TOWARD AN ALGEBRAIC CLASSIFICATION OF
MODULE SPECTRA

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ABSTRACT. The category of modules over an $S$-algebra ($A_\infty$ or $E_\infty$ ring spectrum) has many of the good properties of the category of spectra. When the homotopy groups of the $S$-algebra in question form a sufficiently nice ring, it is possible to see the deviation of the category of modules over an $S$-algebra from the corresponding algebraic module category. In particular, many algebraic modules are realized as homotopy groups of topological modules over $S$-algebras. Examples studied include real and complex $K$-theory, both connective and periodic. Further, Bousfield localization by a smashing spectrum is shown to yield a category of modules over the localized sphere. For periodic $K$-theory, these methods yield an algebraic criterion to determine when a local spectrum is a module over the $K$-theory $S$-algebra, real or complex.

CONTENTS

(1) Introduction
(2) Bousfield localization and modules over $S$-algebras
(3) Categories related to the $K$-theory spectra
(4) A spectral sequence for $R$-modules
(5) Realization of projective and injective modules
(6) Realization of modules of dimension at most one
(7) Classification of $R$-modules of projective dimension at most two
(8) Modules over additive categories
(9) United homology theories
(10) The operation algebra for connective united $K$-theory
(11) A spectral sequence for united homology
(12) Classification of modules over real connective $K$-theory
(13) Modules over periodic $K$-theory and $K_n$-local spectra
(14) Realizing modules of dimension higher than two
(15) A change of rings isomorphism
(16) Future directions

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1. Introduction

The stable category of spectra is known to have good algebraic properties; this
has been expanded and made more precise by Elmendorf, Kriz, Mandell, and May
(EKMM) [12]. Any strict commutative ($E_\infty$) ring spectrum is essentially a com-
mutative $S$-algebra, for which the appropriate diagrams commute on the point-set level.
Here, all $S$-algebras are assumed to be commutative.

Let $R$ be an $S$-algebra. An $R$-module is defined in terms of diagrams that commute
on the point-set level. The classical definition of module spectrum requires that the
diagrams which define the module merely commute up to homotopy; here, such a
spectrum will be called a naive module spectrum.

The category of $R$-modules has all the standard constructions used in stable ho-
motopy theory, yielding many stable homotopy theories. The sphere spectrum $S$
is one example of an $S$-algebra, and the classical stable homotopy category is equivalent
to the derived category of $S$-modules. The algebra in other examples often involves
(known) rings simpler than $\pi_* S$.

Working in the derived category, $\mathcal{D}_R$, of $R$-modules (weak equivalences inverted),
we will not distinguish between an object or map and its weak homotopy type. The
derived category is equivalent to the homotopy category of cell modules and cellular
maps. There is a smash product of $R$-modules, $\wedge_R$, which is analogous to the derived
tensor product of $R_*$-modules. In the context of $S$-modules, $\wedge$ denotes $\wedge_S$, the smash
product over $S$.

Modules over an $S$-algebra $R$ can be closely related to algebraic modules over the
ring $R_*$ of homotopy groups of $R$. For example, the derived category of a discrete
commutative ring is equivalent to the derived category of modules over the Eilenberg-
MacLane spectrum of the ring.

Given an $R_*$-module $M_*$, $M^*$ denotes the $R^*$ module obtained by regrading: $M^n = M_{-n}$. Note that, for any naive $R$-module spectrum or $R$-module $X$, $\pi_* X$ has a
canonical $R_*$-module structure. Realization of an $R_*$-module refers to finding an
$R$-module with homotopy isomorphic to the given module.

This paper investigates the algebraic classification of modules in these stable ho-
motopy categories of $R$-modules. The main classification results follow Bousfield
[7, 8].

Theorem. Let $R$ be an $S$-algebra. Then every $R_*$-module of projective dimension at
most two can be realized as the module of homotopy groups of some $R$-module. Such
an $R$-module is unique up to homotopy if the $R_*$-module has projective or injective
dimension at most one. When $M_*$ is an $R_*$-module of projective dimension two, there
is an equivalence relation finer than homotopy equivalence so that equivalence classes
of $R$-modules with homotopy $M_*$ are in bijective correspondence with the elements
of $\text{Ext}^{2-1}_{R^*}(M^*, M^*)$. 

Ext is relevant since the main tool used to prove this theorem is a spectral sequence

\[ E_2^{s,t}(M, N) = \text{Ext}^{s,t}_{R_\ast}(M^\ast, N^\ast) \implies [M, N]_{\ast}^{s+t} \]

converging from algebraic Ext groups to homotopy classes of \( R \)-module maps.

In the dimension one case, the spectral sequence reduces to a short exact sequence describing the homotopy classes of maps of \( R \)-modules. Complex periodic \( K \)-theory, \( KU \), is an example of an \( S \)-algebra for which this is true.

Further, when \( R_\ast \) has global dimension at most two and is sufficiently sparse, the category of \( R \)-modules can be described completely algebraically. One example is complex connective \( K \)-theory, \( ku \). This result was first presented in [24].

For real \( K \)-theory, the classification is more complex: the relevant ring has infinite cohomological dimension; however, generalizing the concept of ring allows construction of an algebraic category with dimension two. Note that, as \( ko \)-modules, \( ku \simeq ko \wedge C(\eta) \). We also use \( kt \), defined here as \( ko \wedge C(\eta^2) \), which is a connective version of self-conjugate \( K \)-theory [2, 8].

Let \( crt \) be the category with objects \( ko, ku, \) and \( kt \) and maps all maps of \( ko \)-modules. The category of additive functors from \( crt \) into the category of abelian groups has many of the properties of a module category: We call this the category of united modules or \( crt \)-modules. Any \( ko \)-module \( X \) gives a \( crt \)-module \( \pi_crt(X) \) by smashing with \( S, C(\eta), \) and \( C(\eta^2) \) respectively, and taking homotopy. When the image of a \( crt \)-module fits into certain exact sequences, it is called \( crt \)-acyclic. The exactness condition is motivated by the topology and is necessary for realizability.

**Theorem.** The category of \( crt \)-acyclic \( crt \)-modules has enough projectives and all objects have projective dimension at most two. Any \( crt \)-acyclic \( crt \)-module can be realized as \( \pi_crt(X) \) for some \( ko \)-module \( X \). This \( ko \)-module is unique up to homotopy if the \( crt \)-module has projective or injective dimension at most one. For a fixed \( crt \)-module \( M \) of projective dimension two, there is an equivalence relation finer than homotopy equivalence so that equivalence classes of \( ko \)-modules \( X \) with \( \pi_crt(X) = M \) are in bijective correspondence with the elements of \( \text{Ext}^{2,-1}_{crt}(M, M) \).

Again, the main tool used to prove the theorem is a spectral sequence

\[ E_2^{s,t}(M, N) = \text{Ext}^{s,t}_{ko_\ast}(M^\ast, N^\ast) \implies [M, N]_{ko}^{s+t} \]

converging from a generalization of Ext for united modules to homotopy classes of maps of \( ko \)-modules.

A similar result holds for the periodic theory, which has dimension one and is thus simpler. The periodic version of \( crt \) is called \( CRT \).

The algebraic description of \( \mathcal{D}_{KO} \) and \( \mathcal{D}_{KU} \) given by the above theorems, together with Bousfield’s description of the category of \( K_\ast \)-local spectra in terms of objects called \( ACRT \)-modules, yields an algebraic criterion for when a \( K_\ast \)-local spectrum can be given the structure of a \( KO \)- or \( KU \)-module. The notation for localization with
respect to periodic $K$-theory is not ambiguous, since both $KO$ and $KU$ give the same localization functor.

Let $U$ be the right adjoint to the forgetful functor from $ACRT$-modules to $CRT$-modules. Note that the complexification map $c: KO \rightarrow KU$ is a map of $S$-algebras: Any $KU$-module is a $KO$-module.

**Theorem.** Let $X$ be a $K_s$-local spectrum. Then $X$ is equivalent to a $KO$-module if and only if $K_s^{CRT}(X) \cong U \pi_s^{CRT}(X)$, where $\pi_s^{CRT}(X)$ is a $CRT$-acyclic $CRT$-module. Further, $X$ is a $KU$-module if, in addition, $X_*$ can be given the structure of a $KU_s$-module.

The criterion shows that any naive module spectrum over these $S$-algebras is homotopic to a $KO$- or $KU$-module respectively.

**Corollary.** Any naive module spectrum over $KO$ or $KU$ is homotopic to a $KO$- or $KU$-module, respectively. Further, the derived category of naive $KO$- or $KU$-module spectra is equivalent to the derived category of $KO$- or $KU$-modules.

In fact, Bousfield localization at any smashing spectrum is closely related to topological module categories.

**Theorem.** Given a smashing $R$-module $E$, the derived category $\mathcal{D}_R[E^{-1}]$ of $E_*$-local $R$-modules is equivalent to the derived category $\mathcal{D}_{RE}$ of $RE$-modules.

In particular, this holds for $R = S$ and $E = KU$.

Returning to $ko$, we show that any $ko_s$-module $M_*$ can be realized as the homotopy of some $ko$-module.

**Theorem.** Given any $ko_s$-module $M_*$, it is possible to construct a $crt$-acyclic $crt$-module with $ko$ part $M_*$. Thus, by the classification theorem for $ko$-modules, $M_*$ can be realized as the homotopy of some $ko$-module.

In fact, we can define a general notion of a united theory. The result generalizes to any united theory all of whose acyclic objects have dimension at most two.

**Theorem.** Let $R$ be an $S$-algebra with a united theory $R\mathcal{F}$. Then given any $R_s$-module $M_*$ it is possible to construct an acyclic united module $M$ such that the $R$-part of $M$ is $M_*$. Thus, by the classification theorem, when the united theory $R\mathcal{F}$ is of dimension at most two, any $R_s$-module can be realized as the homotopy of some $R$-module.

In particular, this can also be done for $KO$.

In order to use these united theories for calculations, it is useful to understand how to compute Ext in these functor categories. The following is a change of rings theorem.
Let $\rho$ and $\rho'$ denote the left and right adjoints to the forgetful functor from $crt$-modules to $[ku, ku]_{ko}$-modules. For a $crt$-module $M$, $M^U$ denotes the $ku$-part of $M$; $M^O$, the $ko$-part.

**Theorem.** For $crt$-acyclic $crt$-modules $L$ and $M$ with $\eta = 0$ in $M^O$ and $A = [ku, ku]_{ko}$, there are natural isomorphisms

$$\rho'(M^U) \cong M \cong \rho(M^U),$$
$$\text{Ext}^{s,t}_{crt}(M, L) \cong \text{Ext}^{s,t}_{A}(M^U, L^U),$$
$$\text{Ext}^{s,t}_{crt}(L, M) \cong \text{Ext}^{s,t}_{A}(L^U, M^U).$$

Motivation beyond the desire to reduce all homotopy theory to algebra lies in various places. The focus here is on connective topological $K$-theory: complex $K$-theory, for example, for its relation to $C^*$-algebras; real $K$-theory, for example, toward greater understanding of the corresponding Adams spectral sequence.

Acknowledgments and thanks are due to Peter May, my advisor; to Jim McClure for first suggesting this line of research as well as for valuable guidance; to Pete Bousfield, Bob Bruner, Igor Kriz, Stephan Stolz, Maria Basterra, Michael Maltenfort, and Michael Mandell for helpful discussions; and to the homotopy theory community more widely for making this a hospitable, indeed friendly, area in which to do research.

2. **Bousfield localization and modules over $S$-algebras**

One method of studying stable homotopy theory is to study the category of spectra through the eyes of a homology theory, smashing all objects with a fixed spectrum. A more sophisticated method is to use Bousfield localization. Bousfield localization at a spectrum $E$ constructs a category in which $E$-homology isomorphisms are inverted. When Bousfield localization coincides with smashing all objects with the localization of the sphere, $E$ is called smashing.

We recall the basic definitions.

For a spectrum $E$, a map $f: A \longrightarrow B$ which induces an isomorphism $E_*A \longrightarrow E_*B$ is called an $E_*$-equivalence. If $E_*A = 0$, then $A$ is $E_*$-acyclic; $X$ is $E_*$-local if $X^*A = 0$ for any $E_*$-acyclic $A$, or equivalently, if any $E_*$ equivalence $f: A \longrightarrow B$ induces a bijection $f^*: [B, X]_* \longrightarrow [A, X]_*$. The $E_*$-localization of $X$, $XE$, is the terminal $E_*$-equivalence out of $X$. In fact, each spectrum $X$ can be decomposed naturally into $E_*$-local and $E_*$-acyclic spectra $XE$ and $EX$ via the cofibration

$$EX \longrightarrow X \longrightarrow XE \longrightarrow \Sigma(EX)$$

in the stable category (of $S$-modules or equivalently of spectra). The $E_*$-localization functor is sometimes denoted $L_E$. 

There are analogous definitions and localizations in the category of modules over any $S$-algebra. Bousfield’s methods to construct local spectra also generalize to this context.

Note that any $R$-module $X$ (or even a naive module spectrum over a naive ring spectrum $R$ more generally) is automatically $R_\ast$-local since $R_\ast Y = 0$ implies $X_\ast Y = 0 = X^\ast Y$ since any element of $X_\ast Y$ factors through $S^{X_\ast Y} \to R^{X_\ast Y}$ via the module structure map [1].

The Bousfield localization of an $S$-algebra is again an $S$-algebra [12]. In particular, this is true of the sphere spectrum. If $E_\ast$-localization is equivalent to smashing with the $E_\ast$-local sphere, then $E$ is called smashing. When $E$ is smashing, the derived category of $E_\ast$-local spectra is equivalent to that of $S_E$-modules.

**Theorem 1.** If $E$ is smashing, then the derived category of $E_\ast$-local spectra is equivalent to the derived category of $S_E$-modules.

This comparison of categories is a nice observation, but not yet very useful for calculations since $S_E$ is usually quite nasty.

The same holds for localization in the category of $R$-modules for an arbitrary $S$-algebra $R$. For an $R$-module $E$, we define

$$E^R_\ast(X) = [R, E \wedge_R X]^R_\ast = \pi_*(E \wedge_R X).$$

Now, $L_E$ is replaced by $L^E_R$, which denotes $E^R_\ast$-localization. We denote the localization $L^E_R X$ of an $R$-module $X$ by $X_E$. $E$ is smashing in $\mathcal{D}_R$ if $X_E \simeq X \wedge_R R_E$.

The notation $\mathcal{D}_R[E^{-1}]$ denotes the category of $E_\ast$-local $R$-modules, which is equivalent to the category $\mathcal{D}_R$ after inverting $E^R_\ast$-equivalences.

**Theorem 2.** Given a smashing $R$-module $E$, the derived category $\mathcal{D}_R[E^{-1}]$ of $E^R_\ast$-local $R$-modules is equivalent to the derived category $\mathcal{D}_{R_E}$ of $R_E$-modules.

