ON PRODUCTS IN THE MINIMAL SIMPLICIAL SETS

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ABSTRACT. The product structures of the minimal subcomplexes of loop spaces are studied. The relations among the homotopy groups, the homology groups, and the total space of a connected strong homotopy associative minimal simplicial $H$-set are given. Some minimal simplicial groups are also considered.

1. Introduction

It is well known that every fibrant simplicial set $X$ is homotopy equivalent to a minimal simplicial set $Y$. If $X$ is a loop space, then there is an induced multiplication on $Y$ given by the composite

$$Y \times Y \to X \times X \to X \to Y.$$ 

In this case, $Y$ is a minimal simplicial set together with a multiplication. It is an old story in topology to look for a product in the minimal simplicial sets. J.F. Adams showed that the two-stage Postnikov system $X$ with $\pi_n(X) \neq 0$ and $\pi_{n+1}(X) \neq 0$ is homotopy equivalent to a minimal simplicial group $[?]$. J. Milnor gave a counterexample that $\Omega(S^{n+1}(n+1, n+2, n+3))$ is not homotopy equivalent to a minimal simplicial group, where $S^{n+1}(n+1, n+2, n+3)$ is the 3-stage Postnikov system by taking the first three homotopy groups of $S^{n+1}$. G. Whitehead gave some non-associative minimal simplicial abelian groupoids.

In this paper, we study minimal simplicial $H$-sets, i.e., minimal simplicial sets with multiplications. We always assume that a simplicial $H$-set $X$ has a strict unit element $e$. A simplicial $H$-set $X$ is called strong homotopy associative if

1. $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu) \colon X \times X \times X \to X$ rel. $e \times X \times X$;
2. $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu) \colon X \times X \times X \to X$ rel. $X \times e \times X$; and
3. $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu) \colon X \times X \times X \to X$ rel. $X \times X \times e$, 

where $\mu \colon X \times X \to X$ is the multiplication. It was pointed out by J. Stasheff that if a multiplication $\mu \colon X \times X \to X$ is homotopy associative, then there exists a multiplication $\mu' \colon X \times X \to X$ which is strong homotopy associative $[?]$. Notice that the homotopy groups $\pi_*(X)$ can be identified with the cycles in $X$ if $X$ is minimal, where $x \in X_n$ is a cycle if $d_j x = *$ for all $j$. A simplicial $H$-set is said to be right

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(left) group-like if $X$ has a strict right (left) inverse map. Our main theorem is as follows.

**Theorem 1.1.** Let $X$ be a connected strong homotopy associative minimal simplicial $H$-set. Then:

1. The associativity
   $$ (ab)c = a(bc) $$
holds if one of $a,b,c$ is in $\pi_*(X)$.
2. For $a \in X_n$, there exists a unique left inverse $b$ in $X_n$ such that $ba = e$ and there exists a unique right inverse $c$ in $X_n$ such that $ac = e$.
3. The commutativity
   $$ ab = ba $$
holds if $a \in \pi_*(X)$.
4. $X$ is generated by $\pi_*(X)$ as a simplicial $H$-set.
5. The fibration
   $$ F_n(X) \to P_n(X) \to P_{n+1}(X) $$
is a central extension and it is also a principal $F_n(X) \cong K(\pi_n(X), n)$ bundle, where $\{P_n(X)\}_{n \geq 0}$ is the Moore-Postnikov system of $X$.
6. Let $H: \pi_*(X) \to \check{H}_*(X; Z)$ be the Hurewicz map. Then there exists a (graded) subset $S$ of $\pi_*(X)$ such that
   
   a. $H(x) \neq 0$ for $x \in S$;
   
   b. $H(x_1) \neq H(x_2)$ for $x_1 \neq x_2$ in $S$; and
   
   c. $X$ is generated by $S$ as a right (or left) group-like simplicial $H$-set.

Assertions (1) to (4) give a general description of the relations between the homotopy groups and the total space. Assertion (5) shows that the Postnikov system of a connected homotopy associative minimal simplicial $H$-set is very nice. Assertion (6) gives a relation between the total space and its homology. This supports the Moore conjecture in some sense although it is still very unclear if the exponents of the homotopy groups are related to the homology groups. We should point out that the minimal subcomplex of a loop space is non-associative in general. This paper is our starting work to understand the product structures in the “minimal models” for the loop spaces. The understanding of these product structures may help us to study some homotopy problems such as the Freyd conjecture, the Moore conjecture, and the Kervaire invariants problem.

As an example, the complete answer when a two-stage Postnikov system $X$ with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+1}(X) = \mathbb{Z}/2$ is homotopy equivalent to a minimal simplicial group is given as follows.
Theorem 1.2. Let $n, i > 0$ and let $X$ be a two-stage Postnikov system with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+i}(X) = \mathbb{Z}/2$. Then $X$ is homotopy equivalent to a minimal simplicial group if and only if the Postnikov invariant of $X$ is trivial or $Sq^{i+1}$.

The article is organized as follows. In Section 2, we give a modified Moore-Postnikov system of a simplicial set. In Section 3, we give some basic properties of minimal simplicial sets. The minimal simplicial $H$-sets are studied in Section 4. The proofs of the assertions (1)–(5) in Theorem 1.1 are given in this section, where assertions (1) and (2) are given in Proposition 4.2, where assertion (3) is Proposition 4.7, where assertion (4) is Theorem 4.12, and where assertion (5) is Theorem 4.9 and Proposition 4.11. In Section 5, we show that the strong homotopy associative is a homotopy invariant and so every loop space is homotopy equivalent to a strong homotopy associative minimal simplicial $H$-set. We should mention Stashef’s comments, after we wrote down this section, that, for any homotopy associative $H$-space $X$, there exists a strong homotopy associative multiplication in $X$. This means somehow this section is not important in this paper. His comments show that every homotopy associative $H$-space is homotopy equivalent to a strong homotopy associative minimal simplicial $H$-set. In Section 6, we give a minimal simplicial cosets’ construction for the loop spaces. This is a generalization of Dold’s work on abelian simplicial groups. In Section 7, we study the relations between the minimal simplicial $H$-sets and their abelianizers. The proof of assertion (6) in Theorem 1.1 is given in this section, where assertion (6) is Corollary 7.5. In Section 8, we give a proof of Theorem 1.2, where Theorem 1.2 is Theorem 8.5.

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2. Modified Moore-Postnikov Systems

The modified Moore-Postnikov systems of simplicial groups were introduced in [?]. In this section, we give a modification of the Moore-Postnikov system of a fibrant simplicial set.

Recall that a fibrant simplicial set (or Kan complex) is a simplicial set $X$ such that the constant map $X \to *$ is a fibre map. Let $X$ be a fibrant simplicial set. The standard Moore-Postnikov system $\{P_n X\}_{n \geq 0}$ is defined as follows.

Let $x, y \in X_q$. The equivalence relation $\sim_n$ on $X$ is defined by setting

\[ x \sim_n y \iff d_{i_{q-n}} \cdots d_{i_1} x = d_{i_{q-n}} \cdots d_{i_1} y \]

for all of the sequences $I = (i_1, \ldots, i_{q-n})$. Equivalently, $x \sim_n y \iff$ the representing maps $f_x, f_y : \Delta[q] \to X$ agree on $\Delta[q]^{(n)}$. Now the simplicial set $P_n X$ is defined to be $P_n X = X/ \sim_n$ [?, ?].
The modified Moore-Postnikov system \( \{ P_n X, \tilde{P}_n X \}_{n \geq 0} \) is defined as follows. Let \( x, y \in X_q \). The equivalence relation \( \approx_n \) on \( X \) is defined by setting
\[
x \approx_n y \iff d_{i_1} \cdots i_{q-n} \cdots d_{i_1} y
\]
for all of the sequences \( I = (i_1, \ldots, i_{q-n}) \). Equivalently, \( x \approx_n y \iff \) the restriction of the representing maps \( f_x|\Delta[q]^{[n]} \) and \( f_y|\Delta[q]^{[n]} : \Delta[q]^{[n]} \to X \) are homotopic relative to the \((n-1)\)-skeleton \( \Delta[q]^{[n-1]} \). The natural sequence of maps
\[
X \to \cdots \to P_n(X) \to \tilde{P}_n(X) \to P_{n-1}(X) \to \cdots \to P_0(X) \to \tilde{P}_0(X) \to *
\]
is called the modified Moore-Postnikov system for \( X \).

**Theorem 2.1.** Let \( X \) be a fibrant simplicial set. Then

1. For each \( n \geq 0 \), \( f_n : X \to \tilde{P}_n(X) \) is a fibre map.
2. Each \( \tilde{P}_n(X) \) is a fibrant simplicial set.
3. For each \( n \geq 0 \), \( q_n : P_n(X) \to \tilde{P}_n(X) \) is a fibre map.
4. For each \( n \geq 0 \), \( q_n : P_n(X) \to \tilde{P}_n(X) \) is a homotopy equivalence.
5. For each \( n \geq 1 \), \( p_n : \tilde{P}_n(X) \to P_{n-1}(X) \) is a minimal fibre map.
6. For each \( n \geq 1 \), the fibre \( \tilde{F}_n(X) \) of \( p_n : \tilde{P}_n(X) \to P_{n-1}(X) \) is isomorphic to the minimal simplicial set \( K(\pi_n(X), n) \).

**Proof.** Assertion (1) follows from assertion (3) and the well known result that \( X \to P_n(X) \) is a fibre map. Assertion (2) also follows from assertion (3) because the fibre map \( q_n : P_n(X) \to \tilde{P}_n(X) \) is onto and \( P_n(X) \) is a fibrant simplicial set.

(3). Consider the commutative diagram
\[
\begin{array}{ccc}
\Lambda^k[q] & \xrightarrow{f} & P_n(X) \\
\downarrow \quad q_n & & \downarrow \\
\Delta[q] & \xrightarrow{g} & \tilde{P}_n(X).
\end{array}
\]
Notice that \( P_n(X)_q = \tilde{P}_n(X)_q = X_q \) for \( q \leq n - 1 \). We may assume that \( q > n \).

Case I: \( q = n + 1 \). Let \( x = g(i_{n+1}) \) and let \( z \in P_n(X)_{n+1} \) such that \( q_0(z) = x \), where \( i_{n+1} \) is the only nondegenerate element of dimension \( n + 1 \) in \( \Delta[n+1] \). Notice that \( q_n \circ f = q_n \circ f_z|\Lambda^k[n+1] \), where \( f_z \) is the representing map of the element \( z \). Thus there exists a homotopy \( F : \Lambda^k[n+1] \times I \to P_n(X) \) such that \( F_0 = f_z \), \( F_1 = f_z|\Lambda^k[n+1] \), and \( F_t \) is constant on the \((n-1)\)-skeleton \( \Lambda^k[n+1]^{[n-1]} = \Delta[n + 1]^{[n-1]} \). Now let \( G \)
be the extension map in the following diagram

\[ \Delta[n+1] \times I \]

\[ \Lambda^k[n+1] \times \bigcup \Delta[n+1] \times \text{I} \]

Then the map \( G_0 : \Delta[n+1] \to P_n(X) \) is an extension of the map \( f \) which is also a lifting of the map \( g \).

Case II: \( q > n + 1 \). Let \( f' : \Delta[q] \to P_n(X) \) be an extension of the map \( f \). Notice that \( \Delta[q][n] = \Lambda^k[q][n] \). It is easy to check that \( q_n \circ f' = g \).

