DICKSON INVARIANTS HIT BY THE STEENROD SQUARES

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Abstract. Let D_3 be the Dickson invariant ring of $\mathbb{F}_2[X_1, X_2, X_3]$ by $\mathrm{GL}(3, \mathbb{F}_2)$. In this paper, we prove each element in D_3 is hit by the Steenrod square in $\mathbb{F}_2[X_1, X_2, X_3]$. Our method provides a clue in attacking the question in the general case.

1. Introduction

Throughout this paper, we work on the polynomials over $\mathbb{F}_2$. Let $\mathrm{GL}(n, \mathbb{F}_2)$ be the $n \times n$-general linear group over $\mathbb{F}_2$. Then the standard group action:

$$\mathrm{GL}(n, \mathbb{F}_2) \times \mathbb{F}_2[X_1, X_2, \dots, X_n] \to \mathbb{F}_2[X_1, X_2, \dots, X_n]$$

gives rise to the ring of Dickson invariants

$$\mathbb{F}_2[X_1, X_2, \dots, X_n]^{\mathrm{GL}(n,\mathbb{F}_2)},$$

which is isomorphic to the polynomial ring $\mathbb{F}_2[Q_{n,0}, Q_{n,1}, ..., Q_{n,n-1}]$, where each generator $Q_{n,i}$ ($i = 1, ..., n-1$) is a certain polynomial over $\mathbb{F}_2$.

Definition 1.1. *A polynomial $f(X_1, X_2, \dots, X_n)$ is hit if it can be written as*

$$f(X_1, X_2, , \dots, X_n) = \sum_{i \geq 1} \mathrm{Sq}^i f_i(X_1, X_2, \dots, X_n),$$

for some polynomials $f_i(X_1, X_2, \dots, X_n)$.

Question 1.2. *Is each Dickson invariant hit?*

The motivation of presenting this question was illustrated in detail in an excellent expository paper [9], p501. In general, this question is hard to answer. But when $n = 1$ and 2, the answer is negative. For example, when $n = 1$, the Dickson invariant X_1 is not hit; when $n = 2$, $X_1^2 + X_2^2 + X_1X_2$ is a Dickson invariant but is not hit.

Conjecture 1.3 (Hung, [1, 2, 3, 4, 8])**.** *When $n \geq 3$, all Dickson invariants are hit.*

The main result of this paper is to prove the above conjecture when $n = 3$. The method we use is elementary though a bit tedious. We hope that it provides a clue in attacking the conjecture in the general case.

Write $V_1 = X_1$, $V_2 = X_2(X_2 + X_1)$ and $V_3 = X_3(X_3 + X_2)(X_3 + X_1)(X_3 + X_2 + X_1)$. Then $Q_{3,0} = V_3V_2V_1$, $Q_{3,1} = Q_{2,0}^2 + V_3Q_{2,1} = (V_2V_1)^2 + V_3(V_1^2 + V_2)$ and $Q_{3,2} = Q_{2,1}^2 + V_3 = (V_1^2 + V_2)^2 + V_3$. Clearly, $\deg(Q_{3,0}) = 7$, $\deg(Q_{3,1}) = 6$ and $\deg(Q_{3,2}) = 4$. The Steenrod operation acts on the Dickson invariants in the following way,

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Theorem 1.4 (Hung [1]). *The Steenrod operation acts trivially on $Q_{3,0}, Q_{3,1}$ and $Q_{3,2}$ except for the following cases,*

$$\begin{array}{rl} \mathrm{Sq}^4 Q_{3,0} = Q_{3,0}Q_{3,2}, & \mathrm{Sq}^6 Q_{3,0} = Q_{3,0}Q_{3,1}, \\ \mathrm{Sq}^7 Q_{3,0} = Q_{3,0}^2, & \mathrm{Sq}^1 Q_{3,1} = Q_{3,0}, \\ \mathrm{Sq}^4 Q_{3,1} = Q_{3,1}Q_{3,2}, & \mathrm{Sq}^5 Q_{3,1} = Q_{3,0}Q_{3,2}, \\ \mathrm{Sq}^6 Q_{3,1} = Q_{3,1}^2, & \mathrm{Sq}^2 Q_{3,2} = Q_{3,1} \\ \mathrm{Sq}^3 Q_{3,2} = Q_{3,0}, & \mathrm{Sq}^4 Q_{3,2} = Q_{3,2}^2. \end{array}$$

. □

Now we define a function $\gamma : \mathbb{N} \cup \{0\} \to \mathbb{N}$ as follows: $\gamma(0) = 0$ and for $k \geq 1$, $\gamma(k) = 2^k - 1$. Write $\mu(n)$ for

$$\min\{m \in \mathbb{N} \mid n = \sum_{i=1}^{m} \gamma(k_i) \quad \text{for some } k_i \in \mathbb{N}\}.$$

Let E and F be homogeneous polynomials of degrees e and f, respectively. Then we have the following,

Theorem 1.5 (Silverman [5]). *Suppose that $e < (2^{k+1} - 1)\mu(f)$ for some $k \geq 0$. Then the polynomial $EF^{2^{k+1}}$ is hit.*

. □

For an integer t, let $\alpha(t)$ be the number of digits 1 in the binary expansion of t and let U be an n-variable homogeneous polynomial. Then we have the following theorem, known as the Peterson Conjecture,

Theorem 1.6 (Wood [8]). *If $\alpha(\deg(U) + n) > n$, then U is hit.*

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We will frequently use the above theorem in the next section.

John Milnor showed that there is an anti-automorphism χ from the Steenrod algebra to itself, satisfying the following condition,

$$\chi(\mathrm{Sq}^k) = \sum_{i=1}^{k} \mathrm{Sq}^i \chi(\mathrm{Sq}^{k-i}) \quad \text{for } k \geq 1.$$

We will frequently use the so-called χ-*trick* in the following sections, which is based on the following,

Proposition 1.7. *For any polynomials E_1 and E_2, $E_1[\mathrm{Sq}^m(E_2)] - [\chi(\mathrm{Sq}^m)(E_1)]E_2$ is hit, where m is any non-negative integer.*

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2. Proofs

Clearly, to prove our result, we only need show that $Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2}$ is hit for any non-negative integers n_0, n_1 and n_2. The proof will split into the following cases,

(1) At least two of n_0, n_1 and n_2 are even;
(2) Exactly one of n_0, n_1 and n_2 is even;
(3) n_0, n_2 are odd and $n_1 \equiv 1 \mod 4$;
(4) n_0 is odd, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 1 \mod 4$;
(5) $n_0 \equiv 1 \mod 4$, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 3 \mod 4$;
(6) $n_0 \equiv 3 \mod 4$, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 3 \mod 4$.

Now we will prove each of the above six cases.

Remark 2.1. The proof of Cases(1)-(4) is a part of the first author's MSc thesis[7].

Notation 2.2. *For simplicity, we use the following notation: for any polynomials E_1 and E_2, $E_1 \equiv E_2$ means that $E_1 - E_2$ is hit.*

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2.1. Proof of Case(1).

If n_0, n_1 and n_2 are all even, then the result is a direct consequence of the definition of the Steenrod square.