**Proof.** Let $E$ be a smashing $R$-module; the map $X \cong R \wedge_R X \to R_E \wedge_R X$ gives the localization functor, so that any $E^R_\ast$-local $R$-module is homotopic to one of the form $R_E \wedge_R X$. Also, $R_E \wedge_R R_E \to R_E$ is an isomorphism since the idempotent functor $L^R_E$ is smashing: Localization gives an inverse to the multiplication map.

Next we show that any $R_E$-module $X$ has $X \cong R_E \wedge_R X$ as $R_E$-modules. First note that $L^R_E = L^R_{R_E}$, that is, $R_{E_\ast}$-isomorphisms are the same as $E_\ast$-isomorphisms: Any $R_{E_\ast}$-isomorphism has an $E_{E_\ast}$-acyclic cofiber, but any $R_{E_\ast}$-acyclic has trivial $E_{E_\ast}$-localization, so it is $E_{E_\ast}$-acyclic. Thus, $X$ has $X \cong R_E \wedge_R X$ as $R$-modules. The forgetful functor and $R_E \wedge_R (-)$ are adjoint functors between $\mathcal{D}_R$ and $\mathcal{D}_{R_E}$; the counit $R_E \wedge_R X \to X$ is by definition a map of $R_E$-modules, giving the desired weak equivalence on the level of $R_E$-modules.
Further, for $E^R_*$-local $R$-modules $W \simeq R_E \wedge_R X$ and $Z \simeq R_E \wedge_R Y$,

$$[W, Z]^R_* \cong [R_E \wedge_R X, R_E \wedge_R Y]^R_* = [X, R_E \wedge_R Y]^R_*$$

$$\cong [R_E \wedge_R X, R_E \wedge_R Y]^R_* \cong [W, Z]^R_*$$

where the second isomorphism is by $X \simeq R_E \wedge_R X$ and the third follows from freeness. The isomorphisms are all natural.

Therefore, the derived category of $E^R_*$-local $R$-modules is equivalent to that of $R_E$-modules.

EKMM [12] has generalized this theorem to describe $E_*^*$-local $R$-modules for arbitrary $E$: The categories $\mathcal{D}_E^R[E^{-1}]$ and $\mathcal{D}_{R_E}[(R_E \wedge_R E)^{-1}]$ are equivalent.

Although Bousfield localization does not require $E$ to be a ring spectrum or an $S$-algebra, it would be interesting to know, for an $S$-algebra $E$, how $E$-modules and $S_E$-modules are related. The example of $E = KU$ mentioned in the introduction is discussed in the next section.

3. Categories related to the $K$-theory spectra

The connective complex $K$-theory spectrum $ku$ is an $S$-algebra by infinite loop space technology [18]. As shown in [12], $KU$ is an $S$-algebra by localization in the category of $ku$-modules; $S_K$, the $K_*$-local sphere, is therefore an $S$-algebra, as noted above.

The $K$-local sphere, $S_K$, is closely related to the (periodic) Image of $J$ spectrum. Localized at a prime $p$, we have the cofibration (where $r = 3$ if $p = 2$ and $r$ generates the units mod $p^2$ for an odd prime)

$$J(p) \rightarrow KO \xrightarrow{\psi^{-1}} KO \rightarrow \Sigma J(p)$$

and the $K(p)_*$-local sphere is the homotopy fiber of the map

$$J(p) \rightarrow \Sigma^{-1}S\mathbb{Q}$$

which is a rational isomorphism on $\pi_{-1}$. Thus it is possible to calculate $\pi_*S_K$ [6].

Since localization of $S$- or $ku$-modules with respect to $KU$ is smashing, Theorem 1 shows that each local category $\mathcal{D}_{S[KU^{-1}]}$ and $\mathcal{D}_{ku}[KU^{-1}]$ is a module category; $\mathcal{D}_{ku}[KU^{-1}]$ is actually the derived category of $KU$-modules.

We have the diagram of categories

$$\begin{array}{ccc}
\mathcal{D}_S & \xrightarrow{L_{KU}} & \mathcal{D}_S[KU^{-1}] \\
F & \downarrow & \downarrow F \\
\mathcal{D}_{ku} & \xrightarrow{L_{KU}} & \mathcal{D}_{ku}[KU^{-1}]
\end{array} \approx \begin{array}{c}
\mathcal{D}_{S_K} \\
\mathcal{D}_{KU}
\end{array}$$
where $L$ is a localization functor and $F$ is a free functor. For an $S$-module $X$, the free functor to $ku$-modules is given by $ku \wedge_S X$; similarly, the free functor from $S_K$-modules to $KU$-modules is given by $KU \wedge_{S_K} X$, which is equivalent to $KU \wedge_S X$, since $KU$ is $K_*$-local. It is shown below that the categories $\mathcal{D}_S$, $\mathcal{D}_{S_K}$, $\mathcal{D}_{ku}$, and $\mathcal{D}_{KU}$ are all distinct.

We have three categories to compare: $K_*$-local spectra (i.e., $S_K$-modules), $K_*$-local $ku$-modules, and $KU$-modules, or $K_{ku}$-local $ku$-modules. Note that, as one would expect, not all $ku$-modules are $K_*$-local: $L_K ku \simeq S_K \wedge_S ku \not\simeq ku$ by direct calculation.

Further, since they are not rationally periodic (see [6] Corollary 4.4), the homotopy groups of $L_K ku$ are not periodic; not all $K_*$-local $ku$-modules are $KU$-modules. This is in sharp contrast to localization in the category of $ku$-modules, which is simply the free functor from $ku$-modules to $KU$-modules.

To see that not all $S_K$-modules are $ku$-modules, we note that no element of $\pi_*(S_K)$ of positive degree can be represented by a map of $ku$-modules. If $S_K$ were a $ku$-module, then consider any element $2 [S_K; S_K] = [ku, S_K] \simeq [S, S_K] \subseteq [S, S]_q$, $q > 0$.

The isomorphisms are given by freeness; the inclusion is a theorem relying on the relation of $S_K$ to the image of $J$. The element $\alpha$ would be represented by a map $S_K \rightarrow S_K$ and also as a map $S \rightarrow S$. Since $\alpha$ is of positive degree, $\alpha: S \rightarrow S$ gives the zero map $ku \rightarrow ku$ after smashing with $ku$; $1_{ku} \wedge \alpha$ is null. Now the commutative diagram

\[ S \wedge S_K \xrightarrow{\eta \wedge 1} ku \wedge S_K \xrightarrow{1 \wedge \alpha} ku \wedge S_K \]

would guarantee that $\alpha$ itself be null; but $S_{K_*}$ has non-trivial elements in positive degree, so $S_K$ cannot be a $ku$-module.

A similar result involving real $K$-theory holds as well.

4. A Spectral Sequence for $R$-modules

All algebraic maps from the homotopy of a free $R$-module to that of any $R$-module can be realized as the homotopy of some map of naive module spectra. The theory of $R$-modules actually allows the realization of many more maps. The method is to use a spectral sequence to show that maps with certain algebraic properties exist.

EKMM [12] exhibit a spectral sequence (EKMSS)

\[ E_2^{s,t}(X, Y) = E_{R^*}(X^*, Y^*) \implies \pi_{-(s+t)}(F_R(X, Y)), \]
with differentials $d_{s,t}^r : E_{r}^{s,t} \rightarrow E_{r+1}^{s+r,t-r+1}$. Here, $\text{Ext}_{R}^{s,t}(P,Q) = \text{Ext}_{R}^{s,0}(\Sigma^t P,Q)$, $(\Sigma^t L) = L_{s-t}$. The filtration on $\pi_*(F_R(X,Y))$, is given by letting $F^*\pi_*(F_R(X,Y))$ be the image of $\pi_*(F_R(X_s,Y)) \rightarrow \pi_*(F_R(X,Y))$, where $X_s$ are constructed from a free resolution of $X_s$. Thus, the $(s,t)$-th term of the associated bigraded group of the filtration is

$$E_0^{s,t}\pi_*(F_R(X,Y)) = F^s\pi_{-s-t}(F_R(X,Y))/F^{s+1}\pi_{-s-t}(F_R(X,Y)).$$

One can define $\text{Ext}^*_R(X,Y) = \pi_{-s}(F_R(X,Y))$.

Composition pairings are also discussed in [12]. The pairing

$$F_R(Y,Z) \wedge_R F_R(X,Y) \rightarrow F_R(X,Z)$$

induces a pairing of spectral sequences of differential $R_*$-modules that coincides with the algebraic Yoneda pairing on the $E_2$-level and converges to the pairing induced by composition. This is proven by taking free resolutions of $X$ and $Y$ in the contravariant side of each function spectrum:

$$F_R(Y_s,Z) \wedge_R F_R(X_s,Y) \rightarrow F_R(Y,Z) \wedge_R F_R(X_s,Y) \rightarrow F_R(X_s,Z).$$

For $a \in \text{Ext}_{R}^{s,t}(Y^*,Z^*)$ and $b \in \text{Ext}_{R}^{u,v}(X^*,Y^*)$,

$$d_r(ab) = (d_r a)b + (-1)^{t+s}a(d_r b).$$

The EKMSS is always conditionally convergent; with additional information, it is often strongly convergent, for example, when $X$ has a finite length cellular resolution as an $R$-module.

This spectral sequence is essential to understanding the difference between $R$-modules and $R_*$-modules. When an $R_*$-module—for example, the homotopy of an $R$-module—is placed in the $E_2$ term of this spectral sequence, we regrade it as an $R^*$-module.

5. Realization of projective and injective modules

5.1. Projective modules. Any projective module $P$ over any ring $A$ is the direct summand of a free module $F \cong P \oplus Q \cong Q \oplus P$. Thus, Eilenberg’s swindle gives us a two step free resolution of $P$:

$$0 \rightarrow P \oplus Q \oplus F^\oplus \rightarrow P \oplus Q \oplus F^\oplus \rightarrow P \rightarrow 0,$$

where the map between free modules moves each copy of $P$ to the next $P$-coordinate to the right.

See [17, 20] for a discussion of modules over graded rings.

When $A$ is the ring of homotopy groups $R_*$ for an $S$-algebra $R$, any free module is easily realized as the wedge of copies of $R$ (up to suspensions). The EKMSS (Section 4) shows that any map between free modules can be realized in topology; the cofiber of the realization of this map is the realization of our projective module.
Proposition 3. Let $R$ be an $S$-algebra. Then any projective $R_*$-module $P_*$ can be realized as the $R_*$-module of homotopy groups of some $R$-module $X$: $\pi_*(X) = X_* = P_*$. For any other $R$-module $Y$, $[X,Y]_*^R \cong \text{Hom}_{R_*}(P^*, Y^*)$.

Proof. The vanishing of higher Ext groups shows that any map from a projective module is uniquely realizable. □

5.2. Injective modules. For a Noetherian ring $A$, the direct sum of injective modules is injective, and we can decompose any injective module as the direct sum of certain indecomposables.

Recall that the injective hull of a module is the minimal injective extension of the module; it is unique up to non-canonical isomorphism.

Proposition 4. ([15, 16, 20]) Let $A$ be a Noetherian ring, $p$ a prime ideal in $A$. Then the injective hull of $A/p$ is indecomposable with respect to direct sum. Further, any indecomposable injective $A$-module is the injective hull of $A/p$ for some prime $p$.

It is an easy exercise to show that, for any prime $p$ in $A$, the injective hull of $A/p$ is the quotient field of $A/p$, taken as an $A$-module. But [12] allows all such constructions on the spectrum level, so long as all primes are generated by regular sequences. Thus, when the primes of $A$ are generated by regular sequences, all injectives can be realized. Note that this is true without restricting the dimension of the Noetherian ring $R_*$.

Proposition 5. Let $R$ be an $S$-algebra such that $R_*$ is a Noetherian ring with all primes generated by regular sequences. Then any injective $R_*$-module $I_*$ can be realized as the $R_*$-module of homotopy groups of some $R$-module $Y$: $\pi_*(Y) = Y_* = I_*$. For any other $R$-module $X$, $[X,Y]_*^R \cong \text{Hom}_{R_*}(X^*, I^*)$.

Proof. The vanishing of higher Ext groups shows that any map into an injective module is uniquely realizable. □

6. Realization of modules of dimension at most one

Given an $R_*$-module $M_*$ of projective dimension one, there is a projective resolution

$$0 \rightarrow P_1 \xrightarrow{\alpha} P_0 \rightarrow M_* \rightarrow 0$$

and $M_*$ can be realized as the homotopy cofiber of the (unique) realization of $\alpha$. Given an $R_*$-module $M_*$, let $|M_*|$ denote an $R$-module with homotopy $M_*$. In this case, the realization $|M_*|$ is unique up to homotopy since the identity map on $M_*$ lifts to a comparison of resolutions

$$0 \rightarrow P_1 \xrightarrow{\alpha} P_0 \rightarrow M_* \rightarrow 0$$

and

$$0 \rightarrow P'_1 \xrightarrow{\alpha'} P'_0 \rightarrow M_* \rightarrow 0.$$
Since the two maps from $P_1$ to $P'_0$ are the same in algebra, their realizations in topology are the same in homotopy and we get an equivalence between the two realizations of $M_\ast$.

When $R_\ast$ is a Noetherian ring with all primes generated by regular sequences, we can use the dual construction to realize any module of injective dimension at most one.

Note that, without further hypotheses, we can describe the Hom set between two $R$-modules of dimension at most one only up to extension:

$$0 \rightarrow \text{Ext}^{1,-1}_{R^*}(X^\ast, Y^\ast) \rightarrow [X,Y]_0^R \rightarrow \text{Hom}_{R^*}(X^\ast, Y^\ast) \rightarrow 0.$$  

The difference of any two maps $X \rightarrow Y$ with the same effect in homotopy is measured by an element of $\text{Ext}^{1,-1}_{R^*}(X^\ast, Y^\ast)$. Of course, only one module need be of (projective or injective, depending on the variable) dimension at most one.

Composition is given by the naturality of the EKMSS and the fact that the product in Ext corresponds to the composition product on the associated graded of the Hom sets in the derived module spectrum category.

In the case where $R_\ast$ is of global dimension at most one, this is an almost complete algebraic description of the category, including the corollary that an $R$-module is determined by its homotopy groups as an $R_\ast$-module. One example, mentioned above, is periodic $K$-theory.

Let $R$ be an $S$-algebra with $R_\ast$ of global dimension at most one and concentrated in even degrees. Since realizations of $R_\ast$-modules are unique up to homotopy, any $R$-module $X$ is the wedge of its even and odd pieces: $X = X_{\text{even}} \vee X_{\text{odd}}$. Let $Y$ be another $R$-module. By additivity, we can assume $X$ and $Y$ are each concentrated in either even or in odd degrees. The group $[X,Y]^R_0$ depends on the relative parities of $X$ and $Y$. Using the suspension functor, it suffices to calculate $[X,Y]^R_0$.

