU_{[2]}^* (\cdot) \xrightarrow{\kappa \ast} MU_{[2]}^* (\cdot),
\]

we have \( \kappa \circ \kappa = \kappa \), and \( \kappa \) fixes an element in \( MU_{[2]}^* (\cdot) \) if and only if it is in the image of \( \rho^S_{U} \). Furthermore, the map \( \kappa \ast \) annihilates all the higher coefficients of the power series \( \kappa_\ast (x) \) and all the elements in \( MU_{[2]}^* \) in degrees congruent to 2 modulo 4.

The cohomology theory \( MS\mathbb{P}_{[2]}(\cdot) \) is complex oriented with the complex orientation \( \kappa (x) = \sqrt{-x/x} \in MS\mathbb{P}_{[2]}^2 (\mathbb{C}P^\infty) \). Hence \( MS\mathbb{P}_{[2]}^* (\mathbb{C}P^\infty) \cong MS\mathbb{P}_{[2]}^* [\{ \kappa(x) \}] \).

In short, the spectrum \( MS\mathbb{P}_{[2]} \) is a multiplicative summand of \( MU_{[2]} \). Let \( \pi \) and \( \iota \) be the projection map onto this summand and the inclusion map from this summand, respectively:

\[
(3-5) \quad \kappa : MU_{[2]} \xrightarrow{\pi} MS\mathbb{P}_{[2]} \xrightarrow{\iota} MU_{[2]}.
\]

Recall that \( m_k = |\mathbb{C}P^k|/(k+1) \in MU_Q^* \) for \( k \geq 1 \). Modulo decomposables, the power series \( \kappa(x) \) and the behavior of \( \kappa_\ast : MU_{[2]}^* \to MU_{[2]}^* \) are described as follows.

Lemma 3.3. The power series \( \kappa_\ast (x) = \kappa(x) = x\sqrt{-x/x} \in MS\mathbb{P}_{[2]}^*[\{ x \}] \subset MU_Q^*[\{ x \}] \) is such that

\[
\kappa(x) \equiv x + \sum_{k \geq 1} m_{2k-1} x^{2k} \quad \text{mod (decomposables)},
\]

\[
\log^{MU} (\kappa^{-1}(T)) \equiv x + \sum_{k \geq 1} m_{2k} T^{2k+1} \quad \text{mod (decomposables)}.
\]

Furthermore, \( \kappa_\ast \) kills all the generators in degrees congruent to 2 mod 4, and \( \kappa_\ast (m_{2k}) \equiv m_{2k} \) modulo decomposables in \( MU_Q^* \) for all \( k \geq 1 \).

Proof.\ Since \( \log^{MU} (x) = x + \sum_{k \geq 1} m_k x^{k+1} \), its inverse power series \( \exp^{MU}(x) \) is such that \( \exp^{MU}(x) \equiv x - \sum_{k \geq 1} m_k x^{k+1} \) modulo decomposables. So it follows that \( \mathfrak{f} = [-1]_{MU}(x) = \exp^{MU}(-\log^{MU}(x)) \equiv -x - 2 \sum_{k \geq 1} m_{2k-1} x^{2k} \). Hence \( \kappa(x) = x\sqrt{-\mathfrak{f}/x} \equiv x \sqrt{1 + 2 \sum_{k \geq 1} m_{2k-1} x^{2k-1}} \equiv x + \sum_{k \geq 1} m_{2k-1} x^{2k} \) modulo decomposables. From this we have \( \kappa^{-1}(x) \equiv x - \sum_{k \geq 1} m_{2k-1} x^{2k} \). Thus,

\[
\log^{MU} (\kappa^{-1}(x)) = \kappa^{-1}(x) + \sum_{k \geq 1} m_k (\kappa^{-1}(x))^{k+1}
\]

\[
\equiv x - \sum_{k \geq 1} m_{2k-1} x^{2k} + \sum_{k \geq 1} m_k x^{k+1} \quad \text{mod (decomposables)}
\]

\[
\equiv x + \sum_{k \geq 1} m_{2k} x^{2k+1} \quad \text{mod (decomposables)}.
\]
By (2-3), we see that \( \kappa_*(m_{2k}) \equiv m_{2k} \mod \text{decomposables} \). Since \( MSp^*[2] \) is generated by elements having degrees congruent to 0 mod 4, \( \kappa_* \) kills all the odd degree elements. So we have \( \kappa_*(m_{2k-1}) = 0 \) for all \( k \geq 1 \). This completes the proof of Lemma 3-3. \( \square \)

Now we consider multiplicative cohomology transformations \( \tau_* : MSp^*[2]^* (\cdot) \to MSp^*[2]^*(\cdot) \) which factor through \( MSp^*[2]^* (\cdot) \). These transformations are characterized as follows.

**Proposition 3-4.** Let \( \tau : MU[2] \to MU[2] \) be a ring spectra map corresponding to a power series \( \tau_*(x) = \tau(x) \in MU[2][[x]] \). Then the ring map \( \tau \) factors through \( MSp[2] \) if and only if \( \tau(x) \) is of the form

\[
(3-7) \quad \tau(x) = x \sqrt{-g(xT)} / x^2,
\]

where \( g(w) \in MSp^*[2][[w]] \) is a power series whose leading term is \( w \), and with coefficients in \( MSp^*[2] \).

**Proof.** Suppose a ring map \( \tau \) from \( MU[2] \) to itself factors through \( MSp[2] \) as follows:

\[
\tau : MU[2] \xrightarrow{\pi} MSp[2] \xrightarrow{\tau} MSp[2] \xrightarrow{\pi} MU[2],
\]

for some ring map \( \pi \) from \( MSp[2] \) to \( MSp[2] \). By Proposition 2-3, to such \( \pi \), there corresponds a unique power series \( \pi_*(w) = g(w) \in MSp^*[2][[w]] \) in \( w \), where \( \pi_* : MSp^*[2](\mathbb{HP}^\infty) \to MSp^*[2](\mathbb{HP}^\infty) \). By pulling back this calculation to \( \mathbb{CP}^\infty \), we have \( \pi_*(xT) = g(xT) \) in \( MSp^*[2](\mathbb{CP}^\infty) = MSp^*[2][[x^{MSp}]] \), where \( x^{MSp} = \kappa(x) = x \sqrt{-1/x} \). Thus, \( \tau_* = j_* \circ \pi_* \circ \kappa_* \) for some ring map \( \kappa_* : MSp[2] \to MSp[2] \) which factors through \( MSp[2] \). Consequently, the effects of \( \tau_* \) on \( x \) are the same. Hence, by Proposition 2-1, we must have \( \tau = \tau_* = j \circ \kappa \circ \pi \) and the map \( \tau \) factors through \( MSp[2] \). \( \square \)

We study more properties of those ring maps \( \tau : MU[2] \to MU[2] \) which factor through \( MSp[2] \). Let \( \log^\tau(T) \) be the power series obtained from \( \log^{MU}(T) \) by applying \( \tau_* \) to its coefficients. That is,

\[
(3-8) \quad \log^\tau(T) = T + \sum_{k \geq 1} \tau_*(m_k)T^{k+1} \in MU[2][[T]].
\]
Proposition 3-5. Let $\tau : MU_{[2]} \to MU_{[2]}$ be the ring map factoring through $MSP_{[2]}$. Let $g(w) \in MSP_{[2]}[[w]]$ be the power series associated to $\tau$ as in (3-7).

(i) Let $\tau_*(x) = \tau(x) \in MU_{[2]}[[x]]$ for some power series $\tau(x)$. Then

$$\tau_*(x) = \tau(x) = \tau(x)$$

Here, the power series $\tau((x))$ is the one obtained by replacing $x$ by $\tau(x)$.

(ii) The power series $\log^\tau(T)$ given in (3-8) is an odd power series in $T$. That is, $\tau_* (m_{2k-1}) = 0$ for all $k \geq 1$. Furthermore, let $\varphi(T) \in MSP_{[2]}[[T]]$ be the power series such that $\varphi(T) \equiv T \mod (T^2)$ and $\varphi(T) = \sqrt{1 + g(T)}$. Then $\varphi(T)$ is an odd power series such that $\tau(x) = \varphi(\kappa(x))$ and in $MU^*_Q$ we have

$$T + \sum_{k \geq 1} \tau_*(m_{2k}) T^{2k+1} = \varphi^{-1}(T) + \sum_{k \geq 1} \kappa_*(m_{2k}) (\varphi^{-1}(T))^{2k+1}.$$  

In other words, $\log^\tau(T) = \log^\kappa(\varphi^{-1}(T))$.

(iii) Let $g(w) = w + \sum_{k \geq 1} \gamma_k w^{2k+1} \in MSP_{[2]}[[w]]$. Then we have the following congruence relation in $MU^*_Q$ for any $k \geq 1$:

$$\tau_* (m_{2k}) \equiv m_{2k} - \gamma_k \mod \text{decomposables}.$$  

If, for some $n$, $\gamma_k = 0$ for $k < n$ in the above expression of $g(w)$, then in $MU^*_Q$,

$$\tau_* (m_{2k}) = \kappa_*(m_{2k}) \quad \text{for} \quad k < n, \quad \text{and} \quad \tau_* (m_{2n}) = \kappa_*(m_{2n}) - \gamma_n.$$  

(iv) The kernel of the induced map $\tau_* : MSP_{[2]} \to MSP_{[2]}$ is contained in the ideal generated by the higher coefficients of $g(w)$. That is,

$$\ker \tau_* \subset (\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots) \subset MSP_{[2]}.$$  

Proof. By (3-3), we have $x + \tau = \sum_{i \geq 1} \alpha_i (x \tau)^i$ for some $\alpha_i \in MU^{2-4i}$. Since $\tau_*$ annihilates all the elements in $MU^*$ in degrees congruent to 2 mod 4 because $\tau_*$ factors through $MSP_{[2]}$, applying $\tau_*$ to the above identity, we have $\tau_* (x) + \tau_* (\tau) = 0$.

Since $\tau_* (x) = \tau(x)$, we have $\tau_* (\tau) = -\tau(x)$. On the other hand, replacing $x$ by $\tau$ in the power series $\tau(x)$,

$$\tau(\tau) = \tau \left( \frac{-g(x \tau)}{\tau^2} \right) = -\tau \left( \frac{-\tau}{x} \right) \sqrt{\frac{-g(x \tau)}{\tau^2}} = -\tau \sqrt{\frac{-g(x \tau)}{x^2}} = -\tau(x).$$

Here, we are still using the convention that $\sqrt{1 + \text{higher order terms}} = 1 + \text{(higher order terms)}$. In the above calculation, since $-\tau/x$ starts with 1, we can move its square inside of the square root. This proves (3-9). For (ii), from (2-3), we have $\log^\tau(T) = \log^{MU}(\tau^{-1}(T))$. From the definition of $\tau$, we have $\log^{MU}(\tau) = -\log^{MU}(x)$. Let $\tau(x) = T$. Then from $\tau(x) + \tau(\tau) = 0$, we have $\tau = \tau^{-1}(-T)$. Thus $\log^\tau(-T) = \log^{MU}(\tau^{-1}(-T)) = \log^{MU}(\tau) = -\log^{MU}(x) = -\log^{MU}(\tau^{-1}(T)) = -\log^\tau(T)$. This shows that the power series $\log^\tau(T)$ in (3-8)
is an odd power series. Consequently, the coefficients of even powers of $T$ vanishes, i.e., $\tau_r(m_{2k-1}) = 0$ for $k \geq 1$. Next, we have $\log^r(T) = \log^{s_i} (\tau^{-1}(T)) = \log^{s_i} (\varphi^{-1}(T)))$. This proves (3.10).

