

HOMOTOPY THEORY OF THE SUSPENSIONS OF THE PROJECTIVE PLANE

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Abstract. The homotopy theory of the suspensions of the real projective plane is largely investigated. The homotopy groups are computed up to certain range. The decompositions of the self smashes and the loop spaces are studied with some applications to the Stiefel manifolds.

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1. Introduction

Let $\mathbb{R}P^2$ be the projective plane. The $n$-dimensional mod 2 Moore space $P^n(2)$ is defined by $P^n(2) = \Sigma^{n-2} \mathbb{R}P^2$ for $n \geq 2$. As a space, $P^n(2)$ is the homotopy cofibre of the degree 2 map $[2] : S^{n-1} \to S^{n-1}$. In other words, $P^n(2)$ is the cell complex obtained by attaching an $n$-cell to $S^{n-1}$ where the attaching map is of degree 2. In geometry, $P^n(2)$ is the $2n$-skeleton of the Stiefel manifold $V_{2n+1,2}$ (or the associated spherical bundle $\tau(S^{2n})$ of the tangent bundle of $S^{2n}$). More precisely, let $x$ be any point in $V_{2n+1,2}$. Then the open manifold $V_{2n+1,2} \setminus \{x\}$ is homotopy equivalent to $P^n(2)$. Also an even dimensional projective plane admits a cellular decomposition in terms of mod 2 Moore spaces.

In this paper, we study homotopy theoretic aspects of mod 2 Moore spaces. Particularly we will compute the homotopy groups up to $8 +$ dimension. By knowing these homotopy groups, we obtain some information on three-cell complexes. For instance, by using the fact that $\pi_{10}(P^6(2)) = \mathbb{Z}/8$, we show that if a simply connected complex $X$ has the same mod 2 cohomology ring as that of $(S^6)$, then $X$ is homotopy equivalent to $P^6(2)$. Also an even dimensional projective plane admits a cellular decomposition in terms of mod 2 Moore spaces.

Our methods for computing the homotopy groups are: (1). to use the results on the homotopy groups of spheres, (2). to investigate various fibre sequences and (3). to study splittings of spaces.
homology of the loop space of \( X \), we obtain a map from a sphere to the loop space of \( X \). This map then induces a group homomorphism from the homotopy groups of a sphere to that of \( X \). In our case, the mod 2 homology of the loop space \( \Omega P^n(2) \) is the tensor algebra on two generators for \( n \geq 3 \). Many spherical classes can be obtained by considering the Samelson products. We should point out that the existence of certain spherical classes in the mod 2 homology \( H_*(\Omega P^n(2)) \) is equivalent to the strong form of Kervaire invariant one conjecture. (See Section 2.4 for details.)

The canonical method to study the homotopy groups of a space \( X \) is to consider certain fibration resolution of \( X \) or its loop spaces. The resolution produces a spectral sequence in general. Depending on the type of fibration resolution considered, there are several types of the Adams spectral sequences using different homology theories. In this paper, roughly speaking, we consider the spherical fibration resolutions of \( P^n(2) \). These resolutions will be considered case by case depending on special properties of the considered mod 2 Moore space. For instance, there are several ways to produce fibration resolutions for \( P^4(2) \): (1). Consider the pinch map \( P^4(2) \rightarrow S^4 \) and then do analysis on the homotopy fibre \( F^2 \). (2). Consider the canonical inclusion \( P^4(2) \rightarrow \tau(S^4) \) and study its homotopy fibre. (3). Consider the map \( \phi: P^4(2) \rightarrow \mathbb{H}P^\infty \) given by the composition of the pinch map \( P^4(2) \rightarrow S^4 \) and the inclusion \( S^4 \rightarrow \mathbb{H}P^\infty \). In the last case, the homotopy fibre of the map \( \phi \) is homotopy equivalent to the 7-skeleton of the double loop space \( \Omega^2 S^5 \). Since our purpose is to compute low homotopy groups, we are not going to study the “entire fibration resolutions” and their associated spectral sequences in this paper. On the other hand, we freely consider different fibrations until the homotopy groups are determined, where the different fibrations will be used particularly for determining the group structure of the homotopy groups.

Although the space \( P^n(2) \) itself is indecomposable up to homotopy, its loop space \( \Omega P^n(2) \) admits various product decompositions for \( n \geq 3 \). By applying the functor \( \pi_* \) to decompositions of loop spaces, we obtain decompositions of the homotopy groups and then the homotopy groups can be obtained by studying the factors in the decompositions. For instance, the homotopy group \( \pi_1(\Omega P^3(2)) = \mathbb{Z}/2^{14} \oplus \mathbb{Z}/4^{12} \) is obtained by taking the direct sum of the corresponding homotopy groups of many factors in a decomposition of the triple loop space of \( P^3(2) \).