If $X$ and $Y$ are both even (or both odd), then note that $\text{Ext}^{1,-1}_{R^*}(X^\ast, Y^\ast)$ is zero.

If the parities of $X$ and $Y$ differ, then $\text{Hom}_{R^*}(X^\ast, Y^\ast) = 0$ and

$$[X,Y]^R_0 \cong \text{Ext}^{1,-1}_{R^*}(X^\ast, Y^\ast).$$

More generally, we have the following theorem.

**Theorem 6.** For an $S$-algebra $R$ of global dimension at most one with $R_\ast$ concentrated in degrees congruent to zero mod $k$, $k > 1$, each $R$-module $X$ splits as the wedge of $k$ pieces $X = \vee_{j=1}^k X_j$ such that $X_{j_\ast}$ is concentrated in degrees congruent to $j$ mod $k$. For two $R$-modules $X$ and $Y$,

$$[X,Y]_{ij}^R \cong \begin{cases} \text{Hom}_{R^*}(X_{i\ast}^\ast, Y_{j\ast}^\ast) & i = j, \\ \text{Ext}^{1,-1}_{R^*}(X_{i\ast}^\ast, Y_{j\ast}^\ast) & i + 1 = j, \\ 0 & \text{otherwise.} \end{cases}$$
Proof. There are no non-trivial maps between two modules $X$ and $Y$ if $X$ is concentrated in degrees congruent to $m \mod k$, $Y$ in $n \mod k$, $m \neq n$. Similarly, there are no non-trivial extensions

$$0 \to Y_* \to M \to \Sigma^{-1} X_* \to 0$$

when $m + 1 \neq n$. \qed

7. Classification of $R$-modules of projective dimension at most two

7.1. Realization. The previous sections showed how to realize any $R_*$-module of projective dimension at most one as the homotopy of an $R$-module. When the projective dimension of $M_*$ is two, $M_*$ has a projective resolution

$$0 \to P_2 \to P_1 \to P_0 \to M_* \to 0,$$

which can be split into two short exact sequences

$$0 \to P_2 \to P_1 \to K \to 0,$$

$$0 \to K \to P_0 \to M_* \to 0,$$

where $K = \ker (P_0 \to M_*)$. Using the EKMSS, any $R_*$-module of projective dimension two can be realized as the homotopy of some $R$-module: The modules $K$ and $P_0$, as well as the map whose cokernel is $M_*$, can be realized as in the above section. One realization of $M_*$ is the cofiber of the realization of the map $K \to P_0$.

Now, when $R_*$ has global dimension at most 2, $\text{Ext}_{R_*}^{s,t}(M^*, N^*)$ vanishes for $s > 2$. Thus the only possible non-trivial differentials are

$$d_2 : \text{Ext}_{R_*}^{0,t}(M^*, N^*) \to \text{Ext}_{R_*}^{2,t-1}(M^*, N^*),$$

the spectral sequence collapses at $E_3$, and a map $\theta : M_* \to N_*$ of $R_*$-modules yields an obstruction $d_2(\theta) \in \text{Ext}_{R_*}^{2,t-1}(M^*, N^*)$ which vanishes if and only if $\theta$ can be realized as the effect in homotopy of a map $X \to Y$ of $R$-modules with $\pi_* X \cong M_*$ and $\pi_* Y = N_*$. 

7.2. The Difference Between Two Realizations of an $R_*$-module. A realization of an $R_*$-module $M_*$ consists of an $R$-module $X$ and an isomorphism

$$\alpha : \pi_*(X) = X_* \xrightarrow{\sim} M_*.$$

Following Bousfield, two realizations $(X, \alpha)$ and $(Y, \beta)$ are said to be strictly equivalent if there is an equivalence $f : X \simeq Y$ with $\beta f_* = \alpha$. Then $D(\alpha, \beta) = \beta d_2(\beta^{-1} \alpha)\alpha^{-1}$ defines the difference in $\text{Ext}_{R_*}^{2,t-1}(M^*, M^*)$. This difference satisfies:

1. $D(\alpha, \alpha) = 0$;
2. $D(\alpha, \beta) = 0$ iff $\alpha$ is strictly equivalent to $\beta$;
3. $D(\alpha, \gamma) = D(\alpha, \beta) + D(\beta, \gamma)$;
4. $D(\alpha, \beta) = -D(\beta, \alpha)$;
5. $D(g\alpha, g\beta) = gD(\alpha, \beta)g^{-1}$ for each $g \in \text{Aut} M_*$. 


Call the collection of all strict equivalence classes of realizations of $M_*$ $\mathcal{R}(M_*)$. Adapting the methods of Bousfield [8], $\mathcal{R}(M_*)$ can be determined algebraically.

**Theorem 7.** For each $R_*$-module $M$ and realization $\alpha: X_* \twoheadrightarrow M_*$, the difference function gives a bijection $D(\alpha, -): \mathcal{R}(M_*) \cong \text{Ext}^{2}_{R_*}(M^*, M^*)$.

**Proof.** Injectivity of the difference function follows from the above properties; only surjectivity needs proof. Let

$$0 \rightarrow K \rightarrow P_0 \xrightarrow{\varepsilon} M_* \rightarrow 0$$

be a short exact sequence, where $P_0$ is projective and $K$ is the kernel of $\varepsilon$. The map $\varepsilon$ can be chosen so that $X$ is the cofiber of the realization of the map $K \rightarrow P_0$. Identify $X$ with $M_*$ by the isomorphism $\alpha$.

Any element $u \in \text{Ext}^{2}_{R_*}(M^*, M^*) = \text{Ext}^{1}_{R_*}(K, M^*)$ lifts to some $\pi \in \text{Ext}^{1}_{R_*}(K, P_0)$, since $K$ has projective dimension at most 1; the EKMSS then yields a (unique) map $\pi: |K| \rightarrow |P_0|$, where $|J|$ is a realization of the $R_*$-module $J$. If the cofibration

$$\Sigma^{-1}X \rightarrow |K| \xrightarrow{f} |P_0| \rightarrow X$$

gives the realization $\alpha$, then

$$\Sigma^{-1}Y \rightarrow |K| \xrightarrow{f + \pi} |P_0| \rightarrow Y$$

yields a realization $\beta: Y_* \twoheadrightarrow M_*$ with $D(\alpha, \beta) = u$. \qed

This strict equivalence of realizations is a finer equivalence than homotopy type; $\text{Aut}(M_*)$ acts on $\mathcal{R}(M_*)$ by composition. There is, however, a forgetful bijection from the orbit set $\mathcal{R}(M_*)/\text{Aut}(M_*)$ to the set of realizations of $M_*$. A crossed homomorphism $d: G \rightarrow B$ for a group $G$ acting on an abelian group $B$ is a function with $d(gh) = d(g) + g \cdot d(h)$; the associated crossed homomorphism action $G \times B \rightarrow B$ carries $(g, b)$ to $d(g) + g \cdot b$. Now, $\text{Aut}(M_*)$ acts on $\text{Ext}^{2}_{R_*}(M^*, M^*)$ by $g \cdot u = gug^{-1}$; given a realization $\alpha$ of $M^*$, we have a crossed homomorphism $d: \text{Aut}(M^*) \rightarrow \text{Ext}^{2}_{R_*}(M^*, M^*)$ given by $d(g) = D(\alpha, g\alpha)$. The associated crossed homomorphism action of $\text{Aut}(M_*)$ on the Ext group corresponds to the composition action of $\text{Aut}(M_*)$ on $\mathcal{R}(M_*)$ by the bijection of the above theorem.

Thus, we have the following theorem.

**Theorem 8.** The homotopy types of $R$-modules with homotopy $M_*$ are in bijective correspondence with the elements of $\text{Ext}^{2}_{R_*}(M^*, M^*)/\text{Aut}(M_*)$.

This classification is *not* purely algebraic, however, since in general we lack an algebraic description of the $d_2$ differential of the EKMSS. One possible approach toward a more algebraic classification is to use Toda brackets. Alternatively, given a functorial realization of $R$-modules with zero differential in the EKMSS, the methods of [7] give an easier way to construct a category equivalent to the derived category of $R$-modules. We describe these next.
7.3. Sparse Graded Rings. Let $R$ be an $S$-algebra of global dimension at most two with $R_*$ concentrated in degrees congruent to zero mod $k$, $k \geq 2$. Examples of such $S$-algebras are $ku$ and $ko(p)$ for an odd prime $p$; since $\eta$ is null after inverting $2$, $\pi_* ko(p) = \mathbb{Z}[u], |u| = 4$. Any $R_*$-module $M_*$ is then the homotopy of a wedge $X = \bigvee_{i=1}^{\infty} X_i$ with the homotopy of $X_i$ concentrated in degrees congruent to $i$ mod $m$. Call this the wedge realization of $M_*$. By the classification of $R$-modules, each wedge summand is unique up to homotopy, since any $N_*$ concentrated in degrees congruent to $i$ mod $m$ can be resolved by modules concentrated in the same degrees mod $m$.

The analysis of [7] applies to this situation; the category of $R_*$-modules has a completely algebraic structure. The key is to note that there is a functorial realization of any $R_*$-module as an $R$-module for which the differentials in the EKMSS between any two such $R$-modules are all zero: Realize any $R_*$-modules $M_*, N_*$ as wedges $X$ and $Y$ as above. Since the wedge realization is homotopy unique, there is a right inverse to the map $[X,Y]_0 \longrightarrow \text{Hom}_{R^*}(M^*, N^*)$ given by the natural realizations of maps between the wedge summands of like degree together with the zero map between non-matching summands. Thus, the map of Hom sets is onto, and the differential is necessarily zero.

The next step is to construct a Bousfield $k$-invariant to measure the difference between a given $R$-module and these realizations with trivial EKMSS differential. Let $X$ be an $R$-module. Define

$$k_X \in E^{2,1}_2(X, X) \cong \text{Ext}^{2,1}_R(X^*, X^*)$$

as follows: Let $X'$ be an $R$-module with homotopy isomorphic to $X_*$, equivalent to the wedge realization above. Choose an isomorphism $\alpha: X'_* \longrightarrow X_*$. Define $k_X = (d_2\alpha)\alpha^{-1}; \alpha \in E^2_2(X', X), \alpha^{-1} \in E^2_2(X, X')$. The element $k_X$ is independent of the choice of $X'$ and $\alpha$, for if $X''$ is another choice of $X'$, with $\beta$ an isomorphism from $X''$ to $X$, choose an isomorphism $\gamma: X'' \longrightarrow X_*$ such that $\beta = \alpha \gamma$. Since $X''$ and $X'$ are equivalent to wedge realizations, $d_2(\gamma) = 0$, with $d_2(\beta)\beta^{-1} = d_2(\alpha)\gamma^{-1}\alpha^{-1} = d_2(\alpha)\alpha^{-1}$. Thus, $k_X$ is well-defined.

The differential in the EKMSS can now be expressed algebraically.

**Proposition 9.** Let $R$ be an $S$-algebra of global dimension at most two with $R_*$ concentrated in degrees congruent to zero mod $m$, $m \geq 2$. For $R$-modules $X$ and $Y$, the EKMSS differential

$$d_2: E^{0,1}_2(X, Y) \longrightarrow E^{2,1}_2(X, Y)$$

is given by $d_2 f = k_Y f + (-1)^{t-1} f k_X$ for each $f \in E^{0,1}_2(X, Y)$.

**Proof.** The proof is exactly as in [7], Proposition 8.10. Let $X'$ denote the wedge realization of $X_*$. For isomorphisms $\alpha: X'_* \longrightarrow X_*$ and $\beta: Y'_* \longrightarrow Y_*$, consider $\alpha$ and
β as elements of the $E_2$ term: $\alpha \in E_2^{0,0}(X', X)$, $\alpha^{-1} \in E_2^{0,0}(X, X')$, $\beta \in E_2^{0,0}(Y', Y)$. Choose $f' \in E_2^{0,t}(X', Y')$ such that $\beta f' \alpha^{-1} = f \in E_2^{0,t}(X, Y)$. Since $d_2 f' = 0$,

$$
\begin{align*}
    d_2 f &= d_2(\beta f' \alpha^{-1}) = (d_2 \beta) f' \alpha^{-1} + (-1)^t \beta f'(d_2 \alpha^{-1}) \\
         &= (d_2 \beta) \beta^{-1} \beta f' \alpha^{-1} + (-1)^t \beta f' \alpha^{-1} \alpha (d_2 \alpha^{-1}) \\
         &= \alpha (d_2 \alpha^{-1} - (d_2 \alpha) \alpha^{-1} = d_2 (1) - k_X = -k_X.
\end{align*}
$$

Now,

$$
\alpha (d_2 \alpha^{-1}) = d_2 (\alpha \alpha^{-1}) - (d_2 \alpha) \alpha^{-1} = d_2 (1) - k_X = -k_X
$$

and this proves the result. \(\square\)

This allows us to prove the following characterization of maps in the $R$-module category.

**Corollary 10.** For $R$-modules $X$ and $Y$, a homomorphism $f$ of degree $t$ from $X_*$ to $Y_*$ is the homotopy of a map of $R$-modules if and only if $k_Y f = (-1)^t f k_X$.

Now let $\mathbb{M}k$ denote the category of pairs $(M_*, k)$, with $M_*$ an $R_*$-module and $k$ in $\operatorname{Ext}^{2,-1}_R(M_*, M_*)$; morphisms $f$ from $(M_*, k)$ to $(N_*, k')$ satisfy $k' f = f k$ in $\operatorname{Ext}^{2,-1}_R(M_*, N_*)$.

**Theorem 11.** Let $R$ be an $S$-algebra of global dimension at most two, with $R$ concentrated in degrees congruent to zero mod $m$, $m \geq 2$. Then for any $(M_*, k)$ in $\mathbb{M}k$, there is an $R$-module $Y$ such that $(M_*, k) = (Y_*, k_Y)$. Thus, the homotopy types of $R$-modules correspond to isomorphism types in $\mathbb{M}k$.

**Proof.** This is essentially the same as [7], Theorem 9.1. We already have a full additive functor from $R$-modules to $\mathbb{M}k$, as noted above. The wedge realization provides a realization of any $R_*$-module with zero $k$-invariant. That any $k$-invariant can be obtained can be seen by lifting $\operatorname{Ext}$ elements, as in the proof of the more general classification above. Again, let $X'$ denote the wedge realization of $X_*$. Given $(M_*, k)$ in $\mathbb{M}k$, let

$$
0 \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} M_* \longrightarrow 0
$$

be a projective resolution of $M_*$. Let $K = \ker \epsilon$; the exact sequence splits into two short exact sequences

$$
0 \longrightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{a} K \longrightarrow 0, \quad 0 \longrightarrow K \xrightarrow{b} P_0 \xrightarrow{\epsilon} M_* \longrightarrow 0.
$$

Given any $\xi : K' \to P'_0$ such that $\xi_* = 0$, the homotopy cofiber of $b + \xi$ gives a realization $Y_\xi$ of $M_*$. It suffices to choose $\xi$ so that $k_{Y_\xi} = k$; $Y_\xi$ is then the desired realization of $(M_*, k)$.