(4). Let \( F'_n(X) \) be the fibre of the fibre map \( q_n : P_n(X) \to \tilde{P}_n(X) \). Let \( x \in F'_n(X)_q \) such that \( d_j x = * \) for all \( j \). If \( q > n \), then \( x = * \) in \( P_n(X) \). If \( q = n \), then \( x \approx * \). Notice that \( F'_n(X)_q = * \) for \( q \leq n - 1 \). Thus \( \pi_*(F'_n(X)) = 0 \) and the assertion follows.

(5). Similar to (3), it is easy to check that \( p_n : \tilde{P}_n(X) \to P_{n-1}(X) \) is a fibre map. We need to show that the fibre map \( p_n \) is minimal. Consider the commutative diagram

\[ \Lambda^k[q] \xrightarrow{f} \tilde{P}_n(X) \]

\[ \Delta[q] \xrightarrow{g} P_{n-1}(X), \]

where \( f_k = f_0, f_1 \) are two extensions in the diagram. Notice that \( \tilde{P}_n(X)_q = P_{n-1}(X)_q \) for \( q \leq n - 1 \). We may assume that \( q > n \).

Notice that \( d_j f_0(i_q) = d_j f_1(i_q) \) for \( j \neq k \). Thus

\[ d_k f_0(i_q) \simeq d_k f_1(i_q) \]

and so \( d_k f_0(i_q) = d_k f_1(i_q) \) by the equivalence relation \( \approx_n \).

(6). This assertion follows from (4) and (5). \( \square \)

3. Minimal Simplicial Sets

Many nice properties of minimal simplicial sets are well known. In this section, we pay more attention to the relations between the homotopy groups and the total spaces of minimal simplicial sets.

Let \( X \) be a fibrant simplicial set and let \( x, y \in X_n \) such that \( d_j x = d_j y \) for all \( j \). Let \( f_x, f_y : \Delta[n] \to X \) be the representing maps of \( x, y \) respectively. Then the map
Lemma 3.1. There is a homotopy \( F : f_x \simeq f_y \) rel. \( \Delta[n] \) if and only if the map 
\[ f_x \cup f_y : \Delta[n] \cup_{\Delta[n]} \Delta[n] \to X \] 
is well defined. The following lemma is a standard exercise.

**Theorem 3.2.** Let \( g : X \to Y \) be a simplicial map of simplicial sets. Then

1. If \( X \) is a minimal simplicial set and \( g_* : \pi_*(X,v) \to \pi_*(Y,g(v)) \) is one to one for each \( v \in X_0 \), then the map \( g : X \to Y \) is also one to one.

2. If \( X \) is a connected fibrant simplicial set, if \( Y \) is a connected minimal simplicial set, and if \( g_* : \pi_*(X) \to \pi_*(Y) \) is onto, then the map \( g : X \to Y \) is also onto.

**Proof.** (1). The proof is given by induction on the dimension \( n \) of \( X_n \). Notice that \( X_0 = \pi_0(X) \). Thus \( g : X_0 \to Y_0 \) is 1-1. Suppose that \( g : X_q \to Y_q \) is 1-1 for \( q < n \) with \( n > 0 \). Let \( x_1, x_2 \in X_n \) such that \( g(x_1) = g(x_2) \). Then \( d_j x_1 = d_j x_2 \) for all \( j \) since \( g : X_{n-1} \to Y_{n-1} \) is 1-1 by induction. Consider the map \( f_{x_1} \cup f_{x_2} : \Delta[n] \cup_{\Delta[n]} \Delta[n] \to X \) and the composite \( g \circ (f_{x_1} \cup f_{x_2}) = (g \circ f_{x_1}) \cup (g \circ f_{x_2}) : \Delta[n] \cup_{\Delta[n]} \Delta[n] \to Y \), where \( f_{x_i} \) is the representing map of the element \( x_i \). Notice that \( g \circ f_{x_1} = g \circ f_{x_2} \). The map 
\[ |g \circ (f_{x_1} \cup f_{x_2})| : \left| \Delta[n] \cup_{\Delta[n]} \Delta[n] \right| \to |Y| \] 
is null, where \( |X| \) is the geometric realization of the simplicial set \( X \). Notice that 
\[ \left| \Delta[n] \cup_{\Delta[n]} \Delta[n] \right| \simeq S^n \]
and \( g_* : \pi_*(X) \to \pi_*(Y) \) is one to one. Thus the map \( f_{x_1} \cup f_{x_2} : \Delta[n] \cup_{\Delta[n]} \Delta[n] \to X \) is null and so \( x_1 \simeq x_2 \). Notice that \( X \) is minimal. Thus \( x_1 = x_2 \) and the assertion follows.

(2). The proof is given by induction on the Moore-Postnikov systems of \( X \) and \( Y \). Notice that \( Y \) is a minimal simplicial set. Then \( P_n(Y) = \bar{P}_n(Y) \) for each \( n \geq 0 \). Notice that \( \bar{P}_0(X) = K(\pi_0(X),0) \) and \( P_0(Y) = \bar{P}_0(Y) = K(\pi_0(Y),0) = 0 \). The map \( P_0(g) : P_0(X) \to P_0(Y) \) is onto. Suppose that the map \( P_q(g) : P_q(X) \to P_q(Y) \) is onto for \( q < n \) with \( n > 0 \). Consider the following commutative diagram of minimal fibrations

\[
\begin{array}{ccc}
\bar{F}_n(Y) & \rightarrow & P_n(Y) \\
\downarrow & & \downarrow p_n \\
\bar{F}_n(X) & \rightarrow & \bar{P}_n(X) \\
\end{array}
\]

By induction, the map \( P_{n-1}(g) : P_{n-1}(X) \to P_{n-1}(Y) \) is onto. By Theorem 2.1 the minimal simplicial set \( \bar{F}_n(X) \) is isomorphic to the minimal simplicial set \( K(\pi_n(X),n) \).
Notice that $\pi_n(X)$ is a group for $n \geq 1$. There is an induced simplicial group structure on $\tilde{F}_n(X)$ such that $\tilde{F}_n(X)$ is isomorphic to $K(\pi_n(X), n)$ as minimal simplicial groups. Now the homomorphism $g_*: \pi_n(X) \to \pi_n(Y)$ is an epimorphism of groups for $n \geq 1$. Thus the map $\tilde{F}_n(g): \tilde{F}_n(X) \to \tilde{F}_n(Y)$ is onto. Now both $p_n: P_n(Y) \to P_{n-1}(Y)$ and $p_n: \tilde{P}_n(X) \to \tilde{P}_{n-1}(X)$ are minimal fibre maps over connected bases. Thus both $p_n: P_n(Y) \to P_{n-1}(Y)$ and $p_n: \tilde{P}_n(X) \to \tilde{P}_{n-1}(X)$ are fibre bundles ([?], Proposition 6.2). Now it is easy to check that $\tilde{P}_n(g): \tilde{P}_n(X) \to \tilde{P}_n(Y)$ is onto and so the composite

$$\xymatrix{ P_n(X) \ar[r] & \tilde{P}_n(X) \ar[r] & P_n(Y) }$$

is onto. The assertion follows.

By inspecting the proof, we have a slight generalization.

**Corollary 3.3.** Let $g: X \to Y$ be a simplicial map of fibrant simplicial sets. If $X$ is minimal and $g_*: \pi_k(X, v) \to \pi_k(Y, g(v))$ is one to one for $k \leq n$ and each $v \in X_0$, then $g: X^{[k]} \to Y^{[k]}$ is one to one for $k \leq n$, where $X^{[k]}$ is the $k$-skeleton of $X$.

The following Lemma will be used.

**Lemma 3.4.** Let $p: E \to B$ be a fibre bundle with a fibre $F$. Suppose that

1. Both $B$ and $E$ are minimal; and
2. $p_*: \pi_*(E, v) \to \pi_*(B, p(v))$ is onto for each $v$ in $E_0$.

Then $E$ is minimal.

**Proof.** Notice that $B$ is fibrant. Thus $E$ is fibrant. Let $x, y \in E_n$ such that $x \simeq y$. Then $f_x|_{\Delta[n]} = f_y|_{\Delta[n]}$ and $f_x \simeq f_y$ rel. $\Delta[n]$, where $f_x$ and $f_y: \Delta[n] \to E$ are representing maps of $x$ and $y$, respectively. Notice that $B$ is minimal. Thus $p(x) = p(y)$. Let $E'$ be the simplicial set in the following diagram

$$\xymatrix{ E' \ar[r]^i \ar[d]^{p'} & E \ar[d]^p \\
\Delta[n] \ar[r]_{f_p(*)} & B }$$

There are simplicial maps $f'_x$ and $f'_y: \Delta[n] \to E$ such that $i \circ f'_x = f_x$, $i \circ f'_y = f_y$, and $f'_x|_{\Delta[n]} = f'_y|_{\Delta[n]}$. Notice that $p: E \to B$ is a fibre bundle. Thus there is a
commutative diagram

\[
\begin{array}{ccc}
\Delta[n] \times F & \xrightarrow{g} & E' \\
\downarrow q & & \downarrow p' \\
\Delta[n] & \xrightarrow{=} & \Delta[n],
\end{array}
\]

where \( q \) is the first coordinate projection and \( g: \Delta[n] \times F \to E' \) is an isomorphism. By the condition (2), the composite

\[
\pi_*(\Delta[n] \times F) \xrightarrow{g*} \pi_*(E') \xrightarrow{i*} \pi_*(E)
\]

is one to one, where the base point in \( \Delta[n] \times F \) is chosen to be a vertex in \( g^{-1} \circ f'_x(\Delta[n]) \). Thus \( g^{-1} \circ f'_x \simeq g^{-1} \circ f'_y \) rel. \( \Delta[n] \). Let \( q': \Delta[n] \times F \to F \) be the second coordinate projection. Thus \( q' \circ g^{-1} \circ f'_x \simeq q' \circ g^{-1} \circ f'_y \) rel. \( \Delta[n] \). Notice that \( F \) is minimal. Thus \( q' \circ g^{-1} \circ f'_x = q' \circ g^{-1} \circ f'_y \). Notice that \( q \circ g^{-1} \circ f'_x = q \circ g^{-1} \circ f'_y \). Thus \( g^{-1} \circ f'_x = g^{-1} \circ f'_y \) or \( f'_x = f'_y \) and so \( f_x = f_y \) or \( x = y \). The assertion follows.

\[ \square \]

4. Minimal Simplicial \( H \)-sets

In this section, we always assume that a simplicial \( H \)-set has a strict unit element \( e \).

Let \( X \) and \( Y \) be simplicial \( H \)-sets. An \( H \)-map \( f \) is a simplicial map \( f: X \to Y \) which preserves the multiplication up to homotopy. A homomorphism \( f \) is a simplicial map \( f: X \to Y \) which preserves the multiplication strictly. A simplicial homomorphism is certainly an \( H \)-map. A minimal simplicial \( H \)-set \( X \) is a minimal simplicial set together with a multiplication such that \( X \) is also a simplicial \( H \)-set.

Let \( X \) be a fibrant simplicial \( H \)-set. Then we have \( \mu \circ (\mu \times 1) = \mu \circ (1 \times \mu) \) restricted to the subsimplicial set \( e \times X \times X \cup X \times e \times X \cup X \times X \times e \) of \( X \times X \times X \), where \( \mu: X \times X \to X \) is the multiplication of \( X \).

**Definition 4.1.** A fibrant simplicial \( H \)-set \( X \) is said to be left (middle; right) strong homotopy associative if

\[
\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu) \text{ rel. } e \times X \times X
\]

(rel. \( X \times e \times X \); rel. \( X \times X \times e \), respectively). A fibrant simplicial \( H \)-set \( X \) is said to be strong homotopy associative if \( X \) is left, middle, and right strong homotopy associative.