Suppose that n_1, n_2 are even and n_0 is odd. Then putting $n_0 = 2l_0 + 1$, $n_1 = 2l_1$ and $n_2 = 2l_2$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0+1}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2} \\ &= Q_{3,0}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &= [\mathrm{Sq}^1 Q_{3,1}](Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &= \mathrm{Sq}^1[Q_{3,1}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \equiv 0 \quad \text{(see Notation 2.2).} \end{aligned}$$

Suppose that n_0, n_2 are even and n_1 is odd. Then by letting $n_0 = 2l_0$, $n_1 = 2l_1 + 1$ and $n_2 = 2l_2$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0}Q_{3,1}^{2l_1+1}Q_{3,2}^{2l_2} \\ &= Q_{3,1}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &= [\mathrm{Sq}^2 Q_{3,2}](Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &= \mathrm{Sq}^2[Q_{3,2}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] + Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &\equiv Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2. \end{aligned}$$

Notice that

$$Q_{3,2} = Q_{2,1}^2 + V_3 \text{ and } V_3 = \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3). \tag{2.1.1}$$

Then

$$\begin{aligned} Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 &= (Q_{2,1}^2 + V_3)\mathrm{Sq}^2[(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &= [Q_{2,1}\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + \mathrm{Sq}^1[(Q_{2,1}X_3 + X_3^3)(\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}))^2] \\ &\equiv 0 \end{aligned}$$

Suppose that n_0, n_1 are even and n_2 is odd. Putting $n_0 = 2l_0$, $n_1 = 2l_1$ and $n_2 = 2l_2 + 1$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2+1} \\ &= Q_{3,2}Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2} \\ &= (Q_{2,1}^2 + \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3))(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \text{ using (2.1.1)} \\ &= (Q_{2,1}Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 + \mathrm{Sq}^1[(Q_{2,1}X_3 + X_3^3))(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &\equiv 0 \end{aligned}$$

So Case (1) has been done.

2.2. **Proof of Case(2).** Suppose that $n_0 = 2l_0$, $n_1 = 2l_1 + 1$ and $n_2 = 2l_2 + 1$. We will use the χ-*trick* (1.7) to prove the result. Notice that

$$Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} = [\mathrm{Sq}^4(Q_{3,1})](Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \equiv Q_{3,1}\chi(\mathrm{Sq}^4)(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \qquad (2.2.1)$$

Using the formulae $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^2$ and $Q_{3,1} = \mathrm{Sq}^2(Q_{3,2})$, we can easily see that the last term of (2.2.1) can be reduced to

$$\mathrm{Sq}^2(Q_{3,2})[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + Q_{3,1}[\mathrm{Sq}^1(\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}))]^2.$$

Noting that $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, so the last term is zero. Again we use the χ-*trick*, the first term is hit if and only if $Q_{3,2}\chi(\mathrm{Sq}^2)[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2$ is hit. Now

$$\begin{aligned} &Q_{3,2}\chi(\mathrm{Sq}^2)[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= Q_{3,2}\mathrm{Sq}^2[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= (Q_{2,1}^2 + V_3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= [Q_{2,1}\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + V_3[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &\equiv V_3[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= \mathrm{Sq}^1\{(Q_{2,1}X_3 + X_3^3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2\} \\ &\equiv 0. \end{aligned}$$

Suppose that $n_0 = 2l_0 + 1$, $n_1 = 2l_1$ and $n_2 = 2l_2 + 1$. Then

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0+1}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2+1} \\ &= \mathrm{Sq}^1(Q_{3,1})[Q_{3,2}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &= \mathrm{Sq}^1[Q_{3,1}Q_{3,2}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \qquad (\text{since } \mathrm{Sq}^1Q_{3,2} = 0) \\ &\equiv 0. \end{aligned}$$

Suppose that $n_0 = 2l_0 + 1$, $n_1 = 2l_1 + 1$ and $n_2 = 2l_2$. Then

$$Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} = Q_{3,0}^{2l_0+1}Q_{3,1}^{2l_1+1}Q_{3,2}^{2l_2} = \mathrm{Sq}^2(Q_{3,2})[Q_{3,0}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2].$$

Since $\mathrm{Sq}^i(Q_{3,0}) = 0$ for $i = 1, 2$ and $\mathrm{Sq}^1Q_{3,2} = 0$, it is sufficient to show that

$$Q_{3,2}Q_{3,0}\mathrm{Sq}^2[(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2]$$

is hit. Indeed,

$$\begin{aligned} &Q_{3,2}Q_{3,0}\mathrm{Sq}^2[(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &= (Q_{2,1}^2 + V_3)Q_{3,0}[\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= V_3V_2V_1Q_{2,1}^2[\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + V_3Q_{3,0}[\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \ (\text{since } Q_{3,0} = V_3V_2V_1) \\ &= \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3)V_2V_1Q_{2,1}^2[\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &\quad + \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3)Q_{3,0}[\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2. \end{aligned}$$

The last two terms are hit, since Sq^1 acts on V_2V_1 and $Q_{3,0}$ trivially.

2.3. **Proof of Case(3).** Suppose that $n_0 = 4l_0+1$, $n_1 = 4l_1+1$ and $n_2 = 2l_2+1$. Then

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{4l_0+1}Q_{3,1}^{4l_1+1}Q_{3,2}^{2l_2+1} \\ &= Q_{3,0}Q_{3,1}Q_{3,2}(Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{l_2})^2 \end{aligned}$$

Write B for $Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{l_2}$. Then

$$\begin{aligned} &Q_{3,0}Q_{3,1}Q_{3,2}(Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{l_2})^2 \\ &= Q_{3,0}Q_{3,1}Q_{3,2}B^2 \\ &= V_3V_2^3V_1^7B^2 + V_3V_2^5V_1^3B^2 + V_3^2V_2V_1^7B^2 + V_3^3V_2^1V_1^3B^2 + V_3^2V_2^2V_1^5B^2 \\ &\quad + V_3^2V_2^4V_1^1B^2 + V_3^3V_2^2V_1^1B^2 \end{aligned}$$

Now we will show that each term after the last equality is hit. Using the following lemma, we will know that the first four terms are hit.

Lemma 2.3. *For any positive odd integers k_0, k_1 and k_2, the polynomial $V_3^{k_0}V_2^{k_1}V_1^{k_2}B^2$ is hit.*

Proof. Put $k_0 = 2a+1, k_1 = 2b+1$ and $k_2 = 2c+1$. Then we have

$$\begin{aligned} V_3^{k_0}V_2^{k_1}V_1^{k_2}B^2 &= V_3^{2a+1}V_2^{2b+1}V_1^{2c+1}B^2 = [\mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3)]V_3^{2a}V_2^{2b+1}V_1^{2c+1}B^2 \\ &= \mathrm{Sq}^1[(Q_{2,1}X_3 + X_3^3)(V_3^{2a}V_2^{2b+1}V_1^{2c+1}B^2)] \quad (\text{since } \mathrm{Sq}^1(V_2V_1) = 0). \end{aligned}$$

□

Since $V_3^2V_2^2V_1^5B^2 = V_1(V_3V_2V_1^2B)^2$ and $V_3^2V_2^4V_1^1B^2 = V_1(V_3V_2^2B)^2$, we verify that these two terms satisfy the condition in Theorem 1.5. So they are hit.

It remains to show that $V_3^3V_2^2V_1^1B^2$ is hit. In fact

$$\begin{aligned} V_3^3V_2^2V_1^1B^2 = \; & X_1^3X_2^2X_3^{12}B^2 + X_1X_2^4X_3^{12}B^2 + X_1^7X_2^2X_3^8B^2 + X_1X_2^8X_3^8B^2 \\ & + X_1^8X_2^6X_3^3B^2 + X_1^5X_2^2X_3^{10}B^2 + X_1^4X_2^3X_3^{10}B^2 + X_1^2X_2^5X_3^{10}B^2 \\ & + X_1X_2^6X_3^{10}B^2 + X_1^9X_2^2X_3^6B^2 + X_1^8X_2^3X_3^6B^2 + X_1^2X_2^9X_3^6B^2 \\ & + X_1^4X_2^{10}X_3^3B^2 + X_1X_2^{10}X_3^6B^2 + X_1^9X_2^4X_3^4B^2 + X_1^5X_2^8X_3^4B^2 \\ & + X_1^8X_2^5X_3^4B^2 + X_1^4X_2^9X_3^4B^2 + X_1^7X_2^6X_3^4B^2 + X_1^3X_2^{10}X_3^4B^2 \\ & + X_1^5X_2^3X_3^9B^2 + X_1^3X_2^5X_3^9B^2 + X_1^8X_2^4X_3^5B^2 + X_1^2X_2^{10}X_3^5B^2 \\ & + X_1^9X_2^5X_3^3B^2 + X_1^5X_2^9X_3^3B^2. \end{aligned}$$

Using Theorem 1.5, we can prove all but the following terms are hit: $X_1^3X_2^5X_3^9B^2$, $X_1^5X_2^3X_3^9B^2$, $X_1^9X_2^3X_3^5B^2$, $X_1^9X_2^5X_3^3B^2$, $X_1^3X_2^9X_3^5B^2$ and $X_1^5X_2^9X_3^3B^2$. Indeed we only need the following proposition to conclude the result.