Our decompositions of \( \Omega P^n(2) \) for \( n \geq 3 \) are the special case of general results in [36, 37]. Let \( X \) be a path connected \( p \)-local space. Then there is a homotopy decomposition

\[
\Omega \Sigma X \simeq A^{\text{min}}(X) \times \Omega(\bigvee_{k=2}^{\infty} Q_k^{\text{max}}(X)),
\]

where \( A^{\text{min}} \) and \( Q_k^{\text{max}} \) are functors and \( Q_k^{\text{max}}(X) \) is a functorial retract of the suspension of the \( k \)-fold self smash product \( X^{(k)} \). By applying the Hilton-Milnor Theorem to the second factor, one gets further decompositions of \( \Omega \Sigma X \). The study of this decomposition for the special case where \( X = P^{n-1}(2) \) is given in chapter 4.

Since the space \( Q_k^{\text{max}}(X) \) is a retract of \( \Sigma X^{(k)} \), the study of splittings of self smash products of \( P^n(2) \) is important for our purpose to compute the homotopy groups. In chapter 3, we study splittings of the self smash products of \( \mathbb{R}P^2 \). The homology of indecomposable factors in \( (\mathbb{R}P^2)^{(k)} \) has been determined in [38]. The cell structure of
the indecomposable factors is the main issue that we are going to study in this paper. One of the interesting points is that these indecomposable factors admits smash product decomposition. For instance, the smallest retract of \( (\mathbb{R}P^2)^{(7)} \) which contains the bottom cell is homotopy equivalent \( \mathbb{H}P^2 \land \mathbb{C}P^2 \land \mathbb{R}P^2 \). (See Corollary 3.11.) This gives a relation among different projective planes. We should point out that the splittings of self smash products of more general spaces play a role in Morava \( K \)-theory, see [32]. Furthermore, the problem on determining functorial splittings of self smashes of suspensions is equivalent to the fundamental problem in the modular representation theory of the symmetric groups, see [38]. This problem remains open in both homotopy theory and the representation theory in general. However much progress has been achieved recently in both areas. In homotopy theory, the homology of functorial indecomposable retracts of self smashes of a two-cell suspension has been determined in [38] with applications given in [10]. The corresponding results in the representation theory was obtained in [13, 14].

The \( EHP \) sequence is one of the main tools for computing the homotopy groups of spheres. In Section 4.4, we study an analogue for mod 2 Moore spaces. More precisely, we consider the reduced James-Hopf map \( \tilde{H}_2: \Omega P^{n+1}(2) \rightarrow \Omega P^{2n+1}(2) \) for \( n \geq 2 \) which is given by the second James-Hopf map \( \tilde{H}_2: \Omega P^{n+1}(2) \rightarrow \Omega \Sigma (P^n(2))^{(2)} \) composing with the loop of the pinch map \( \Sigma (P^n(2))^{(2)} \rightarrow P^{2n+1}(2) \). We show that the atomic retract of the homotopy fibre of the map \( \tilde{H}_2 \) is homotopy equivalent to \( S^n \{2\} \), the homotopy fibre of the degree 2 map \( [2]: S^n \rightarrow S^n \). (See Theorem 4.21.) Also a product decomposition of the homotopy fibre of the map \( \tilde{H}_2 \) is given in Section 4.4 up to certain dimension.

The space \( P^3(2) = \Sigma \mathbb{R}P^2 \) will be treated as a special case. There is a decomposition of the triple loop space of \( P^3(2) \) which is obtained as follows. Let \( \gamma_2: \Sigma \mathbb{R}P^2 \rightarrow BSO(3) \) be the adjoint map of the canonical inclusion \( \mathbb{R}P^2 \rightarrow SO(3) \). Then the homotopy fibre of \( \gamma_2 \) is homotopy equivalent to \( \Sigma \mathbb{R}P^4 \lor P^6(2) \), where \( \mathbb{R}P^a_b = \mathbb{R}P^b / \mathbb{R}P^a-1 \) is a truncated projective space. Let \( \Omega_0 X \) denote the path-connected component of \( \Omega X \) which contains the base-point and let \( X(n) \) denote the \( n \)-connected cover of \( X \). Then we have a decomposition

\[
\Omega_0^3 P^3(2) \simeq \Omega_0^3 (\Sigma \mathbb{R}P^4 \lor P^6(2)) \times \Omega_2^2 (S^3(3)),
\]

see Theorem 6.7. This induces a decomposition of \( \pi_r(P^3(2)) \) for \( r \geq 4 \). By applying the Hilton-Milnor theorem to \( \Omega (\Sigma \mathbb{R}P^4 \lor P^6(2)) \), we are able to compute \( \pi_r(P^3(2)) \) for \( r \leq 11 \).

The main part of this paper, particularly the calculation of the homotopy groups, is from my Ph.D. thesis [45] of the University of Rochester under the supervise of Professor Frederick Cohen. The table of the homotopy groups computed in this paper have been announced without proofs in [8, pp.1202]. Some of the homotopy groups of the mod 2 Moore spaces have been computed by Mukai and others with applications [27, 28, 29] and so there are some overlap between our work and theirs. An application to construct non-suspension \( co-H \)-spaces is given in [48].