**Proposition 4.2.** Let \( X \) be a minimal simplicial \( H \)-set and let \( a, b, c \) be elements in \( X_n \). Suppose that \( X \) is left (middle; right) strong homotopy associative. Then
(1) If \( a \in \pi_n(X) \) (if \( b \in \pi_n(X) \); if \( c \in \pi_n(X) \), respectively), then
\[
(ab)c = a(bc),
\]
where we identify the elements in the homotopy groups \( \pi_n(X) \) with the cycles.

(2) If \( X \) is connected, then each element in \( X \) has a unique left strict inverse (each element in \( X \) has a unique right strict inverse if \( X \) is right strongly homotopy associative).

**Proof.** (1). Let \( F : X^3 \times I \to X \) be a homotopy such that \( F_0 = \mu \circ (\mu \times 1) \), \( F_1 = \mu \circ (1 \times \mu) \), and \( F_t \) is constant on \( e \times X \times X \) for \( t \in I \). Let \( f_a \), \( f_b \), and \( f_c : \Delta[n] \to X \) be the representing maps of \( a \), \( b \), and \( c \), respectively. Define \( G : \Delta[n] \times I \to X \) to be the composite
\[
\Delta[n] \times I \xrightarrow{\Delta \times 1} \Delta[n]^3 \times I \xrightarrow{f_a \times f_b \times f_c \times 1} X^3 \times I \xrightarrow{F} X,
\]
where \( \Delta : \Delta[n] \to \Delta[n]^3 \) is the diagonal map. Notice that
\[
G(x,t) = F(f_a(x), f_b(x), f_c(x), t) = F(e, f_b(x), f_c(x), t) = f_b(x)f_c(x)
\]
for \( x \in \Delta[n] \) and \( t \in I \). Notice that \( G_0 = (ab)c \) and \( G_1 = a(bc) \). Thus \( (ab)c \simeq a(bc) \) and so \( (ab)c = a(bc) \). Similarly, if \( X \) is middle (right) strongly homotopy associative, then \( (ab)c = a(bc) \) if \( b \in \pi_n(X) \) (if \( c \in \pi_n(X) \)).

(2). The proof is given by induction on the dimension \( n \) of \( X_n \) starting with \( X_0 = \{e\} \). Suppose that each element in \( X_q \) has a unique left strict inverse for \( q < n \) with \( n > 0 \). Let \( a \in X_n \). Then there exists a unique element \( (d_ja)^{-1} \) such that \( (d_ja)^{-1}(d_ja) = e \) for \( 0 \leq j \leq n \). By the uniqueness, the elements \( (d_1a)^{-1}, \ldots, (d_na)^{-1} \) have matching faces. Notice that \( X \) is a Kan complex. There exists an element \( b \in X_n \) such that \( d_jb = (d_ja)^{-1} \) for \( 1 \leq j \leq n \). Thus
\[
d_j(ba) = d_jb \cdot d_ja = (d_ja)^{-1} \cdot (d_ja) = e
\]
for \( 1 \leq j \leq n \) and so \( d_0(ba) \simeq e \). Thus \( d_0(ba) = e \) and the element \( c = ba \in \pi_n(X) \).

Now, by assertion (1), we have
\[
e = c^{-1}c = e^{-1} \cdot (ba) = (e^{-1}b)a
\]
and so \( c^{-1}b \) is a left inverse of \( a \). Now let \( b_1 \) and \( b_2 \) in \( X_n \) such that \( b_1a = b_2a = e \). Notice that \( (d_jb_1)(d_ja) = (d_jb_2)(d_ja) = e \) for \( 0 \leq j \leq n \). By induction, we have \( d_jb_1 = d_jb_2 \) for \( 0 \leq j \leq n \). Let \( f_a, f_{b_1}, f_{b_2} : \Delta[n] \to X \) be representing maps of \( a, b_1, \) and \( b_2 \), respectively. Define \( \varphi \) to be the composite
\[
\Delta[n] \cup_{\Delta[n]} \Delta[n] \xrightarrow{\Delta} (\Delta[n] \cup_{\Delta[n]} \Delta[n])^2 \xrightarrow{(f_{b_1} \cup f_{b_2}) \times (f_a \cup f_a)} X^2 \xrightarrow{\mu} X.
\]
Then $\varphi$ restricted to the first copy of $\Delta[n]$ is $f_{b_1} \cdot f_a = e$ and $\varphi$ restricted to the second copy of $\Delta[n]$ is $f_{b_2} \cdot f_a = e$. Thus $\varphi$ is the constant map $e$. Notice that, in the homotopy group

$$[\Delta[n] \cup_{\Delta[n]} \Delta[n], X] \cong \pi_n(X),$$

$[\varphi] = [f_{b_1} \cup f_{b_2}] + [f_a \cup f_a]$. Since both $[\varphi]$ and $[f_a \cup f_a]$ are zero, $[f_{b_1} \cup f_{b_2}] = 0$ and so $f_{b_1} \cup f_{b_2}$ is null. Thus $b_1 \simeq b_2$ and so $b_1 = b_2$. Similarly, if $X$ is right strong homotopy associative, then every element in $X$ has a unique right inverse. □

**Notation 4.3.** Let $x \in X$. The left inverse of $x$ is denoted by $^l x^{-1}$ and the right inverse of $x$ is denoted by $^r x^{-1}$. If $X$ is left strong homotopy associative, the *left inverse simplicial map* $\chi_l: X \to X$ is defined by

$$\chi_l(x) = ^l x^{-1}$$

and the *right inverse simplicial map* $\chi_r: X \to X$ is defined by

$$\chi_r(x) = ^r x^{-1}.$$

**Corollary 4.4.** Any connected minimal simplicial monoid is a minimal simplicial group.

**Lemma 4.5.** If $X$ is a connected strong homotopy associative minimal simplicial $H$-set, then the left inverse map $\chi_l$ is homotopic to the right inverse map $\chi_r$.

**Proof.** Consider the homotopy classes $[X, X]$ which is a group since $X$ is a simplicial $H$-set. Notice that $[1_X] \cdot [\chi_r] = e$ and $[\chi_l] \cdot [1_X] = e$. Thus $[\chi_l] = [\chi_r]$. The assertion follows. □

**Proposition 4.6.** Let $X$ be a connected strong homotopy associative minimal simplicial $H$-set and let $a \in \pi_n(X)$ and $b \in X_n$ with $n > 0$. Then

1. $((ab)a^{-1})^r b^{-1} = e$,
2. $((ba)^r b^{-1}) a^{-1} = e$,
3. $((a^r b^{-1})a^{-1}) b = e$, and
4. $((^rb^{-1} a)b)a^{-1} = e$.

**Proof.** (1). Consider the composite

$$\psi: X \times X \xrightarrow{\Delta \times \Delta} X^2 \times X^2 \xrightarrow{1 \times T \times 1} X^4 \xrightarrow{1 \times 1 \times \chi_l \times \chi_r} X^4 \xrightarrow{\varphi} X,$$

where $\varphi(x_1, x_2, x_3, x_4) = ((x_1 x_2) x_3) x_4$. Notice that $\psi(e, x) = ((ex)e)^r x^{-1} = x^r x^{-1} = e$ and $\psi(x, e) = ((xe)^r x^{-1}) e = e$. The map $\psi: X \times X \to X$ factors through $X \wedge X$.
and so there is a commutative diagram

\[
\begin{array}{ccc}
X \times X & \xrightarrow{\psi} & X \\
\downarrow & & \downarrow \phi \\
X \wedge X & & X
\end{array}
\]

Let \( f_a : S^n \to X \) be the representing map of \( a \) and let \( f_b : \Delta[n] \to X \) be the representing map of \( b \). Let \( g \) be the composite

\[
\Delta[n] \xrightarrow{\Delta} \Delta[n] \times \Delta[n] \xrightarrow{p \times 1} S^n \times \Delta[n] \xrightarrow{f_a \times f_b} X \times X \xrightarrow{\psi} X,
\]

where \( p : \Delta[n] \to S^n \) is the pinch map. Then \( g(\tau_n) = ((ab)a^{-1})^i b^{-1} \). Consider the commutative diagram

\[
\begin{array}{ccc}
\Delta[n] & \xrightarrow{(p \times 1) \circ \Delta} & S^n \times \Delta[n] \\
\downarrow & & \downarrow \\
S^n = \Delta[n]/\Delta[n] & \xrightarrow{f_a \wedge f_b} & S^n \wedge \Delta[n] \\
\downarrow & & \downarrow \\
X \wedge X & \xrightarrow{\bar{\psi}} & X.
\end{array}
\]

Let \( \bar{g} \) be the composite

\[
\begin{array}{ccc}
S^n & \xrightarrow{f_a \wedge f_b} & X \wedge X \\
\downarrow & & \downarrow \bar{\psi} \\
\Delta[n]/\Delta[n] & \xrightarrow{f_a \wedge f_b} & X.
\end{array}
\]

Then \( \bar{g}(\sigma_n) = ((ab)a^{-1})^i b^{-1} \), where \( \sigma_n \) is the only nondegenerate element of dimension \( n \) in \( S^n \). Thus the map \( \bar{g} \) is the representing map of the cycle \((ab)a^{-1})^i b^{-1} \). Notice that the space \( S^n \wedge \Delta[n] \) is contractible. Thus \( \bar{g} \) is null and so \((ab)a^{-1})^i b^{-1} \simeq e\) or \((ab)a^{-1})^i b^{-1} = e\).

Assertions (2), (3), and (4) follow in a similar way.

**Proposition 4.7.** Let \( X \) be a connected strong homotopy associative minimal simplicial \( H \)-set and let \( a \in \pi_n(X) \) and \( b \in X_n \) with \( n > 0 \). Then \( ab = ba \).

**Proof.** By Proposition 4.6, we have \((ab)a^{-1})^i b^{-1} = e\). Notice that \( b^i b^{-1} = e\). By the uniqueness of the left inverse, we have \( b = (ab)a^{-1} \). Thus \( ba = ((ab)a^{-1})a \). By Proposition 4.2, \((ab)a^{-1}a = (ab)(a^{-1}a) = ab \) and the assertion follows.

**Definition 4.8.** A short sequence \( F \xrightarrow{i} E \xrightarrow{p} B \) is called a minimal central extension if

1. \( F, E, \) and \( B \) are simplicial \( H \)-sets.
(2). The map \( i : F \to E \) is a simplicial monomorphism.
(3). The map \( p : E \to B \) is a simplicial epimorphism.
(4). The map \( p : E \to B \) is a minimal fibre map.
(5). For each \( x \in F \) and each \( y \in E \), we have
\[ i(x)y = yi(x) \]
in \( E \).

**Theorem 4.9.** Let \( X \) be a connected strong homotopy associative minimal simplicial \( H \)-set. Let \( \{P_n(X)\}_{n \geq 0} \) be the Moore-Postnikov system of \( X \). Then the short exact sequence
\[
K(\pi_n(X), n) \longrightarrow P_n(X) \longrightarrow P_{n-1}(X)
\]
is a minimal central extension for each \( n \).