Proposition 2.4. *$X_1^3X_2^5X_3^9B^2 + X_1^5X_2^3X_3^9B^2$, $X_1^9X_2^3X_3^5B^2 + X_1^9X_2^5X_3^3B^2$ and $X_1^3X_2^9X_3^5B^2 + X_1^5X_2^9X_3^3B^2$ are hit.*

Proof. Careful observations reveal the following,

$$X_1^3X_2^5X_3^9 + X_1^5X_2^3X_3^9 = \mathrm{Sq}^2(X_1^9X_2^3X_3^3) + \mathrm{Sq}^1(X_1^{10}X_2^3X_3^3) + X_1^9X_2^4X_3^4, \tag{2.3.1}$$

$$X_1^9X_2^3X_3^5 + X_1^9X_2^5X_3^3 = \mathrm{Sq}^2(X_1^3X_2^9X_3^3) + \mathrm{Sq}^1(X_1^3X_2^{10}X_3^3) + X_1^4X_2^9X_3^4$$

and

$$X_1^3X_2^9X_3^5 + X_1^5X_2^9X_3^3 = \mathrm{Sq}^2(X_1^3X_2^3X_3^9) + \mathrm{Sq}^1(X_1^3X_2^3X_3^{10}) + X_1^4X_2^4X_3^9.$$

We shall show that $X_1^3X_2^5X_3^9B^2 + X_1^5X_2^3X_3^9B^2$ is hit; the remaining two cases can be proved similarly.

By Theorem 1.5, $X_1^9X_2^4X_3^4B^2 = X_1(X_1^4X_2^2X_3^2B)^2$ in (2.3.1) is hit. So we are left to show that $\mathrm{Sq}^2(X_1^9X_2^3X_3^3)B^2 + \mathrm{Sq}^1(X_1^{10}X_2^3X_3^3)B^2$ is hit. This can be done by using the χ-*trick*. In fact,

$$X_1^3X_2^3X_3^9[\chi(\mathrm{Sq}^2)(B^2)] + X_1^3X_2^3X_3^{10}[\chi(\mathrm{Sq}^1(B^2)] = 0,$$

where we use the facts that $\mathrm{Sq}^2B^2 = 0$ and $\mathrm{Sq}^1B^2 = 0$. □

To finish the proof of Case(3), we are left to show that $Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+1}Q_{3,2}^{2l_2+1}$ is hit. The proof can be done as follows,

$$\begin{aligned}
&Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+1}Q_{3,2}^{2l_2+1}\\
&= Q_{3,0}^3Q_{3,1}Q_{3,2}Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}\\
&= \mathrm{Sq}^2[Q_{3,0}Q_{3,1}^3Q_{3,2}](Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}) + Q_{3,0}Q_{3,1}^4(Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2})\\
&\equiv Q_{3,0}Q_{3,1}^3Q_{3,2}\chi(\mathrm{Sq}^2)(Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}) + Q_{3,0}^{4l_0+1}Q_{3,1}^{4l_1+4}Q_{3,2}^{2l_2}
\end{aligned}$$

It is easy to see that the first term is zero and that the second term is hit by using the result of Case(1).

2.4. **Proof of Case(4).** Put $n_0 = 2l_0 + 1$, $n_1 = 4l_1 + 3$ and $n_2 = 4l_2 + 1$. It is easy to verify that

$$Q_{3,0}Q_{3,1}^3Q_{3,2} = \mathrm{Sq}^4[Q_{3,0}Q_{3,1}Q_{3,2}^3] + Q_{3,0}^3Q_{3,2}^2 + Q_{3,0}Q_{3,1}Q_{3,2}^4.$$

Thus

$$\begin{aligned}
Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0+1}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+1} = Q_{3,0}^3Q_{3,1}Q_{3,2}(Q_{3,0}^{2l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2})\\
&= \mathrm{Sq}^4[Q_{3,0}Q_{3,1}Q_{3,2}^3](Q_{3,0}^{2l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}) + Q_{3,0}^{2l_0+3}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2+2} + Q_{3,0}^{2l_0+1}Q_{3,1}^{4l_1+1}Q_{3,2}^{4l_2+4}
\end{aligned}$$

The last two terms are hit by the results of Cases(1) and (2). Using the χ-*trick* and the result of Case(3), it can be shown that the first term is also hit. We will leave the detail to the reader to verify the conclusion.

2.5. **Proof of Case(5).** When $n_0 = 4l_0 + 1, n_1 = 4l_1 + 3$ and $n_2 = 4l_2 + 3$,

$$\begin{aligned}
Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{4l_0+1}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+3}\\
&= P_1Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2},
\end{aligned}$$

where $P_1 = Q_{3,0}Q_{3,1}^3Q_{3,2}^3$, which is a symmetric polynomial in X_1, X_2 and X_3. Let $X_1^aX_2^bX_3^c$ be a term in the expansion form of P_1. For dimmesion reason, $a + b + c = 37$. Suppose that a is odd, b and c are even. Then setting $k = 0$, $E = X_1$ and

$$F = X_1^{\frac{a-1}{2}}X_2^{\frac{b}{2}}X_3^{\frac{c}{2}}Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2},$$

$$\deg F = \frac{(a-1)+b+c}{2} + (7\cdot 2l_0) + (6\cdot 2l_1) + (4\cdot 2l_2) = 18 + 2(7l_0 + 6l_1 + 4l_2).$$

So $\mu(\deg(F)) \geq 2$. Then we can use Theorem 1.5 to conclude that $X_1^aX_2^bX_3^cQ_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$ is hit. Suppose that a, b and c are all odd. Then after expanding P_1, we know that up to the permutation, (a, b, c) can only have the following choices,

$$(3,13,21), (3,9,25), (5,11,21), (5,13,19), (7,9,21), (7,11,19), (7,13,17), (9,11,17). \qquad (2.5.1)$$

Referring to the above list, we wish to show that $X_1^3X_2^{13}X_3^{21}Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$ is hit. Check the condition of Theorem 1.5 as follows. Setting $k = 1$, $E = X_1^3X_2X_3$ and $F = X_2^3X_3^5Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}$, then

$$\deg E = 5 \text{ and } \deg F = 8 + 7l_0 + 6l_1 + 4l_2.$$

Hence $(2^{k+1}-1)\mu(\deg F) = 3\mu(8+7l_0+6l_1+4l_2)$. Suppose that l_0 is even. Then the condition of Theorem 1.5 is satisfied. Hence we are done. Suppose that l_0 is odd, if the condition of the Peterson Conjecture(Wood's Theorem) is satisfied, namely,

$$3 < \alpha(37+4\cdot 7l_0+4\cdot 6l_1+4\cdot 4l_2+3),$$

then we have proved the result. If the condition is not satisfied, then the Minimum Spike exists [6], p578. So there exist non-negative integers $m \geq r \geq s$, such that $37+4\cdot 7l_0+4\cdot 6l_1+4\cdot 4l_2 = 2^m-1+2^r-1+2^s-1$. This implies that $7l_0+6l_1+4l_2 = 2^{m-2}+2^{r-2}-9$, $s=2$ and $r \geq 3$. The property for Minimum Spike implies that we can assume $m > r$ if $r > s$. So $\mu(8+7l_0+6l_1+4l_2) = \mu(8+2^{m-2}+2^{r-2}-9) \geq 2$. Hence the condition for Theorem 1.5 is satisfied. Therefore $X_1^3X_2^{13}X_3^{21}Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$ is hit. Using the same method, we can show that all the polynomials corresponding to the list (2.5.1) are hit except for the case: $(a,b,c)=(7,11,19)$ and its permutations.