Any element $u \in \operatorname{Ext}^{2,-1}_R(M_*, M_*) = \operatorname{Ext}^{1,-1}_R(K, M_*)$ lifts to some $\overline{u} \in \operatorname{Ext}^{1,-1}_R(K, P_0)$, the group which classifies maps from $|K|$ to $|P_0|$ which are zero on homotopy groups. Any such $\overline{u}$ is equivalent to a map $\xi$ as above, since $K$ has
projective dimension at most one. If $b$ is the map which gives the wedge realization $Y'$ of $M_\ast$, then $b + \pi$ yields the desired $Y_\xi$.  \\It is possible to construct additive bigraded categories of $R$-modules and of $R_\ast$-modules paired with $k$-invariants. These categories are additively equivalent, as in [7].

Note that this analysis requires only a natural realization of $R_\ast$-modules as $R$-modules with zero differential in the EKMSS, which is a weaker condition than sparseness.

8. Modules over additive categories

In order to use algebra to classify modules over the real $K$-theory $S$-algebras $ko$ and $KO$, it is necessary to generalize the algebraic concept of a module to that of a functor over an appropriate generalization of a ring.

A ring can be considered [19] as a suitably structured category with only one object. Much homological algebra generalizes to “modules” over a small additive category $\mathcal{C}$, specifically, additive functors from $\mathcal{C}$ to abelian groups; the morphisms are the natural transformations. One can consider the Hom sets in $\mathcal{C}$ as giving additive operations on the modules. The category $\mathcal{A}b^\mathcal{C}$ of additive functors from $\mathcal{C}$ to $\mathcal{A}b$ is actually equivalent to a category of modules over an honest ring; we will, however, keep our additive category while also considering our functors naively as sets of modules with operations.

Since our functor category is abelian, we have kernels and cokernels and can define exact sequences and resolutions. Although the definition of Ext is well-known for any abelian category, this section outlines the exposition of [19] in order to have an explicit description of the derived functors.

As an example, a graded ring $R = \{R_n\}_{n \in \mathbb{Z}}$ is an additive category $\mathcal{C}$ with object set $\mathbb{Z}$ and morphisms $\mathcal{C}(m, n) = R_{n-m}$; composition is given by multiplication in the ring. Note that it only makes sense to add elements of the same degree, as is standard for topologists. Then, $\mathcal{A}b^\mathcal{C}$ is the category of graded $R$-modules.

8.1. The Yoneda lemma. Given a functor $F : \mathcal{C} \to \mathcal{A}b$, and an object $X$ of $\mathcal{C}$, the isomorphism
\[
\text{Nat}(\mathcal{C}(X, -), F) \cong FX
\]
sends each natural transformation to the image of the identity map on $X$. This isomorphism is natural in $X$ and $F$. In terms of $\mathcal{A}b^\mathcal{C}$, we have the isomorphism $\mathcal{A}b^\mathcal{C}(\mathcal{C}(X, -), F(-)) \cong F(X)$.

8.2. Projective and injective functors. An object (i.e. functor) $F$ of $\mathcal{A}b^\mathcal{C}$ is called projective if $\mathcal{A}b^\mathcal{C}(F, -)$ preserves epis. This is equivalent to requiring that, for a projective $F$ and any epi natural transformation $A \longrightarrow B$, any natural transformation
F \to B lifts to one \( F \to A \); thus, \( \text{Nat}(F, A) \to \text{Nat}(F, B) \). Dually, \( G \) is injective if \( \mathcal{A}b^\mathcal{C}(\_, G) \) preserves monics.

**8.3. Free objects in \( \mathcal{A}b^\mathcal{C} \).** For \( X \) in \( \mathcal{C} \), \( M \) in \( \mathcal{A}b^\mathcal{C} \), consider \( M \) as the disjoint union of the various \( M^X = M(X) \); an element \( x \) of \( M^X \) is then considered to be an element of \( M \). For an index set \( I \), \( \{x_i\}_{i \in I} \subseteq M \) is a family of generators for \( M \) if for every \( x \in M \) there are objects \( X_i \) of \( \mathcal{C} \) and \( \lambda_i \in \mathcal{C}(X_i, X) \), where only finitely many \( \lambda_i \) are non-zero, such that

\[
x = \sum_{i \in I} M^\lambda_i(x_i).
\]

That is, any element of an abelian group in the image of the functor \( M \) is the sum of the images of the \( x_i \) under maps which are in the image of \( M \). Equivalently, the natural transformation

\[
\bigoplus_{i \in I} \mathcal{C}(X_i, \_) \to M
\]

taking \( 1_{X_i} \) to \( x_i \) should be an epi in \( \mathcal{A}b^\mathcal{C} \); when this natural transformation is iso, the family \( \{x_i\} \) is called a basis for \( M \). This is true exactly when the \( \lambda_i \) above are unique. In this case, \( M \) is called free. Thus, the free objects in \( \mathcal{A}b^\mathcal{C} \) are exactly the sums of representable functors from \( \mathcal{C} \) to \( \mathcal{A}b \). When the objects of \( \mathcal{C} \) are graded, then all sums of suspensions of representable functors are free.

**8.4. A Hom functor.** For an additive category \( \mathcal{C} \), we have the symbolic Hom functor

\[
\text{Hom}_\mathcal{C} : (\mathcal{A}b^\mathcal{C})^{\text{op}} \otimes \mathcal{A}b^\mathcal{C} \to \mathcal{A}b.
\]

This is, up to natural equivalence, the unique limit preserving functor satisfying

\[
\text{Hom}_\mathcal{C}(\mathcal{C}(X, \_), F(-)) = F(X) = \mathcal{A}b^\mathcal{C}(\mathcal{C}(X, \_), F(-)),
\]

the last isomorphism by the Yoneda lemma. The Kan extension theorem [19] then gives that \( \text{Hom}_\mathcal{C}(G, F) = \mathcal{A}b^\mathcal{C}(G, F) \) for any \( F \) and \( G \).

This Hom functor has a left adjoint

\[
\otimes_\mathcal{C} : \mathcal{A}b^\mathcal{C} \otimes \mathcal{A}b^{\mathcal{C^{\text{op}}}} \to \mathcal{A}b
\]

which satisfies

\[
F \otimes_\mathcal{C} \mathcal{C}(\_, X) = F(X).
\]

That is, \( (\_) \otimes_\mathcal{C} G \) is a left adjoint to \( \text{Hom}_\mathcal{C}(G, \_) \).
8.5. Derived functors. For $F$ and $G$ in the functor category $\mathcal{Ab}$, define

$$\text{Ext}^n_{\mathcal{C}}(G, F) = H^n(\text{Hom}_{\mathcal{C}}(X, F))$$

where $X$ is a projective resolution of $G$. The standard properties of Ext hold, and since coproducts in $\mathcal{Ab}$ preserve monics, Ext can be computed by simultaneously resolving both variables, with $F$ resolved by an acyclic right complex.

Similarly, $\text{Tor}^n_{\mathcal{C}}(F, G) = H^n(F \otimes_{\mathcal{C}} Y)$, where $Y$ is a projective resolution of $G$ in $\mathcal{Ab}_{\mathcal{C}op}$, gives the derived functors of $\otimes_{\mathcal{C}}$.

9. United homology theories

Bousfield, in [8, 9], defines an additive category $\text{CRT}$ (which he calls $\text{Alg}(\text{CRT})$) and the corresponding category of $\text{CRT}$-modules. Bousfield’s method generalizes to other united homology theories such as the one below, as well as to united cohomology theories.

9.1. United homology theories. Let $R$ be an $S$-algebra and $\mathcal{F}$ some collection of finite spectra. Define the category $R\mathcal{F}$ to be the category with objects

$$\{R \wedge F_\alpha | F_\alpha \in \mathcal{F}\}$$

and morphisms all homotopy classes of $R$-module maps between any pair of objects. $R\mathcal{F}$ is an additive category which yields two united homology theories: First, let $M = R^\mathcal{F}(X)$ denote the united theory on spectra $X$ given by the functors from $R\mathcal{F}$ to abelian groups with

$$M^F = M(R \wedge F) = \pi_*(R \wedge F \wedge X).$$

Second, there is a united theory on $R$-modules $X$ given by the functors $N = \pi_*^\mathcal{F}(X)$ from $R\mathcal{F}$ to abelian groups with

$$N^F = N(R \wedge F) = \pi_*(X \wedge_R (R \wedge F)) \cong \pi_*(X \wedge F).$$

This has a naive variant. If $R$ is merely a ring spectrum, then the maps in $R\mathcal{F}$ should be all maps in the derived category of naive module spectra. The theory $R_*^\mathcal{F}(X)$ is as defined above; $N = \pi_*^\mathcal{F}(X)$ is defined by $N^F = \pi_*(X \wedge F)$.

The first type of united theory is used by Bousfield [8] to classify $K_*$-local spectra. The second type is used below to classify both $KO$-modules and $ko$-modules.

Since $[R \wedge F, R \wedge F]^R_* \cong [S, R \wedge F']_* \cong [S, R \wedge F' \wedge DF]_*$, the Spanier-Whitehead duals of the finite spectra in $\mathcal{F}$ yield the representable functors (free objects). Thus the free objects under such a theory are given by the united homology of all suspensions of the Spanier-Whitehead duals of the finite spectra in $\mathcal{F}$. Note that the objects of $R\mathcal{F}$ need only be semi-finite as $R$-modules when $R$ is an $S$-algebra, that is, $D^2_R F \simeq F$ for $F$ in $R\mathcal{F}$.

A united module, or $R\mathcal{F}$-module, $M$ is called $\mathcal{F}$-acyclic if it takes cofibrations of $R$-modules in $R\mathcal{F}$ to long exact sequences.
9.2. Connective united $K$-theory. This is described in more detail in the next section. The obstacle to using the same techniques as for $ku$ to classify $ko$-modules is precisely that $ko$ has infinite global dimension. By killing nilpotent elements, we obtain a theory $R\mathcal{F} = \text{crt}$ with global dimension 2 and it suffices to construct an appropriate spectral sequence in order to obtain the desired classification. An appropriate choice of finite spectra is $S$, $C(\eta)$, and $C(\eta^2)$, yielding $ko$, $ku$, and $kt$.

One can check that using only $S$ and $C(\eta)$ over $ko$ yields an algebraic category with infinite homological dimension; adding $C(\eta^2)$, or $kt$, eliminates this difficulty.

9.3. Periodic united $K$-theory. The inspiration for studying connective united $K$-theory came from Bousfield’s use of a periodic version, $\text{CRT}$, in [8]; the categories $\text{crt}$ and $\text{CRT}$ use the same finite spectra $S$, $C(\eta)$, and $C(\eta^2)$. The periodic theory has global dimension one. Bousfield adjusts the category of modules by adding Adams operations to classify $K_*$-local spectra. The united category without operations classifies $KO$-modules.

10. The operation algebra for connective united $K$-theory

10.1. Anderson cofibrations with spheres. To calculate the algebra of operations, we use the cofibration sequences of [2, 8] ($\eta$ denotes the Hopf map)

$$
\Sigma S \xrightarrow{\eta} S \xrightarrow{c} C(\eta) \xrightarrow{r} \Sigma^2 S,
$$

$$
\Sigma^2 S \xrightarrow{\eta^2} S \xrightarrow{\varepsilon} C(\eta^2) \xrightarrow{k} \Sigma^3 S,
$$

and

$$
C(\eta^2) \xrightarrow{h} C(\eta) \xrightarrow{\varphi} \Sigma^2 C(\eta) \xrightarrow{j} \Sigma C(\eta^2),
$$

which, when smashed with $ko$, give cofibration sequences relating $ko$, $ku$, and $kt$.

First, describe and fix names for maps among $S$, $C(\eta)$, and $C(\eta^2)$:

- $\eta: S^1 \to S$ is the Hopf map, the non-zero element of $\pi_1 S$;
- $c: S \to C(\eta)$, $\varepsilon: S \to C(\eta^2)$ are given by inclusion of the zero cell;
- $h: C(\eta^2) \to C(\eta)$ is any map of degree one on the zero cells;
- $j: C(\eta) \to \Sigma^{-1} C(\eta^2)$ is degree one on the 2-cells;
- $k: \Sigma^{-1} C(\eta^2) \to S^2$ is degree one on the 2-cells;
- $r: C(\eta) \to S^2$ is top cell projection;
- $\psi_1 = \psi: C(\eta) \to C(\eta)$ and $\psi_2 = \psi: C(\eta^2) \to C(\eta^2)$ are degree one on 0-cells and degree −1 on top cells;
- $\varphi: C(\eta) \to \Sigma^2 C(\eta)$ kills the zero cell and sends the 2-cell to the bottom cell of $\Sigma^2 C(\eta)$ by a degree one map, as $C(\eta) \xrightarrow{1-\psi} S^2 \xrightarrow{\varepsilon} \Sigma^2 C(\eta)$.
Note that the definitions of $h$ and $r$ imply that the diagrams

\[
\begin{array}{c}
S \xrightarrow{\varepsilon} C(\eta^2) \\
\downarrow c \\
C(\eta)
\end{array} \quad \text{and} \quad \begin{array}{c}
C(\eta) \xrightarrow{j} \Sigma^{-1}C(\eta^2) \\
\downarrow r \\
S^2
\end{array}
\]

commute.

10.2. Anderson cofibrations of $ko$-modules. The maps above induce, upon smashing with the identity on $ko$,

- $\eta: \Sigma ko \rightarrow ko$;
- $c: ko \rightarrow ku$ and $\varepsilon: ko \rightarrow kt$, unit maps;
- $\zeta: kt \rightarrow ku$, with $c = \zeta \varepsilon$;
- $\gamma: ku \rightarrow \Sigma^{-1}kt$, from $j$;
- $\tau: \Sigma^{-1}kt \rightarrow \Sigma^2ko$, from $k$;
- $r: ku \rightarrow \Sigma^2ko$;
- $\psi_U = \psi: ku \rightarrow ku$ and $\psi_T = \psi: kt \rightarrow kt$;
- $\varphi: ku \rightarrow \Sigma^2ku$.

Note that some of the degrees of maps differ from those of the same name in Bousfield’s periodic version [8]; the difference comes from the lack of an inverse to the Bott element. To simplify the number of generating operations, note that $r = \tau \gamma$ and $c = \zeta \varepsilon$. The map $c$ is complexification; $r$ is realification.

