Polynomials Maps and Even Dimensional Spheres

Francisco-Javier TURIEL
Geometría y Topología, Facultad de Ciencias Ap. 59,
29080 Málaga, Spain
email: turiel@agt.cie.uma.es

Abstract. We construct, for every even dimensional sphere \( S^n, n \geq 2 \), and every odd integer \( k \), a homogeneous polynomial map \( f : S^n \rightarrow S^n \) of Brouwer degree \( k \) and algebraic degree \( 2 | k | - 1 \).

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A polynomial map from \( X \subset \mathbb{R}^m \) to \( Y \subset \mathbb{R}^r \) is the restriction to \( X \) of a polynomial map \( F : \mathbb{R}^m \rightarrow \mathbb{R}^r \) such that \( F(X) \subset Y \). When each component of \( F \) is homogeneous of degree \( k \), we will say that the polynomial map from \( X \subset \mathbb{R}^m \) to \( Y \subset \mathbb{R}^r \) is homogeneous of degree \( k \). As usual \( S^n \) is the sphere on \( \mathbb{R}^{n+1} \) defined by the equation \( x_1^2 + \ldots + x_{n+1}^2 = 1 \), in short \( ||x||^2 = 1 \), whereas \( S^n_r, 1 \leq r \leq n \), will be the differentiable manifold, diffeomorphic to \( S^n \), defined by the equation \( (x_1^2 + \ldots + x_r^2)^2 + x_{r+1}^2 + \ldots + x_{n+1}^2 = 1 \). In this work we show:

Theorem 1. Suppose \( n \) even and \( \geq 2 \). Let \( k \) be an integer. Then:
(a) If \( k \) is odd, there exists a homogeneous polynomial map from \( S^n \) to \( S^n \) of Brouwer degree \( k \) and algebraic degree \( 2 | k | - 1 \).
(b) If \( k \) is even there exists, for each \( 2 \leq 2r \leq n \), a polynomial map from \( S^n_{2r} \) to \( S^n \) of Brouwer degree \( k \).

Representing elements of \( \pi_n(S^n) \) by polynomial maps is an old question [1] which was affirmatively solved by Wood, in 1968, provided that \( n \) is odd (theorem 1 of [3], see[4] as well for the complex sphere). Nevertheless, as far as I know, this problem is still
open for \( n \) even; our theorem settles it when the Brouwer degree is odd. In both cases the polynomial maps constructed are homogeneous; therefore the problem of representing elements of \( \pi_n(S^n) \) by homogeneous polynomial maps is solved now, since only zero and the odd topological degrees may be represented in this way when \( n \) is even \([2]\).

The proof of theorem 1 of \([3]\) makes use of a natural polynomial map of topological degree 2 (lemmas 4 and 5). Nothing similar is known for \( n \) even; however the polynomial map \( x \in \mathbb{R}^3 \to (x_1^2 - x_2^2, 2x_1x_2, x_3) \in \mathbb{R}^3 \) send \( S_2^2 \) into \( S^2 \) with topological degree 2. Part (b) of our theorem generalizes this map.

For proving theorem 1 we start constructing a family of real polynomials in one variable. Let \( \varphi_\ell = \sum_{j=0}^{\ell} a_j t^j \) be the Taylor expansion of \( \varphi = (1 - t)^{-1/2} \), at zero, up to order \( \ell \); that is to say \( a_j = \frac{(2j - 1)(2j - 3) \cdots 1}{2^j \cdot j!} \). Since the radius of convergence of the power series \( \sum_{j=0}^{\infty} a_j t^j \) is 1 and each \( a_j > 0 \), we have \( a_0 = 1 \leq \varphi_\ell \leq (1 - t)^{-1/2} \), \( t \in [0, 1] \), whence \((t - 1)\varphi_\ell^2(t) + 1 \geq 0 \) and \( \varphi_\ell(t) \geq 1 \) when \( t \geq 0 \) (both inequalities are obvious if \( t \geq 1 \)).