Lemma 2.5. *For any non-negative integers l_0, l_1 and l_2, the following statements are true.*

(1) $\mathrm{Sq}^i[Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}] = 0$ *for i not divisible by 4;*
(2) *For any non-negative integer t,* $\mathrm{Sq}^{4t}[Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}] = \sum Q_{3,0}^{4s_0}Q_{3,1}^{4s_1}Q_{3,2}^{4s_2}$ *for some integers s_0, s_1 and s_2;*
(3) $\mathrm{Sq}^4\mathrm{Sq}^4[Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}] = 0$.

Proof. The proposition is a basic consequence of the Cartan formula and Theorem 1.4. We leave the detail to the reader. □

In the following, we will frequently use the above lemma without mentioning each time. Denote by Y the product $Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$.

Lemma 2.6.

$$(X_1^7X_2^{11}X_3^{19}+X_1^{11}X_2^7X_3^{19})Y \equiv [X_1^7X_2^{11}X_3^7+X_1^{11}X_2^7X_3^7]\mathrm{Sq}^8\mathrm{Sq}^4Y.$$

. □

We will provide the proof in the appendix, since it involves very elementary and tedious computation.

Let P be the set consisting of the index $(1,2,3)$ and all its permutations. Then we have the following,

$$\sum_P X_1^7X_2^{11}X_3^{19}Y = (X_1^7X_2^{11}X_3^{19}+X_1^{11}X_2^7X_3^{19})Y + (X_1^7X_2^{19}X_3^{11}+X_1^{11}X_2^{19}X_3^7)Y$$
$$+ (X_1^{19}X_2^{11}X_3^7+X_1^{19}X_2^7X_3^{11})Y \equiv 0 \quad \text{(using Lemma 2.6)}.$$

2.6. **Proof of Case(6).** When $n_0 = 4l_0+3, n_1 = 4l_1+3$ and $n_2 = 4l_2+3$,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+3} \\ &= P_2Y, \end{aligned}$$

where $Y = Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$ and $P_2 = Q_{3,0}^3Q_{3,1}^3Q_{3,2}^3$. After expanding P_2, by applying Theorem 1.5 and Peterson's Conjecture(Wood's Theorem), we can conclude that all the corresponding terms are hit, except the following terms, up to the permutations,

$$X_1^7X_2^{19}X_3^{25}Y, X_1^7X_2^{11}X_3^{33}Y \text{ and } X_1^{11}X_2^{19}X_3^{21}Y \text{ (up to the permutation)}. \qquad (2.6.1)$$

For example, $X_1^5X_2^{21}X_3^{25}Y$ appears in the expansion of P_2Y we will show that it is hit. Set $k=1$, $E = X_1X_2X_3$ and $F = X_1X_2^5X_3^6Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}$. If $\deg F$ is even, then the condition of Theorem 1.5 satisfied. So $EF^4(=X_1^5X_2^{21}X_3^{25}Y)$ is hit. If $\deg F$ is odd, then we can assume that $\alpha(3+EF^4) < 3$

(otherwise the result is proved using the Peterson Conjecture (Wood's Theorem)). So the Minimum Spike exists [6], p578. Hence $\deg(EF^4) = 2^m - 1 + 2^r - 1 + 2^s - 1$ for some integers $m \geq r \geq s$. Therefore we have the following,

$$3 + 48 + 4(7l_0 + 6l_1 + 4l_2) = \deg(EF^4) = 2^m - 1 + 2^r - 1 + 2^s - 1.$$

This implies that $7l_0 + 6l_1 + 4l_2 = 2^{m-2} + 2^{r-2} - 13$, $s = 1$ and $r \geq 2$. Using the property of the Minimum spike, we may assume that $m > r$. So

$$\mu(\deg F) = \mu(12 + 7l_0 + 6l_1 + 4l_2) = \mu(2^{m-2} + 2^{r-2} - 1) \geq 2.$$

Therefore the inequality: $\deg E < (2^2 - 1)\mu(F)$ is true. Hence the condition of Theorem 1.5 is satisfied. Using the same method we can show that all the terms of P_2Y corresponding to the expansion of P_2 are hit, except the terms listed in (2.6.1).

Recall that Y denotes $Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$. Since

$$X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25} = X_1^7\mathrm{Sq}^8[X_2^{11}X_3^{25}],$$

we have

$$\begin{aligned}
&(X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25})Y \\
&= X_1^7\mathrm{Sq}^8[X_2^{11}X_3^{25}]Y \\
&\equiv X_2^{11}X_3^{25}\chi(\mathrm{Sq}^8)[X_1^7Y] \\
&= X_2^{11}X_3^{25}(\mathrm{Sq}^7\mathrm{Sq}^1 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^8)[X_1^7Y] \\
&= X_1^{11}X_2^{11}X_3^{25}\mathrm{Sq}^4Y + X_1^7X_2^{11}X_3^{25}\mathrm{Sq}^8Y \\
&\equiv Y\chi(\mathrm{Sq}^4)[X_1^{11}X_2^{11}X_3^{25}] + X_1^7X_2^{11}X_3^{25}\mathrm{Sq}^8Y \\
&= [{X_1}^{11}{X_2}^{12}{X_3}^{28} + {X_1}^{12}{X_2}^{11}{X_3}^{28} + {X_1}^{12}{X_2}^{13}{X_3}^{26} + {X_1}^{13}{X_2}^{12}{X_3}^{26} \\
&\quad + {X_1}^{13}{X_2}^{13}{X_3}^{25}]Y + X_1^7X_2^{11}X_3^{25}\mathrm{Sq}^8Y.
\end{aligned}$$

Using Theorem 1.5, we obtain the following,

$$(X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25})Y \equiv X_1^7X_2^{11}X_3^{25}\mathrm{Sq}^8Y \tag{2.6.2}$$

It is easy to show that (2.6.2) implies that

$$\sum_P(X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25})Y \equiv \sum_P X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y, \tag{2.6.3}$$

where P is the set consisting of the index $(1, 2, 3)$ and all its permutations. In fact

$$\begin{aligned}
\mathrm{Sq}^4[X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^8Y] &= \mathrm{Sq}^4[X_1^7X_2^{11}X_3^{21}]\mathrm{Sq}^8Y + X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&= [{X_1}^7{X_2}^{11}{X_3}^{25} + {X_1}^7{X_2}^{14}{X_3}^{22} \\
&\quad + {X_1}^8{X_2}^{13}{X_3}^{22} + {X_1}^8{X_2}^{14}{X_3}^{21} + {X_1}^9{X_2}^{12}{X_3}^{22} \\
&\quad + {X_1}^9{X_2}^{13}{X_3}^{21} + {X_1}^{10}{X_2}^{11}{X_3}^{22} + {X_1}^{10}{X_2}^{12}{X_3}^{21} \\
&\quad + {X_1}^{11}{X_2}^{11}{X_3}^{21}]\mathrm{Sq}^8Y + X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\equiv [X_1^7X_2^{11}X_3^{25} + X_1^{11}X_2^{11}X_3^{21}]\mathrm{Sq}^8Y \\
&\quad + X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \text{ (using Theorem 1.5)}
\end{aligned}$$

Lemma 2.7.

$$X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \equiv (X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y.$$

. □

Again we will leave the proof to the appendix.