The following cofibrations are obtained:

\[
\begin{align*}
\Sigma ko & \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{r} \Sigma^2 ko, \\
\Sigma^2 ko & \xrightarrow{\eta^2} ko \xrightarrow{\varepsilon} kt \xrightarrow{\Sigma \tau} \Sigma^3 ko, \\
kt & \xrightarrow{\zeta} ku \xrightarrow{\varphi} \Sigma^2 ku \xrightarrow{\Sigma^2 \gamma} \Sigma kt.
\end{align*}
\]

Here, $ku$ is merely shorthand for $ko \wedge C(\eta)$ and similarly for $kt$, however it is possible to prove that these are equivalent as $ko$-modules to the usual $ku$ and $kt$ by the classification below (Theorem 17). There are other proofs of this fact as well, for example, using the homology of $ko$ and $ku$ [Bruner, personal correspondence].

10.3. Determining the operations. Recall the ring structures of $ko_*$, $ku_*$, and $kt_*$:

$ko_* = \mathbb{Z}[\eta, \omega, \beta_O]/(2\eta, \eta^3, \eta \omega, \omega^2 - 4\beta_O)$, $|\eta| = 1$, $|\omega| = 4$, $|\beta_O| = 8$;

$ku_* = \mathbb{Z}[\beta_U]$, $|\beta_U| = 2$; and

$kt_* = \mathbb{Z}[\eta, \xi, \beta_T]/(2\eta, \eta^2, \eta \xi, \xi^2)$, $|\eta| = 1$, $|\xi| = 3$, $|\beta_T| = 4$. 
In the world of \( ko \)-modules, \( X_* = [ko, X]^ko_* \), so these rings give all maps from \( ko \) to each of the three objects. Explicitly, for \( n \geq 0 \) (other groups are all zero),

\[
[ko, ko]_{3n}^{ko} = \langle \beta^3_O \rangle = \mathbb{Z} \\
[ko, ko]_{3n+1}^{ko} = \langle \beta^n_O \eta \rangle = \mathbb{Z}/2 \\
[ko, ko]_{3n+2}^{ko} = \langle \beta^n_O \eta^2 \rangle = \mathbb{Z}/2 \\
[ko, ko]_{3n+4}^{ko} = \langle \beta^n_O \omega \rangle = \mathbb{Z}
\]

\( [ko, ko]_{3n+i}^{ko} = 0, \ i \in \{3, 5, 6, 7\} \)

\[
[ko, ku]_{2n}^{ko} = \langle \beta^n_U \rangle = \mathbb{Z} \\
[ko, ku]_{2n+1}^{ko} = 0 \\
[ko, kt]_{4n}^{ko} = \langle \beta^n_T \rangle = \mathbb{Z} \\
[ko, kt]_{4n+1}^{ko} = \langle \beta^n_T \eta \rangle = \mathbb{Z}/2 \\
[ko, kt]_{4n+2}^{ko} = 0
\]

Except where noted, the maps given below are proven to be generators by looking at the action on homotopy groups.

Since \( ku_*(\eta) = 0 \) and the duals of \( C(\eta) \) and \( C(\eta^n) \) are \( DC(\eta) = \Sigma^{-2}C(\eta) \) and \( DC(\eta^n) = \Sigma^{-3}C(\eta^n) \), the cofibrations in 10.2 give

\[
[ku, ku]_{n}^{ko} = ku_{n+2}(C(\eta)) = ku_{n+2} \oplus ku_{n} = \begin{cases} 
\mathbb{Z} = \langle \phi \rangle & n = -2, \\
\mathbb{Z} \oplus \mathbb{Z} = \langle \beta^k_U, \beta^k_T \psi \rangle & n = 2k, \ k \geq 0, \\
0 & \text{otherwise};
\end{cases}
\]

Similarly, since \( kt_{n+2}(C(\eta)) \cong ku_{n+2}(C(\eta^n)) = ku_{n+2} \oplus ku_{n-1} \)

\[
[ku, ku]_{n}^{ko} = ku_{n+3}(C(\eta^n)) = \begin{cases} 
\mathbb{Z} = \langle \gamma \beta^k_U \rangle & n = 2k - 3, \ k \geq 0, \\
\mathbb{Z} = \langle \beta^k_U \rangle & n = 2k, \ k \geq 0, \\
0 & \text{otherwise};
\end{cases}
\]

and since \( kr_*(\eta^n) = 0 \), \( kr_{n+3}(C(\eta^n)) = kr_{n+3} \oplus kr_{n} \),

\[
[ku, ku]_{n}^{ko} = \begin{cases} 
\mathbb{Z} = \langle \epsilon \tau \rangle & n = -3, \\
\mathbb{Z}/2 = \langle \beta^1_T \eta \epsilon \tau \rangle & n = 4k - 2, \ k \geq 0, \\
\mathbb{Z} \oplus \mathbb{Z} = \langle \beta^k_T, \beta^k_T \psi \rangle & n = 4k, \ k \geq 0, \\
\mathbb{Z} \oplus \mathbb{Z}/2 = \langle \beta^{k+1}_T \epsilon \tau, \beta^k_T \eta \rangle & n = 4k + 1, \ k \geq 0, \\
\mathbb{Z} = \langle \beta^k_T \xi \rangle & n = 4k + 3, \ k \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
For $[kt, kt]_{k_0}$, note that it suffices to prove that $\eta \epsilon \tau$ is essential. But this is clear: replace $S$-cells in $\eta \epsilon k: C(\eta^2) \to \Sigma^2 C(\eta^2)$ with $ko$-cells; one sees the composition is not null.

10.4. Relations between operations. Note that $\omega = \tau \beta_T \gamma \beta_U \zeta \epsilon$ and $\xi = \gamma \beta_U \zeta$, further reducing our generating maps.

The effect of the various maps from $ko$ on homotopy is clear from the above ring structure; for example, $\eta f = f \eta$ for any map $f$ (when the composition makes sense).

Composing these maps with those in 10.2, we obtain the following relations:

$$
\begin{align*}
\psi_U \psi_U &= 1, & \psi_T \psi_T &= 1, & \psi_T \epsilon &= \epsilon, \\
\psi_U \beta_U &= -\beta_U \psi_U, & \psi_T \beta_T &= \beta_T \psi_T, & \psi_T \xi &= -\xi, \\
\psi_T \eta &= \eta, & \psi_U \zeta &= \zeta \psi_T &= \zeta, & \beta_U \varphi &= 1 - \psi_U \\
\psi_T \gamma &= -\gamma \psi_U = \gamma, & \tau \psi_T &= -\tau, & \epsilon \beta_O &= \beta_T^2 \epsilon, \\
\beta_T \gamma &= \gamma \beta_U, & \beta_T \gamma &= \gamma \beta_U, & \tau \beta_T^2 &= \beta_O \tau, \\
\tau \epsilon &= \eta, & \tau \beta_T \epsilon &= 0, & \zeta \epsilon &= 0, \\
\gamma \beta_U \zeta &= \eta, & \beta_T \epsilon \tau &= \epsilon \tau \beta_T + \eta \beta_T, & \zeta \epsilon \tau \gamma \beta_U &= 1 + \psi_U, \\
\epsilon \tau \gamma \beta_U \zeta &= 1 + \psi_T, & \tau \gamma \beta_U \zeta \epsilon &= 2, & \gamma \beta_U \zeta \epsilon \tau &= 1 - \psi_T,
\end{align*}
$$

with additional relations involving other elements of $ko$, $ku$, and $kt$. These relations are more than sufficient, however, to prove the desired results. The proof that the $crt$-module category has projective dimension two relies on a reduction to the complex part of the module, regarded as a module over $ku$. Knowing some of the relations in the operation algebra is useful, but it is not necessary to know all of them to use the united theory.

10.5. Operations on united modules. Rephrasing the terminology of functors in terms of modules, the algebra of operations on a triple $M = \{M^O_*, M^U_*, M^T_*\}$ of $\mathbb{Z}$-graded abelian groups which form a module over connective united $K$-theory is generated by homomorphisms

$$
\begin{align*}
\beta_O: M^O_* &\to M^{O+8}_*, & \beta_U: M^U_* &\to M^{U+2}_*, & \beta_T: M^T_* &\to M^{T+4}_*, \\
\eta: M^O_* &\to M^{O+1}_*, & \psi_U: M^U_* &\cong M^U_*, & \psi_T: M^T_* &\cong M^T_*, \\
\eta: M^T_* &\to M^{T+1}_*, & \epsilon: M^O_* &\to M^T_*, & \zeta: M^T_* &\to M^U_*, \\
\gamma: M^U_* &\to M^{T+1}_*, & \tau: M^T_* &\to M^{O+3}_*, & \varphi: M^U_* &\to M^{U+2}_*,
\end{align*}
$$

satisfying appropriate relations as in 10.4.

Note that unlike $CRT$, objects in $crt$ need not have periodicity; $\beta_O$, $\beta_U$, and $\beta_T$ need not be isomorphisms.
Define $\Sigma M$ by $(\Sigma M)^X = \Sigma(M^X) = M^X_{s-1}$ for $X$ one of $O$, $U$, or $T$. A crt-module $M$ is crt-acyclic if its operations give rise to long exact sequences analogous to those of 10.2

\[
\begin{align*}
M_{n-1}^O & \xrightarrow{\eta} M_n^O \xrightarrow{c} M_n^U \xrightarrow{r} M_{n-2}^O, \\
M_{n-2}^O & \xrightarrow{\eta^2} M_n^O \xrightarrow{\varepsilon} M_n^T \xrightarrow{\Sigma r} M_{n-3}^O,
\end{align*}
\]

and

\[
M_n^T \xrightarrow{\iota} M_n^U \xrightarrow{\varphi} M_{n-2}^U \xrightarrow{\Sigma^2 \gamma} M_{n-1}^T.
\]

Note that $\pi_*^{crt} X$ is crt-acyclic for any ko-module $X$. A CRT-module is CRT-acyclic if the same sequences are exact (see [8]).

11. A spectral sequence for united homology

The object of this section is to construct a spectral sequence analogous to that of [12]. Let $R$ be an $S$-algebra; $R\mathcal{F}$, a united homology theory over $R$; $M$ and $N$, $R$-modules. This spectral sequence

\[
E_2 = \text{Ext}^s_t(\pi_*^\mathcal{F}(M), \pi_*^\mathcal{F}(N)) \Longrightarrow E_0^{st} \pi_*(F_R(M, N))
\]

converges from Ext over the united theory to the $R_*$-module of homotopy classes of $R$-module maps from $M$ to $N$. Notation, in fact most of the exposition, follows theirs.

When the united theory is small, the algebraic category is abelian, so that derived functors such as Ext are well known. We present an explicit description here which is useful for understanding the realization of united modules and maps between them.

11.1. Realizing Hom-sets of free objects. As noted above, the monogenic free objects for united $\mathcal{F}$-theory are suspensions of $\pi_*^\mathcal{F}(R \wedge DF)$ for $F \in \mathcal{F}$, where $D$ denotes the Spanier-Whitehead dual. Now, the functor

\[
\text{Hom}_\mathcal{F} : (\mathcal{A}b^\mathcal{F})^{op} \otimes \mathcal{A}b^\mathcal{F} \to \mathcal{A}b
\]

satisfies the Yoneda relation $\text{Hom}_\mathcal{F}(\mathcal{F}(R \wedge F, -), M(-)) = M^F$. Thus, for any functor $M$ realizing the united $\mathcal{F}$-theory of an $R$-module $X$,

\[
\text{Hom}_\mathcal{F}(R_*(-), M(-)) = M^S = X_* = \pi_*F_R(R, X), \quad \text{and}
\]

\[
\text{Hom}_\mathcal{F}((R \wedge DF)_*(-), M(-)) = M^F = \pi_*(X \wedge F) = \pi_*F_R(R \wedge DF, X).
\]

It is thus clear that, for an $R$-module $X$ and a representable functor $M$, any element of the Hom set $\text{Hom}_\mathcal{F}(M, \pi_*^\mathcal{F}(X))$ is realizable as a map of $R$-modules. The $R \wedge DF$ become the analogues of spheres in a new cellular theory for $R$-modules.
11.2. Constructing the spectral sequence. Given a (cell) $R$-module $M$, choose a free resolution of $\pi_*^F M$ by $\mathcal{F}$-modules

$$\cdots \longrightarrow F_s \xrightarrow{d_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\epsilon} \pi_*^F M \longrightarrow 0.$$ 

Set $Q_0 = \ker \epsilon$, $Q_s = \ker d_s$ for $s \geq 1$. Let $K_s$ denote the wedge of (de)suspensions of $F_s \in \mathcal{F}$ so that $FK_s = R \wedge K_s$ yields $F_s$ under $\pi_*^F$. Set $M_0 = M$. Inductively construct cofibrations of $R$-modules

$$FK_s \xrightarrow{k_s} M_s \xrightarrow{i_s} M_{s+1} \xrightarrow{j_{s+1}} \Sigma FK_s$$

for $s \geq 0$ such that

1. $k_0$ realizes $\epsilon$ on $\pi_*^F$,
2. $\pi_*^F M_s = \Sigma^s Q_{s-1}$, $s \geq 1$,
3. $k_s$ realizes $\Sigma^s d_s : \Sigma^s F_s \longrightarrow \Sigma^s Q_{s-1}$ on $\pi_*^F$ for $s \geq 1$ (thus $\pi_*^F k_s$ is surjective),
4. $i_s$ induces the zero homomorphism on $\pi_*^F$ for $s \geq 0$,
5. $j_{s+1}$ realizes the inclusion $\Sigma^{s+1} Q_s \longrightarrow \Sigma^{s+1} F_s$ on $\pi_*^F$ for $s \geq 0$.

Note that (iii) gives (iv), (v), and the $s+1$ case of (ii).

Obtain an exact couple by defining

$$D_1^{s,t} := \pi_{-s-t}(F_R(M_s, N)), \quad E_1^{s,t} := \pi_{-s-t}(F_R(FK_s, N)).$$

The cofibrations obtained inductively above induce maps

$$i := (i_s)^* : D_1^{s,t} \longrightarrow D_1^{s-1,t+1}$$

$$j := (k_s)^* : D_1^{s,t} \longrightarrow E_1^{s,t}$$

$$k := (j_{s+1})^* : E_1^{s,t} \longrightarrow D_1^{s+1,t}.$$ 

Since $E_1^{s,t} = \pi_{-s-t}(F_R(FK_s, N)) \cong \text{Hom}_{\mathcal{F}}^{\cdot}(F_s, \pi_*^F N)$ and $d_1 = jk = \text{Hom}_{\mathcal{F}}(d, 1)$,

$$E_2^{s,t} = \text{Ext}_{\mathcal{F}}^{s,t}(\pi_*^F M, \pi_*^F N).$$

Note that the edge homomorphism is induced by the map $FK_0 \longrightarrow M$.