*Proof.* Let \( F_n(X) \) be the fibre of the fibre map \( p_n : P_n(X) \to P_{n-1}(X) \). Notice that the equivalence relation \( \sim_n \) in Section 2 preserves the multiplication. Thus \( F_n(X) \), \( P_n(X) \), and \( P_{n-1}(X) \) are minimal simplicial \( H \)-sets. Let \( \tilde{F}_n(X) \) be the quotient simplicial set of \( F_n(X) \) modulo the product equivalence relation generated by the relations \((ab)c \sim a(bc)\) for \( a, b, c \in F_n(X) \). Then \( \tilde{F}_n(X) \) is a simplicial monoid with the projection map \( q : F_n(X) \to \tilde{F}_n(X) \). Notice that the left inverse and the right inverse are well defined in \( F_n(X) \). Thus the left inverse and the right inverse are well defined in \( \tilde{F}_n(X) \) and so \( \tilde{F}_n(X) \) is a simplicial group. Notice that \( \tilde{F}_n(X) \cong F_n(X) \cong \pi_n(X) \). There exists a simplicial homomorphism \( \varphi : \tilde{F}_n(X) \to K(\pi_n(X), n) \) such that \( \varphi_n(\tilde{F}_n(X)) \cong \pi_n(K(\pi_n(X), n)) \). Now the composite \( p \circ q : F_n(X) \to K(\pi_n(X), n) \) is a simplicial homomorphism which is a weak homotopy equivalence. Thus \( p \circ q : F_n(X) \to K(\pi_n(X), n) \) is an isomorphism of minimal simplicial \( H \)-sets and so \( F_n(X) \) is the minimal abelian simplicial group \( K(\pi_n(X), n) \). Now consider the commutator map \( \psi_2 = [ , ] : F_n(X) \wedge P_n(X) \to P_n(X) \) defined by
\[
\psi_2(a, b) = ((ab)^r a^{-1})^t b^{-1}.
\]
Let \( S = \psi_2(F_n(X) \wedge P_n(X)) \) be the image of \( \psi_2 \) in \( P_n(X) \). Notice that \( p_n \circ \psi_2 = e \). Thus \( S \subseteq F_n(X) \) as a subsimplicial set. Let \( \tilde{S} \) be the subsimplicial group of \( F_n(X) \) generated by \( S \). Then \( \tilde{S} \) is also a minimal simplicial group and \( \pi_*(\tilde{S}) \to \pi_*(F_n(X)) \) is a monomorphism. By Proposition 4.7, \( S_n = \{e\} \) and so \( \tilde{S}_n = \{e\} \). Notice that \( F_n(X) \) is \( K(\pi_n(X), n) \). Thus \( \pi_*(\tilde{S}) = \{e\} \) and therefore \( \tilde{S} = \{e\} \). Thus \( S = \psi_2(F_n(X) \wedge P_n(X)) \) is \( \{e\} \), or
\[
((ab)^r a^{-1})^t b^{-1} = e
\]
for $a \in F_n(X)$ and $b \in P_n(X)$. Notice that $b^rb^{-1} = e$. By the uniqueness of the left inverse, we have

$$(ab)^{r}a^{-1} = b$$

for $a \in F_n(X)$ and $b \in P_n(X)$. We need the following lemma.

**Lemma 4.10.** Let $a, b, c \in P_n(X)$. Then

$$(ab)c = a(bc)$$

if one of $a, b, c$ is in $F_n(X)$.

**Continuation of the Proof of Theorem 4.9.** Now we have

$$(ab)^{r}a^{-1}a = ba.$$ 

By Lemma 4.10, we have $((ab)^{r}a^{-1})a = (ab)^{r}(a^{-1}a)$. Notice that $F_n(X)$ is an abelian simplicial group. Thus $1 = r^{-1} = a^{-1}$ and the assertion follows. $\square$

**Proof of Lemma 4.10.** The proof is given by induction on the dimension $q$ of $P_n(X)_q$. Notice that $F_n(X)_q = \{e\}$ for $q < n$. The assertion holds for $q < n$. By Proposition 4.2, the assertion holds for $q = n$. Now suppose that the assertion holds for $k < q$ with $q > n$. Let $a, b, c \in P_n(X)_q$ such that one of $a, b, c$ is in $F_n(X)_q$. By induction, $d_j((ab)c) = d_j(a(bc))$ for all $j$. Then $f_{(ab)c}\Delta[q] = f_{a(bc)}\Delta[q]$, where $f_x$ is the representing map of $x$. Notice that $q > n$ and $\pi_q(P_n(X)) = 0$. The map

$$f_{(ab)c} \cup f_{a(bc)} : \Delta[q] \cup \Delta[q] \to P_n(X)$$

is null. Thus $(ab)c \simeq a(bc)$ and so $(ab)c = a(bc)$. The assertion follows. $\square$

**Proposition 4.11.** Let $X$ be a connected strong homotopy associative simplicial $H$-set and let $\{P_n(X)\}_{n \geq 0}$ be the Moore-Postnikov system of $X$. Then

$$p_n : P_n(X) \to P_{n-1}(X)$$

is a principal $K(\pi_n(X), n)$-bundle.

**Proof.** Let $\alpha, \beta \in F_n(X)$ and let $b_1, b_2 \in P_n(X)$. By Lemma 4.10, we have

$$\alpha(\beta b_1) = (\alpha\beta)b_1.$$ 

Suppose that $ab_1 = ab_2$. By Lemma 4.10, we have

$$b_1 = eb_1 = (\alpha^{-1}\alpha)b_1 = \alpha^{-1}(\alpha b_1) = \alpha^{-1}(\alpha b_2) = (\alpha^{-1}\alpha)b_2 = eb_2 = b_2.$$ 

Thus $P_n(X)$ is a free $K(\pi_n(X), n)$-space. Notice that $p_n : P_n(X) \to P_{n-1}(X)$ is a homomorphism. The map $p_n : P_n(X) \to P_{n-1}(X)$ factors through $P_n(X)/F_n(X)$. Notice that $P_n(X)/F_n(X)$ is a Kan complex and $P_n(X)/F_n(X) \to P_{n-1}(X)$ is a weak homotopy equivalence. Thus $P_n(X)/F_n(X)$ is isomorphic to $P_{n-1}(X)$. The assertion follows. $\square$
Theorem 4.12. Let \( X \) be a connected strong homotopy associative simplicial \( H \)-set and let \( \bar{X} \) be the simplicial \( H \)-set of \( X \) generated by \( \pi_*(X) \). Then
\[
\bar{X} = X.
\]

Proof. We show that the composite \( \psi_n : \bar{X} \to X \to P_n(X) \) is onto for each \( n \) by induction on \( n \). Notice that \( X_n = P_n(X)_n \). Thus the assertion will follow from this statement. Notice that \( P_0(X) = \{e\} \). The statement holds for \( n = 0 \). Suppose that the composite \( \psi_{n-1} : \bar{X} \to X \to P_{n-1}(X) \) is onto. Let \( P_n(\bar{X}) = \psi_n(\bar{X}) \), the image of \( \psi_n \). Notice that \( \psi_n \) is a homomorphism. Thus \( P_n(\bar{X}) \) is a subsimplicial \( H \)-set of \( P_n(X) \). Notice that \( \pi_n(X) \subseteq P_n(\bar{X})_n \) and so \( F_n(X) \subseteq P_n(\bar{X}) \). By the proof of Proposition 4.11, \( F_n(X) \) acts on \( P_n(\bar{X}) \) freely. Thus \( F_n(X) \to P_n(\bar{X}) \to P_n(\bar{X})/F_n(X) \) is a principal \( F_n(X) \)-bundle. By induction, the composite \( X \xrightarrow{\psi_n} P_n(X) \xrightarrow{p_n} P_{n-1}(X) \) is onto. Thus
\[
P_n(\bar{X})/F_n(X) \to P_n(X)/F_n(X) = P_{n-1}(X)
\]
is onto. Consider the commutative diagram of principal \( F_n(X) \)-bundles
\[
\begin{array}{ccc}
F_n(X) & \xrightarrow{\psi_n} & F_n(X) \\
\downarrow & & \downarrow \\
P_n(\bar{X}) & \xrightarrow{} & P_n(X) \\
\downarrow & & \downarrow \\
P_n(\bar{X})/F_n(X) & \to & P_n(X)/F_n(X).
\end{array}
\]
Thus \( P_n(\bar{X}) \to P_n(X) \) is onto and therefore \( P_n(\bar{X}) = P_n(X) \). The assertion follows.

\[\square\]

Corollary 4.13. Let \( R \) be a commutative ring with a unit element and let \( X \) be a strong homotopy associative minimal simplicial \( H \)-set. Then the simplicial algebra \( R(X) \) is generated by \( \pi_*(X) \).

Remark 4.14. (1). The simplicial algebra \( R(X) \) with the multiplication induced by the one in \( X \) is not associative in general. However, the associative and commutative laws hold for some special elements by Propositions 4.2 and 4.7.

(2). For \( x \in \pi_*(X) \), the element \( x - e \) is already a cycle in \( R(X) \). Thus \( \pi_*(X) \) can be identified with certain elements on the cycles of \( R(X) \). The map \( \pi_*(X) \to \)
$H_*(R(X)) \approx H_*(X; R)$ is exactly the Hurewicz map. One may expect to replace $R(X)$ by a chain algebra without missing the elements in $\pi_*(X)$.

(3). One may also give a quotient map from $X$ to a minimal simplicial group by modulo the relation generated by the associators. If $X \simeq \Omega Y$, we do believe that the homology of the abelianizer of the resulting minimal simplicial group is exactly the desuspension of the homology of $Y$.

Let $f : X \to Y$ be a simplicial map of simplicial $H$-sets. The map $f$ is said to be an $H$-map if $f$ preserves the multiplication up to homotopy. The map $f$ is called a simplicial homomorphism if $f$ preserves the multiplication strictly. A simplicial homomorphism is certainly a simplicial $H$-map. But a simplicial $H$-map is not a simplicial homomorphism in general.

**Corollary 4.15.** Let $X$ and $Y$ be connected strong homotopy associative minimal simplicial $H$-sets and let $f, g : X \to Y$ be simplicial homomorphisms. Suppose that $f_* = g_* : \pi_*(X) \to \pi_*(Y)$. Then $f = g$.

Somehow, this statement is related to the Freyd Conjecture. Thus we give a conjecture as follows.

**Conjecture 4.16.** Let $X$ and $Y$ be minimal simplicial sets such that both $X$ and $Y$ are homotopic to the infinite loop spaces of finite spectra with induced multiplications. Let $f : X \to Y$ be an infinite loop map. Then $f$ is homotopic to a simplicial homomorphism.

If this conjecture is true, we can give a proof of the Freyd Conjecture as follows. Let $A$ and $B$ be finite complex and let $f : A \to B$ be a stable map such that $f_* : \pi^S_*(A) \to \pi^S_*(B)$ is zero. Let $X$ and $Y$ be minimal simplicial sets such that $X \simeq Q(A)$ and $Y \simeq Q(B)$, with induced multiplications. Now the stable map $f : A \to B$ induces an infinite loop map $g : X \to Y$. Suppose that our conjecture is true. Then the map $g$ is homotopic to a simplicial homomorphism $g' : X \to Y$. By Corollary 4.15, the map $g'$ should be trivial and so $f$ is null. However, we should point out that we will meet very complicated obstruction problems whence we want to check if an $H$-map is homotopic to a simplicial homomorphism.