For (iii), given $g(w)$ as above, the corresponding power series $\varphi(T)$ is of the form $\varphi(T) \equiv T + \sum_{k \geq 1} \gamma_k T^{2k+1}$ modulo decomposables. (This is the reason for our choice of the multiplicative factor $(-1)^{k/2}$ in the coefficients of $g(w)$.) So, $\varphi^{-1}(T) \equiv T - \sum_{k \geq 1} \gamma_k T^{2k+1}$, again modulo decomposables. Hence, modulo decomposables, the R.H.S. of (3.10) is congruent to

$$\varphi^{-1}(T) + \sum_{k \geq 1} \kappa_*(m_{2k}) (\varphi^{-1}(T))^{2k+1} \equiv (T - \sum_{k \geq 1} \gamma_k T^{2k+1}) + \sum_{k \geq 1} \kappa_*(m_{2k}) T^{2k+1} \equiv T + \sum_{k \geq 1} (m_{2k} - \gamma_k) T^{2k+1}.$$  

Here, we used a congruence relation in Lemma 3.3. By (3.10), this means that $\tau_r(m_{2k}) \equiv m_{2k} - \gamma_k$ for all $k \geq 1$ modulo decomposables. This proves (3.11).

For some $n$, if $\gamma_k = 0$ for all $k < n$, then modulo the ideal $(T^{2n+3})$, we have $\varphi^{-1}(T) \equiv T - \gamma_n T^{2n+1}$. Substituting this into the R.H.S. of (3.10), we obtain

$$T + \sum_{k=1}^n \tau_r(m_{2k}) T^{2k+1} \equiv T - \gamma_n T^{2n+1} + \sum_{k=1}^n \kappa_*(m_{2k}) T^{2k+1} \mod (T^{2n+3}).$$

Comparing the coefficients of $T^{2k+1}$ for $k \leq n$, we obtain (3.12).

For (iv), since $\varphi^{-1}(T) \equiv T \mod (\gamma_1, \ldots, \gamma_k, \ldots)$ in $\text{MSp}^*_2$, reducing (3.10) modulo the same ideal extended to $\text{MU}^*_Q$, we have $\tau_r(m_{2k}) \equiv \kappa_*(m_{2k})$ for all $k \geq 1$. Since $\tau_r$ preserves the subring $\text{MSp}^*_2$ of $\text{MU}^*_Q$ and $\kappa_*$ is the identity on this subring, for any $z \in \text{MSp}^*_2$, we have $\tau_r(z) \equiv z$ modulo $(\gamma_1, \ldots, \gamma_k, \ldots)$. Thus, if $\tau_r(z) = 0$ for some $z \in \text{MSp}^*_2$, then $z \equiv 0$ modulo the ideal $(\gamma_1, \ldots, \gamma_k, \ldots) < \text{MSp}^*_2$.

This completes the proof of Proposition 3.5. □

Remark. Later in Corollary 5.4, we will show that the inclusion relation in (3.13) is actually an equality when $\tau$ is an idempotent.

§4. Constructing Multiplicative Idempotents in $\text{MSp}^*_2$

In this section, we construct multiplicative idempotents acting on $\text{MSp}^*_2$. It turns out that no polynomial generator in degree $-4n$ such that $2n + 1$ is a prime power can be annihilated by multiplicative idempotents on $\text{MSp}^*_2$ with only 1 inverted. On the other hand, for any collection of polynomial generators of $\text{MSp}^*_2$ in non-prime-power degrees, there exists a multiplicative idempotent on $\text{MSp}^*_2$ which annihilates exactly the given collection of polynomial generators, and no more.

To describe our idempotents, recall that a cobordism class of a real $4n$ dimensional oriented closed manifold $M^{4n}$ can be taken as a polynomial generator of the oriented cobordism ring $\Omega_*^{\text{SO}}$ if and only if $s_n([M^{4n}]) = \sigma_n$, where

$$\sigma_n = \begin{cases} p & \text{if } 2n + 1 \text{ is a power of the prime } p, \\ 1 & \text{if } 2n + 1 \text{ is not a prime power}. \end{cases}$$
Here, \( s_n(\cdot) \) is a Pontryagin characteristic number which vanishes on decomposable elements [M2]. For example, \( s_n([\mathbb{C}P^{2n}]) = 2n + 1 \) for all \( n \geq 1 \), in other words, \( s_n(m_{2n}) = 1 \) for all \( n \geq 1 \).

Since the composition of natural ring maps \( MSp \xrightarrow{\rho_{Sp}^*} MU \xrightarrow{\rho_{Sp}^*} MSO \) is a homotopy equivalence after inverting 2, we can use the same criteria to identify polynomial generators of \( MSp_{[2]}^* \) over \( \mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}] \). Since \( \rho_{Sp}^* : MSp_{[2]}^* \to MU_{[2]}^* \) is an inclusion map, we regard \( MSp_{[2]}^* \) as a subring of \( MU_{[2]}^* \).

For each \( n \geq 1 \), let \( \lambda_n \in MSp_{[2]}^{-4n} \) be a polynomial generator such that \( s_n(\lambda_n) = \sigma_n \). Since \( MSp_{[2]}^* \) is a subring of \( MU_{[2]}^* = \mathbb{Q}[m_1, m_2, \ldots, m_n, \ldots] \), we can write

\begin{equation}
\lambda_n = \sigma_n m_{2n} + h(n)(m_1, m_2, \ldots, m_{2n-1}) \in MSp_{[2]}^{-4n}, \quad n \geq 1,
\end{equation}

for some weighted homogeneous polynomial \( h(n) \) with \( \mathbb{Q} \)-coefficients. The choice of the generators \( \lambda_n \) is not unique. Other choice of generators have different decomposable parts in terms of \( m_k \)'s. But we choose one set of such generators.

Recall that by Proposition 2-3, ring spectra maps \( \tau : MSp_{[2]} \to MSp_{[2]} \) and power series \( g(w) \) correspond in \( 1 : 1 \) manner.

**Proposition 4-1.** The ring map \( \tau = [\lambda_n] : MSp_{[2]} \to MSp_{[2]} \) corresponding to any power series \( g(w) \) of the form

\begin{equation}
g(w) \equiv w + (-1)^n 2 \lambda_n w^{n+1} \mod (\lambda_n w^{n+2})
\end{equation}

has the following properties:

\begin{equation}
\begin{aligned}
\tau_*(\lambda_k) &= \lambda_k \quad \text{for} \quad k < n, \\
\tau_*(\lambda_n) &= (1 - \sigma_n) \lambda_n, \\
\tau_*(\lambda_k) &\equiv \lambda_k \mod (\lambda_n) \cap (\text{decomposables}) \quad \text{for} \quad k > n.
\end{aligned}
\end{equation}

For \( k \neq n \), \( \tau_*(\lambda_k) \) is a polynomial generator in degree \(-4k\) in \( MSp_{[2]}^* \), and the kernel of \( \tau_* : MSp_{[2]}^* \to MSp_{[2]}^* \) is such that \( \text{Ker} \tau_* \subset (\lambda_n) \).

Furthermore, \( \tau = [\lambda_n] : MSp_{[2]} \to MSp_{[2]} \) is an idempotent if and only if \( 2n + 1 \) is not a prime power, which is the case if and only if \( \text{Ker} \tau_* = (\lambda_n) \).

**Proof.** When \( k < n \), in the right hand side of (4-2) with \( n \) replaced by \( k \), the effect of \( \tau_* \) is the same as the effect of \( \kappa_* \) by the first formula in (3-12) in Proposition 3-5 (iii), because of the form of the power series \( g(w) \) in (4-3). Thus \( \tau_*(\lambda_k) = \kappa_*(\lambda_k) = \lambda_k \) for \( k < n \) since \( \kappa_* \) fixes elements in \( MSp_{[2]}^* \). This proves the first formula in (4-4).

The effect of \( \tau_* \) on \( \lambda_n \) is calculated using (3-12) and (4-2) as follows:

\begin{align*}
\tau_*(\lambda_n) &= \sigma_n \cdot \tau_*(m_{2n}) + h(n)(\tau_*(m_1), \ldots, \tau_*(m_{2n-1})) \\
&= \sigma_n \cdot (\kappa_*(m_{2n}) - \lambda_n) + \kappa_*(h(n)(m_1, \ldots, m_{2n-1})) \\
&= \kappa_*(\sigma_n m_{2n} + h(n)(m_1, \ldots, m_{2n-1})) - \sigma_n \lambda_n \\
&= \kappa_*(\lambda_n) - \sigma_n \lambda_n = (1 - \sigma_n) \lambda_n.
\end{align*}

Here, again we used the fact that \( \kappa_*(\lambda_n) = \lambda_n \) because the Buhşaber idempotent \( \kappa_* \) fixes \( MSp_{[2]}^* \). This shows the second formula in (4-4).
For the third formula, note that modulo the ideal \((\lambda_n)\), we have \(g(w) \equiv w\). So we have \(\varphi^{-1}(T) \equiv T\) modulo \((\lambda_n)\). Thus, from (3-10) we have \(\tau_*(m_{2k}) \equiv \kappa_*(m_{2k})\) mod \((\lambda_n)\) for all \(k \geq 1\). Hence \(\tau_*(\lambda_k) \equiv \kappa_*(\lambda_k) = \lambda_k\) for all \(k \geq 1\), modulo the ideal \((\lambda_n)\). By dimensional reason, when \(k > n\), the difference \(\tau_*(\lambda_k) - \lambda_k \in (\lambda_n)\) must also lie in the ideal of decomposable elements. This proves the third formula in (4-4).

Since the characteristic number \(s_k(\cdot)\) vanishes on decomposable elements, (4-4) shows that \(s_k(\tau_*(\lambda_k)) = s_k(\lambda_k)\) for all \(k\) different from \(n\). So \(\tau_*(\lambda_k)\) for \(k \neq n\) can be taken as polynomial generators of \(MSP^{*}_{[2]}\) in degree \(-4k\).

By the second part of Proposition 2-3, the ring map \(\tau = \tau[\lambda_n]\) associated to a power series \(g(w)\) such that \(g(w) \equiv w + (-1)^n 2\lambda_n w^{n+1}\) mod \((\lambda_n) \cap \text{(decomposables)}\) is a multiplicative idempotent if and only if \(\tau_*\) annihilates all the higher coefficients of \(g(w)\), that is, if and only if \(\tau_*(\lambda_n) = 0\). By the second formula in (4-4), this happens if and only if \(\sigma_n = 1\). By the definition of \(\sigma_n\) in (4-1), we see that \(\tau[\lambda_n]\) is a multiplicative idempotent if and only if \(2n + 1\) is not a prime power. This completes the proof of Proposition 4-1.