Combining the above lemma with (2.6.3), we have the following,

$$\begin{aligned}
&\textstyle\sum_P (X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25} + X_1^{11}X_2^{21}X_3^{19})Y \\
&\equiv \textstyle\sum_P [X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^{11}X_2^{21}X_3^{19}Y] \\
&\equiv \textstyle\sum_P [(X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y + X_1^{11}X_2^{21}X_3^{19}Y] \\
&\equiv 0.
\end{aligned}$$

2.7. **Appendix.**

2.7.1. *Proof of Lemma 2.6.* Recall that $Y := Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$. Then

$$(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y = X_3^{19}\mathrm{Sq}^4[X_1^7X_2^7]Y + R$$

where

$$R = (X_1^8X_2^{10}X_3^{19} + X_1^9X_2^9X_3^{19} + X_1^{10}X_2^8X_3^{19})Y.$$

It is easy to see that R is hit by using Theorem 1.5. Then

$$\begin{aligned}
(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y &\equiv X_1^7X_2^7\chi(\mathrm{Sq}^4)[X_3^{19}Y] \\
&\equiv X_1^7X_2^7X_3^{19}\mathrm{Sq}^4Y \\
&= \mathrm{Sq}^{15}[X_1^3X_2^3X_3^9]X_1X_2X_3\mathrm{Sq}^4Y \\
&= X_1^3X_2^3X_3^9\chi(\mathrm{Sq}^{15})[X_1X_2X_3\mathrm{Sq}^4Y] \\
&\equiv X_1^3X_2^3X_3^9\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1[X_1X_2X_3]\mathrm{Sq}^4Y \\
&\quad + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y + (X_1^5X_2^7X_3^{13} \\
&\quad + X_1^7X_2^5X_3^{13} + X_1^4X_2^4X_3^{17} + X_1^4X_2^{11}X_3^{10} \\
&\quad + X_1^{11}X_2^4X_3^{10})\mathrm{Sq}^8\mathrm{Sq}^4Y.
\end{aligned}$$

Using Theorem 1.5, we can show that the last term is hit. So

$$\begin{aligned}
&(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y \\
&\equiv X_1^3X_2^3X_3^9\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1[X_1X_2X_3]\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&\equiv X_1^{11}X_2^{11}X_3^{11}\mathrm{Sq}^4Y + (X_1^5X_2^{11}X_3^{17} + X_1^{11}X_2^5X_3^{17} + X_1^4X_2^4X_3^{25} + X_1^4X_2^{19}X_3^{10} \\
&\qquad + X_1^{19}X_2^4X_3^{10})\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y.
\end{aligned}$$

Again using Theorem 1.5, we can prove that the middle term is hit. Hence

$$\begin{aligned}
&(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2} \\
&\equiv X_1^{11}X_2^{11}X_3^{11}\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&= \mathrm{Sq}^{15}[X_1^5X_2^5X_3^5]X_1X_2X_3\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&\equiv X_1^5X_2^5X_3^5\chi(\mathrm{Sq}^{15})[X_1X_2X_3\mathrm{Sq}^4Y] + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&= X_1^5X_2^5X_3^5(\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)[X_1X_2X_3\mathrm{Sq}^4Y] + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&= X_1^5X_2^5X_3^5(\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)[X_1X_2X_3]\mathrm{Sq}^4Y \\
&\qquad + X_1^5X_2^5X_3^5(\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)[X_1X_2X_3]\mathrm{Sq}^8\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&= [X_1^7X_2^{13}X_3^{13} + X_1^{13}X_2^7X_3^{13} + X_1^{13}X_2^{13}X_3^7 + X_1^6X_2^6X_3^{21} + X_1^6X_2^{21}X_3^6 \\
&\qquad + X_1^{21}X_2^6X_3^6]\mathrm{Sq}^4Y + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y + [X_1^7X_2^9X_3^9 + X_1^9X_2^7X_3^9 \\
&\qquad + X_1^9X_2^9X_3^7 + X_1^6X_2^6X_3^{13} + X_1^6X_2^{13}X_3^6 + X_1^{13}X_2^6X_3^6]\mathrm{Sq}^8\mathrm{Sq}^4Y \\
&\qquad + X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y.
\end{aligned}$$

Expanding it term by term, using Theorem 1.5, we can conclude that

$$(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y \equiv X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y. \tag{2.7.1}$$

On the other hand,

$$
\begin{aligned}
&X_1^7X_2^7X_3^{11}\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&=\mathrm{Sq}^9[X_1^2X_2^2X_3^5]X_1^3X_2^3X_3\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&\equiv X_1^2X_2^2X_3^5\chi(\mathrm{Sq}^9)[X_1^3X_2^3X_3\mathrm{Sq}^8\mathrm{Sq}^4Y]\\
&=X_1^2X_2^2X_3^5(\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)[X_1^3X_2^3X_3\mathrm{Sq}^8\mathrm{Sq}^4Y]\\
&\equiv X_1^2X_2^2X_3^5(\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&\qquad+X_1^7X_2^7X_3^7\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y\qquad\text{(using Lemma 2.5 and Theorem 1.5)}\\
&=[{X_1}^{11}{X_2}^5{X_3}^9+{X_1}^5{X_2}^{11}{X_3}^9+{X_1}^7{X_2}^{11}{X_3}^7\\
&\qquad+{X_1}^{10}{X_2}^6{X_3}^9+{X_1}^{11}{X_2}^7{X_3}^7+{X_1}^6{X_2}^{10}{X_3}^9\\
&\qquad+{X_1}^6{X_2}^6{X_3}^{13}+{X_1}^7{X_2}^5{X_3}^{13}+{X_1}^5{X_2}^7{X_3}^{13}]\mathrm{Sq}^8\mathrm{Sq}^4Y+X_1^7X_2^7X_3^7\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&\equiv[X_1^7X_2^{11}X_3^7+X_1^{11}X_2^7X_3^7]\mathrm{Sq}^8\mathrm{Sq}^4Y+X_1^7X_2^7X_3^7\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y.
\end{aligned}
$$

So to prove the lemma, it suffices to show that $X_1^7X_2^7X_3^7\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y$ is hit. This can be done as follows,

$$
\begin{aligned}
&X_1^7X_2^7X_3^7\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&=\mathrm{Sq}^9(X_1^3X_2^3X_3^3)X_1X_2X_3\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y\\
&=X_1^3X_2^3X_3^3\chi(\mathrm{Sq}^9)[X_1X_2X_3\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y]\\
&=X_1^3X_2^3X_3^3(\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)[X_1X_2X_3\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y]\\
&\equiv(X_1^4X_2^5X_3^4+X_1^4X_2^4X_3^5+X_1^5X_2^4X_3^4)\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8\mathrm{Sq}^4Y\qquad\text{(using Lemma 2.5 and Theorem 1.5)}\\
&\equiv 0.
\end{aligned}
$$

2.7.2. *Proof of Lemma 2.7.* First of all we claim the following,

Proposition 2.8.

$$
\begin{aligned}
&X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv X_1^{19}X_2^9X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y+[X_1^{11}X_2^9X_3^{11}+X_1^7X_2^{13}X_3^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\qquad+X_1^5X_2^7X_3^{11}\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y+X_1^7X_2^{21}X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}
$$

. □

We will leave the proof to the next section.