On the other hand if we set \( \varphi = t^{\ell+1}R + \varphi_\ell \) then \( \sum_{j=0}^{\ell} t^j = (1 - t)^{-1} = \varphi^2 = t^{\ell+1}R + \varphi_\ell^2 \) on \((-1, 1)\); Therefore \( \varphi_\ell^2 = t^{\ell+1}S + \sum_{j=0}^{\ell} t^j \) where \( S \) is a polynomial in \( t \). It follows, from that, the existence of a polynomial \( \lambda_\ell \) of degree \( \ell \) such that \((t - 1)\varphi_\ell^2(t) + 1 = t^{\ell+1}\lambda_\ell \).

**Lemma 1.** For every \( \ell \) one has \((t - 1)\varphi^2 + 1 = t^{\ell+1}\lambda_\ell \) where \( \lambda_\ell \) is a polynomial of degree \( \ell \). Moreover \( \lambda_\ell(t) \geq 0 \) and \( \varphi(t) > 0 \) for each \( t \in \mathbb{R} \) if \( \ell \) is even, and for any \( t \geq 0 \) if \( \ell \) is odd.

**Proof.** It will suffice to show that \( \lambda_\ell(t) \geq 0 \) and \( \varphi_\ell(t) > 0 \) if \( \ell \geq 2 \) is even and \( t < 0 \). First we will prove, by induction on \( \ell \), the existence of a \( \delta_\ell > 0 \) such that \( \varphi_\ell \) is strictly decreasing on \((-\infty, -1 + \delta_\ell)\). Note that \( \varphi_\ell = a_\ell t^\ell + a_{\ell-1} t^{\ell-1} + \varphi_{\ell-2} = a((2\ell - 1)t^\ell + 2\ell t^{\ell-1}) + \varphi_{\ell-2} \) where \( a > 0 \).

By induction hypothesis or because \( \varphi_0 = 1 \), the polynomial \( \varphi_{\ell-2} \) is decreasing on
\((-\infty, -1 + \delta_{\ell-2})\), or on \(\mathbb{R}\) if \(\ell = 2\). But the derivative \(((2\ell - 1)t^\ell + 2\ell t^{\ell-1})' = ((2\ell - 1)\ell t^{\ell-1} + 2\ell(\ell - 1)t^{\ell-2}) < 0\) on \((-\infty, -1]\), so \(\varphi_\ell\) is strictly decreasing on some interval \((-\infty, -1 + \delta_\ell)\).

We show now that \(\varphi_\ell(t) > (1-t)^{-1/2} > 0\) if \(t < 0\). As \((1-t)^{-1/2}\) is strictly increasing, it is enough to prove our result on \((-1, 0)\). On this interval \(\lim_{t \to -\infty} \{\varphi_\ell(t)\} = \sum_{j=0}^{\infty} a_j t^j = (1-t)^{-1/2}\). But the series \(\sum_{j=0}^{\infty} a_j t^j\) is alternating and the sequence \(\{a_j \mid t \mid^j\}_{j \in \mathbb{N}}\), whose limit is zero, strictly decreasing; then \(\varphi_\ell(t) > (1-t)^{-1/2} > 0\) for \(\ell\) even.

Finally, if \(\varphi_\ell(t) > (1-t)^{-1/2} > 0\) for any \(t < 0\), a straightforward calculation shows that \((t-1)\varphi_\ell^2(t) + 1 < 0\), whence \(\lambda_\ell(t) \geq 0\) since \(t^{\ell+1} < 0\). \(\square\)

Recall that any polynomial \(\mu\) which do not takes negative values has even degree and can be write \(\mu = \mu_1^2 + \mu_2^2\), where \(\mu_1\) and \(\mu_2\) are polynomials of degree \(\leq \) half of degree of \(\mu\). Therefore by setting \(k = \ell + 1, \alpha = \varphi_\ell\), \(\lambda_\ell = \mu, \beta_1 = \mu_1\) and \(\beta_2 = \mu_2\) one has:

**Corollary 1.** For any odd natural number \(k\) there exist three polynomials \(\alpha, \beta_1, \beta_2\), the first one of degree \(k - 1\) and the other two with degree \(\leq \frac{k-1}{2}\), such that \(\alpha(t) > 0\) and \((1-t)\alpha^2(t) + t^k(\beta_1^2(t) + \beta_2^2(t)) = 1\) anywhere.