Note that for an integer \(n\) such that \(2n + 1\) is not a prime power, the above multiplicative idempotent \(\tau[\lambda_n]\) depends on the specific choice of the power series \(g(w)\) of the form (4-3), although the kernel of the induced homomorphism \(\tau[\lambda_n]*\) on the homotopy groups is completely determined by the choice of the indecomposable element \(\lambda_n\).

The above multiplicative idempotents do not commute with each other in general. So in general their compositions are not multiplicative idempotents, either. However, we can show that if we compose the above type of multiplicative idempotents in a specific order, the resulting map is also a multiplicative idempotent.

**Theorem 4-2.** Let \(n_1 < n_2 < \cdots < n_r\) be positive integers such that \(2n_i + 1\)’s are not prime powers for \(1 \leq i \leq r\). Let \(\lambda_{n_i} \in MSP^{*}_{[2]}\) be a polynomial generator such that \(s_{n_i}(\lambda_{n_i}) = 1\), and let \(\tau[n_1] = \tau[\lambda_{n_1}]: MSP_{[2]} \rightarrow MSP^{*}_{[2]}\) be the multiplicative idempotent associated to any power series \(g_{[n_1]}(w)\) of the form

\[
g_{[n_1]}(w) \equiv w + (-1)^{n_1} 2\lambda_{n_1} w^{n_1+1} \mod (\lambda_{n_1}) \cap \text{(decomposables)}, \quad 1 \leq i \leq r,
\]

for \(1 \leq i \leq r\). Then the successive composition of these idempotents on the right

\[
\tau[n_1, \ldots, n_r] = \tau[n_1] \circ \cdots \circ \tau[n_r]: MSP_{[2]} \rightarrow MSP^{*}_{[2]}
\]

is again a multiplicative idempotent such that

\[
\tau[n_1, \ldots, n_r]_*(\lambda_{n_1}) = 0, \quad \text{for} \quad 1 \leq i \leq r,
\]

\[
\tau[n_1, \ldots, n_r]_*(\lambda_k) \equiv \lambda_k \mod (\lambda_{n_1}, \ldots, \lambda_{n_r}) \cap \text{(decomposables)},
\]

where the second formula holds for all \(k\) different from \(n_i\)’s. Thus, the kernel of the ring map \(\tau[n_1, \ldots, n_r]*: MSP_{[2]} \rightarrow MSP^{*}_{[2]}\) is the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r})\).

The associated power series \(g_{[n_1, \ldots, n_r]}(w) = \tau[n_1, \ldots, n_r]*(w)\) has coefficients in the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}) \subset MSP^{*}_{[2]}\), and we have

\[
g_{[n_1, \ldots, n_r]}(w) = \tau[n_1, \ldots, n_r]*(w) \equiv w + \sum_{i=1}^{r} (-1)^{n_i} 2\lambda_{n_i} w^{n_i+1}.
\]
modulo the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}) \cap \text{(decomposables)}\).

**Proof.** We prove this theorem by induction on \(r\). The case \(r = 1\) is taken care of by Proposition 4.1. Let \(r \geq 1\) and suppose that the above statements hold for a sequence \((n_1, n_2, \ldots, n_r)\) of the above type. Let \(n_{r+1}\) be such that \(2n_{r+1} + 1\) is not a prime power and \(n_r < n_{r+1}\). Let \(\tau[n_{r+1}] : MSp_{[2]} \rightarrow MSp_{[2]}\) be a ring spectra map associated to the power series of the form \(g_{(r+1)}(w) = w + (-1)^{n_{r+1}}2\lambda_{n_{r+1}}w^{n_{r+1} + 1} + \lambda_{n_{r+1}}w^{n_{r+1} + 2}f(w)\) for some \(f(w) \in MSp_{[2]}[[w]]\). By Proposition 4.1, \(\tau[n_{r+1}]\) is a multiplicative idempotent. Note that the leading term of \(f(w)\) is of degree \(-4\). By abbreviating \((n_1, n_2, \ldots, n_r)\) by \(\tilde{n}^{(r)}\), we have

\[
\begin{align*}
g_{(r+1)}(w) &= \tau[n_1, \ldots, n_r, n_{r+1}]_*(w) = \tau[\tilde{n}^{(r)}]_* \circ \tau[n_{r+1}]_* (w) \\
&= \tau[\tilde{n}^{(r)}]_* (w + (-1)^{n_{r+1}}2\lambda_{n_{r+1}}w^{n_{r+1} + 1} + \lambda_{n_{r+1}}w^{n_{r+1} + 2}f(w)) \\
&= g_{[\tilde{n}^{(r)}]}(w) + (-1)^{n_{r+1}}2\tau[\tilde{n}^{(r)}]_* (\lambda_{n_{r+1}}) \cdot (g_{[\tilde{n}^{(r)}]}(w))^{n_{r+1} + 1} \\
&\quad + \tau[\tilde{n}^{(r)}]_* (\lambda_{n_{r+1}}) \cdot (g_{[\tilde{n}^{(r)}]}(w))^{n_{r+1} + 2} \cdot (\tau[\tilde{n}^{(r)}]_*) (g_{[\tilde{n}^{(r)}]}(w)),
\end{align*}
\]

where \(\tau[\tilde{n}^{(r)}]_* f\) is obtained by applying \(\tau[\tilde{n}^{(r)}]_*\) to the coefficients of \(f(w)\). By our inductive hypothesis, \(g_{[\tilde{n}^{(r)}]}(w) \equiv w + \sum_{i=1}^{r+1} (-1)^{n_i}2\lambda_i w^{n_i + 1}\) modulo the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}) \cap (\text{decomposables})\) by (4.8), and \(\tau[\tilde{n}^{(r)}]_* (\lambda_{n_{r+1}}) \equiv \lambda_{n_{r+1}}\) modulo \((\lambda_{n_1}, \ldots, \lambda_{n_r}) \cap (\text{decomposables})\) by (4.7). All the coefficients in the last term in (\*\) are in the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}}) \cap (\text{decomposables})\), because all the coefficients of \(f(w)\) have strictly negative degrees and consequently \(\lambda_{n_{r+1}}\) cannot appear on its own as a coefficient of a power of \(w\). The second term from the last is congruent to \((-1)^{n_{r+1} + 1}2\lambda_{n_{r+1}}w^{n_{r+1} + 1}\) modulo the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}}) \cap (\text{decomposables})\). Hence modulo the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}}) \cap (\text{decomposables})\),

\[
g_{(r+1)}(w) \equiv g_{[\tilde{n}^{(r)}]}(w) + (-1)^{n_{r+1} + 1}2\lambda_{n_{r+1}}w^{n_{r+1} + 1} \equiv w + \sum_{i=1}^{r+1} (-1)^{n_i}2\lambda_i w^{n_i + 1}.
\]

This proves the inductive step for (4.8).

Next, we prove the inductive step for (4.7). From the first and the second formula in (4.4), we have \(\tau[n_{r+1}]_*(\lambda_{n_i}) = \lambda_{n_i}\) for \(1 \leq i \leq r\) and \(\tau[n_{r+1}]_*(\lambda_{n_{r+1}}) = 0\). From these formulae, we have

\[
\begin{align*}
\tau[n_1, \ldots, n_{r+1}]_*(\lambda_{n_{r+1}}) &= \tau[\tilde{n}^{(r)}]_* (\tau[n_{r+1}]_*(\lambda_{n_{r+1}})) = 0, \\
\tau[n_1, \ldots, n_{r+1}]_*(\lambda_{n_i}) &= \tau[\tilde{n}^{(r)}]_* (\tau[n_{r+1}]_*(\lambda_{n_i})) = \tau[\tilde{n}^{(r)}]_* (\lambda_{n_i}) = 0.
\end{align*}
\]

The last identity is due to the inductive hypothesis (4.7). This proves the inductive step for the first formula in (4.7).

At this point, note that the ring map \(\tau[n_1, \ldots, n_{r+1}]_*\) annihilates all the higher coefficients of the associated power series \(g_{[n_1, \ldots, n_{r+1}]}(w)\), since all the higher coefficients are in the ideal \((\lambda_{n_1}, \ldots, \lambda_{n_{r+1}})\) by (4.8). Thus, by Proposition 2.3, the ring spectra map \(\tau[n_1, \ldots, n_{r+1}]\) is a multiplicative idempotent.

If \(\lambda_k\) is any polynomial generator of degree \(-4k\), then by Proposition 4.1 we have \(\tau[n_{r+1}]_*(\lambda_k) \equiv \lambda_k\mod (\lambda_{n_{r+1}})\), which is also valid for \(k = n_{r+1}\). Thus,

\[
\begin{align*}
\tau[n_1, \ldots, n_{r+1}]_*(\lambda_k) &= \tau[\tilde{n}^{(r)}]_* (\lambda_k) \equiv \tau[\tilde{n}^{(r)}]_* (\lambda_{n_{r+1}}) \\
&= \tau[\tilde{n}^{(r)}]_* ((\lambda_{n_{r+1}}),)
\end{align*}
\]
By our inductive hypothesis, $\tau[\tilde{n}^{(r)}]_*(\lambda_k) \equiv \lambda_k \mod (\lambda_{n_1}, \ldots, \lambda_{n_r})$ for all $k$, by the second formula in (4-7). Letting $k = n_{r+1}$, we see that $\tau[\tilde{n}^{(r)}]_*(\lambda_{n_{r+1}}) \in (\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}})$. So we have $\tau[\tilde{n}^{(r)}]_*((\lambda_{n_{r+1}})) \subset (\lambda_{n_1}, \ldots, \lambda_{n_{r+1}})$. Combining these formulae, (***) gives $\tau[\tilde{n}]_*(\lambda_{n_{r+1}}) \equiv \lambda_k \mod (\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}})$ for all $k$. This completes the inductive step for the second formula in (4-7).

We have now completed all the inductive steps and Theorem 4-2 is proved. \hfill \Box

We now consider compositions of infinitely many multiplicative idempotents of the form described in Proposition 4-1. For this, we have to deal with convergence in the cohomology group $\text{MSp}_{[2]}^*(\text{MSp}_{[2]})$ with respect to the filtration topology defined as follows. Let $\iota(k) : \text{MSp}_{[2]}^*(\text{MSp}_{[2]}) \to \text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4k)})$. This map induces a restriction map

$$\iota(k) : \text{MSp}_{[2]}^*(\text{MSp}_{[2]}) \to \text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4k)}).$$

Let $F^{(k)} = \text{Ker} \iota(k) \subset \text{MSp}_{[2]}^*(\text{MSp}_{[2]})$ for $k \geq 0$. Then $F^{(k)}$’s define a decreasing filtration on $\pi^*(\text{MSp}_{[2]}^*(\text{MSp}_{[2]}))$:

$$\text{MSp}_{[2]}^*(\text{MSp}_{[2]}) = F^{(0)} \supset F^{(1)} \supset \cdots \supset F^{(k)} \supset \cdots \supset \bigcap_n F^{(n)} = F^{(\infty)}.$$

By the standard argument, it is easy to see that this defines a complete Hausdorff topology on this cohomology group, that is, $\bigcap_n F^{(n)} = \{0\}$.