Using the χ-*trick* and Theorem 1.5, we can easily verify that

$$
\begin{aligned}
&X_1^{19}X_2^9X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv\chi(\mathrm{Sq}^4)[X_1^{19}X_2^9X_3^{11}]\mathrm{Sq}^8Y\\
&=[{X_1}^{19}{X_2}^{12}{X_3}^{12}+{X_1}^{20}{X_2}^{12}{X_3}^{11}+{X_1}^{20}{X_2}^{10}{X_3}^{13}\\
&\qquad+{X_1}^{21}{X_2}^9{X_3}^{13}+{X_1}^{21}{X_2}^{10}{X_3}^{12}]\mathrm{Sq}^8Y\equiv 0.
\end{aligned}
$$

So

$$
\begin{aligned}
&X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv[X_1^{11}X_2^9X_3^{11}+X_1^7X_2^{13}X_3^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\qquad+X_1^5X_2^7X_3^{11}\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y+X_1^7X_2^{21}X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}
$$

Now we use the χ-*trick* and Theorem 1.5 again as follows,

$$\begin{aligned}
&[X_1^{11}X_2^9X_3^{11} + X_1^7X_2^{13}X_3^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv \chi(\mathrm{Sq}^8)[X_1^{11}X_2^9X_3^{11} + X_1^7X_2^{13}X_3^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&= [{X_1}^{16}{X_2}^{10}{X_3}^{13} + {X_1}^{13}{X_2}^{10}{X_3}^{16} + {X_1}^{11}{X_2}^{12}{X_3}^{16}\\
&\quad +{X_1}^{12}{X_2}^{10}{X_3}^{17} + {X_1}^{13}{X_2}^{9}{X_3}^{17} + {X_1}^{16}{X_2}^{12}{X_3}^{11}\\
&\quad +{X_1}^{17}{X_2}^{9}{X_3}^{13} + {X_1}^{17}{X_2}^{10}{X_3}^{12} + {X_1}^{12}{X_2}^{16}{X_3}^{11}\\
&\quad +{X_1}^{19}{X_2}^{9}{X_3}^{11} + {X_1}^{11}{X_2}^{9}{X_3}^{19} + {X_1}^{8}{X_2}^{14}{X_3}^{17}\\
&\quad +{X_1}^{9}{X_2}^{13}{X_3}^{17} + {X_1}^{7}{X_2}^{16}{X_3}^{16} + {X_1}^{7}{X_2}^{13}{X_3}^{19}\\
&\quad +{X_1}^{7}{X_2}^{21}{X_3}^{11} + {X_1}^{8}{X_2}^{18}{X_3}^{13} + {X_1}^{9}{X_2}^{17}{X_3}^{13}\\
&\quad +{X_1}^{9}{X_2}^{18}{X_3}^{12} + {X_1}^{9}{X_2}^{14}{X_3}^{16}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv [X_1^7X_2^{13}X_3^{19} + X_1^7X_2^{21}X_3^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}$$

$$\begin{aligned}
&X_1^5X_2^7X_3^{11}\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv \chi(\mathrm{Sq}^{16})[X_1^5X_2^7X_3^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&= [{X_1}^{8}{X_2}^{19}{X_3}^{12} + {X_1}^{16}{X_2}^{7}{X_3}^{16} + {X_1}^{17}{X_2}^{9}{X_3}^{13}\\
&\quad +{X_1}^{10}{X_2}^{8}{X_3}^{21} + {X_1}^{10}{X_2}^{9}{X_3}^{20} + {X_1}^{9}{X_2}^{9}{X_3}^{21}\\
&\quad +{X_1}^{9}{X_2}^{11}{X_3}^{19} + {X_1}^{9}{X_2}^{17}{X_3}^{13} + {X_1}^{16}{X_2}^{11}{X_3}^{12}\\
&\quad +{X_1}^{8}{X_2}^{11}{X_3}^{20} + {X_1}^{17}{X_2}^{11}{X_3}^{11} + {X_1}^{9}{X_2}^{19}{X_3}^{11}\\
&\quad +{X_1}^{18}{X_2}^{8}{X_3}^{13} + {X_1}^{18}{X_2}^{9}{X_3}^{12} + {X_1}^{5}{X_2}^{17}{X_3}^{17}\\
&\quad +{X_1}^{10}{X_2}^{17}{X_3}^{12} + {X_1}^{10}{X_2}^{16}{X_3}^{13} + {X_1}^{6}{X_2}^{17}{X_3}^{16}\\
&\quad +{X_1}^{6}{X_2}^{16}{X_3}^{17}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv X_1^{17}X_2^{11}X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y.\\
&\equiv [\chi(\mathrm{Sq}^8)\chi(\mathrm{Sq}^4)X_1^{17}X_2^{11}X_3^{11}]Y\\
&= {X_1}^{18}{X_2}^{21}{X_3}^{12} + {X_1}^{18}{X_2}^{20}{X_3}^{13} + {X_1}^{18}{X_2}^{13}{X_3}^{20} + {X_1}^{18}{X_2}^{12}{X_3}^{21}\\
&\quad +{X_1}^{17}{X_2}^{21}{X_3}^{13} + {X_1}^{17}{X_2}^{13}{X_3}^{21} + {X_1}^{17}{X_2}^{16}{X_3}^{18} + {X_1}^{17}{X_2}^{18}{X_3}^{16}\\
&\quad +{X_1}^{20}{X_2}^{13}{X_3}^{18} + {X_1}^{20}{X_2}^{18}{X_3}^{13} + {X_1}^{24}{X_2}^{11}{X_3}^{16} + {X_1}^{24}{X_2}^{13}{X_3}^{14}\\
&\quad +{X_1}^{24}{X_2}^{14}{X_3}^{13} + {X_1}^{24}{X_2}^{16}{X_3}^{11} + {X_1}^{20}{X_2}^{11}{X_3}^{20} + {X_1}^{20}{X_2}^{12}{X_3}^{19}\\
&\quad +{X_1}^{20}{X_2}^{19}{X_3}^{12} + {X_1}^{20}{X_2}^{20}{X_3}^{11} + {X_1}^{17}{X_2}^{17}{X_3}^{17} + {X_1}^{20}{X_2}^{14}{X_3}^{17}\\
&\quad +{X_1}^{20}{X_2}^{17}{X_3}^{14} \equiv 0 \text{ (using Theorem 1.5).}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv [X_1^7X_2^{13}X_3^{19} + X_1^7X_2^{21}X_3^{11} + X_1^7X_2^{21}X_3^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&= X_1^7X_2^{13}X_3^{19}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv \chi(\mathrm{Sq}^8)\chi(\mathrm{Sq}^4)[X_1^7X_2^{13}X_3^{19}]Y \quad \text{(using the } \chi\text{-}\textit{trick}\text{)}\\
&\equiv (X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y \quad \text{(using Theorem 1.5).}
\end{aligned}$$

Hence we have

$$X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \equiv (X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y. \tag{2.7.2}$$

2.7.3. *Proof of Proposition 2.8.* In this section, we will give a proof of Proposition 2.8.

$$\begin{aligned}
X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y &= X_1^3X_2^3X_3\mathrm{Sq}^{16}[X_1^2X_2^4X_3^{10}]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv X_1^2X_2^4X_3^{10}\chi(\mathrm{Sq}^{16})[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]\\
&= X_1^2X_2^4X_3^{10}[\mathrm{Sq}^9\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{12}\mathrm{Sq}^3\mathrm{Sq}^1\\
&\quad +\mathrm{Sq}^{12}\mathrm{Sq}^4 + \mathrm{Sq}^{13}\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^{15}\mathrm{Sq}^1 + \mathrm{Sq}^{16}\\
&\quad +\mathrm{Sq}^{10}\mathrm{Sq}^4\mathrm{Sq}^2 + \mathrm{Sq}^{14}\mathrm{Sq}^2\\
&\quad +\mathrm{Sq}^{11}\mathrm{Sq}^4\mathrm{Sq}^1][X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y].
\end{aligned}$$

After expanding, we have nine terms and we wish to look into these terms in detail:

TERM 1

$$
\begin{aligned}
&= X_1^2X_2^4X_3^{10}[\mathrm{Sq}^9\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1][X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y] \\
&= X_1^2X_2^4X_3^{10}[\mathrm{Sq}^9\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1][X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +X_1^2X_2^4X_3^{10}[\mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1][X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&= [{X_1}^{12}{X_2}^{13}{X_3}^{14} + {X_1}^{11}{X_2}^{14}{X_3}^{14} + {X_1}^{7}{X_2}^{14}{X_3}^{18} + {X_1}^{7}{X_2}^{20}{X_3}^{12} \\
&\quad +{X_1}^{8}{X_2}^{13}{X_3}^{18} + {X_1}^{8}{X_2}^{20}{X_3}^{11}{X_1}^{11}{X_2}^{10}{X_3}^{18} + {X_1}^{12}{X_2}^{9}{X_3}^{18} \\
&\quad +{X_1}^{18}{X_2}^{9}{X_3}^{12} + {X_1}^{18}{X_2}^{10}{X_3}^{11} + {X_1}^{5}{X_2}^{8}{X_3}^{26} + {X_1}^{5}{X_2}^{22}{X_3}^{12} \\
&\quad +{X_1}^{6}{X_2}^{7}{X_3}^{26} + {X_1}^{6}{X_2}^{22}{X_3}^{11} + {X_1}^{20}{X_2}^{7}{X_3}^{12} + {X_1}^{20}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +[{X_1}^{7}{X_2}^{10}{X_3}^{14} + {X_1}^{7}{X_2}^{12}{X_3}^{12} + {X_1}^{8}{X_2}^{9}{X_3}^{14} + {X_1}^{8}{X_2}^{12}{X_3}^{11} \\
&\quad +{X_1}^{10}{X_2}^{9}{X_3}^{12} + {X_1}^{10}{X_2}^{10}{X_3}^{11} + {X_1}^{5}{X_2}^{8}{X_3}^{18} + {X_1}^{5}{X_2}^{14}{X_3}^{12} \\
&\quad +{X_1}^{6}{X_2}^{7}{X_3}^{18} + {X_1}^{6}{X_2}^{14}{X_3}^{11} + {X_1}^{12}{X_2}^{7}{X_3}^{12} + {X_1}^{12}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\equiv 0.
\end{aligned}
$$

TERM 2

$$
\begin{aligned}
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{12}\mathrm{Sq}^3\mathrm{Sq}^1[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y] \\
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{12}\mathrm{Sq}^3\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^4\mathrm{Sq}^3\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^3\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&= 0 + [{X_1}^{7}{X_2}^{10}{X_3}^{14} + {X_1}^{7}{X_2}^{12}{X_3}^{12} + {X_1}^{8}{X_2}^{9}{X_3}^{14} + {X_1}^{8}{X_2}^{12}{X_3}^{11} \\
&\quad +{X_1}^{10}{X_2}^{9}{X_3}^{12} + {X_1}^{10}{X_2}^{10}{X_3}^{11} + {X_1}^{5}{X_2}^{8}{X_3}^{18} + {X_1}^{5}{X_2}^{14}{X_3}^{12} \\
&\quad +{X_1}^{6}{X_2}^{7}{X_3}^{18} + {X_1}^{6}{X_2}^{14}{X_3}^{11} + {X_1}^{12}{X_2}^{7}{X_3}^{12} \\
&\quad +{X_1}^{12}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +[{X_1}^{5}{X_2}^{8}{X_3}^{14} + {X_1}^{5}{X_2}^{10}{X_3}^{12} + {X_1}^{6}{X_2}^{7}{X_3}^{14} + {X_1}^{6}{X_2}^{10}{X_3}^{11} \\
&\quad +{X_1}^{8}{X_2}^{7}{X_3}^{12} + {X_1}^{8}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\equiv 0.
\end{aligned}
$$

TERM 3

$$
\begin{aligned}
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{12}\mathrm{Sq}^4[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y] \\
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{12}\mathrm{Sq}^4[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^4\mathrm{Sq}^4[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^4[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&= 0 + [{X_1}^{5}{X_2}^{12}{X_3}^{14} + {X_1}^{5}{X_2}^{14}{X_3}^{12} + {X_1}^{6}{X_2}^{13}{X_3}^{12} + {X_1}^{6}{X_2}^{14}{X_3}^{11} \\
&\quad +{X_1}^{7}{X_2}^{10}{X_3}^{14} + {X_1}^{7}{X_2}^{13}{X_3}^{11} + {X_1}^{8}{X_2}^{9}{X_3}^{14} + {X_1}^{8}{X_2}^{12}{X_3}^{11} \\
&\quad +{X_1}^{10}{X_2}^{7}{X_3}^{14} + {X_1}^{10}{X_2}^{10}{X_3}^{11} + {X_1}^{11}{X_2}^{8}{X_3}^{12} + {X_1}^{11}{X_2}^{9}{X_3}^{11} \\
&\quad +{X_1}^{12}{X_2}^{7}{X_3}^{12} + {X_1}^{12}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\quad +[{X_1}^{5}{X_2}^{10}{X_3}^{12} + {X_1}^{6}{X_2}^{9}{X_3}^{12} + {X_1}^{6}{X_2}^{10}{X_3}^{11} + {X_1}^{7}{X_2}^{8}{X_3}^{12} \\
&\quad +{X_1}^{7}{X_2}^{9}{X_3}^{11} + {X_1}^{8}{X_2}^{7}{X_3}^{12} + {X_1}^{8}{X_2}^{8}{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y \\
&\equiv (X_1^7X_2^{13}X_3^{11} + X_1^{11}X_2^9X_3^{11})\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^7X_2^9X_3^{11}\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}
$$

TERM 4
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{13}\mathrm{Sq}^2\mathrm{Sq}^1[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]$
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{13}\mathrm{Sq}^2\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+0+[{X_1}^7{X_2}^{10}{X_3}^{14}+{X_1}^7{X_2}^{12}{X_3}^{12}+{X_1}^8{X_2}^9{X_3}^{14}+{X_1}^8{X_2}^{12}{X_3}^{11}$
$+{X_1}^{10}{X_2}^9{X_3}^{12}+{X_1}^{10}{X_2}^{10}{X_3}^{11}+{X_1}^5{X_2}^8{X_3}^{18}+{X_1}^5{X_2}^{14}{X_3}^{12}$
$+{X_1}^6{X_2}^7{X_3}^{18}+{X_1}^6{X_2}^{14}{X_3}^{11}+{X_1}^{12}{X_2}^7{X_3}^{12}$
$+{X_1}^{12}{X_2}^8{X_3}^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+[{X_1}^5{X_2}^8{X_3}^{14}+{X_1}^5{X_2}^{10}{X_3}^{12}+{X_1}^6{X_2}^7{X_3}^{14}+{X_1}^6{X_2}^{10}{X_3}^{11}$
$+{X_1}^8{X_2}^7{X_3}^{12}+{X_1}^8{X_2}^8{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$\equiv 0.$

TERM 5
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{15}\mathrm{Sq}^1[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]$
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{15}\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^7\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^3\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$= 0+[{X_1}^7{X_2}^{10}{X_3}^{14}+{X_1}^7{X_2}^{12}{X_3}^{12}+{X_1}^8{X_2}^9{X_3}^{14}$
$+{X_1}^8{X_2}^{12}{X_3}^{11}+{X_1}^{10}{X_2}^9{X_3}^{12}+{X_1}^{10}{X_2}^{10}{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+[{X_1}^5{X_2}^8{X_3}^{14}+{X_1}^5{X_2}^{10}{X_3}^{12}+{X_1}^6{X_2}^7{X_3}^{14}+{X_1}^6{X_2}^{10}{X_3}^{11}$
$+{X_1}^8{X_2}^7{X_3}^{12}+{X_1}^8{X_2}^8{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$\equiv 0.$