Let us proof part (a) of theorem 1. Since topological degrees \(\pm 1\) may be represented by linear maps, we can assume \(k \geq 1\). On \(\mathbb{C} \times \mathbb{R}^{n-1} = \mathbb{R}^{n+1}\), endowed with coordinates \((z, y) = (z, y_1, ..., y_{n-1})\) for which \(S^n = \{(z, y) ; |z|^2 + y_1^2 + ... + y_{n-1}^2 = 1\}\), we define

\[ F(z, y) = ((\beta_1 |z|^2) + i\beta_2 |z|^2)z^k, \alpha(|z|^2)y \]

where \(\alpha, \beta_1\) and \(\beta_2\) are as in corollary 1. Then \(F(S^n) \subset S^n\).

Set \(S^1 = \{(z, 0) ; |z|^2 = 1\} \subset S^n\). As \(\alpha(t) > 0\) for each \(t \in \mathbb{R}\), \(F^{-1}(S^1) = S^1\) and \(F\) preserves the orientation transversely to \(S^1\). Hence the maps \(F|_{S^1}\) and \(F|_{S^n}\) have the same topological degree, that is to say \(k\).

By construction all the monomials of \(F\) have odd degree \(\leq 2k-1\). Multiplying everyone
of them by a suitable power of $|z|^2 + y_1^2 + ... + y_{n-1}^2$ the map $F$ becomes homogeneous of algebraic degree $2k - 1$, whereas $F|_{S^n}$ do not change.

For proving (b), first we set $\tilde{\lambda}_\ell(t) = \lambda_\ell(t^2)$ and $\tilde{\varphi}_\ell(t) = \varphi_\ell(t^2)$. By lemma 1 we have $(t^2 - 1)\tilde{\varphi}_\ell(t) + 1 = t^{2\ell+2}\tilde{\lambda}_\ell(t)$, $\tilde{\varphi}_\ell(t) > 0$ and $\tilde{\lambda}_\ell(t) \geq 0$ for any $t \in \mathbb{R}$. This allows us to find out, for every natural number $\tilde{k} \geq 1$, three polynomials $\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2$ such that $\tilde{\alpha}(t) > 0$ and $(1 - t^2)\tilde{\alpha}^2(t) + t^{2\tilde{k}}(\tilde{\beta}_1^2(t) + \tilde{\beta}_2^2(t)) = 1$ anywhere.

Consider on $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-2r+1}$ coordinates $(x, y) = (x_1, ..., x_{2r}, y_1, ..., y_{n-2r+1})$. Let $f : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ be a homogeneous polynomial map of algebraic degree $2\tilde{k}$, sending $S^{2r-1}$ into $S^{2r-1}$ with topological degree $k = \pm 2\tilde{k}$, which always exists (see [3]) and $J : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ the isomorphism given by $Jx = (-x_2, x_1, ..., -x_{2r}, x_{2r-1})$, that is to say the canonical complex structure of $\mathbb{R}^{2r}$. One defines (if $\tilde{k} = 0$ just consider a constant map):

$$F(z, y) = (\tilde{\beta}_1(||x||^2)f(x) + \tilde{\beta}_2(||x||^2)Jf(x), \tilde{\alpha}(||x||^2)y)$$

Then $F(S^n_{2r}) \subset S^n$ and the same argument as in part (a), applied to $S^{2r-1} = \{(x, 0); ||x||^2 = 1\} \subset S^n_{2r}$, shows that the topological degree of $F : S^n_{2r} \to S^n$ equals $k$.

References