**Lemma 4-3.** Let $\tau_1, \tau_2 : \text{MSp}_{[2]} \to \text{MSp}_{[2]}$ be two ring maps and let $\tau_{i*} : \text{MSp}_{[2]}^*(\text{HP}^\infty) \to \text{MSp}_{[2]}^*(\text{HP}^\infty)$ be the induced maps for $i = 1, 2$. Suppose $\tau_{i*}(w) \equiv \tau_{2*}(w) \mod (w^{n+1})$ for some $n \geq 1$. Then the corresponding ring spectra maps $\tau_1$ and $\tau_2$ agree on the $4(n - 1)$-skeleton. That is, we have

$$\iota(n-1)^*(\tau_1) = \iota(n-1)^*(\tau_2) \in \text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4n-4)}),$$

**Proof.** For $i = 1, 2$, let $\theta_i : \text{MSp}_{[2]}^*(\text{MSp}_{[2]}) \to \text{MSp}_{[2]}^*$ be the algebra map corresponding to $\tau_i$ as in Lemma 2-2. From the proof of Lemma 2-3, the coefficients of the power series $\tau_{i*}(w)$ determine the images of the algebra generators $Q_k$ under the maps $\theta_i$. By our assumption, we have $\theta_1(Q_j) = \theta_2(Q_j)$ for $1 \leq j \leq n-1$. Hence $\theta_1$ and $\theta_2$ agree as $\text{MSp}_{[2]}^*$ module maps from $\text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4n-4)})$ to $\text{MSp}_{[2]}^*$. Due to the isomorphism

$$\text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4n-4)}) \cong \text{Hom}_{\text{MSp}_{[2]}^*}(\text{MSp}_{[2]}^*(\text{MSp}_{[2]}^{(4n-4)}), \text{MSp}_{[2]}^*),$$

the above agreement means that $\iota(n-1)^*(\tau_1) = \iota(n-1)^*(\tau_2) : \text{MSp}_{[2]}^{(4n-4)} \to \text{MSp}_{[2]}^*$. That is, $\tau_1$ and $\tau_2$ agree on the $4(n - 1)$-skeleton. \hfill \Box

The above lemma can be used to control the convergence of a sequence of ring maps in terms of the convergence of the associated power series.
Proposition 4-4. Let \( \tau_i : \text{MSp}_{[2]} \to \text{MSp}_{[2]} \) for \( i \geq 1 \) be a sequence of ring spectra maps such that the associated power series \( \tau_i(w) \in \text{MSp}_{[2]}[[w]] \) converges to \( g(w) \) as \( i \to \infty \). If \( \tau : \text{MSp}_{[2]} \to \text{MSp}_{[2]} \) is the ring map corresponding to \( g(w) \), then the sequence of ring spectra maps \( \tau_i \) converges to \( \tau \) with respect to the filtration topology in \( \text{MSp}_{[2]}^*(\text{MSp}_{[2]}) \).

Furthermore, if \( \tau_i \)’s are multiplicative idempotents, then their limit \( \tau \) is also a multiplicative idempotent.

Proof. Since the power series \( g(w) \) and \( \tau_i(w) \) agree more and more as \( i \) increases by our assumption, the corresponding ring maps \( \tau \) and \( \tau_i \) agree on higher and higher skeletons by Lemma 4-3. Hence their difference \( \tau - \tau_i \) belongs to higher and higher filtration as \( i \to \infty \). By definition, this means that \( \lim_i \tau_i = \tau \) with respect to the filtration topology in \( \text{MSp}_{[2]}^*(\text{MSp}_{[2]}) \).

Next, suppose that \( \tau_i \) is a multiplicative idempotent for all \( i \geq 1 \). Then by Proposition 2-3, the induced ring map \( \tau_i : \text{MSp}_{[2]}^* \to \text{MSp}_{[2]}^* \) annihilates all the higher coefficients of \( \tau_i(w) \in \text{MSp}_{[2]}^*[w] \) for all \( i \). We let

\[
\tau_i(w) = w + \sum_{k \geq 1}(-1)^{k}2\gamma_kw^{k+1}, \quad \tau_i(w) = w + \sum_{k \geq 1}(-1)^{k}2\gamma_i^{(i)}w^{k+1}, \quad i \geq 1.
\]

For each \( k \geq 1 \), there exists an integer \( N_k \) such that \( \tau_i(w) \equiv \gamma_i(w) \mod (w^{k+2}) \) for all \( i \geq N_k \) since the power series \( \tau_i(w) \) converge to \( \tau_i(w) \). From Lemma 4-3, the ring spectra maps \( \tau \) and \( \tau_i \) agree on the \( 4k \)-skeleton of \( \text{MSp}_{[2]} \). Thus on the \( k \)th coefficient \( \gamma_k \) of \( \tau_i(w) \), we have \( \tau_i(\gamma_k) = \tau_i(\gamma_k) \) for all \( i \geq N_k \). Since \( \tau_i \) is an idempotent, we have \( \tau_i(\gamma_k^{(i)}) = 0 \) for all \( k \geq 1 \). Since \( \gamma_k = \gamma_k^{(i)} \) for \( i \geq N_k \), we have \( \tau_i(\gamma_k) = \tau_i(\gamma_k) = \tau_i(\gamma_k^{(i)}) = 0 \). Since \( k \) is arbitrary, we see that \( \tau_i \) annihilates all the higher coefficients of \( \tau_i(w) \). Thus, by Proposition 2-3, \( \tau \) is a multiplicative idempotent. \( \square \)

Let \( \mathbb{L} \subset \mathbb{N} \) be any subset of \( \mathbb{N} \) consisting of integers \( \ell \) such that \( 2\ell + 1 \) is not a prime power. The cardinality of \( \mathbb{L} \) may be finite or infinite. For each \( \ell \in \mathbb{L} \), we choose an indecomposable element \( \lambda_\ell \in \text{MSp}_{[2]}^ {4\ell} \) such that \( s_\ell(\lambda_\ell) = 1 \). Such elements always exist by Milnor’s criterion. For each \( \ell \in \mathbb{L} \), we consider a power series of the form (4-3) using \( \lambda_\ell \) in place of \( \lambda_\ell \), and let the corresponding ring spectra map be \( \tau[\ell] = \tau[\lambda_\ell] : \text{MSp}_{[2]} \to \text{MSp}_{[2]} \) for \( \ell \in \mathbb{L} \). We order integers in \( \mathbb{L} \) from the smallest one as follows:

\[
\mathbb{L} : \ell_1 < \ell_2 < \cdots < \ell_r < \cdots.
\]

We consider a sequence of finite compositions of the corresponding ring maps of the following form:

(4-9) \[
\tau^{(r)} = \tau[\ell_1] \circ \tau[\ell_2] \circ \cdots \circ \tau[\ell_r] : \text{MSp}_{[2]} \to \text{MSp}_{[2]}
\]

for \( r \geq 1 \). Please note the order of the compositions. By Theorem 4-2, \( \tau^{(r)} \) is a multiplicative idempotent for all \( r \geq 1 \). We consider the limit as \( r \to \infty \). To describe our result, for each \( k \in \mathbb{N} \setminus \mathbb{L} \) we choose an element \( \lambda_k \in \text{MSp}_{[2]}^ {4\ell} \) such that \( s_k(\lambda_k) = 1 \).
Theorem 4-5. With the above notation, the multiplicative idempotents \( \tau^{(r)} \) for \( r \geq 1 \) converge to a multiplicative idempotent \( \tau = \tau_L : MSp_{[2]} \to MSp_{[2]} \) such that

\[
(4-10) \begin{align*}
\tau_*(\lambda_\ell) &= 0, & \text{for } \ell \in \mathbb{L}, \\
\tau_*(\lambda_k) &\equiv \lambda_k, & \text{mod } (\lambda_\ell \mid \ell \in \mathbb{L}) \cap (\text{decomposables}).
\end{align*}
\]

Furthermore, the associated power series \( \tau_*(w) \in MSp_{[2]}^+[[w]] \) is such that

\[
(4-11) \quad \tau_*(w) \equiv w + \sum_{\ell \in \mathbb{L}} (-1)^\ell 2\lambda_\ell w^{\ell+1},
\]

modulo the ideal \( (\lambda_\ell \mid \ell \in \mathbb{L}) \cap (\text{decomposables}) \subset MSp_{[2]}^+ \). That is, all the higher coefficients of the power series \( \tau_*(w) \) are in the ideal \( (\lambda_\ell \mid \ell \in \mathbb{L}) \), and only indecomposable coefficients which can appear are congruent to \( \lambda_\ell \) for some \( \ell \in \mathbb{L} \).

Proof. Since the multiplicative idempotent \( \tau[\ell, r] \) fixes the \( 4(\ell \ell - 1) \)-skeleton of \( MSp_{[2]} \) by Lemma 4-3, the sequence of multiplicative idempotents \( \tau^{(r)} \) converges to a multiplicative idempotent \( \tau \) with respect to the skeletal filtration topology in \( MSp_{[2]}^+ \) by Proposition 4-4. The formulae (4-10) and (4-11) follow by taking the limit of (4-7) and (4-8). \( \Box \)

In Theorems 4-2 and 4-5, we dealt with compositions on the right of basic multiplicative idempotents described in Proposition 4-1. Next we deal with compositions of these idempotents on the left and we describe differences and similarities between these two ways of composing idempotents.

Let \( 0 < n_1 < n_2 < \cdots < n_r \) be positive integers such that \( 2n_i + 1 \) is not a prime power for all \( 1 \leq i \leq r \). As before, we choose indecomposable elements \( \lambda_n \in MSp_{[2]}^{2n} \) such that \( s_n(\lambda_n) = 1 \), and multiplicative idempotents \( \tau[\lambda_n] \) as in Proposition 4-1 for \( 1 \leq i \leq r \). We consider the composition of these idempotents on the left:

\[
(4-12) \tau[\lambda_{n_r}, \lambda_{n_{r-1}}, \cdots, \lambda_{n_1}] = \tau[\lambda_{n_r}] \circ \tau[\lambda_{n_{r-1}}] \circ \cdots \circ \tau[\lambda_{n_1}] : MSp_{[2]} \to MSp_{[2]}.
\]