TERM 6
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{16}[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]$
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{16}[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^2X_2^4X_3^{10}\mathrm{Sq}^8[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^4[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^2X_2^4X_3^{10}X_1^3X_2^3X_3\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$= 0+0+[{X_1}^5{X_2}^{10}{X_3}^{12}+{X_1}^6{X_2}^9{X_3}^{12}+{X_1}^6{X_2}^{10}{X_3}^{11}+{X_1}^7{X_2}^8{X_3}^{12}$
$+{X_1}^7{X_2}^9{X_3}^{11}+{X_1}^8{X_2}^7{X_3}^{12}+{X_1}^8{X_2}^8{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+[{X_1}^5{X_2}^7{X_3}^{11}]\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y$
$\equiv X_1^7X_2^9X_3^{11}\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^5X_2^7X_3^{11}\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y.$

TERM 7
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{10}\mathrm{Sq}^4\mathrm{Sq}^2[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]$
$= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{10}\mathrm{Sq}^4\mathrm{Sq}^2[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+X_1^2X_2^4X_3^{10}\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$= [{X_1}^5{X_2}^{20}{X_3}^{14}+{X_1}^{18}{X_2}^7{X_3}^{14}+{X_1}^7{X_2}^{20}{X_3}^{12}+{X_1}^{18}{X_2}^9{X_3}^{12}$
$+{X_1}^5{X_2}^{22}{X_3}^{12}+{X_1}^6{X_2}^{21}{X_3}^{12}+{X_1}^6{X_2}^{22}{X_3}^{11}+{X_1}^7{X_2}^{21}{X_3}^{11}$
$+{X_1}^{19}{X_2}^8{X_3}^{12}+{X_1}^{19}{X_2}^9{X_3}^{11}+{X_1}^{20}{X_2}^7{X_3}^{12}+{X_1}^{20}{X_2}^8{X_3}^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y$
$+[{X_1}^7{X_2}^{12}{X_3}^{12}+{X_1}^{10}{X_2}^9{X_3}^{12}+{X_1}^5{X_2}^{12}{X_3}^{14}+{X_1}^5{X_2}^{14}{X_3}^{12}$
$+{X_1}^6{X_2}^{13}{X_3}^{12}+{X_1}^6{X_2}^{14}{X_3}^{11}+{X_1}^7{X_2}^{13}{X_3}^{11}+{X_1}^{10}{X_2}^7{X_3}^{14}$
$+{X_1}^{11}{X_2}^8{X_3}^{12}+{X_1}^{11}{X_2}^9{X_3}^{11}+{X_1}^{12}{X_2}^7{X_3}^{12}+{X_1}^{12}{X_2}^8{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y$
$\equiv (X_1^7X_2^{21}X_3^{11}+X_1^{19}X_2^9X_3^{11})\mathrm{Sq}^4\mathrm{Sq}^8Y + (X_1^7X_2^{13}X_3^{11}+X_1^{11}X_2^9X_3^{11})\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y.$

TERM 8

$$
\begin{aligned}
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{14}\mathrm{Sq}^2[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]\\
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{14}\mathrm{Sq}^2[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^2X_2^4X_3^{10}\mathrm{Sq}^6\mathrm{Sq}^2[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^2\mathrm{Sq}^2[X_1^3X_2^3X_3]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&= 0 + [{X_1}^7{X_2}^{12}{X_3}^{12} + {X_1}^{10}{X_2}^9{X_3}^{12} + {X_1}^5{X_2}^{12}{X_3}^{14}\\
&\quad +{X_1}^5{X_2}^{14}{X_3}^{12} + {X_1}^6{X_2}^{13}{X_3}^{12} + {X_1}^6{X_2}^{14}{X_3}^{11} + {X_1}^7{X_2}^{13}{X_3}^{11}\\
&\quad +{X_1}^{10}{X_2}^7{X_3}^{14} + {X_1}^{11}{X_2}^8{X_3}^{12} + {X_1}^{11}{X_2}^9{X_3}^{11} + {X_1}^{12}{X_2}^7{X_3}^{12}\\
&\quad +{X_1}^{12}{X_2}^8{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y + [{X_1}^5{X_2}^8{X_3}^{14} + {X_1}^5{X_2}^{10}{X_3}^{12}\\
&\quad +{X_1}^6{X_2}^7{X_3}^{14} + {X_1}^6{X_2}^{10}{X_3}^{11} + {X_1}^8{X_2}^7{X_3}^{12} + {X_1}^8{X_2}^8{X_3}^{11}]\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv (X_1^{11}X_2^9X_3^{11} + X_1^7X_2^{13}X_3^{11})\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}
$$

TERM 9

$$
\begin{aligned}
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{11}\mathrm{Sq}^4\mathrm{Sq}^1[X_1^3X_2^3X_3\mathrm{Sq}^4\mathrm{Sq}^8Y]\\
&= X_1^2X_2^4X_3^{10}\mathrm{Sq}^{11}\mathrm{Sq}^4\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\quad +X_1^2X_2^4X_3^{10}\mathrm{Sq}^3\mathrm{Sq}^4\mathrm{Sq}^1[X_1^3X_2^3X_3]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&= [{X_1}^{11}{X_2}^{14}{X_3}^{14} + {X_1}^{12}{X_2}^{13}{X_3}^{14} + {X_1}^7{X_2}^{14}{X_3}^{18} + {X_1}^7{X_2}^{20}{X_3}^{12}\\
&\quad +{X_1}^8{X_2}^{13}{X_3}^{18} + {X_1}^8{X_2}^{20}{X_3}^{11} + {X_1}^{11}{X_2}^{10}{X_3}^{18} + {X_1}^{12}{X_2}^9{X_3}^{18}\\
&\quad +{X_1}^{18}{X_2}^9{X_3}^{12} + {X_1}^{18}{X_2}^{10}{X_3}^{11}]\mathrm{Sq}^4\mathrm{Sq}^8Y + [{X_1}^7{X_2}^{10}{X_3}^{14}\\
&\quad +{X_1}^7{X_2}^{12}{X_3}^{12} + {X_1}^8{X_2}^9{X_3}^{14} + {X_1}^8{X_2}^{12}{X_3}^{11} + {X_1}^{10}{X_2}^9{X_3}^{12}\\
&\quad +{X_1}^{10}{X_2}^{10}{X_3}^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
&\equiv 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\textstyle\sum_{i=1}^9 \text{TERM } i \quad \equiv \quad & X_1^{19}X_2^9X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y + [X_1^{11}X_2^9X_3^{11} + X_1^7X_2^{13}X_3^{11}]\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^8Y\\
& +X_1^5X_2^7X_3^{11}\mathrm{Sq}^{16}\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^7X_2^{21}X_3^{11}\mathrm{Sq}^4\mathrm{Sq}^8Y.
\end{aligned}
$$

This completes the proof of Proposition 2.8.

References

[1] Hung, H.H.V., *The action of the Steenrod squares on the modular invariants of linear groups,* Proc. Amer. Math. Soc. **113**(1991),1097-1104.

[2] Hung, H.H.V., *Spherical classes and the algebraic transfer*, Trans. Amer. Math. Soc. **10**(1997),3893–3910.

[3] Hung, H.H.V., *The weak conjecture on spherical classes*, Math. Z. **231**(1999),727–743.

[4] Hung, H.H.V. and Minh, P.A. *The action of the mod p Steenrod operations on the modular invariants of linear groups*, Vietnam J. Math. **23**(1995),39–56.

[5] Silverman, J., *Hit polynomials and conjugation in the dual Steenrod algebra,* Math. Proc. Caambridge Philos. Soc. **123**(1998), 531-547.

[6] Singer, W.M., *On the action of Steenrod squares on polynomial algebras,* Proc. Amer. Math. Soc. **111**(1991),577-583.

[7] Tan, K.F, *Hit polynomials, Dickson invariants and related topics,* MSc thesis (National University of Singapore)(1999).

[8] Wood, R.M.W, *Steenrod squares of polynomials and the Peterson conjecture,* Math. Proc. Cambridge Philos. Soc. **105**(1989), 307-309.

[9] Wood, R.M.W, *Problems in the Steenrod algebra,* Bull. London Math. Soc. **30**(1998), 449–517.

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