All that remains is to prove appropriate convergence.

11.3. Determining convergence. Let $i^{0,s} : F_R(M_s, N) \longrightarrow F_R(M, N)$ be the map induced by the iterate $M \longrightarrow M_s$; filter $\pi_*(F_R(M, N))$ by

$$F^s(\pi_*(F_R(M, N))) := \text{Image}((i^{0,s})_* : \pi_*(F_R(M_s, N)) \longrightarrow \pi_*(F_R(M, N))).$$

Then the $(s,t)$-th term of the associated bigraded group of $\pi_*(F_R(M, N))$ with this filtration is given by

$$E_{0,t}^{s,t} \pi_*(F_R(M, N)) = F^s \pi_{-s-t}(F_R(M, N))/F^{s+1} \pi_{-s-t}(F_R(M, N)).$$
The group $E_{s,t}^\infty$ is the subquotient of $E_1^{s,t}$ given by $Z_\infty^{s,t}/B_\infty^{s,t}$, with $B_\infty^{s,t} = j(\ker(i^{0,s})_*)$. The definition of the spectral sequence implies that the additive relation $(i^{0,s})_* \circ j^{-1}$ induces an isomorphism

$$E_{s,t}^\infty \cong E_0^{s,t} \pi_*(F_R(M, N)).$$

Note that (iv) above implies $\text{Tel } M_s$ is trivial, so the homotopy limit

$$\text{holim } F_R(M_s, N) \simeq F_R(\text{Tel } M_s, N)$$

is also trivial. The $\text{Lim}^1$ exact sequence for computing $\pi_*(\text{holim } F_R(M_s, N))$ then gives that

$$\text{Lim} \pi_*(F_R(M_s, N)) = 0, \quad \text{and } \text{Lim}^1 \pi_*(F_R(M_s, N)) = 0.$$ 

Thus, the spectral sequence is conditionally convergent in Boardman’s sense [5]. When the category of $\mathcal{F}$-modules has finite projective dimension, the united homology spectral sequence converges strongly. For example, the category of $\text{crt}$-acyclic $\text{crt}$-modules has projective dimension two; the united $\text{crt}$-homology spectral sequence converges strongly.

Exactly the same words used in [12] yield the naturality of this spectral sequence.

11.4. A Yoneda pairing. Consider the pairing

$$F_R(M, N) \wedge_R F_R(L, M) \longrightarrow F_R(L, N).$$

The maps $M \longrightarrow M_s$ induce compatible pairings

$$F_R(M_s, N) \wedge_R F_R(L_r, M) \longrightarrow F_R(M, N) \wedge_R F_R(L_r, M) \longrightarrow F_R(L_r, N),$$

where the $L_r$ are constructed from $L$ in the same manner as the $M_s$ are constructed from $M$. As usual, these pairings induce the required pairing of spectral sequences, giving a Yoneda product on $\text{Ext}$ at $E_2$ and the product induced by composition at $E_\infty$.

11.5. Naive module spectra. Since much of this theory relies only on free objects and extended module spectra, and in particular since the definition of united homology theory only requires a ring spectrum and not an $S$-algebra, one might hope for this spectral sequence to generalize to that wider setting. This, however, is unlikely: the point-set level category is necessary to construct the free resolutions of the $R$-modules; since the homotopy cofiber or fiber of a map of naive module spectra cannot be given an obvious module structure, we cannot make resolutions in the naive module spectrum category.
11.6. A dual spectral sequence. For an $S$-algebra $R$, a united theory $R\mathcal{F}$ has a dual category $DR\mathcal{F}$ found by taking the Spanier-Whitehead dual in the category of $R$-modules of every object and morphism in $R\mathcal{F}$. Note that this satisfies all the properties of $R\mathcal{F}^{op}$: $DR\mathcal{F}$ is equivalent to the opposite category of $R\mathcal{F}$. Thus, we can understand the tensor product as a functor

$$\otimes_{\mathcal{F}} : \mathcal{A}b^{R\mathcal{F}} \otimes \mathcal{A}b^{DR\mathcal{F}} \rightarrow \mathcal{A}b$$

with derived functor $\text{Tor}(M, N)$ defined on a pair of functors $M$ covariant on $R\mathcal{F}$ and $N$ covariant on $DR\mathcal{F}$.

Let $M_\ast \mathcal{F}$ denote $\pi_\ast^\mathcal{F}(M)$ and $N^\ast \mathcal{F}$, the cohomological version obtained from the modules $[R \wedge F, N]$, $F \in \mathcal{F}$. Thus, $N^\ast \mathcal{F} \cong N_\ast D\mathcal{F}$.

To construct the spectral sequence, let $M$ be a right (cell) $R$-module, $N$ a left (cell) $R$-module. The exact couple defined by

$$D_{s+t}^1 := \pi_{s+t+1}(M_{s+1} \wedge R N), \quad E_{s+t}^1 := \pi_{s+t}(FK_s \wedge R N)$$

yields a spectral sequence with $E_{s+t}^2 = \text{Tor}_{s+t}^{\mathcal{F}}(M_\ast \mathcal{F}, N^\ast \mathcal{F})$, converging to $E_{\infty} = E_0^{s,t} \pi_\ast(M \wedge R N)$. The filtration on $\pi_\ast(M \wedge R N)$ is given by the resolution of $M$.

Alternately, one can resolve in the $N$ variable.

The $E^2$ term can be identified by noting that $E_{s+t}^1 \cong F_k \otimes_{\mathcal{F}} N$ by construction. Convergence and functoriality are proven as for the Ext spectral sequence. The Tor spectral sequence always converges strongly: see [12].

12. Classification of modules over real connective $K$-theory

For an $S$-algebra $R$ with homotopy groups forming a Noetherian ring of dimension at most two, the $R$-modules are classified by their homotopy groups and a difference in Ext, as seen above. For other ring spectra, the same method can be used once an algebraic category with sufficient structure and global dimension at most two is built [9]. As noted above, additional free objects provided by a united homology theory can lower the dimension of the algebraic category.

This classification is carried out here for $ko$.

12.1. Homological dimension of $crt$-modules. The following theorems rely on work of Bousfield [8] regarding abelian groups with involution, and essentially parallel his similar theorems for the periodic case. In the category of $crt$-modules, all projectives are free and all $crt$-acyclic objects have projective dimension at most two.

Theorem 12. Given a $crt$-module $M$, the following are equivalent:

1. $M$ has projective dimension (as a $crt$-module) at most 2,
2. $M$ has finite projective dimension, and
3. $M$ is $crt$-acyclic.

This theorem relies on the following determining property of projectives.
Theorem 13. Given a crt-module $M$, the following are equivalent:

1. $M$ is projective (as a crt-module),
2. $M$ is crt-acyclic and $M^U$ is free over $\mathbb{Z}[\beta_U] = [ko, ku]^{ko}$, and
3. $M$ is free.

Proof of Theorem 12. Since $\eta$ is nilpotent, equivalences under $ko$, $ku$, and $kt$-homology are all the same, and the projective dimension of a crt-module is bounded by the $ku$-projective dimension of its $U$-part. Thus, Theorem 12 follows from 13 and the fact that whenever two crt-modules in a short exact sequence are crt-acyclic, the third is as well. □

Proof of Theorem 13. (i) $\implies$ (ii): Given $M$ projective, there is another projective $P$ such that $M \oplus P \cong F$, where $F$ is free, hence the $\pi^crt_*$ of a wedge of suspensions of $ko$, $ku$, and $kt$. Since free objects are all crt-acyclic, so are their direct summands. Now, $M^U$ is a direct summand of $F^U$, hence projective over $ku\subset [ku, ku]^{ko}$ since $F^U$ is. If $M^U$ is finitely generated over $ku$, then it is free. If not, note that $ku$ is commutative and Noetherian and has no non-zero idempotents; by [3] this implies that any non-finitely generated projective is also free.

(ii) $\implies$ (iii): Given $M$ satisfying (2), we want to decompose $M$ as the direct sum of free objects. By Proposition 3.7 of [8] (Lemma 14 below), each $M_n$ decomposes as

$$M_n^U = G_n \oplus \psi G_n \oplus i^+ H_n \oplus i^- I_n$$

for $G_n$, $H_n$, $I_n$ free abelian, where $\psi$ interchanges summands on $G \oplus \psi G$, $\psi = 1$ on $i^+ G$, and $\psi = -1$ on $i^- G$.

For $a$ of degree $n$, let $FU(a)$ denote the monogenic free crt-module generated by $a$ isomorphic to $\Sigma^n \pi^crt_*(Dku)$. Similarly, $FO \cong \pi^crt_*(Dko)$ and $FT \cong \pi^crt_*(Dkt)$.

First, we find a direct summand of $M$ which is isomorphic to the direct sum of copies of $FU$. Select elements $\{a_{\alpha,n}\}$ with $a_{\alpha,n} = a_\alpha$ in $i^+ H_n \subseteq M_n^U$, $a_\alpha$ not in the image of $\beta_U$, and such that $\{a_{\alpha,n}\}$ is a $\mathbb{Z}$-basis for the set of elements $x$ in $i^+ H_n$ with $\beta_U x$ in $G_{n+2} \oplus \psi G_{n+2}$. That such a basis is possible (disjoint from image $\beta_U$) is shown in Lemma 15. Let $K$ be the kernel of the map

$$\bigoplus_{\alpha} \Sigma^n FU(a_{\alpha,n}) \longrightarrow M.$$

$K^U = 0$, since each element $a$ generates a copy of $\mathbb{Z}[\beta_U] a \oplus \Sigma^2 \mathbb{Z}[\beta_U]b$ in $M^U$ with $\psi$-action given by $\psi^2 b = \psi(\beta_U a - b) = b$, isomorphic to the $\psi$-action on $FU$. Since $K$ is a submodule of a direct sum of copies of $FU$, $\eta K^O = 0$, which gives $K^O = 0$. Thus, $K = 0$ and the map is monic. In fact (see Lemma 15), the free module surjects onto a direct summand of $M^U$, so the cokernel $N$ also satisfies (2). Further, $N^U = (N^U)^+ \oplus (N^U)^-$ as graded abelian groups, where

$$G^+ = \{ x \in G : \psi x = x \}$$
and
\[ G^- = \{ x \in G : \psi x = -x \}. \]

Next, we find a direct summand of \( N \) which is the direct sum of copies of \( FO \).

Note that \( \eta^2 = r \beta_U^2 c \) by the relations in 10.2 and 10.5. Thus, \( \eta^2 \) factors as the composition
\[ N^O_n \to (N^U_n)^+ \overset{r \beta^2}{\to} r \beta_U^2(N^U_n)^+ \subseteq N^O_{n+2}. \]
The identities \( cr \beta_U = 1 + \psi \) and \( \beta_U \psi = -\psi \beta_U \) give \( cr \beta_U^2(N^U_n)^+ = (1 + \psi)(N^U_{n+2})^- = 0 \), so \( r \beta_U^2(N^U_n)^+ \) is a \( \mathbb{Z}/2 \)-module (i.e., is contained in \( \eta N^O \)).

Choose a \( \mathbb{Z} \)-basis \( \{b \}_{\gamma} \cup \{c_{\gamma} \} \cup \{d \}_{\delta} \cup \{e \}_{\epsilon} \) for \( N^U \) by extending a basis for the image of \( \beta_U \) to the whole of \( N^U \) such that the \( b \)'s project to a basis for \( r \beta_U^2(N^U)^+/\eta^2 N^O \), the \( c \)'s project to a basis for \( \eta^2 N^O \), the \( d \)'s are trivial in \( r \beta_U^2(N^U)^+ \), and the \( e \)'s are in \( (N^U)^- \).

For any \( c_{\gamma,n} \) not in the image of \( \beta_U \), define \( m_{\gamma,n} \) as follows: Since \( 2(N^U)^+ \subseteq c(N^O) \) and \( \eta^2 = r \beta_U^2 c \), there is an \( m_{\gamma,n} \in N^O_n \) with \( c(m_{\gamma}) - c_\gamma \in \text{Span}\{d\}_{\delta} \), that is, \( \eta^2 m_{\gamma} = r \beta_U^2 c_{\gamma} \). Then \( c_{\gamma,n} \) can be replaced by \( c(m_{\gamma,n}) \). The construction of the \( m_{\gamma} \) implies that
\[ \bigoplus \Sigma^a FO(m_{\gamma,n}) \to N \]
is monic with \( \text{crt-acyclic cokernel } P \) such that \( P^U \) is free over \( \mathbb{Z}[\beta_U] \), \( P^U = (P^U)^+ \oplus (P^U)^- \) as graded abelian groups, and \( \eta^2 P^O = 0 \). The map is monic since the \( m_{\gamma} \) hit elements which are non-zero under \( \eta^2 \); everything else is forced by \( \beta_O \neq 0 \) on \( N^O \), since \( N^U \) is free over \( \mathbb{Z}[\beta_U] \). Also, the map from the free module generated by the \( m_{\gamma} \) hits generators \( c(m_{\gamma}) \) in a basis for \( N^U \), so the cokernel has \( U \)-part free over \( \mathbb{Z}[\beta_U] \). Since \( \eta^2 N^O \subseteq r \beta_U^2(N^U)^+ \), all of \( \eta^2 N^O \) is hit; \( \eta^2 P^O = 0 \).

Lastly, we see that \( P \) is a direct sum of copies of \( FT \) by using the exact sequence
\[ \cdots \to P^U_{n+1} \overset{\gamma}{\to} P^T_{n} \overset{\zeta}{\to} P^*_n \overset{\varphi}{\to} P^U_{n-2} \overset{\gamma}{\to} P^T_{n-1} \to \cdots. \]
Since \( \beta_U \) is injective on \( P^U \), \( 0 = 1 - \psi_U = \beta_U \varphi \) implies \( \varphi = 0 \). So, since \( P^U \) is free over \( \mathbb{Z}[\beta_U] \), the exact sequence
\[ 0 \to (P^U_{n-1})^+ \oplus (P^U_{n-1})^-/2 \overset{\gamma}{\to} P^T_{n} \overset{\zeta}{\to} (P^U_n)^+ \to 0 \]
reduces mod torsion to
\[ 0 \to (P^U_{n-1})^+ \overset{\gamma}{\to} P^T_{n}/\text{tors} \overset{\zeta}{\to} (P^U_n)^+ \to 0, \]
with \( \gamma(P^U_{n-1})^+ = (P^T_{n}/\text{tors})^- \) (to see this, look at the relations between \( \psi_T, \psi_U \), and \( \gamma \)). The exact sequence for \( \eta^2 \) gives the short exact sequence
\[ 0 \to P^O_{n}/\text{tors} \overset{\epsilon}{\to} P^T_{n}/\text{tors} \overset{\tau}{\to} P^O_{n+3}/\text{tors} \to 0, \]

since
\[ \text{tors}(P^O_{n-3}) = \ker(c: P^O_{n-3} \to P^U_{n-3}) = \eta P^O_{n-4} \]
Lemma 14. Any object $i$-summands;

Proof. This is Proposition 3.7 of [8].