Unfortunately, unlike the successive compositions on the right as in (4-6), the ring map (4-12) above is not an idempotent in general. To see the reason, it is enough to consider the case \( r = 2 \). Let \( \tau[\lambda_{n_1}] = \tau_1 \) and \( \tau[\lambda_{n_2}] = \tau_2 \), and let their associated power series be \( \tau_1*(w) = g_1(w) \) and \( \tau_2*(w) = g_2(w) \). Then the power series associated to \( \tau[\lambda_{n_2}, \lambda_{n_1}] \) is given by \( (\tau_2 \circ \tau_1_*)(w) = \tau_2*(\tau_1*(w)) = \tau_2*(g_1(w)) = (\tau_2 \circ g_1)(g_2(w)) \). Since all the higher coefficients of \( g_1(w) \) are in the ideal \( (\lambda_{n_2}) \) and \( \tau_2 \) moves elements within the ideal \( (\lambda_{n_2}) \), all the higher coefficients of \( (\tau_2 \circ g_1)(w) \) are in the ideal \( (\lambda_{n_1}, \lambda_{n_2}) \). So all the higher coefficients of \( (\tau_2 \circ \tau_1_*) \) are also in the ideal \( (\lambda_{n_1}, \lambda_{n_2}) \). Since \( \tau[\lambda_{n_1}] \circ \lambda_{n_1} \) is an idempotent, the map \( \tau[\lambda_{n_2}, \lambda_{n_1}] \), \( \circ \tau[\lambda_{n_1}] \), annihilates \( \lambda_{n_1} \). But since \( \tau[\lambda_{n_1}] \circ \lambda_{n_2} \equiv \lambda_{n_2} \) modulo the ideal \( (\lambda_{n_1}) \), and \( \tau[\lambda_{n_1}] \) annihilates \( \lambda_{n_2} \) and fixes \( \lambda_{n_1} \) by degree reason, \( \tau[\lambda_{n_2}, \lambda_{n_1}] \) may not annihilate \( \lambda_{n_2} \), and consequently \( \tau[\lambda_{n_2}, \lambda_{n_1}] \) may not be a multiplicative idempotent. This is why we used compositions on the right as in (4-6) rather than compositions on the left as in (4-12) to produce multiplicative idempotents. However, we can modify the choice of indecomposable elements \( \lambda_n \)'s so that the successive compositions as in (4-12) does give rise to an idempotent.
Given indecomposable elements $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_r}$ as above, we inductively construct indecomposable elements $\overline{\lambda}_{n_1}, \overline{\lambda}_{n_2}, \ldots, \overline{\lambda}_{n_r}$ such that $s_{n_i}(\overline{\lambda}_{n_i}) = 1$ for $1 \leq i \leq r$ as follows. First let $\overline{\lambda}_{n_1} = \lambda_{n_1}$. Then we let

$$
\begin{align*}
\overline{\lambda}_{n_2} &= \tau[\overline{\lambda}_{n_1}](\lambda_{n_2}), \\
\overline{\lambda}_{n_3} &= \tau[\overline{\lambda}_{n_2}] \circ \tau[\overline{\lambda}_{n_1}](\lambda_{n_3}), \\
& \vdots \\
\overline{\lambda}_{n_r} &= \tau[\overline{\lambda}_{n_{r-1}}] \circ \cdots \circ \tau[\overline{\lambda}_{n_1}](\lambda_{n_r}),
\end{align*}
$$

(4-13)

Here, from one line to the next line, we choose a multiplicative idempotent $\tau[\overline{\lambda}_{n_i}]$ of the form described in Proposition 4-1 using the newly constructed indecomposable element $\overline{\lambda}_{n_i}$, then we define $\overline{\lambda}_{n_{i+1}}$ using $\tau[\overline{\lambda}_{n_i}]$. And we repeat this process.

We now show that the left compositions of basic idempotents constructed for these newly constructed elements $\overline{\lambda}_{n_1}, \ldots, \overline{\lambda}_{n_r}$ is an idempotent whose kernel is precisely the ideal generated by indecomposable elements $\lambda_{n_1}, \ldots, \lambda_{n_r}$ given at the beginning.

**Theorem 4-6.** Let $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_r}$ be indecomposable elements in $MSP_{[2]}^n$ such that $s_{n_i}(\lambda_{n_i}) = 1$ for positive integers $0 < n_1 < n_2 < \cdots < n_r$ such that $2n_i + 1$ is not a prime power for $1 \leq i \leq r$. We choose $\overline{\lambda}_{n_i}$ and multiplicative idempotents $\tau[\overline{\lambda}_{n_i}]$ as in (4-13) for $1 \leq i \leq r$. Then

$$
\begin{align*}
\overline{\lambda}_{n_i} &\equiv \lambda_{n_i} \mod (\lambda_{n_1}, \ldots, \lambda_{n_{i-1}}) & & \text{for } 2 \leq i \leq r, \\
(\overline{\lambda}_{n_1}, \ldots, \overline{\lambda}_{n_r}) &= (\lambda_{n_1}, \ldots, \lambda_{n_r}) & & \text{for } 1 \leq i \leq r.
\end{align*}
$$

(4-14)

The successive left compositions of the basic multiplicative idempotents $\tau[\overline{\lambda}_{n_i}]$

$$
\tau[\overline{\lambda}_{n_r}, \ldots, \overline{\lambda}_{n_1}] = \tau[\overline{\lambda}_{n_r}] \circ \cdots \circ \tau[\overline{\lambda}_{n_1}]: MSP_{[2]}^n \to MSP_{[2]}^n
$$

(4-15)

is a multiplicative idempotent. Its kernel is the ideal generated by $\lambda_{n_1}, \ldots, \lambda_{n_r}$:

$$
\text{Ker} \tau[\overline{\lambda}_{n_r}, \ldots, \overline{\lambda}_{n_1}] = (\lambda_{n_1}, \ldots, \lambda_{n_r}) \subset MSP_{[2]}^n
$$

(4-16)

The power series associated to the ring map $\tau[\overline{\lambda}_{n_r}, \ldots, \overline{\lambda}_{n_1}]$ is such that

$$
\tau[\overline{\lambda}_{n_r}, \ldots, \overline{\lambda}_{n_1}](w) \equiv w + \sum_{i=1}^{r} (-1)^n 2\lambda_{n_i}w^{n_i+1}
$$

modulo the ideal $(\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_r}) \cap (\text{decomposables})$.

**Proof.** We prove this theorem by induction on $r$. When $r = 1$, Theorem 4-6 is the same as Proposition 4-1. Now we assume Theorem 4-6 for any $r$ positive integers $0 < n_1 < n_2 < \cdots < n_r$, and we prove Theorem 4-6 for any $r + 1$ positive integers $0 < n_1 < n_2 < \cdots < n_r < n_{r+1}$.

Suppose that we are given $r + 1$ indecomposable elements $\lambda_{n_1}, \ldots, \lambda_{n_r}, \lambda_{n_{r+1}}$ such that $s_{n_i}(\lambda_{n_i}) = 1$ for $1 \leq i \leq r + 1$, and that we have constructed indecomposable elements $\overline{\lambda}_{n_i}$ and multiplicative idempotents $\tau[\overline{\lambda}_{n_i}]: MSP_{[2]}^n \to MSP_{[2]}^n$ for $1 \leq i \leq r + 1$ as in (4-13).
The identity $\tau_{n+1} = \tau[\tau_{n}, \ldots, \tau_{1}]_{*}(\lambda_{n+1})$ and (4-16) imply that $\tau_{n+1} \equiv \lambda_{n+1}$ modulo the ideal $(\lambda_{n+1})$. This proves the inductive step for the first formula in (4-14).

Using this newly obtained congruence relation and the second formula in (4-14), we have $(\lambda_{n}, \ldots, \lambda_{r}, \tau_{n+1}) = (\lambda_{n}, \ldots, \lambda_{r}, \lambda_{n+1})$. This proves the inductive step for the second formula in (4-14).

By the first formula in (4-4) in Proposition 4-1, we have $\tau[\tau_{n+1}]_{*}(\lambda_{n}) = \lambda_{n}$ for $1 \leq i \leq r$ by degree reason. So we see that $\tau[\tau_{n+1}]_{*}$ preserves the ideal $(\lambda_{n+1})$. Hence using the inductive hypothesis (4-16) and the identity Ker $\tau[\tau_{n+1}]_{*} = (\lambda_{n+1})$ from Proposition 4-1, we see that the kernel of the ring map $\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*} = \tau[\tau_{n+1}]_{*} \circ \tau[\tau_{n}, \ldots, \tau_{1}]_{*}$ is precisely equal to the ideal $(\lambda_{n+1}, \ldots, \lambda_{r}, \tau_{n+1})$. By what we have proved as the inductive step for (4-14), this ideal is equal to $(\lambda_{n+1}, \ldots, \lambda_{r}, \lambda_{n+1})$. This proves the inductive step for (4-16).

Using the inductive hypothesis (4-17), we have

$$\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*}(w) = \tau[\tau_{n+1}]_{*} \circ \tau[\tau_{n}, \ldots, \tau_{1}]_{*}(w)$$

modulo the ideal $\tau[\tau_{n+1}]_{*}((\lambda_{n+1}, \ldots, \lambda_{n}) \cap \text{ (decomposables)}) = (\lambda_{n+1}, \ldots, \lambda_{n})$ by degree reason. We also have

$$\tau[\tau_{n+1}]_{*}(w) \equiv w + (-1)^{n+1}2\lambda_{n}w^{n+1} \mod (\lambda_{n+1}) \cap \text{ (decomposables)}$$

modulo $(\lambda_{n+1}, \ldots, \lambda_{n}) \cap \text{ (decomposables)}$, by the inductive step for (4-14). Combining these two congruences, we have

$$\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*}(w) \equiv w + \sum_{i=1}^{r+1}(-1)^{n+1}2\lambda_{n}w^{n+1},$$

modulo the ideal $(\lambda_{n+1}, \ldots, \lambda_{r}, \lambda_{n+1}) \cap \text{ (decomposables)}$. This proves the inductive step for (4-17).

Since Ker $\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*} = (\lambda_{n+1}, \ldots, \lambda_{r}, \lambda_{n+1})$ by what we have proved as the inductive step above, $\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*}$ annihilates all the higher coefficients of $\tau[\tau_{n+1}, \lambda_{n}, \ldots, \lambda_{1}]_{*}(w)$. Hence the ring map $\tau[\tau_{n+1}, \tau_{n}, \ldots, \tau_{1}]_{*}$ is a multiplicative idempotent.

This completes all the inductive steps and the proof of Theorem 4-6 is now complete. □

Letting $r \rightarrow \infty$, we obtain a statement corresponding to Theorem 4-5.

§5. General Multiplicative Idempotents in $MSp_{[2]}$

The multiplicative idempotents on $MSp_{[2]}$ considered in Theorem 4-5 have associated power series $g(w)$ of the form (4-11), in which an indecomposable element $\lambda_{i}$ in $MSp_{[2]}$ can appear as the coefficient of $w^{\ell+1}$ only when $2\ell + 1$ is not a prime power. In fact, we can show that any multiplicative idempotent acting on $MSp_{[2]}$ has this property.
Lemma 5-1. Let \( \tau : MSp_{[2]} \to MSp_{[2]} \) be a multiplicative idempotent, and let
\[
(5-1) \quad \tau_s(w) = w + \sum_{n \geq 1} (-1)^n 2 \gamma_n w^{n+1} \in MSp^*_2[[w]]
\]
be the associated power series. If the coefficient \( \gamma_\ell \in MSp_{[2]}^{-4\ell} \) is indecomposable for some integer \( \ell \in \mathbb{N} \), then \( 2\ell + 1 \) is not a prime power and in fact \( s_\ell(\gamma_\ell) = 1 \).

If an indecomposable element \( \lambda_\ell \in MSp_{[2]}^{-4\ell} \) is annihilated by the above multiplicative idempotent \( \tau \), then the corresponding \( \ell \)th coefficient \( \gamma_\ell \) is indecomposable and thus \( 2\ell + 1 \) is not a prime power.