5. **Homotopy Type of Strong Homotopy Associative $H$-spaces**

In Section 4, we introduced strong homotopy associative simplicial $H$-sets. The condition of strong homotopy associativity is slightly stronger than that of homotopy associativity. In this section, we will show that every loop space is homotopy equivalent to a strong homotopy associative minimal simplicial $H$-set. Thus the strong homotopy associative minimal simplicial $H$-sets can be considered as the minimal models of the loop spaces.
Lemma 5.1. Let $B$ and $C$ be simplicial sets and let $A \subseteq B \cap C$ be a subsimplicial set such that $A$ is a retract of both $B$ and $C$. Let $X$ be a group-like simplicial $H$-set and let $f : C \cup_A C \to X$ be a simplicial map. Suppose that the composites $B \to B \cup_A C \xrightarrow{f} X$ and $C \to B \cup_A C \xrightarrow{f} X$ are null. Then the map $f : B \cup_A C \to X$ is null.

Proof. Consider the pushout diagram

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & B \cup_A C \\
\downarrow & & \downarrow \\
C/A & = & C/A \\
\end{array}
\]

There is a commutative diagram of short exact sequences of groups

\[
\begin{array}{ccc}
[A, X] & \to & [B, X] \\
\downarrow & & \downarrow \\
[C, X] & \to & [B \cup_A C, X] \\
\downarrow & & \downarrow \\
[C/A, X] & = & [C/A, X].
\end{array}
\]

The assertion follows by inspecting the element $[f] \in [B \cup_A C, X]$ in the diagram. □

Lemma 5.2. Let $f : X \to X'$ be a simplicial $H$-map of group-like simplicial $H$-sets. Suppose that $f$ is a homotopy equivalence and $X'$ is a strong homotopy associative simplicial $H$-set. Then $X$ is a strong homotopy associative simplicial $H$-set.
Proof. Let \( A = e \times X \times X, X \times e \times X, X \times X \times e \) and \( A' = e \times X' \times X', X' \times e \times X' \), \( X' \times X' \times e \), respectively. By Lemma 5.1, the diagram

\[
\begin{array}{ccc}
X^3 \cup_A X^3 & \xrightarrow{\varphi_1 \cup \varphi_2} & X \\
\downarrow f^3 \cup f^3 & & \downarrow f \\
X^{i3} \cup_{A'} X^{i3} & \xrightarrow{\varphi_1 \cup \varphi_2} & X'
\end{array}
\]

commutes up to homotopy, where \( \varphi_1(x, y, z) = (xy)z \) and \( \varphi_2(x, y, z) = x(yz) \). Notice that \( X^3 \cup_A X^3 \simeq X^3 \times 0 \cup A \times I \cup X^3 \times 1 \) and the diagram

\[
\begin{array}{ccc}
X^3 \cup_A X^3 & \rightarrow & X^3 \times 0 \cup A \times I \cup X^3 \times 1 \\
\downarrow f^3 \cup f^3 & & \downarrow f^3 \times 0 \cup f^3|_A \times I \cup f^3 \times 1 \\
X^{i3} \cup_A X^{i3} & \rightarrow & X^{i3} \times 0 \cup A' \times I \cup X^{i3} \times 1
\end{array}
\]

commutes. Thus the diagram

\[
\begin{array}{ccc}
X^3 \times 0 \cup A \times I \cup X^3 \times 1 & \xrightarrow{\varphi_1 \cup (\varphi_1|_A \times I) \cup \varphi_2} & X \\
\downarrow f^3 \times 0 \cup (f^3|_A \times I) \cup f^3 \times 1 & & \downarrow f \\
X^{i3} \times 0 \cup A' \times I \cup X^{i3} \times 1 & \xrightarrow{\varphi_1 \cup (\varphi_1|_A' \times I) \cup \varphi_2} & X'
\end{array}
\]

commutes up to homotopy. Notice that \( X' \) is strong homotopy associative. The map \( \varphi_1 \cup (\varphi_1|_A \times I) \cup \varphi_2 : X^{i3} \times 0 \cup A' \times I \cup X^{i3} \times 1 \rightarrow X' \) extends to a map \( X^{i3} \times I \rightarrow X' \). Thus the map

\[\varphi_1 \cup (\varphi_1|_A \times I) \cup \varphi_2 : X^3 \times 0 \cup A \times X^3 \times 1 \rightarrow X\]

extends to a map \( X^3 \times I \rightarrow X \) up to homotopy. Notice that the injection \( X^3 \times 0 \cup A \times X^3 \times 1 \rightarrow X^3 \times I \) is a cofibre map. The assertion follows.

Corollary 5.3. Every loop space is homotopy equivalent to a group-like strong homotopy associative minimal simplicial \( H \)-set.

6. Minimal Simplicial Sets

It is well known that, for a free abelian group \( G \), there exists a sub-contractible simplicial group \( H \) such that \( G/H \) is minimal \([?, ?, ?, ?]\). In this section, we will
show that, for any simplicial free group $G$, there exists a sub-contractible simplicial group $H$ such that $G/H$ is minimal, where a simplicial free group $G$ is a simplicial group such that each group $G_n$ is free. In the non-commutative case, the subsimplicial group $H$ is not normal in general and so $G/H$ is a minimal simplicial and so $G/H$ is a minimal simplicial coset in general. This shows that, for any loop space, we can pick a minimal simplicial set which has a simplicial group action in addition to the product structure.

Since our definition of simplicial free group is slightly different from the standard definition of free simplicial group [1], we give a proof of the following lemma.

**Lemma 6.1.** Let $G$ be a simplicial group such that each group $G_n$ is free. Then there is an isomorphism

$$
\pi_n(G^{ab}) \cong H_{n+1}(BG; \mathbb{Z})
$$

for each $n \geq 0$, where $G^{ab}$ is the abelianizer of $G$.

**Proof.** First we assume that $G$ is a discrete free group. The natural map $BG \to BG^{ab}$ can extend to a simplicial homomorphism $Z(BG) \to BG^{ab}$, where $Z(BG)$ is the reduced free abelian simplicial group generated by the pointed simplicial set $BG$. The resulting map $Z(BG) \to BG^{ab}$ is a homotopy equivalence.

In the general case, consider the bisimplicial sets

$$(BG)_{m,n} = (BG_m)_n$$

and

$$B(G^{ab})_{m,n} = (BG^{ab}_m)_n.$$ 

The natural map $B(G)_{**} \to B(G^{ab})_{**}$ extends to a bisimplicial homomorphism $Z(B(G)_{**}) \to B(G^{ab})_{**}$, where $Z(B(G)_{**})$ is the reduced free abelian bisimplicial group generated by the pointed bisimplicial set $B(G)_{**}$. Notice that

$$Z(B(G)_{m**}) \to B(G^{ab})_{m**}$$

is a homotopy equivalence for each $m$. Thus $Z(B(G)_{**}) \to B(G^{ab})_{**}$ is a homotopy equivalence ([1], Lemma 4.2, p. 335; also [2, ?]). Notice that the diagonal simplicial set $B^{\Delta}(G)$ of $B(G)_{**}$ is homotopy equivalent to $BG$ ([1], p. 88; or [2], Proposition 3.1). The assertion follows. 

**Theorem 6.2.** Let $G$ be a simplicial group such that each $G_n$ is free. Then there exists a subsimplicial group $H$ such that

1. $H$ is contractible.
2. $G/H$ is minimal.
Proof. Let \( \{P_n(G), \tilde{P}_n(G)\}_{n \geq 0} \) be the modified Moore-Postnikov system of \( G \) introduced in Section 2. First we construct a sequence of simplicial groups \( \{H^n\}_{n \geq 0} \) such that

1. \( H^n \) is a subsimplicial group of \( P_n(G) \) for each \( n \geq 0 \).
2. \( H^n \) is contractible for each \( n \geq 0 \).
3. \( P_n(G)/H^n \) is minimal for each \( n \geq 0 \).
4. \( p_n(H^n) = H^{n-1} \) for each \( n \geq 1 \), where \( p_n: P_n(G) \to P_{n-1}(G) \) is the canonical quotient homomorphism.
5. \( p_n: H^n \to H^{n-1} \) is an isomorphism for \( q \leq n - 1 \) and \( n \geq 1 \).

The construction of \( H^n \) is given by induction on \( n \). The simplicial group \( H^0 \) is defined to be the kernel of \( P_0(G) \to \tilde{P}_0(G) = K(\pi_0(G), 0) \). Notice that \( \tilde{P}_0(G) \) is minimal and \( P_0(G) \to \tilde{P}_0(G) \) is a homotopy equivalence. Thus \( H^0 \) is contractible. Suppose that \( H^{n-1} \) is defined to satisfy the conditions (1) to (5) with \( n \geq 1 \). Let \( H^n \) be the simplicial group in the following pull-back diagram

\[
\begin{align*}
\tilde{H}^n & \to \tilde{P}_n(G) \\
\text{pull} & \downarrow \tilde{p}_n \\
H^{n-1} & \to P_{n-1}(G),
\end{align*}
\]

where \( \tilde{p}_n: \tilde{P}_n(G) \to P_{n-1}(G) \) is the canonical quotient homomorphism in the modified Moore-Postnikov system of \( G \). By a result of the author (Propositions 2.5 and 2.6, [?]), the short exact sequence

\[
\tilde{F}_n(G) \to \tilde{P}_n(G) \xrightarrow{\tilde{p}_n} P_{n-1}(G)
\]

is a central extension and the kernel \( \tilde{F}_n(G) \) is the minimal simplicial group \( K(\pi_n(G), n) \). Thus there is a central extension

\[
K(\pi_n(G), n) \overset{i}{\to} \tilde{H}^n \to H^{n-1}.
\]

Now let \( G' \) be the simplicial group in the following pull-back diagram

\[
\begin{align*}
G' & \to G \\
\text{pull} & \downarrow \bar{q}_n \\
\tilde{H}^n & \to \tilde{P}_n(G),
\end{align*}
\]
where \( \tilde{q}_n : G \rightarrow \tilde{P}_n G \) is the canonical quotient homomorphism. Notice that \( \pi_q (G) \cong \pi_q (\tilde{P}_n G) \) for \( q \leq n \), \( \pi_n (\tilde{H}^n) \cong \pi_n (\tilde{P}_n G) \), and \( \tilde{H}^n \cong K (\pi_n (G), n) \). Thus \( \pi_q (G') = 0 \) for \( q < n \) and \( \pi_n (G') \cong \pi_n (\tilde{H}^n) \cong \pi_n (G) \). Notice \( \tilde{H}^n \) is a subsimplicial group of \( \tilde{P}_n (G) \). Thus \( G' \) is also a subsimplicial group of \( G \) and so \( G' \) is a simplicial free group, \( i.e. \), each \( G' \) is a free group. Thus \( \pi_q (G'^{ab}) = 0 \) for \( q < n \) and the map \( \pi_n (G') \rightarrow \pi_n (G'^{ab}) \) is an isomorphism by Lemma 6.1 and the Hurewicz Theorem. Now \( G'^{ab} \) is a free abelian simplicial group. There is a simplicial homomorphism \( \varphi : G'^{ab} \rightarrow K (\pi_n (G), n) \) such that \( \varphi_* : \pi_n (G'^{ab}) \rightarrow \pi_n (K (\pi_n (G), n)) \) is an isomorphism. Let \( \tilde{R}_n (G) \) be the kernel of \( \tilde{q}_n : G \rightarrow \tilde{P}_n (G) \). Then there is a short exact sequence

\[
\tilde{R}_n (G) \rightarrow G' \rightarrow \tilde{H}^n.
\]