12.2. Technical lemmas used in the proof of Theorem 13. First we define the abelian category $\text{Inv}$ of involutive modules, as in [8], to be the category of $\mathbb{Z}\{1, \psi\}$ modules where $\psi^2 = 1$; this is the category of abelian groups with involution. For an abelian group $G$, $G \oplus \psi G$ denotes the involutive module in which $\psi$ interchanges summands; $i^+G$ denotes $G$ with involution $\psi = 1$; $i^-G$ has involution $\psi = -1$. For an involutive module $M$, $M^+ = \{x \in M | \psi x = x\}$ and $M^- = \{x \in M | \psi x = -x\}$.

Lemma 16, the analogue of [8] Proposition 3.11 for $\text{crt}$-acyclics, then gives that $Q = 0$, so $M$ is a free $\text{crt}$-module. □
Lemma 15. Given a crt-module $M$ such that $M^U$ is free over $\mathbb{Z}[\beta_U]$, with decomposition as abelian groups 

$$M^U_n = G_n \oplus \psi G_n \oplus i^+ H_n \oplus i^- I_n,$$

any element in $G_n \oplus \psi G_n$ must be in a submodule generated under $\psi$ and $\beta_U$ by a generator of a copy of $ku_*(C(\eta))$ taken as a $[ku,ku]^{ko}_\ast$-module. Such a generator is necessarily in $i^+ H_n$ for some $n$. Further, the image of $ku_*(C(\eta))$ in $M^U$ is a direct summand over $\mathbb{Z}[\beta_U, \psi]$.

Proof of Lemma 15. We want to prove that any element of $G_n \oplus \psi G_n$ must be generated under $\psi$ and $\beta_U$ by a generator of a copy of $ku_*(C(\eta))$.

Let $\mathbb{Z}[\beta_U]x$ be a $\mathbb{Z}[\beta_U]$-free submodule and direct summand of $M^U$ generated by $x$ and closed under $\psi$. Note that the united operation $\varphi$ requires $\psi x = x$, so that $x$ can be in the kernel of $\varphi$.

For a module with $M^U = \mathbb{Z}[\beta_U]x \oplus \mathbb{Z}[\beta_U]y$ as $\mathbb{Z}[\beta_U]$-modules and indecomposable over $\mathbb{Z}[\beta_U, \psi]$, $x$ and $y$ must have different degrees of the same parity, since any group $\mathbb{Z}e_1 \oplus \psi \mathbb{Z}e_2$ has an element $e_1 - e_2$ in the image of $\beta_U$. Let $|y| > |x|$ and suppose $|x| = 0$, so $|y| \geq 2$. To rule out $|y| \geq 4$, use the $\eta$ exact sequence and $\eta^3 = 0$ to show that $M^O$ is $(-2)$-connected, since $M^U$ is $(-1)$-connected. Then use the exact sequence with $\varphi$ to see that $MT_1$ includes torsion $\mathbb{Z}/2(\text{coker } \varphi)$; the $\varepsilon$ exact sequence shows then that $M^O_{-1} = \mathbb{Z}/2$. The $\eta$ exact sequence now forces $M^U_{-1} = \mathbb{Z}/2$, a contradiction.

This covers all possible cases, since $\psi^2 = 1$ and $\psi \beta_U = -\beta_U \psi$.

Let $A = [ku,ku]^{ko}_\ast$. For any crt-module $M$, $M^U$ is an $A$-module. Call an $A$-module $\varphi$-acyclic if the complex 

$$\cdots \longrightarrow M^U_{*+2} \overset{\varphi}{\longrightarrow} M^U_* \overset{\varphi}{\longrightarrow} M^U_{*-2} \longrightarrow \cdots$$

given by $\varphi \in A_{-2}$ is exact.

Lemma 16. Let $M$ be a crt-acyclic crt-module. The following are equivalent:

1. $M^U$ is $\varphi$-acyclic,
2. $\eta = 0$ in $M^O$, and
3. $\eta^2 = 0$ in $M^O$ and $\eta \varepsilon \tau = 0$ in $M^T$.

Proof. This is the $\text{crt}$ analogue of Proposition 3.11 of [8].

(i) implies (ii): Note that $\varphi = cr$. Now, an element $x \in M^O$ is in the image of $r$ if and only if $\eta x = 0$. Also, for any $x \in M^O$, $\varphi(cx) = 0$; by $\varphi$-acyclicity, there is an element $y \in \varphi^{-1}(cx)$. Suppose $x$ is not in the image of $r$. Then $ry \neq x$, but $c ry = \varphi y = cx$, so $x - ry = \eta z$ (ker $c = \text{im } \eta$). Since $ry \neq x$, $\eta z \neq 0$ and $z$ is not in the image of $r$. Thus, we can do for $z = x_1$ what we just did for $x = x_0$, obtaining a sequence $x_n \in M^O$ such that $\eta x_n \neq 0$ and $x_{n-1} - \eta x_n$ is in the image of $r$. Since
\( \eta r = 0, \eta x_0 = \eta^2 x_1 = \eta^3 x_2 = 0, \) so we must have had \( x \) in the image of \( r \) (kernel of \( \eta \)) after all: \( \eta = 0 \) in \( M^O \).

(ii) implies (i): If \( \eta = 0 \), then
\[
\ker \varphi = \ker (c r) = \ker (r) \cup r^{-1} (\ker (c) \cap \im (r)) = \ker (r) \cup r^{-1} (\im (\eta) \cap \ker (\eta)) = \ker (r) = \im (c) = c (\ker (\eta)) = c (\im (r)) = \im (c r) = \im (\varphi).
\]

Thus, \( M^U \) is \( \varphi \)-acyclic.

(ii) implies (iii): Now, \( \eta = 0 \) in \( M^O \) implies \( \eta^2 = 0 \) in \( M^O \). Since the composition
\[
M^T_s \xrightarrow{\tau} M^O_{s-3} \xrightarrow{\varepsilon} M^T_{s-3} \xrightarrow{\eta} M^T_{s-2}
\]
equals the composition
\[
M^T_s \xrightarrow{\tau} M^O_{s-3} \xrightarrow{\eta} M^O_{s-2} \xrightarrow{\varepsilon} M^T_{s-2},
\]
we also get \( \eta \varepsilon \tau = 0 \) in \( M^T \).

(iii) implies (ii): If \( \eta^2 = 0 \) in \( M^O \), then
\[
0 \rightarrow M^O_s \xrightarrow{\varepsilon} M^T_s \xrightarrow{\tau} M^O_{s-3} \rightarrow 0
\]
is exact, but \( \eta \varepsilon \tau = 0 \) in \( M^T \) implies \( \eta \tau = 0 \)
\[
M^T_s \xrightarrow{\tau} M^O_{s-3} \xrightarrow{\eta} M^O_{s-2} \xrightarrow{\varepsilon} M^T_{s-2}
\]
and \( \tau \) (which surjects onto \( M^O \)) maps into the kernel of \( \eta \), that is, \( \eta = 0 \) in \( M^O \).

12.3. Classifying \( ko \)-modules. As in the dimension two case for rings above (Theorem 7), all \( crt \)-modules can be realized as \( \pi^{crt}_* \) of some \( ko \)-module, again using projective resolutions.

The analysis is completed after verifying that the classification of \( ko \)-modules with the same \( \pi^{crt}_* M \) is given by a quotient of \( \Ext^{2,-1}_{crt}(M,M) \), exactly as in [8] and in Theorem 7.

Theorem 17. The category of \( crt \)-acyclic \( crt \)-modules has enough projectives and all objects have projective dimension at most two. Any \( crt \)-acyclic \( crt \)-module can be realized as \( \pi^{crt}_*(X) \) for some \( ko \)-module \( X \). This \( ko \)-module is unique up to homotopy if the \( crt \)-module has projective or injective dimension at most one. For a fixed \( crt \)-module \( M \) of projective dimension two, there is an equivalence relation finer than homotopy equivalence so that equivalence classes of \( ko \)-modules \( X \) with \( \pi^{crt}_*(X) = M \) are in bijective correspondence with the elements of \( \Ext^{2,-1}_{crt}(M,M) \).
13. Modules over periodic $K$-theory and $K_*$-local spectra

A simpler classification results when $\mathcal{F}$-modules have $\mathcal{F}$-projective dimension at most one. One example is Bousfield’s $CRT$-theory [8], noted above. This classification yields an algebraic criterion for when a $K_*$-local spectrum can be given the structure of a $KO$-module or $KU$-module.

13.1. Classifying $KO$-modules. Since Bousfield proves that all $CRT$-acyclic $CRT$-modules have projective dimension at most one and that all projectives are free (hence easily realized), we obtain a classification of $CRT$-modules using the united $CRT$-homology spectral sequence. A $KO$-module $M$ is determined by $\pi_*^{CRT}(M)$ and the group of $KO$-module maps between two $KO$-modules $M$ and $N$ is given by the short exact sequence

$$0 \to \text{Ext}^{1}_{CRT}(\pi_*^{CRT}(M), \pi_*^{CRT}(N)) \to [M, N]_0^{KO} \to \text{Hom}_{CRT}(\pi_*^{CRT}(M), \pi_*^{CRT}(N)) \to 0.$$ 

Note that we have now classified $R$-modules where $R$ is real or complex $K$-theory, either connective or periodic.

13.2. Local spectra with module structures. If $X$ is a $K_*$-local spectrum (or $S$-module), it would be good to know when $X \simeq M$ as spectra for some $KO$- or $KU$-module $M$. Note that $X_*$ for a spectrum $X$ has only the structure of an $S_*$-module; here, we need consider it only as a graded abelian group.

A $CRT$-module enriched with Adams operations is called an $ACRT$-module [8]. These $ACRT$-modules classify $K_*$-local spectra by taking $K_*^{CRT}(X)$ as $CRT$-modules for $X$ $K_*$-local, together with Adams operations induced by those on $KO$.

Note that $\pi_*^{CRT}(X)$ is not in general a $CRT$-module, though the acyclicity condition always holds. If $X$ has the structure of a naive $KO$-module spectrum, however, then $\pi_*^{CRT}(X)$ is a $CRT$-module.

Let $U$ be the right adjoint to the forgetful functor from $ACRT$-modules to $CRT$-modules. Note that the complexification map $c: KO \to KU$ is a map of $S$-algebras: Any $KU$-module is a $KO$-module.

**Theorem 18.** Let $X$ be a $K_*$-local spectrum or $S$-module. Then $X$ is equivalent to a $KO$-module if and only if $K_*^{CRT}(X) \cong U\pi_*^{CRT}(X)$ as $ACRT$-modules, where $\pi_*^{CRT}(X)$ is a $CRT$-acyclic $CRT$-module. Further, $X$ is a $KU$-module if and only if, in addition, $X_*$ can be given the structure of a $KU_*$-module.

**Proof.** Necessity is obvious.

If $\pi_*^{CRT}(X)$ is $CRT$-acyclic, then by the classification theorem for $KO$-modules, there is a $KO$-module $Y$, unique up to homotopy, such that $\pi_*^{CRT}(X) \cong \pi_*^{CRT}(Y)$.

For $X$ and $Y$ to have the same homotopy type as spectra, it suffices to check their $K_*$-local type, since both are $K_*$-local. The question is now answered by the
analysis in [8] relating the categories of CRT-modules and CRT-modules with Adams operations, or ACRT-modules.

Bousfield constructs a right adjoint $U$ to the forgetful functor from ACRT-modules to CRT-modules. Since $U$ is a right adjoint, it preserves injectives; in fact, this is how he shows that the category of ACRT-modules has enough injectives. Further, given a naive $KO$-module spectrum $Y$, $K_*^{CRT}(Y) \cong U\pi_*^{CRT}(Y)$ as ACRT-modules. Thus, $X$ is a $KO$-module if $K_*^{CRT}(X) \cong U\pi_*^{CRT}(Y)$. In particular, although the category of ACRT-modules has global injective dimension two, since $U$ preserves injectives, our prospective $KO$-module $X$ must have $K_*^{CRT}(X)$ with injective dimension at most one as an ACRT-module.

In order for a $K_*$-local spectrum $X$ to be a $KU$-module, $X$ must be a $KO$-module by neglect of structure and $X_*$ must have a $KU_*$-module structure; in fact, since the Bott element must be an isomorphism, any $KU_*$-module structures on $X_*$ are isomorphic. Let $Y_*$ denote $X_*$ with a $KU_*$-module structure. $Y_*$ has an injective resolution

$$0 \to Y_* \to I_0 \to I_1 \to 0$$

of $KU_*$-modules, which we realize (homotopy uniquely) as a cofibration

$$Y \to |I_0| \to |I_1| \to \Sigma Y$$

of $KU$-modules; $\pi_*(Y) \cong Y_*$.

By neglect of structure, $Y$ is a $KO$-module, hence $X$ and $Y$ have the same homotopy type if they satisfy the condition $K_*^{CRT}(X) \cong U\pi_*^{CRT}(Y)$. In this case, $X$ is homotopy equivalent to the $KU$-module $Y$. $\square$

Note that this shows readily that any naive $KU$-module spectrum is equivalent to a $KU$-module; thus, any naive $KU$-module spectrum can be given the structure of a $KU$-module. Further, since any map of naive module spectra is a map between modules of dimension at most one, it can be realized as a strict map. Thus, the homotopy category of naive module spectra over $KU$ (weak equivalences inverted) is equivalent to the derived category of $KU$-modules.

**Corollary 19.** The derived category of $KU$-modules is equivalent to the derived category of naive $KU$-module spectra. The same is true for $KO$.

This answers a question of Mark Hovey [personal communication] of when a $K_*$-local spectrum has the structure of a $KU$-module spectrum.

### 14. Realizing modules of dimension higher than two

Given an $S$-algebra $R$ with a united theory $R\mathcal{F}$ (the $crt$ and $CRT$ theories, for example), and given any $R_*$-module $M_*$ it is possible to construct an $R\mathcal{F}$-module with $M_*$ as its value at $R \wedge S$ (9.1). For the sake of clarity, the exposition here will focus on the example of $ko$. 
Theorem 20. Given any $ko_*$-module $M_*$, it is possible to construct a $crt$-acyclic $crt$-module with $ko$ part $M_*$. Thus, by the classification theorem for $ko$-modules, $M_*$ can be realized as the homotopy of some $ko$-module.