Proof. If a coefficient \( \gamma_\ell \) is indecomposable for some \( \ell \in \mathbb{N} \), then we may write
\[
\gamma_\ell = c \sigma_\ell m_{2\ell} + h(\ell)(m_1, \ldots, m_{2\ell-1})
\]
for some \( c \neq 0 \in \mathbb{Z}[\frac{1}{2}] \), and for some weighted homogeneous polynomial \( h(\ell) \) with rational coefficients. Since \( \tau \) is a multiplicative idempotent, \( \tau_\star \) annihilates all the higher coefficients of \( \tau_s(w) \) by Proposition 2-3. So we have \( \tau_s(\gamma_\ell) = 0 \). By Proposition 3-5, modulo decomposable elements, we have
\[
0 = \tau_s(\gamma_\ell) \equiv c \sigma_\ell \tau_s(m_{2\ell}) \equiv c \sigma_\ell(m_{2\ell} - \gamma_\ell) \equiv c \sigma_\ell(m_{2\ell} - c \sigma_\ell m_{2\ell}) = c \sigma_\ell(1 - c \sigma_\ell)m_{2\ell}.
\]
By considering the characteristic number \( s_\ell(\cdot) \), we must have \( c \sigma_\ell(1 - c \sigma_\ell) = 0 \).

Since \( c \neq 0 \) by our assumption, we must have \( c \sigma_\ell = 1 \). If \( \ell \) is such that \( 2\ell + 1 = p^j \) for some odd prime \( p \) and \( j \geq 1 \), then \( \sigma_\ell = p \) and thus \( c = 1/p \). But this is impossible since \( c \in \mathbb{Z}[\frac{1}{2}] \), thus, we must have that \( 2\ell + 1 \) is not a prime power for any prime.

In this case, \( \sigma_\ell = 1 \) and consequently \( c = 1 \in \mathbb{Z}[\frac{1}{2}] \). Hence \( s_\ell(\gamma_\ell) = c \sigma_\ell = 1 \).

Next, if \( \tau_s(\lambda_\ell) = 0 \) for some indecomposable \( \lambda_\ell \in MSp_{[2]}^{-4\ell} \), write
\[
\lambda_\ell = c \sigma_\ell m_{2\ell} + h(\ell)(m_1, \ldots, m_{2\ell-1})
\]
for some \( c \neq 0 \in \mathbb{Z}[\frac{1}{2}] \). We have \( 0 = \tau_s(\lambda_\ell) \equiv c \sigma_\ell \tau_s(m_{2\ell}) \equiv c \sigma_\ell(m_{2\ell} - \gamma_\ell) \) modulo decomposables by (3-11). By taking \( s_\ell(\cdot) \), we have \( s_\ell(\gamma_\ell) = s_\ell(m_{2\ell}) = 1 \). Hence \( \gamma_\ell \) is indecomposable. By the first part of this Lemma, we see that \( 2\ell + 1 \) is not a prime power. \( \square \)

Let \( k \) be an integer such that the corresponding \( k \)th coefficient \( \gamma_k \) in \( \tau_s(w) \) is decomposable. We examine the behavior of \( \tau_s \) on indecomposable elements in degree \(-4k\).

Lemma 5-2. Let \( \tau : MSp_{[2]} \to MSp_{[2]} \) be a multiplicative idempotent and let
\[
(5-2) \quad \tau_s(w) = w + \sum_{n \geq 1} (-1)^n 2 \gamma_n w^{n+1} \text{ be its associated power series. Let }
\]
\[
L_\tau = \{ \ell \in \mathbb{N} \mid s_\ell(\gamma_\ell) \neq 0 \}.
\]
Then for each \( k \in \mathbb{N} \setminus L_\tau \), there exists an indecomposable element \( \lambda_k \in MSp_{[2]}^{-4k} \) such that \( s_k(\lambda_k) = \sigma_k \) and \( \tau_s(\lambda_k) = \lambda_k \).

Proof. For each \( k \in \mathbb{N} \setminus L_\tau \), we choose an arbitrary indecomposable element \( \lambda'_k \in MSp_{[2]}^{-4k} \) such that \( s_k(\lambda'_k) = \sigma_k \). We may write
\[
\lambda'_k = \sigma_k m_{2k} + h(k)(m_1, \ldots, m_{2k-1}).
\]
If we let \( \lambda_k = \tau_*(\lambda'_k) \), then \( \tau_*(\lambda_k) = (\tau_* \circ \tau_*)(\lambda'_k) = \tau_*(\lambda'_k) = \lambda_k \). We calculate \( s_k(\lambda_k) \). From (3-11), in \( MU_{\mathbb{Q}_s} \) we have \( \tau_*(m_{2k}) \equiv m_{2k} - \gamma_k \mod \text{decomposables} \). Since \( k \in \mathbb{N} \setminus \mathbb{L}_r \), the element \( \gamma_k \) itself is a decomposable element by the definition of \( \mathbb{L}_r \). Hence \( \tau_*(m_{2k}) \equiv m_{2k} \mod \text{decomposables} \) for \( k \in \mathbb{N} \setminus \mathbb{L}_r \). Thus, we have \( \lambda_k = \tau_*(\lambda'_k) = \sigma_k m_{2k} \mod \text{decomposables} \). This implies that \( s_k(\lambda_k) = \sigma_k \). This completes the proof. \( \square \)

Combining the previous two lemmas, we obtain the following description for general multiplicative idempotents.

**Theorem 5-3.** Let \( \tau : MSp_{[2]} \to MSp_{[2]}^* \) be a multiplicative idempotent and let \( \tau_*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}^*[[w]] \) be the associated power series. We let \( \mathbb{L}_r = \{ \ell \in \mathbb{N} \mid s_\ell(\gamma_\ell) \neq 0 \} \). Then the following statements hold.

(i) For any integer \( \ell \in \mathbb{L}_r \), \( 2\ell + 1 \) is not a prime power and \( s_\ell'(\gamma_\ell) = 1 \).

(ii) For any integer \( k \in \mathbb{N} \setminus \mathbb{L}_r \), there exists an indecomposable element \( \lambda_k \in MSp_{[2]^{-k}} \) such that \( s_k(\lambda_k) = \sigma_k \), and we have

\[
MSp_{[2]}^* = \mathbb{Z}[\frac{1}{2}][\gamma_\ell, \lambda_k \mid \ell \in \mathbb{L}_r, k \in \mathbb{N} \setminus \mathbb{L}_r].
\]

Furthermore, the multiplicative idempotent \( \tau_* : MSp_{[2]}^* \to MSp_{[2]}^* \) is given by

\[
\begin{align*}
\tau_*(\gamma_\ell) &= 0, & \ell &\in \mathbb{L}_r, \\
\tau_*(\lambda_k) &= \lambda_k, & k &\in \mathbb{N} \setminus \mathbb{L}_r.
\end{align*}
\] (5-3)

Thus, \( \text{Ker} \tau_* \) is the ideal generated by the indecomposable elements \( \gamma_\ell \) for \( \ell \in \mathbb{L}_r \).

(iii) For any \( k \in \mathbb{N} \setminus \mathbb{L}_r \), \( \gamma_k \) is a decomposable element in the ideal \( (\gamma_\ell \mid \ell \in \mathbb{L}_r) \).

**Proof.** (i) is proved in Lemma 5-1. For (ii), the equation \( s_k(\lambda_k) = \sigma_k \) follows from Lemma 5-2. Also, the elements \( \gamma_\ell \)'s and \( \lambda_k \)'s form a complete set of generators of \( MSp_{[2]}^* \cong MSp_{[2]}^* \) in view of Milnor’s criterion. Since \( \tau \) is a multiplicative idempotent, from Proposition 2-3 we have \( \tau_*(\gamma_n) = 0 \) for all \( n \geq 1 \), in particular for \( n \in \mathbb{L}_r \). This proves the first formula in (5-3). The second formula in (5-3) follows from Lemma 5-2. For (iii), if \( k \in \mathbb{N} \setminus \mathbb{L}_r \), then \( \gamma_k \) is in the ideal of decomposable elements by the definition of the set \( \mathbb{L}_r \). Since \( \tau_*(\gamma_k) = 0 \) by Proposition 2-3, \( \gamma_k \) is in the ideal \( (\gamma_\ell \mid \ell \in \mathbb{L}_r) \) by (ii). This completes the proof of Theorem 5-3. \( \square \)

**Corollary 5-4.** Let \( \tau : MSp_{[2]} \to MSp_{[2]} \) be a multiplicative idempotent. Then the kernel of the induced map \( \tau_* : MSp_{[2]} \to MSp_{[2]}^* \) is precisely the ideal generated by the higher coefficients of the associated power series

\[
\tau_*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}^*[[w]].
\]

Furthermore, as a set of generators of this ideal, we may take indecomposable coefficients \( \{\gamma_\ell \mid \ell \in \mathbb{L}_r \} \) of \( \tau_*(w) \).

**Proof.** This follows from (ii) of Theorem 5-3. \( \square \)
Corollary 5-5. Let $\tau : MSp_{[2]} \to MSp_{[2]}$ be any multiplicative idempotent and let $\theta : MSp_{[2]} \to MSp_{[2]a}$ be the corresponding algebra map. Let $I = (Q_1, \ldots, Q_n, \ldots) \subset MSp_{[2]a}(MSp_{[2]})$ be the augmentation ideal. Then the sequence $I \xrightarrow{\theta} MSp_{[2]a} \xrightarrow{\tau} MSp_{[2]a}$ is exact. In other words, $\ker \tau_s$ is generated by elements $\theta \tau(Q_k)$ for $k \geq 1$.

Proof. This follows from the fact that the higher coefficients of the associated power series $\tau_s(w)$ are exactly the images of $Q_j$’s of the associated algebra map $\theta_s$, as shown in the proof of Proposition 2-3. □

So far in this section, we have assumed that we are given a multiplicative idempotent and we deduced its properties from the associated power series. We consider the converse, and ask the following question.

Question. Let $\tau : MSp_{[2]} \to MSp_{[2]}$ be the ring map associated to a given power series $g(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}[[w]]$. What conditions on $\gamma_n$’s are necessary and sufficient so that $\tau$ is a multiplicative idempotent?

Given a power series $g(w)$, we define a set $\mathbb{L}_g$ as $L_\tau$ given in (5-2). From Theorem 5-3, we know that in order that $\tau$ is a multiplicative idempotent, all the higher coefficients of $g(w)$ have to belong to the ideal $(\gamma_\ell \mid \ell \in \mathbb{L}_g)$, and $s_\ell(\gamma_\ell) = 1$ for $\ell \in \mathbb{L}_g$. It turns out that these conditions are not enough. See Theorem 5-9 below for precise conditions. Our necessary and sufficient condition described there is constructive and it is expressed inductively. Our basic observation is the following.