Notice that \( G_q \cong \tilde{P}_n (G)_q \) for \( q \leq n - 1 \). Thus \( \tilde{R}_n (G)_q = \{ e \} \) for \( q \leq n - 1 \) and so \( N \tilde{R}_n (G)_q = \{ e \} \) for \( q \leq n - 1 \), where \( NG \) is the Moore chain complex of a simplicial group \( G \). Let \( \psi \) be the composite

\[
\tilde{R}_n (G) \rightarrow G' \rightarrow G'^{ab} \xrightarrow{\varphi} K (\pi_n (G), n).
\]

We want to show that the simplicial homomorphism \( \psi : \tilde{R}_n (G) \rightarrow K (\pi_n (G), n) \) is trivial. Notice that any simplicial group is generated by its Moore chain complex as a simplicial group [?]. It suffices to show that \( N \psi : N \tilde{R}_n (G) \rightarrow NK (\pi_n (G), n) \) is trivial. Notice that

\[
NK (\pi_n (G), n)_q = \begin{cases} 
\pi_n (G) & \text{if } q = n \\
0 & \text{if } q \neq n.
\end{cases}
\]

It suffices to show that \( N \psi : N \tilde{R}_n (G)_n \rightarrow NK (\pi_n (G), n)_n \) is trivial. Notice that \( N \tilde{R}_n (G)_{n - 1} = \{ e \} \). Thus all elements in \( N \tilde{R}_n (G)_n \) are boundaries and so \( N \psi : N \tilde{R}_n (G)_n \rightarrow NK (\pi_n (G), n)_n \) is trivial, or \( \psi : \tilde{R}_n (G) \rightarrow K (\pi_n (G), n) \) is trivial. Thus the composite \( G' \rightarrow G'^{ab} \rightarrow K (\pi_n (G), n) \) factors through \( \tilde{H}^n \) and so there is a simplicial homomorphism \( f : \tilde{H}^n \rightarrow K (\pi_n (G), n) \) such that \( f_* : \pi_n (\tilde{H}^n) \rightarrow \pi_n (K (\pi_n (G), n)) \) is an isomorphism. Now the composite homomorphism

\[
K (\pi_n (G), n) \xrightarrow{i} \tilde{H}^n \xrightarrow{f} K (\pi_n (G), n)
\]

is a weak homotopy equivalence of minimal simplicial groups. Thus the composite \( f \circ i : K (\pi_n (G), n) \rightarrow K (\pi_n (G), n) \) is an isomorphism of simplicial groups and so

\[
\tilde{H}^n \cong K (\pi_n (G), n) \times H^{n - 1}
\]
as simplicial groups. Thus there exists a simplicial monomorphism $\theta: H^{n-1} \to \tilde{P}_n(G)$ such that the diagram

$$
\begin{array}{c}
H^{n-1} \xrightarrow{\theta} \tilde{P}_n(G) \\
\parallel \\
H^{n-1} \xrightarrow{\tilde{P}_n(G)} \end{array}
$$

commutes. Let $\tilde{H}^{n-1} = \theta(H^{n-1})$. Then there is a commutative diagram

$$
\begin{array}{c}
K(\pi_n(G), n) \xrightarrow{=} K(\pi_n(G), n) \\
\parallel \\
\tilde{H}^{n-1} \xrightarrow{\tilde{P}_n(G)} \tilde{P}_n(G)/\tilde{H}^{n-1} \\
\parallel \\
H^{n-1} \xrightarrow{\tilde{P}_n(G)} P_{n-1}(G)/H^{n-1} \\
\parallel \\
\tilde{P}_n(G) \xrightarrow{\tilde{P}_n(G)} \end{array}
$$

It is easy to check that $\tilde{p}: P_n(G)/\tilde{H}^{n-1} \to P_{n-1}(G)/H^{n-1}$ is a principal $K(\pi_n(G), n)$-bundle. By Lemma 3.4, $\tilde{P}_n(G)/\tilde{H}^{n-1}$ is minimal. Now define $H^n$ to be the simplicial group in the following pull-back diagram

$$
\begin{array}{c}
H^n \xrightarrow{\tilde{P}_n(G)} \tilde{P}_n(G) \\
\parallel \\
\tilde{H}^{n-1} \xrightarrow{\tilde{P}_n(G)} \tilde{P}_n(G) \\
\parallel \\
H^n \xrightarrow{\tilde{P}_n(G)} \\
\parallel \\
\tilde{P}_n(G). \\
\end{array}
$$

Then $P_n(G)/H^n \cong \tilde{P}_n(G)/\tilde{H}^{n-1}$. Notice that $P_n(G)/H^n$ is minimal and $P_n(G) \cong P_n(G)/H^n$. Thus $H^n$ is contractible. Notice that $P_n(G)_q \cong \tilde{P}_n(G)_q$ for $q \leq n - 1$. Thus $H^n_q \cong \tilde{H}_q^{n-1} \cong H_q^{n-1}$ for $q \leq n - 1$. This gives a construction of $\{H^n\}_{n \geq 0}$.

Now let $H = \lim_{n \geq 0} H^n$, the inverse limit. Then $H_q \cong H_q^n$ for $q \leq n$. Thus $H$ is contractible. Notice that $G_q \cong P_n(G)_q$ for $q \leq n$. Thus $(G/H)_q \cong (P_n(G)/H^n)_q$ for $q \leq n$ and so $G/H$ is minimal. The proof is done. \qed
7. Abelianizer of a Minimal Simplicial $H$-set

In this section, we give a relation between a minimal simplicial $H$-set and its abelianizer.

**Definition 7.1.** Let $X$ be a simplicial $H$-set. The abelianizer $X^{\text{ab}}$ of $X$ is defined to be the quotient simplicial set of $X$ modulo the product equivalence relation generated by the relations

1. $a(bc) \approx a(bc)$
2. $ab \approx ba$

for $a, b, c \in X$. Thus $X^{\text{ab}}$ is an abelian simplicial monoid.

**Definition 7.2.** Let $X$ be a simplicial $H$-set with right (left) inverse. A subsimplicial $H$-set $X_0$ of $X$ is said to be right (left) group-like if $r_x^{-1} \in X_0$ for $x \in X'$, i.e., the right inverse of an element in $X_0$ is in $X_0$.

**Lemma 7.3.** Let $X$ be a connected strong homotopy associative minimal simplicial $H$-set and let $X_0$ be a right (or left) group-like subsimplicial $H$-set of $X$. Then $X_0$ is brant and so $X_0$ is also minimal.

**Proof.** Let $P_n(X')$ be the image of the composite $X' \to X \to P_n(X)$. By the definition of the equivalence relation $\sim_n$ in Section 2, $P_n(X') = X'/\sim_n$. First we show that $P_n(X')$ is fibrant for each $n$ by induction on $n$. Notice that $P_0(X') = P_0(X) = \{e\}$. Thus $P_0(X')$ is fibrant. Now suppose that $P_{n-1}(X')$ is fibrant with $n > 0$. Let $E$ be the simplicial set in the following pull-back diagram

$$
\begin{array}{ccc}
E & \longrightarrow & P_n(X) \\
\downarrow \text{pull} & & \downarrow \\
P_{n-1}(X) & \longrightarrow & P_{n-1}(X).
\end{array}
$$

Then there is a principal $F_n(X)$-bundle

$$F_n(X) \to E \to P_{n-1}(X').$$

Let $F_n(X') = F_n(X) \cap P_n(X')$. Then $F_n(X')$ is a right group-like subsimplicial $H$-set of the minimal simplicial abelian group $F_n(X) \cong K(\pi_n(X), n)$. Thus $F_n(X')$ is a minimal simplicial abelian group. By Theorem 4.9, $P_n(X')$ is a free $F_n(X')$-simplicial
set. Thus there is a commutative diagram

\[
\begin{array}{ccc}
F_n(X') & \rightarrow & F_n(X) \\
\downarrow & & \downarrow \\
P_n(X') & \rightarrow & E \\
\downarrow & & \downarrow \\
P_n(X')/F_n(X') & \rightarrow & P_{n-1}(X').
\end{array}
\]

Notice that \(P_n(X') \rightarrow P_{n-1}(X')\) is onto. Thus \(q: P_n(X')/F_n(X') \rightarrow P_{n-1}(X')\) is onto. Now let \(\{x_1\}, \{x_2\} \in P_n(X')/F_n(X')\) such that \(q(\{x_1\}) = q(\{x_2\})\), where \(x_1, x_2 \in P_n(X')\). Then there exists an \(\alpha \in F_n(X)\) such that \(x_1 = \alpha x_2\) in \(P_n(X')\). By Lemma 4.10, \(x_1 x_2^{-1} = (\alpha x_2) \cdot (x_2^{-1}) = \alpha x_2^{-1} = \alpha\). Notice that \(x_1 \cdot x_2^{-1} \in P_n(X')\). Thus \(\alpha \in F_n(X') = P_n(X') \cap F_n(X)\) and so \(\{x_1\} = \{x_2\}\). Thus \(q: P_n(X')/F_n(X') \rightarrow P_{n-1}(X')\) is an isomorphism and \(P_n(X') \rightarrow P_{n-1}(X')\) is a principal \(F_n(X')\)-bundle. By induction, \(P_{n-1}(X')\) is fibrant. Thus \(P_n(X')\) is fibrant. The induction is finished. Now notice that \(X' = \lim_n P_n(X')\) and \(X'_q \cong P_n(X')_q\) for \(q \leq n\). Thus \(X'\) is fibrant which is the assertion. \(\square\)

**Theorem 7.4.** Let \(X\) be a connected strong homotopy associative minimal simplicial \(H\)-set and let \(p: X \rightarrow X^{ab}\) be the quotient map. Let \(S \subseteq \pi_*(X)\) be a (graded) subset of \(\pi_*(X)\) such that \(p_*(S)\) is a set of generators for \(p_*(\pi_*(X)) = \text{Im}(p_*: \pi_*(X) \rightarrow \pi_*(X^{ab}))\). Then the right (or left) group-like subsimplicial \(H\)-set of \(X\) generated by \(S\) is equal to the total space \(X\).