The general case is stated at the end of this section. The method of proof will be clear from the example of $ko$; essentially, it is to build a complex of copies of $\pi_*^{crt}(ko)$ from a free resolution of the module, then to see that the homology of the complex is concentrated in degree zero and is our desired $crt$-module.

14.1. The construction. Let $M_*$ denote a (graded) $ko_*$-module. Take a free resolution $F^O_\ast \to M_\ast \to 0$ of $M_\ast$; each $F^O_n$ is a free, graded $ko_*$-module. Let $F^U_n$ be a free graded $ku_*$-module on generators corresponding to those of $F^O_n$. Now construct a bicomplex

$$
\cdots \xrightarrow{\eta} F^O_n \xrightarrow{c} F^U_n \xrightarrow{r} F^O_{n-1} \cdots \\
\cdots \xrightarrow{\eta} F^O_{n-1} \xrightarrow{c} F^U_{n-1} \xrightarrow{r} F^O_{n-2} \cdots \\
\cdots \xrightarrow{\eta} F^O_0 \xrightarrow{c} F^U_0 \xrightarrow{r} F^O_{-1} \cdots 
$$

where each $X_k$ has an additional internal grading. Suspension affects only this internal grading: $\Sigma X_{k-1} = X_{k+1}$. The maps $\eta, c, r$ are induced by the maps

$$
\Sigma ko_* \xrightarrow{\eta} ko_* \xrightarrow{c} ku_* \xrightarrow{r} \Sigma^2 ko_*;
$$

and the differentials of $F^U_n$ are given by the following method: $F^O_n \to F^O_{n-1}$ has a unique realization as a map of $ko$-modules; smash with the identity on $C(\eta)$.

This is equivalent to defining $F^U$ as $F^O \otimes_{ko_*} ku_*$, since the modules in $F^O_\ast$ are all free over $ko_*$. Thus, we can construct a complex $F$ of free $crt$-modules as

$$
F = F^O_\ast \otimes_{ko_*} \pi_*^{crt}(ko),
$$

where

$$
F^X_\ast = F^O_\ast \otimes_{ko_*} \pi_*^{crt}(ku),
$$

We obtain diagrams similar to the one above corresponding to the other long exact sequences defining $crt$-acyclicity. The rows are exact complexes by construction. Since free modules are flat, each complex $F^X_\ast$ is a resolution. Define a $crt$-module $M$ by $M^O = M_*$, $M^U = H_\ast(F^U_\ast)$, and $M^T = H_\ast(F^T_\ast)$, operations induced by those on the resolutions $F^X_\ast$. 

Now $M$ can be realized in topology (since $crt$-theory has dimension two) if and only if $M$ is $crt$-acyclic. To see that $M$ is $crt$-acyclic, recall that any bicomplex $\mathbb{F}$ has two spectral sequences [13] (where $\partial$, induced by $\eta$, denotes the horizontal differential and $d$, the vertical)

$$E_2^{p,q} = H_p(H_{q-p}(\mathbb{F}, d), \partial), \quad \text{and} \quad E_2^{p,q} = H_p(H_{q-p}(\mathbb{F}, \partial), d)$$

each converging to the homology of the associated total complex (appropriately filtered). In our case, the spectral sequences are half-plane spectral sequences. Since they are spectral sequences from a bicomplex, they automatically converge conditionally [5]. Further, only finitely many differentials are non-zero since each spectral sequence has differentials which go from the groups in the free resolutions downward (in the above diagram) eventually into the lower half-plane, which is all zero.

In this case, the one spectral sequence converges to zero and the other, since it is a single line, converges from the homology of the complex

$$\cdots \rightarrow \Sigma M^O \rightarrow M^O \rightarrow M^U \rightarrow \Sigma^2 M^O \rightarrow \cdots$$
to zero, forcing the complex in question to be exact. Similar analysis yields exactness for the $M^O-M^T$ and $M^U-M^T$ complexes. Thus, $M$ is $crt$-acyclic and can be realized as $\pi_*^{crt}(X)$ for some $ko$-module $X$.

The details of the generalization of this construction are in the section below. The guarantee that any $F$-module can be realized holds only for theories of dimension at most two; in such a case, the analysis realizing and classifying such modules is exactly as in the $crt$ case.

14.2. General united theories. Let $R\mathcal{F}$ be a united theory for an $S$-algebra $R$, with $X$ an $R$-module in $R\mathcal{F}$. Set $A = [X, X]^R$. Note that $\pi_*^{\mathcal{F}}(DX)$ consists of $A$-modules and $A$-module homomorphisms. Given any $A$-module $M_*$ with $A$-free resolution

$$F_*^X \rightarrow M_* \rightarrow 0,$$

we can form $F = F_*^X \otimes_A \pi_*^{\mathcal{F}}(DX)$, which is a complex of $R\mathcal{F}$-modules such that $F_*^X$ is the complex $F_*^X$. The acyclicity conditions are proven as in the $crt$-case above, and we define the $R\mathcal{F}$-module $M$ by $M^Y = H_*(F_*^Y)$ for $Y$ in $R\mathcal{F}$, operations induced by those on $F$.

Thus we obtain an $\mathcal{F}$-acyclic $R\mathcal{F}$-module $M$.

When $M$ has projective dimension at most two, we can use the techniques of earlier sections to realize $M$ as $\pi_*^{\mathcal{F}}(Z)$ for an $R$-module $Z$.

This finishes the proof of Theorem 20 and its generalization:

**Theorem 21.** Let $R$ be an $S$-algebra with a united theory $R\mathcal{F}$, $X$ an $R$-module in $R\mathcal{F}$. Let $A = [X, X]^R$. Then given any $A$-module $M_*$ it is possible to construct an $\mathcal{F}$-acyclic united module $M$ such that $M^X = M_*$. Thus, by the techniques preliminary to the classification theorem, when the united theory $R\mathcal{F}$ is of dimension at most
two, any $A$-module can be realized as the homotopy of $DX \wedge_R Z$ for some $R$-module $Z$. In particular, any $R_*$-module is the homotopy of some $R$-module.

In particular, any $ko_*$-module is the homotopy of some $ko$-module, and the same is true for $KO$.

Note also that any $[ku, ku]^k_*$-module can be realized as $\pi_*(ku \wedge_{ko} Z)$ for some $ko$-module $Z$, since $ku$ is self-dual up to suspension as a $ko$-module.

15. A change of rings isomorphism


15.1. Universal functors between $A$-modules and $crt$-modules. The forgetful functor $M \mapsto M^U$ from $crt$-modules to (graded) $A$-modules has both a left adjoint and a right adjoint.

Lemma 22. The left adjoint $\rho'$ to the forgetful functor from $crt$-modules to (graded) $A$-modules is given by

\[
\begin{align*}
(\rho'M)^U_* &= M_*, \\
(\rho'M)^O_* &= N_*, \text{ and} \\
(\rho'M)^T_* &= N_* \oplus N_{*+1},
\end{align*}
\]

where $N_*$ is the double desuspension of the cokernel of $\varphi$. The operations on $\rho'M$ are given by $\eta = 0$, $\psi_T[x, y] = [x, -y]$, $\beta_0[x] = [\beta_0^1 x]$, $\beta_T[x, y] = [\beta_0^2 x, \beta_0^2 y]$, $\varepsilon[x] = [x, 0]$, $\zeta[x, y] = [x + \psi x]$, $\gamma x = [0, x]$, and $\tau[x, y] = [y]$.

The right adjoint $\rho$ to the forgetful functor is given by

\[
\begin{align*}
(\rho M)^U_* &= M_*, \\
(\rho M)^O_* &= L_*, \text{ and} \\
(\rho M)^T_* &= L_* \oplus L_{*+1},
\end{align*}
\]

where $L_*$ is the kernel of $\varphi$. The operations on $\rho M$ are given by $\eta = 0$, $\psi_T(x, y) = (x, -y)$, $\beta_0(x) = \beta_0^1 x$, $\beta_T(x, y) = (\beta_0^2 x, \beta_0^2 y)$, $\varepsilon(x) = (0, 0)$, $\zeta(x, y) = x$, $\gamma x = (0, x + \psi x)$, and $\tau(x, y) = y$.

Proof. The adjunctions are verified by using the operation sequences, noting that maps must commute with all $crt$-operations: Given an $A$-module map $\alpha: M_* \to X_*$, where $X$ is a $crt$-module, there should be a unique map $\alpha^{crt}: \rho'(M) \to X$ with $\alpha^U = \alpha$. The definition of $\rho'$ and the $crt$-operations require that $\alpha^U = \alpha$, $\alpha^T$ is determined by $\alpha^O$ (because of the $O$-$T$ and $U$-$T$ sequences), and $\alpha^O[y] = r\alpha(y)$, which is well-defined since $[y] = [z]$ if and only if $\varphi(y - z) = 0$, and $r\varphi = 0$. Similarly for $\alpha': X_*^U \to M_*$, we need a unique map $\alpha'^{crt}: X \to \rho(M)$. This time, $\alpha^O(x) = \alpha c(x) \in \rho(M) = \ker \varphi$ since $\alpha$ is a map of $A$-modules, so $\varphi c(x) = \alpha c(x) = \alpha(0)$. \[\square\]
Lemma 23. If $M$ is a projective $A$-module, then $\rho'M$ is a projective $crt$-module. If $M$ is an injective $A$-module, then $\rho M$ is an injective $crt$-module.

Lemma 24. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of $A$-modules. If $N$ is $\varphi$-acyclic, then $0 \longrightarrow \rho'L \longrightarrow \rho'M \longrightarrow \rho'N \longrightarrow 0$ is an exact sequence of $crt$-modules. If $L$ is $\varphi$-acyclic, then $0 \longrightarrow \rho L \longrightarrow \rho M \longrightarrow \rho N \longrightarrow 0$ is an exact sequence of $crt$-modules.

Proof. Any right adjoint preserves limits, hence is left exact and preserves monics; dually for any left adjoint and epis. Thus, it suffices to check the exactness of $0 \longrightarrow \rho'L \longrightarrow \rho'M$ and $\rho M \longrightarrow \rho N \longrightarrow 0$ where $N$ and $L$ are $\varphi$-acyclic. But this is clear from the definitions of $\rho$ and $\rho'$ and the fact that $A$-module maps commute with $\varphi$. □

Lemma 25. For any $\varphi$-acyclic $A$-module $M$, $\rho'M$ and $\rho M$ are $crt$-acyclic.

Proof. Again, by direct calculation from the definitions of the adjoint functors. □

Theorem 26. For $crt$-acyclic $crt$-modules $L$ and $M$ with $\eta = 0$ in $M^O$, there are natural isomorphisms

$$\rho'(M^U) \cong M \cong \rho(M^U),$$

$$\Ext^s_{crt}(M, L) \cong \Ext^s_A(M^U, L^U),$$

$$\Ext^s_{crt}(L, M) \cong \Ext^s_A(L^U, M^U).$$

Proof. We use the fact that $M^U$ is $\varphi$-acyclic (Lemma 16), the lemmas above, and the long exact sequences which link the $crt$-operations. Since $\eta = 0$, we have the isomorphisms

$$\rho'(M^U) = \text{coker} \varphi = M^U / \text{im} \varphi = M^U / \ker(\varphi) \longrightarrow \Sigma^2 M^O$$

and $M^O \cong \text{im}(c) \cong \ker \varphi = \rho(M^U)^O$. Given these isomorphisms and that, by adjointness, $\Hom_{crt}$ is determined by $\Hom_A$, the isomorphism between $U$-pieces lifts to a $crt$-isomorphism.

Taking a free or injective $A$-resolution of $M$, depending on whether $M$ is in the contravariant or covariant variable, we obtain the desired change of rings isomorphisms. □
16. Future Directions

16.1. C*-algebras and connective K-theory. Module spectra (in the classical, naive sense) over connective K-theory which are of the form $ku \wedge X$ for a compact space $X$ are closely related to C*-algebras. For $X$ and $Y$ compact spaces,

$$[\mathcal{K} \otimes CX, \mathcal{K} \otimes CY] \cong [X, ku \wedge Y]^{Stable} = [ku \wedge X, ku \wedge Y]^{ku},$$

where $\mathcal{K}$ denotes compact operators, and $CZ$ is the space of all maps $Z \to \mathbb{C}$. Dădălărat [11] and Blackadar [4] give more details on this relation between connective K-theory and C*-algebras. Segal [22] has given a more geometric construction of the connective K-theory spectrum.

Thus, it would be good to know which module spectra over $ku$ are actually of the form $ku \wedge X$, at least for $X$ a finite CW-complex.

16.2. Extended modules. The $ko$ Adams spectral sequence is useful for its quick convergence. This leads one to a desire to understand $ko$-modules better, for example, which $ko$-modules can occur as the $ko$-homology of a space or spectrum.

Note that, since $ku$ and $kt$ are finite $ko$-modules, the result above on finitely generated $R_\ast$-modules generalizes to finitely generated $crt$-modules.

In a different vein, a theorem of Jung and Stolz [23] states that a manifold admits a positive scalar curvature metric if and only if the image in $ko$-theory of the spin bordism class given by the classifying map $M \to B_\pi M$ of the universal cover lies in a certain subgroup of $ko_n B_\pi M$ ($n = \dim M \geq 5$). Thus, the study of $ko \wedge B_\pi$ would be interesting. The periodic case reduces to representation theory. While there is no general description of $HZ \wedge B_\pi$, connective K-theory may be a middle ground between this and the periodic K-theory of $B_\pi$.

16.3. Questions. Among other interesting questions is one suggested by Neil Strickland: when is a module over $MU_\ast$ the homotopy of an $MU$-module? Here we have realized all modules of projective dimension at most two, but the question remains for all higher and infinite $MU_\ast$-dimensional modules.

Perhaps more tractable would be to investigate other rings of finite homological dimension first. This might require a better understanding of the relation between the associated graded of the Hom groups from the EKMSS and the actual Hom groups.

Along another tack is the investigation of other united homology theories. It would be interesting to finding situations apart from K-theory where these theories are useful.

The work of T.-Y. Lin included results about modules over $S_\ast$. One hope would be that we could determine $S_\ast$-injectives or obtain more information about maps between 2-cell complexes. Since $S_\ast$ is not concentrated in even (or otherwise sparse) degrees, the algebra over this uncalculated ring is likely to be difficult to approach, but it should be possible to see part of the picture.

The categories $\mathcal{D}_R$ of $R$-modules give alternate worlds of homotopy theory. As this paper shows, these worlds are often simpler than the usual stable category. It would
be interesting to investigate, for general $S$-algebras $R$, whether there is a choice of $R$ such that the every Bousfield class ($R$-modules with the same localization functor) has a complement in the algebra of Bousfield classes of $R$-modules. Also possibly interesting would be analogues of the chromatic filtration and whether the telescope conjecture might be true over some choice of $R$. This might aid in determining the deviation of the telescope conjecture from the truth.

References


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