Lemma 5-6. Given a power series $g(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1}$, let

\begin{equation}
(5-4) \quad g^{(N)}(w) = w + \sum_{n=1}^{N} (-1)^n \gamma_n w^{n+1}
\end{equation}

be the degree $N+1$ truncated polynomial of $g(w)$ for $N \geq 1$. Let $\tau, \tau^{(N)} : MSp_{[2]} \to MSp_{[2]}$ be the ring spectra maps associated to $g(w)$ and $g^{(N)}(w)$, respectively. Then $\tau$ is a multiplicative idempotent if and only if $\tau^{(N)}$ is a multiplicative idempotent for all $N \geq 1$.

Proof. Since the power series $g(w)$ and $g^{(N)}(w)$ agree mod ($w^{N+2}$), the corresponding ring maps $\tau$ and $\tau^{(N)}$ agree on $4N$-skeleton of $MSp_{[2]}$ by Lemma 4-3. Thus,

\begin{equation}
(*) \quad \tau_s(\gamma_n) = \tau^{(N)}_s(\gamma_n) \quad \text{for} \quad 1 \leq n \leq N.
\end{equation}

If $\tau$ is a multiplicative idempotent, then $\tau_s$ annihilates all the higher coefficients of $g(w)$ and we have $\tau_s(\gamma_n) = 0$ for all $n \geq 1$. This implies that for any $N \geq 1$, $\tau^{(N)}_s(\gamma_n) = 0$ for $1 \leq n \leq N$ due to $(*)$. This means that $\tau^{(N)}_s$ annihilates all the higher coefficients of the associated power series $g^{(N)}(w)$. Hence $\tau^{(N)}$ is a multiplicative idempotent for any $N \geq 1$.

Conversely, if $\tau^{(N)}$ is a multiplicative idempotent for all $N \geq 1$, then $(*)$ implies that $\tau_s$ annihilates all the higher coefficients of $g(w)$. Thus $\tau$ is a multiplicative idempotent. □

In view of this lemma, we can construct power series corresponding to multiplicative idempotents inductively in terms of polynomials adding one term at a
Lemma 5-7. Let $\tau_1 : \text{MSp}[2] \to \text{MSp}[2]$ be a ring map with the associated power series $g_1(w) = \tau_1(w) = w + \sum_{n \geq 1} (-1)^n 2\delta_n w^{n+1}$. Suppose that $\delta_N$ is a decomposable element for some $N \geq 1$ and let $g_2(w) = w + (-1)^N 2\gamma_N w^{N+1} + \cdots$ be a power series for some $\gamma_N \in \text{MSp}[4N]$. Let $\tau$ be a ring spectra map corresponding to the composed power series $g(w) = g_2(g_1(w))$. Then the effects of $\tau_*$ and $\tau_\gamma$ on $\text{MSp}[2]$ are related by the following formula:

\[(5-5) \quad T + \sum_{k \geq 1} \tau_*(m_{2k}) T^{2k+1} = \varphi_2^{-1}(T) + \sum_{k \geq 1} \tau_1(m_{2k})(\varphi_2^{-1}(T))^{2k+1},\]

where $\varphi_2(T) = \sqrt{-g_2(-T^2)} \equiv T \mod (T^2)$, and $T$ is a formal variable. In particular, $\tau$ and $\tau_\gamma$ agree on the $4(N-1)$-skeleton of $\text{MSp}[2]$, and on degree $4N$,

\[(5-6) \quad \tau_*(\gamma_N) = \tau_1*(\gamma_N) = s_N(\gamma_N)\gamma_N.\]

Suppose $\tau_1$ is a multiplicative idempotent. Then, for $\tau$ to be a multiplicative idempotent, it is necessary that $\tau_1(\gamma_N) = s_N(\gamma_N)\gamma_N$. In other words, if $\gamma_N$ is decomposable, then for $\tau$ to be an idempotent it is necessary that $\gamma_N \in \text{Ker} \tau_1$. If $\gamma_N$ is indecomposable, then for $\tau$ to be an idempotent it is necessary that $2N+1$ is not a prime power, $s_N(\gamma_N) = 1$, and $\gamma_N \in \text{Im} \tau_1$.

Proof. Let $\varphi(T) = \sqrt{-g(-T^2)}$ and $\varphi_1(T) = \sqrt{-g_1(-T^2)}$. Then we have $\varphi(T) = \varphi_1(\varphi_1(T))$. From (3-10), we have $\log^\gamma(T) = \log^\gamma(\varphi_1^{-1}(T))$. This implies that $\log^\gamma(T) = \log^\gamma(\varphi^{-1}(T) = \log^\gamma(\varphi_2^{-1}(T)) = \log^\gamma(\varphi_2^{-1}(T))$. This proves (5-5).

Since $\varphi_2^{-1}(T) = T - \gamma_N T^{2N+1} + \cdots$, equating the coefficients of $T^{2k+1}$’s, we have

$$\tau_*(m_{2k}) = \tau_1*(m_{2k}) \quad \text{for} \quad k \leq N-1,$$

and

$$\tau_*(m_{2N}) = \tau_1*(m_{2N}) - \gamma_N.$$

Thus, $\tau_*$ and $\tau_\gamma$ agree on elements in degree $-4k$ for $1 \leq k \leq N-1$. If $\gamma_N = cm_{2N} + h_{N}(m_{1}, \ldots, m_{2N-1})$ for some $c \in \mathbb{Z}[\frac{1}{2}]$ and for some weighted homogeneous polynomial $h_{N}$ with rational coefficients, then a similar calculation as in the proof of Proposition 4-1 shows that $\tau_*(\gamma_N) = \tau_1*(\gamma_N) - c\gamma_N$. Since $c = s_N(\gamma_N)$, we obtain (5-6).

If $\tau$ is a multiplicative idempotent, then $\tau_*$ annihilates all higher coefficients of

$$g(w) = g_2(g_1(w)) \equiv w + \sum_{k=1}^{N-1} (-1)^k 2\delta_k w^{k+1} + (-1)^N 2(\delta_N + \gamma_N) w^{N+1} \mod (w^{N+2}).$$

Since $\tau_1$ is a multiplicative idempotent and $\delta_N$ is decomposable, $\delta_N$ must belong to the ideal generated by $\delta_1, \ldots, \delta_{N-1}$ by Corollary 5-4. Since $\tau$ is an idempotent, $\tau_*$ annihilates $\delta_1, \ldots, \delta_{N-1}$. Thus, $\delta_N$ is also annihilated by $\tau_*$. Since $\tau_*$ also annihilates $\gamma_N$ because $\tau_*$ annihilates all the higher coefficients of $g(w)$, it follows that $\tau_*(\gamma_N) = 0$. Combining with (5-6), we have $\tau_1(\gamma_N) = s_N(\gamma_N)\gamma_N$. Since $\tau_1$ is assumed to be an idempotent, $s_N(\gamma_N)^2 = s_N(\gamma_N)$. Thus, $s_N(\gamma_N) = 0, 1$. Hence if $\gamma_N$ is decomposable, we must have $\tau_1(\gamma_N) = 0$. And if $\gamma_N$ is
indecomposable, we must have \( s_N(\gamma_N) = 1 \), which can happen only when \( 2N + 1 \) is not a prime power by Theorem 5-3 (i). In this case, \( \tau_*(\gamma_N) = \gamma_N \) implies that \( \gamma_N \in \text{Im } \tau_* \). □

We apply this lemma to our present context of constructing multiplicative idempotents inductively.

**Lemma 5-8.** Let \( g(w) = w + \sum_{n=1}^{N-1}(-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}[[w]] \). For some \( \gamma_N \in MSp_{[2]}^{-4N} \), we let \( \bar{g}(w) = g(w) + (-1)^N 2\gamma_N w^{N+1} \). Let \( \tau, \bar{\tau} : MSp_{[2]} \rightarrow MSp_{[2]} \) be the ring maps corresponding to \( g(w) \) and \( \bar{g}(w) \), respectively. Then

\[ \bar{\tau} \text{ is an idempotent} \iff \tau \text{ is an idempotent and } \tau_*(\gamma_N) = s_N(\gamma_N) \gamma_N \in MSp_{[2]}^* \]

\[ \iff \begin{cases} \tau \text{ is an idempotent and either } (a) \gamma_N \in \text{Ker } \tau_* \text{ or } \\ (b) 2N + 1 \text{ is not a prime power, } s_N(\gamma_N) = 1, \text{ and } \gamma_N \in \text{Im } \tau_* \end{cases} \]

**Proof.** Let \( h(w) \) be a power series such that \( \bar{g}(w) = h(g(w)) \). \( h(w) \) is of the form

\[ h(w) = w + (-1)^N 2\gamma_N w^{N+1} + \text{(higher order terms).} \]

Suppose \( \bar{\tau} \) is a multiplicative idempotent. Since \( g(w) \) is a truncation of \( \bar{g}(w) \), \( \tau \) is a multiplicative idempotent by Lemma 5-6. By applying Lemma 5-7 with \( g_1(w) \) and \( g_2(w) \) replaced by \( g(w) \) and \( h(w) \), we see that \( \tau_*(\gamma_N) = s_N(\gamma_N) \gamma_N \). In \( MSp_{[2]}^* \), this condition is equivalent to the one stated above due to Theorem 5-3 (i).

Conversely, suppose \( \tau \) is a multiplicative idempotent and the element \( \gamma_N \) is such that \( \tau_*(\gamma_N) = s_N(\gamma_N) \gamma_N \). Then by (5-6), the ring map \( \bar{\tau} \) has the property \( \bar{\tau}_*(\gamma_N) = \tau_*(\gamma_N) - s_N(\gamma_N) \gamma_N = 0 \). Since the power series \( g(w) \) and \( \bar{g}(w) \) agree modulo \( (w^{N+1}) \), the corresponding ring maps \( \tau \) and \( \bar{\tau} \) agree on the \( 4(N-1) \)-skeleton by Lemma 4-3 and we have \( \bar{\tau}_*(\gamma_n) = \tau_*(\gamma_n) \) for \( 1 \leq n \leq N - 1 \). Since \( \tau \) is assumed to be a multiplicative idempotent, \( \tau_*(\gamma_n) = 0 \) for \( 1 \leq n \leq N - 1 \). Hence \( \bar{\tau}_* \) annihilates all the higher coefficients of the associated power series \( \bar{g}(w) \). Thus \( \bar{\tau} \) is a multiplicative idempotent. □

Combining Lemma 5-6 and Lemma 5-8, we get the following characterization for those power series \( g(w) \) which correspond to multiplicative idempotents.