**Proof.** Let \(X'\) be the right group-like subsimplicial \(H\)-set of \(X\) generated by \(S\). It suffices to show that \(P_n(X') = P_n(X)\) for each \(n \geq 0\). The proof of this statement is given by induction on \(n\). Notice that \(P_0(X') = P_0(X) = \{e\}\). Suppose that \(P_{n-1}(X') = P_{n-1}(X)\) with \(n > 0\). By Lemma 7.3, \(X'\) is also a connected strong homotopy associative minimal simplicial \(H\)-set. Thus \(F_n(X') \cong K(\pi_n(X'), n)\) is a minimal abelian simplicial group and \(F_n(X') = P_n(X') \cap F_n(X)\). Let

\[q: F_n(X) \rightarrow F_n(X)/F_n(X')\]
be the quotient simplicial homomorphism. Now we define a function \( \varphi: P_n(X) \to F_n(X)/F_n(X') \) as follows. Consider the commutative diagram

\[
\begin{array}{ccc}
F_n(X) & \longrightarrow & P_n(X) \longrightarrow P_{n-1}(X) \\
\downarrow & & \downarrow \\
F_n(X') & \longrightarrow & P_n(X') \longrightarrow P_{n-1}(X),
\end{array}
\]

where \( p_n: P_n(X) \to P_{n-1}(X) \) and \( p'_n: P_n(X') \to P_{n-1}(X) = P_n(X') \) are principal \( F_n(X) \)- and \( F_n(X') \)-bundles, respectively. Let \( x \in P_n(X) \). There exist \( a \in P_n(X') \) and \( \alpha \in F_n(X) \) such that \( x = \alpha a \) in \( P_n(X) \). Define

\[
\varphi(x) = q(x).
\]

First we need to show that \( \varphi \) is well-defined as a function. Let \( x = \alpha' a' \) with \( \alpha' \in F_n(X) \) and \( a' \in P_n(X') \). Then \( p'_n(a) = p'_n(a') \) and so \( a \cdot a'^{-1} \in F_n(X') \). Now

\[
q(\alpha') = q(\alpha' \cdot (a' \cdot a'^{-1})) \\
= q((\alpha'a') \cdot a'^{-1}) \quad \text{(Lemma 4.10)} \\
= q((\alpha a) \cdot a'^{-1}) \\
= q(\alpha \cdot (a \cdot a'^{-1})) \quad \text{(Lemma 4.10)} \\
= q(\alpha).
\]

Thus \( \varphi: P_n(X) \to F_n(X)/F_n(X') \) is well-defined as a function. It is easy to check that \( \varphi d_j = d_j \varphi \) and \( \varphi s_j = s_j \varphi \). Thus \( \varphi: P_n(X) \to F_n(X)/F_n(X') \) is a simplicial map. Let \( x = \alpha a \) and \( y = \beta b \) in \( P_n(X) \), where \( \alpha, \beta \in F_n(X) \) and \( a, b \in P_n(X') \). Then

\[
xy = (\alpha a) \cdot (\beta b) \\
= \alpha(\alpha (\beta b)) \quad \text{(Lemma 4.10)} \\
= \alpha((\beta a) b) \\
= \alpha((\beta a) b) \quad \text{(Theorem 4.9)} \\
= (\alpha(\beta a)) b \quad \text{(Lemma 4.10)} \\
= ((\alpha \beta)a) b \\
= (\alpha \beta)(ab).
\]

Thus \( \varphi(xy) = q(\alpha \beta) = q(\alpha) q(\beta) = \varphi(x) \varphi(y) \) and so \( \varphi: P_n(X) \to F_n(X)/F_n(X') \) is a simplicial homomorphism. Notice that \( F_n(X)/F_n(X') \) is an abelian simplicial
group. Thus \( \varphi \) factors through \( P_n(X)^{ab} \), i.e., there is a simplicial homomorphism \( \bar{\varphi} : P_n(X)^{ab} \to F_n(X)/F_n(X') \) such that the diagram

\[
\begin{array}{ccc}
P_n(X) & \xrightarrow{\varphi} & F_n(X)/F_n(X') \\
p' & & \\
\downarrow & & \\
P_n(X)^{ab} & \xrightarrow{\bar{\varphi}} & \\
\end{array}
\]

commutes, where \( p' : P_n(X) \to P_n(X)^{ab} \) is the quotient map. Consider the commutative diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{i} & X \\
\downarrow p' & & \downarrow p \\
X'^{ab} & \xrightarrow{q_n} & P_n(X) \\
\end{array}
\]

Notice that \( \pi_n(X) \cong \pi_n(P_n(X)) \) and \( p_* (S) \) is a set of generators for \( p_* (\pi_n(X)) \subseteq \pi_*(X^{ab}) \). Thus, for any \( \alpha \in \pi_n(P_n(X)) \), there exists an element \( \beta \in \pi_n(X') \) such that

\[ p'_*(\alpha) = p'_* \circ q_n \circ i_*(\beta) \]

in \( \pi_n(P_n(X)^{ab}) \). Notice that there is a commutative diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{i} & X \\
\downarrow q_n & & \downarrow q_n \\
P_n(X') & \xrightarrow{i_n} & P_n(X) \\
\end{array}
\]

Thus, for any \( \alpha \in \pi_n(P_n(X)) \), there exists an element \( \beta \in \pi_n(P_n(X')) \) such that

\[ p'_*(\alpha) = p'_* \circ i_*(\beta) \]

in \( \pi_n(P_n(X)^{ab}) \) and so

\[ \varphi_*(\alpha) = \bar{\varphi}_* \circ p'_*(\alpha) = \bar{\varphi}_* \circ p'_* \circ i_*(\beta) = \varphi_* \circ i_*(\beta). \]

Notice that the composite

\[
P_n(X') \xrightarrow{i_n} P_n(X) \xrightarrow{\varphi} F_n(X)/F_n(X')
\]

is trivial. Thus \( \varphi_* : \pi_n(P_n(X)) \to \pi_n(F_n(X)/F_n(X')) \) is trivial and so \( q_* : \pi_n(F_n(X)) \to \pi_n(F_n(X)/F_n(X')) \) is trivial. Notice that \( F_n(X) = K(\pi_n(X), n) \) and \( F_n(X)/F_n(X') = \)
$K(\pi_n(X)/\pi_n(X'), n)$ are minimal simplicial abelian groups. Thus $q_n : \pi_n(F_n(X)) \to \pi_n(F_n(X)/F_n(X'))$ is onto and so

$$\pi_n(F_n(X)/F_n(X')) = 0.$$ 

Thus $F_n(X)/F_n(X') = \{e\}$ or $F_n(X') = F_n(X)$. Now consider the commutative diagram of principal $F_n(X)$-bundles

$$
\begin{array}{ccc}
F_n(X) & \longrightarrow & P_n(X) \\
\downarrow & & \downarrow \\
F_n(X) & \longrightarrow & P_n(X').
\end{array}
$$

Thus $P_n(X') = P_n(X)$ and the assertion follows. \hfill \Box

**Corollary 7.5.** Let $X$ be a connected strong homotopy associative minimal simplicial $H$-set and let $H : \pi_*(X) \to \check{H}_*(X;\mathbb{Z})$ be the Hurewicz map. Then there exists a (graded) subset $S$ of $\pi_*(X)$ such that

1. $H(x) \neq 0$ for each $x \in S$;
2. $H(x_1) \neq H(x_2)$ for $x_1 \neq x_2$ in $S$; and
3. The right (or left) group-like subsimplicial $H$-set of $X$ generated by $S$ is equal to the total space $X$.

**Proof.** Let $Z(x)$ be the reduced free abelian simplicial group generated by $X$, i.e., the free abelian simplicial group generated by $X$ modulo the subsimplicial group generated by the base-point $e$. Notice that $X^{ab}$ is abelian simplicial group. Thus the quotient map $p : X \to X^{ab}$ extends to $Z(X)$. More precisely, there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{p} & X^{ab} \\
\downarrow h & & \downarrow \\
\mathbb{Z}(X), & \xleftarrow{h} &.
\end{array}
$$

where $h(x) = x - e$ for $x \in X$. Notice that $h_* : \pi_*(X) \to \pi_*(\mathbb{Z}(X)) \cong \check{H}_*(X;\mathbb{Z})$ is the Hurewicz map. The assertion follows from Theorem 7.4. \hfill \Box
8. Examples of Minimal Simplicial Groups

In this section, we give some examples and counterexamples of minimal simplicial groups. We will give a complete answer when a two-stage Postnikov system $X$ with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+i}(X) = \mathbb{Z}/2$ is homotopy equivalent to a minimal simplicial group.

**Lemma 8.1.** Let $n \geq 1$ and $i \geq 1$ be positive integers. Then

1. If $i > n$, there is no two-stage Postnikov system $X$, up to homotopy, with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+i}(X) = \mathbb{Z}/2$ such that $X$ has a nontrivial Postnikov invariant and $X$ is homotopy equivalent to a minimal simplicial group.
2. If $1 \leq i \leq n$, there is at most one possible two-stage Postnikov system $X$, up to homotopy, with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+i}(X) = \mathbb{Z}/2$ such that $X$ has nontrivial Postnikov invariant and such that $X$ is homotopy equivalent to a minimal simplicial group.

**Proof.** Let $X$ be a two-stage Postnikov system with $\pi_n(X) = \mathbb{Z}/2$, $\pi_{n+i}(X) = \mathbb{Z}/2$, and a nontrivial Postnikov invariant. Suppose that there exists a minimal simplicial group $G$ such that $X \simeq G$. Then $G^{ab} \simeq K(\mathbb{Z}/2, n)$ and there is a central extension

$$K(\mathbb{Z}/2, n + i) \longrightarrow G \longrightarrow K(\mathbb{Z}/2, n)$$

by Theorem 4.9. Let $f: S^n \to G$ be the inclusion of the nondegenerate bottom cell and let $\varphi: F(S^n) \to G$ be the simplicial homomorphism such that $\varphi|_{S^n} = f$. By Theorem 7.5, $\varphi: F(S^n) \to G$ is a surjection. Notice that

$$\Gamma^3_{(2)}(G) = \{e\},$$

where $\{\Gamma^r_{(p)}(G)\}_{r \geq 1}$ is the (mod $p$) lower central series of $G$ starting with $\Gamma^1_{(p)}(G) = G$. Thus there is a commutative diagram of central extensions

$$K(\mathbb{Z}/2, n + i) \longrightarrow G \longrightarrow K(\mathbb{Z}/2, n)$$

$$\Gamma^2_{(2)}(F(S^n))/\Gamma^3_{(2)}(F(S^n)) \longrightarrow F(S^n)/\Gamma^3_{(2)}(F(S^n)) \longrightarrow K(\mathbb{Z}/2, n).$$

Notice that $[?]$

$$\pi_{n+r}(\Gamma^2_{(2)}(F(S^n))/\Gamma^3_{(2)}(F(S^n))) \simeq \pi_{n+r}(L^2_{(2)}(K(\mathbb{Z}/2, n))) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq r \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where $L^2(V)$ is the free restricted Lie algebra generated by the $\mathbb{Z}/2$-simplicial vector space $V$. Let $x_{n+r}$ be a cycle of dimension $n + r$ in $\Gamma^2_{(2)}(F(S^n))/\Gamma^3_{(2)}(F(S^n))$ which
is a representative of $\pi_{n+r}(\Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n))) \cong \mathbb{Z}/2$. There is a simplicial homomorphism

$$\psi: \bigoplus_{0 \leq r \leq n} K(\mathbb{Z}/2, n + r) \to \Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n))$$

such that $\psi(l_{n+r}) = x_{n+r}$, where $l_{n+r}$ is the nondegenerate generator of dimension $n+r$ in $K(\mathbb{Z}/2, n + r)$. Then $\psi$ is a homotopy equivalence. Let $\theta_{r,i}$ be the composite

$$K(\mathbb{Z}/2, n + r) \xleftarrow{\bigoplus_{0 \leq r \leq n} K(\mathbb{Z}/2, n + r)} \to \Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n)) \xrightarrow{\bar{\phi}} K(\mathbb{Z}/2, n + i).$$

Then $\theta_{r,i}: K(\mathbb{Z}/2, n + r) \to K(\mathbb{Z}/2, n + i)$ is a simplicial homomorphism. Notice that $N(\theta_{r,i}): N(K(\mathbb{Z}/2, n + r)) \to N(K(\mathbb{Z}/2, n + i))$ is trivial for $r \neq i$. Thus $\theta_{r,i}$ is trivial for $r \neq i$.

Case I. $i > n$.