**Theorem 5-9.** Let \( \tau : MSp_{[2]} \rightarrow MSp_{[2]} \) be a ring spectra map with its associated power series \( g(w) = w + \sum_{k=1}^{N-1}(-1)^k 2\gamma_k w^{k+1} \). Let \( \tau^{(n)} : MSp_{[2]} \rightarrow MSp_{[2]} \) be the ring map associated to the truncated polynomial \( g^{(n)}(w) = w + \sum_{k=1}^{n}(-1)^k 2\gamma_k w^{k+1} \) for all \( n \geq 1 \). We let \( \tau^{(0)} \) be the identity map. Then

\[ \tau^{(n)} \text{ is an idempotent} \iff \tau_*(^{(n-1)})(\gamma_n) = s_n(\gamma_n) \gamma_n \text{ for all } n \geq 1 \]

This condition is equivalent to the following conditions for all \( n \geq 1 \):

1. \( \gamma_n \in \text{Ker } \tau_*(^{(n-1)}) \) and \( \gamma_n \) is decomposable, if \( 2n + 1 \) is a prime power,
2. \( \gamma_n \in \text{Ker } \tau_*(^{(n-1)}) \), if \( 2n + 1 \) is not a prime power and \( \gamma_n \) is decomposable,
3. \( \gamma_n \in \text{Im } \tau_*(^{(n-1)}) \) and \( s_n(\gamma_n) = 1 \), if \( 2n + 1 \) is not a prime power and \( \gamma_n \) is indecomposable.
Proof. Suppose $\tau$ is a multiplicative idempotent. Then by Lemma 5-6, $\tau^{(n)}$ are all multiplicative idempotents for all $n \geq 0$. Applying Lemma 5-8 to the pair $\tau^{(n)}$ and $\tau^{(n-1)}$, we see that $\tau^{(n-1)}_*(\gamma_n) = s_n(\gamma_n)\gamma_n$ for all $n \geq 1$.

Conversely, assume that $\tau^{(n-1)}_*(\gamma_n) = s_n(\gamma_n)\gamma_n$ for all $n \geq 1$. We show that $\tau^{(n)}$ is a multiplicative idempotent for all $n \geq 0$ by induction on $n$. When $n = 0$, then $\tau^{(0)}$ is an identity map, and in particular it is an idempotent. Now assume that $\tau^{(n-1)}$ is a multiplicative idempotent for $n \geq 1$. By Lemma 5-8 applied to $\tau^{(n-1)}$ and $\tau^{(n)}$ together with our hypothesis $\tau^{(n-1)}_*(\gamma_n) = s_n(\gamma_n)\gamma_n$, we see that $\tau^{(n)}$ is also a multiplicative idempotent. This completes the inductive step and we have shown that $\tau^{(n)}$ is a multiplicative idempotent for all $n \geq 1$. By Lemma 5-6, this implies that $\tau$ is also a multiplicative idempotent.

By Theorem 5-3 (i), $s_n(\gamma_n) \neq 0$ if and only if $2n + 1$ is not a prime power, and in this case we actually must have $s_n(\gamma_n) = 1$. From this we obtain the above more detailed conditions. \hfill \Box

§6. Multiplicative Indecomposable Splittings of $MSp_{[2]}$

In this final section, we split off minimal ring spectra from $MSp_{[2]}$. Although such subspectra are not unique, they are all homotopically equivalent to each other. First we clarify what we mean by minimal ring spectra.

Definition 6-1. A ring spectrum $E$ is said to be multiplicatively indecomposable if any multiplicative idempotent acting on $E$ is an identity map.

When a ring spectrum $E$ is multiplicatively indecomposable, we cannot split off a subring spectrum. In this sense, $E$ is “minimal”.

We apply Theorem 4-5 for the largest possible $L \subset \mathbb{N}$. We let

$$L = \{ \ell \in \mathbb{N} \mid 2\ell + 1 \text{ is not a prime power} \},$$

$$P = \{ k \in \mathbb{N} \mid 2k + 1 \text{ is a prime power} \}.$$

For each $\ell \in \mathbb{L}$, we choose an indecomposable element $\lambda_\ell \in MSp^{-4\ell}_{[2]}$ such that $s_\ell(\lambda_\ell) = 1$. The choice of these indecomposable elements is not unique.

Theorem 6-2. For any choice of indecomposable elements $\lambda_\ell \in MSp^{-4\ell}_{[2]}$ for $\ell \in \mathbb{L}$, there exists a multiplicative idempotent $\tau : MSp_{[2]} \to MSp_{[2]}$ and indecomposable elements $\lambda_k \in MSp^{-4k}_{[2]}$ for $k \in \mathbb{P}$ such that

$$\tau_*(\lambda_n) = \begin{cases} 0, & \text{if } 2n + 1 \text{ is not a prime power} \\ \lambda_n, & \text{if } 2n + 1 \text{ is a prime power}. \end{cases}$$

Let $LSp_{\tau} = \tau(MSp_{[2]})$ be the multiplicative summand split off from $MSp_{[2]}$ as the image of $\tau$. Then its homotopy groups form a polynomial algebra given by $LSp_{\tau_*} = \mathbb{Z}[\frac{1}{n}]\lambda_n \mid n \in \mathbb{P}$, and the spectrum $LSp_{\tau}$ is a multiplicatively indecomposable ring spectrum.

Proof. The first half of Theorem 6-2 is a special case of Theorem 4-5. To find elements $\lambda_k$ for $k \in \mathbb{P}$ stated above we first choose arbitrary element $\lambda'_k \in MSp^{-4k}_{[2]}$ such that $s_k(\lambda'_k) = \sigma_k$. We then let $\lambda_k = \tau_*(\lambda'_k)$. It then follows that $s_k(\lambda_k) = \sigma_k$ and $\tau_*(\lambda_k) = \lambda_k$ for $k \in \mathbb{P}$, as in the proof of Lemma 5-2.
To see that the ring spectrum $LSp = LSp_*$ is multiplicatively indecomposable, let $\chi : LSp \to LSp$ be an arbitrary multiplicative idempotent. We can extend it to a multiplicative idempotent of $MSp_{[2]}$ by letting $\tilde{\chi} : MSp_{[2]} \xrightarrow{\pi} LSp \xrightarrow{j} MSp_{[2]}$, where $\pi$ and $j$ are the projection and the inclusion maps. Let the associated power series be $\tilde{\chi}_*(w) = w + \sum_{n \geq 1} (-1)^n 2\gamma_n w^{n+1} \in MSp_{[2]}[[w]]$. By Proposition 3-5 (iii), in $MU_{[2]}$ we have $\tilde{\chi}_*(m_{2n}) = m_{2n} - \gamma_n$ modulo the ideal of decomposable elements for any $n \geq 1$. If $k \in \mathbb{P}$, then $\gamma_k$ is decomposable by Lemma 5-1 and we have $\tilde{\chi}_*(m_{2k}) = m_{2k}$ modulo decomposables for $k \in \mathbb{P}$. Thus, $\chi_*(\lambda_k)$ has the same $s_k$'s as $\lambda_k$, and so $\chi_*(\lambda_k)$ is an indecomposable element in $LSp_*$ for all $k \in \mathbb{P}$. It follows that $\chi$ is a multiplicative idempotent which induces an isomorphism on homotopy groups. Hence $\chi^2 = \chi$ and $\chi$ is invertible. It follows that $\chi$ is an identity map. This proves that $LSp$ is multiplicatively indecomposable. □

Remark. (i) The homotopy group of the $BP$ spectrum at a prime $p$ is given by $BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots, v_n, \ldots]$, where $|v_n| = 2(p^n - 1)$. If we use Hazewinkel generators rather than Araki generators (see [R]), then in fact $v_n \in MU_{2(p^n - 1)}$ is integral and $v_n = pm_{2n} + (\text{decomposables})$ for all $n \geq 1$. Hence if $k \in \mathbb{P}$ is of the form $2k + 1 = p^j$ for some odd prime $p$ and for some $j \geq 1$, we may let $\lambda_k = \tau_*(v_j)$ using the $j$th Hazewinkel generator of $BP_*$ for the odd prime $p$.

(ii) When localized at an odd prime $p$, the spectrum $MSp_{(p)}$ splits into a wedge sum of many suspension copies of the $BP$ spectrum. The above theorem says that when only 2 is inverted in $MSp$, copies of $BP$ spectra for all odd primes are bound together in $LSp$ and they cannot be separated, since $LSp$ is multiplicatively indecomposable.

Although $LSp_*$ is uniquely determined by the multiplicative idempotent $\tau$, the ring map $\tau$ itself is not canonically determined by the set of generators $\lambda_\ell$'s chosen for each $\ell \in \mathbb{P}$. The situation is similar to Lemma 4-1: the multiplicative idempotent depends on our choice of power series of the form $(4-3)$ for a given $\lambda_n$.

However, any two such multiplicatively indecomposable ring spectra can be shown to be equivalent, as follows.

**Proposition 6-3.** Let $\tau_1, \tau_2 : MSp_{[2]} \to MSp_{[2]}$ be multiplicative idempotents such that $L\tau_1 = L\tau_2 = \mathbb{L}$. Let $LSp_{\tau_1}$ and $LSp_{\tau_2}$ be the ring spectra split off as the images of the idempotents $\tau_1$, $\tau_2$, respectively. Then these two ring spectra are equivalent.

**Proof.** For $i = 1, 2$, let $\pi_i$ and $j_i$ be the projection and the inclusion maps for $LSp_{\tau_i}$, that is, $MSp_{[2]} \xrightarrow{\pi_i} LSp_{\tau_i} \xrightarrow{j_i} MSp_{[2]}$. Then the composition of ring spectra maps $LSp_{\tau_i} \xrightarrow{j_i} MSp_{[2]} \xrightarrow{\pi_2} LSp_{\tau_2}$ induces an isomorphism on homotopy groups, because the indecomposable generators in $LSp_{\tau_i}^*$ are mapped into indecomposable generators of $LSp_{\tau_2}^*$. Hence it is a homotopy equivalence. □

From now on, we just refer to the spectrum $LSp$ without making the multiplicative idempotent $\tau$ explicit. As remarked above, when localized at $p$, $MSp$ and $MU$ split into wedge sums of many suspension copies of the $BP$ spectrum. A similar decomposition exists when only 2 is inverted.
Corollary 6-4. We have the following decompositions of spectra:

\[ MSp_{[2]} \cong \bigvee_I \Sigma^{2|I|} LSp, \quad MU_{[2]} \cong \bigvee_J \Sigma^{2|J|} LSp, \]

where \( I \) ranges over all finite (possibly empty) sequences of integers from \( 2 \mathbb{L} \):

\[ I = (i_1 \leq i_2 \leq \cdots \leq i_r \mid i_k \in 2 \mathbb{L} \text{ for } 1 \leq k \leq r, r \geq 0), \]

and \( J \) ranges over all finite (possibly empty) sequences of integers which are either odd or from \( 2 \mathbb{L} \):

\[ J = (j_1 \leq j_2 \leq \cdots \leq j_r \mid j_k \in 2 \mathbb{L} \cup (2\mathbb{N} - 1) \text{ for } 1 \leq k \leq r), \]

and \( |I| = \sum_k i_k \) and \( |J| = \sum_k j_k \).

The proof is standard and straightforward, so it is omitted.

Remark. In [BM], a ring spectrum \( \text{odd} \, MU \) is constructed as the image of the multiplicative idempotent \( \varepsilon_2 \) on \( MU_{[2]} \) which annihilates \( m_{2k-1} \) for all \( k \geq 1 \) and which is an identity on \( m_{2k} \) for all \( k \geq 1 \). Since the composition of ring spectra maps \( MSp_{[2]} \xrightarrow{\text{odd}} MU_{[2]} \xrightarrow{\text{odd}} \text{odd} \, MU \) is an equivalence, we also have a decomposition of the form (6-3) for \( \text{odd} \, MU \).

References


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