Notice that $\theta_{r,i}: K(\mathbb{Z}/2, n + r) \to K(\mathbb{Z}/2, n + i)$ is trivial for all $r \leq n$. Thus $\bar{\phi} \circ \psi: \bigoplus_{0 \leq r \leq n} K(\mathbb{Z}/2, n + r) \to K(\mathbb{Z}/2, n + i)$ is trivial and so $B\bar{\phi} \circ B\psi: \bigoplus_{0 \leq r \leq n} K(\mathbb{Z}/2, n + r + 1) \to K(\mathbb{Z}/2, n + i + 1)$ is trivial. Consider the commutative diagram of fibre sequences

$$
\begin{array}{ccc}
K(\mathbb{Z}/2, n) & \xrightarrow{k(G)} & K(\mathbb{Z}/2, n + i + 1) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}/2, n) & \xrightarrow{\partial} & B(\Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n))) \xrightarrow{B(\phi)} B(F(S^n)/\Gamma^3_2(F(S^n))),
\end{array}
$$

where $k(G)$ is the Postnikov invariant of $G$. Thus $k(G)$ is null, which contradicts the fact that $X$ has a nontrivial Postnikov invariant. Assertion (1) follows.

Case II. $1 \leq i \leq n$.

There is a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{0 \leq r \leq n} K(\mathbb{Z}/2, n + r) & \xrightarrow{\psi} & \Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n)) \\
\downarrow{p_i} & & \downarrow{\bar{\phi}} \\
K(\mathbb{Z}/2, n + i) & \xrightarrow{k(G)} & K(\mathbb{Z}/2, n + i)
\end{array}
$$
where \( p_i \) is the projection, and so there is a commutative diagram

\[
\begin{array}{ccc}
K(\mathbb{Z}/2, n) & \xrightarrow{k(G)} & K(\mathbb{Z}/2, n + i + 1) \\
\partial & & \downarrow B\theta \\
K(\mathbb{Z}/2, n) & \xrightarrow{B(\Gamma_2^1(F(S^n)))} & \mathcal{K}(\mathbb{Z}/2, n + r + 1) & \xrightarrow{B\psi} & K(\mathbb{Z}/2, n + i + 1),
\end{array}
\]

where \( \partial: K(\mathbb{Z}/2, n) \to B(\Gamma_2^1(F(S^n))) \) is the boundary of the fibre sequence

\[
B(\Gamma_2^1(F(S^n))) \to B(F(S^n)/\Gamma_2^1(F(S^n))) \to K(\mathbb{Z}/2, n + 1).
\]

Notice that the Postnikov invariant \( k(G) \simeq k(X) \) is nontrivial. Thus \( B\theta: K(\mathbb{Z}/2, n + i + 1) \to K(\mathbb{Z}/2, n + i + 1) \) is the identity map and so \( k(G) \simeq Bp_i \circ B\psi^{-1} \circ \partial \). The assertion (2) follows.

Thus \( F(S^n)/\Gamma_2^1(F(S^n)) \) is universal in some sense. Now we study the minimal simplicial groups produced from \( F(S^n)/\Gamma_2^1(F(S^n)) \). For simplicity, \( \Gamma_2^1(F(S^n)) \) is denoted by \( \Gamma \). Notice that \( \Gamma^2/\Gamma^3 \) is a simplicial vector space over \( \mathbb{Z}/2 \). Thus, for each \( 0 \leq i \leq n \), there exists a subsimplicial vector space \( V^i(n) \) of \( \Gamma^2/\Gamma^3 \) such that \( (\Gamma^2/\Gamma^3)/V^i(n) \) is a minimal simplicial group \( K(\mathbb{Z}/2, n + i) \) and \( \pi_n(\Gamma^2, \Gamma^3) \cong \pi_{n+i}(\Gamma^2/\Gamma^3)/V^i(n)) \). Notice that the short exact sequence of simplicial groups

\[
\Gamma^2/\Gamma^3 \longrightarrow F(S^n)/\Gamma^3 \longrightarrow K(\mathbb{Z}/2, n)
\]
is a central extension. Thus \( V^i(n) \) is a normal subsimplicial group of \( F(S^n)/\Gamma^3 \) and so \( G^i(n) = (F(S^n)/\Gamma^3)/V^i(n) \) is a simplicial group. Notice that there is a short exact sequence of simplicial groups

\[
K(\mathbb{Z}/2, n + i) \longrightarrow G \longrightarrow K(\mathbb{Z}/2, n).
\]

Thus \( G^i(n) \) is a minimal simplicial group for each \( 0 \leq i \leq n \). It is easy to check that \( G^0(n) \cong K(\mathbb{Z}/4, n) \). We will determine the Postnikov invariant of \( G^i(n) \) for \( 0 < i \leq n \).

**Lemma 8.2.** Given \( V^i(n + 1) \) with \( 0 \leq i \leq n \), there is a choice of \( V^i(n) \) such that

\[
G^i(n) \cong \Omega(G^i(n + 1))
\]
as simplicial groups.
Proof. Notice that \( \Omega(F(S^{n+1}))_q = \ker(d_0: F(S^{n+1})_{q+1} \to F(S^{n+1})_q) \). There is a commutative diagram of simplicial groups

\[
\begin{array}{ccc}
\Omega(\Gamma^3_2(F(S^n))) & \longrightarrow & \Omega(F(S^{n+1}))/\Gamma^3_2(F(S^{n+1})) \\
\downarrow s & & \downarrow \bar{s} \\
\Gamma^3_2(F(S^n)) & \longrightarrow & F(S^n)/\Gamma^3_2(F(S^n)),
\end{array}
\]

where \( S \) is the suspension homomorphism and so there is a commutative diagram of short exact sequences of simplicial groups

\[
\begin{array}{ccc}
\Omega(\Gamma^2_2(F(S^{n+1}))/\Gamma^3_2(F(S^{n+1}))) & \longrightarrow & \Omega(F(S^{n+1}))/\Gamma^3_2(F(S^{n+1}))/K(\mathbb{Z}/2, n) \\
\downarrow \bar{s} & & \downarrow \bar{s} \\
\Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n)) & \longrightarrow & F(S^n)/\Gamma^3_2(F(S^n))/K(\mathbb{Z}/2, n).
\end{array}
\]

Notice that \( \bar{s}_*: \pi_r(F(S^n))/\Gamma^3_2(F(S^n)) \to \pi_r(\Omega(F(S^{n+1}))/\Gamma^3_2(F(S^{n+1}))) \) is an isomorphism for \( r \leq 2n \). Let \( V^i(n) \) be the pull-back of the diagram

\[
\begin{array}{ccc}
\Omega(V^i(n+1)) & \longrightarrow & \Omega(\Gamma^2_2(F(S^{n+1}))/\Gamma^3_2(F(S^{n+1}))) \\
\downarrow & & \downarrow \\
\Gamma^2_2(F(S^n))/\Gamma^3_2(F(S^n)) & \longrightarrow & F(S^n)/\Gamma^3_2(F(S^n))
\end{array}
\]

and let \( G^i(n) = (F(S^n)/\Gamma^3_2(F(S^n)))/V^i(n) \). Then \( G^i(n) \cong \Omega(G^i(n+1)) \) as minimal simplicial groups, which is the assertion. \( \square \)

Thus all of \( G^i(n) \) can be chosen to be infinite loop spaces. Let \( X \) be a space. The \((\text{mod } 2)\) (co-)homology of \( X \) is denoted by \((H^*(X)) H_*(X)\).

**Lemma 8.3.** Let \( x_n \) be the generator for \( H_*(G^n(n)) = \mathbb{Z}/2 \). Then \( x_n^2 \neq 0 \) in \( H_*(G^n(n)) \).

**Proof.** Consider the simplicial group ring \( \mathbb{F}_2(F(S^n)) \), where \( \mathbb{F}_2 \) is the field with two elements. Let \( I \) be the augmentation ideal of \( \mathbb{F}_2(F(S^n)) \). Consider the exponential map \( \theta: F(S^n)/I^3 \to I/I^3 \), where \( \theta(x) = x - e \). Notice that \( \theta_*: \pi_*(I^2/I^3) \to \pi_*(I^2/I^3) \) is
onto and $I^2/I^3 \simeq K(\mathbb{Z}/2, 2n)$. Thus the composite

$$
\begin{array}{ccc}
K(\mathbb{Z}/2, 2n) & \longrightarrow & \Gamma^2/\Gamma^3 \\
& \searrow & \searrow \\
& & F(S^n)/\Gamma^3
\end{array}
\longrightarrow
\begin{array}{ccc}
& & I/I^3 \\
& \downarrow & \downarrow \\
& & f
\end{array}
\longrightarrow
\begin{array}{ccc}
& & K(\mathbb{Z}/2, 2n)
\end{array}
$$

is a homotopy equivalence, where $p: I/I^3 \to I^2/I^3 \simeq K(\mathbb{Z}/2, 2n)$ is the retraction, and so $G^n(n) \simeq K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, 2n)$ as simplicial sets. Notice that there is a commutative diagram

$$
\begin{array}{ccc}
F(S^n) & \longrightarrow & F(S^n)/\Gamma^3 \\
\downarrow & & \downarrow \\
I & \longrightarrow & I/I^3
\end{array}
\longrightarrow
\begin{array}{ccc}
& & G^n(n)
\end{array}
$$

where $f: G^n(n) \to K(\mathbb{Z}/2, 2n)$ is the retraction, and the composite

$$
\begin{array}{ccc}
F(S^n) & \longrightarrow & I \\
\downarrow & & \downarrow \\
I/I^3 & \longrightarrow & K(\mathbb{Z}/2, 2n)
\end{array}
$$

is a representative of $H^{2n}(F(S^n)) \cong \mathbb{Z}/2$. Thus $H^r(G^n(n)) \to H^r(F(S^n))$ is onto for $r \leq 2n$ and so $H_r(F(S^n)) \to H_r(G^n(n))$ is 1-1 for $r \leq 2n$. Notice that

$$
H_*(F(S^n)) \cong H_*(\Omega S^{n+1}) \cong P(l_n),
$$

where $\dim(l_n) = n$. Thus $x_n^2 \neq 0$ in $H_*(G^n(n))$, which is the assertion. \qed

**Lemma 8.4.** The Postnikov invariant of $B(G^n(n))$ is $Sq^{n+1}$ for $n > 0$.

**Proof.** Let $x_n$ and $y_{n+1}$ be the generators for $H_*(G^n(n))$ and $H_{n+1}(B(G^n(n)))$, respectively. Consider the (mod 2) Serre spectral sequence for the principal $G^n(n)$-bundle

$$
G^n(n) \longrightarrow E(G^n(n)) \longrightarrow B(G^n(n)).
$$

Then $d_{n+1}(y_{n+1}) = x_n$ and so $d_{n+1}(y_{n+1} \otimes x_n) = x_n^2 \neq 0$. Thus $\bar{y}_{n+1}^2 = 0$ in $H^*(B(G^n(n)))$, where $\bar{y}_{n+1}$ is the dual of $y_{n+1}$, or $S_q^{n+1}(\bar{y}_{n+1}) = 0$. Consider the fibre sequence

$$
K(\mathbb{Z}/2, 2n) \longrightarrow BG^n(n) \longrightarrow K(\mathbb{Z}/2, n+1) \longrightarrow K(\mathbb{Z}/2, 2n+2).
$$

Then $f^*(l_{2n+2}) = Sq^{n+1}(\bar{y}_{n+1})$, where $l_{2n+2}$ is the generator for $H^{2n+2}(K(\mathbb{Z}/2, 2n+2))$. The assertion follows. \qed

**Theorem 8.5.** Let $X$ be a two-stage Postnikov system with $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+1}(X) = \mathbb{Z}/2$. Then $X$ is homotopy equivalent to a minimal simplicial group if and only if the Postnikov invariant of $X$ is trivial or $Sq^{i+1}$. 
Proof. By Lemma 8.1, it suffices to determine the Postnikov invariant of $G^i(n)$. Notice $G^i(n)$ is unique up to homotopy. The assertion follows from Lemmas 8.2 and 8.4.

REFERENCES