Some terminologies and notations used in this paper are as follows. Every space is localized at 2. We always assume that a space is a path-connected \( CW \)-complex with a non-degenerate base point * and any map is a pointed map. Let \( f: X \rightarrow Y \) be a map. We write \( F_f \) for the homotopy fibre of \( f \) and \( C_f \) is the homotopy cofibre of \( f \).
The homotopy fibre of the pinch map $P^n(2) \to S^n$ is denoted by $F^n\{2\}$. The mod 2 homology of a space $X$ will be denoted by $H_*(X)$. On the other hand, the integral homology is denoted by $H_*(X;\mathbb{Z})$ to avoid confusion. The $n$-fold self smash product of $X$ is denoted by $X^{(n)}$. Let $X$ be a co-$H$-space. A map of degree $k$, denoted by $[k] : X \to X$, means the composite $X \xrightarrow{\mu_k} \bigvee_{j=1}^k X \xrightarrow{\text{fold}} X$, where $\mu_k$ is a choice of $k$-fold comultiplication. (Note. If $X$ is not homotopy associative, the map $[k]$ depends on the choice of a $k$-fold comultiplication and so may not be unique in general.) The homotopy fibre of $[k] : S^n \to S^n$ is denoted by $S^n\{k\}$. The homotopy cofibre of $[k] : S^n \to S^n$ is the Moore space $P^{n+1}(k)$. For an $H$-space $X$, a power map of degree $k$, denoted by $k : X \to X$, means the composite $X \xrightarrow{\Delta_k} X^k \xrightarrow{\mu_k} X$, where $\mu_k$ is a choice of $k$-fold multiplication. Let $X$ be a complex. We write $sk_n(X)$ for the $n$-skeleton of $X$. The $n$-connected cover of $X$ is denoted by $X(\pi)$.

The article is organized as follows. In chapter 2, we give some preliminaries and discuss some classical problems related to mod 2 Moore spaces. The study of self smashes of the projective plane $\mathbb{R}P^2$ is given in chapter 3. In chapter 4, we study the loop space of mod 2 Moore spaces. The computation of the homotopy groups of $P^n(2)$ for $n \geq 4$ is given in chapter 5. The homotopy theory of $P^3(2)$ is studied in chapter 6. The computation of the homotopy groups of $P^3(2)$ is given in the last section of this chapter.

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2. Preliminary and the Classical Homotopy Theory

2.1. Action of the Steenrod Algebra. Recall the Steenrod algebra $\mathcal{A}$ can be described as all linear natural transformations of the mod 2 cohomology functor. We refer the reader to [40] for several basic properties of $\mathcal{A}$. The algebra $\mathcal{A}$ is generated by symbols $Sq^n$, $n \geq 1$, $|Sq^n| = n$, subject to the “Adem relations”. That is, if one takes the tensor algebra on these symbols and quotients by the two-sided ideal generated by the Adem relations one obtains $\mathcal{A}$. The fundamental properties of $Sq$ are as follows

1) For all integers $i \geq 0$ and $n \geq 0$, there is a linear natural transformation

$$Sq^i : H^n(X, A) \longrightarrow H^{n+i}(X, A).$$
2) $Sq^0 = 1$ and $Sq^1$ is the Bockstein homomorphism.
3) $Sq^n x = x^2$ if $|x| = n > 0$.
4) $Sq^i x = 0$ if $|x| < n$.
5) The Cartan formula

$$Sq^n(x \otimes y) = \sum_{i=0}^{n} Sq^i x \otimes Sq^{n-i} y.$$  

6) The Adem relations

$$Sq^aSq^b = \sum_{j=0}^{[a/2]} \left( \begin{array}{c} b-1-j \\ a-2j \end{array} \right) Sq^{a+b-j}Sq^j$$

for $0 < a < 2b$, where $[a/2] = \max\{k \in \mathbb{Z} | k \leq a/2\}$.

We write $Sq^n_\ast : H_q(X) \to H_{q-n}(X)$ for the dual-action of $Sq^n$ on the homology functor. By the duality, the Cartan formula and the Adem relations are given by

$$Sq^n_\ast(x \otimes y) = \sum_{i=0}^{n} Sq^i_\ast(x) \otimes Sq^{n-i}_\ast(y)$$  

$$Sq^b_\ast Sq^a_\ast = \sum_{j=0}^{[a/2]} \left( \begin{array}{c} b-1-j \\ a-2j \end{array} \right) Sq^j_\ast Sq^{a+b-j}_\ast$$

for $0 < a < 2b$, respectively.

Recall that the cohomology ring $H^\ast(\mathbb{R}P^2)$ is the quotient of the polynomial algebra on the symbol $x$, $|x| = 1$, by the ideal generated by $x^3$. Thus $H^\ast(\mathbb{R}P^2)$ has a basis $\{1, x, x^2 = Sq^1 x\}$ and so the reduced homology $\tilde{H}_\ast(\mathbb{R}P^2)$ has a basis $\{u, v\}$ with $|v| = 2$ and $Sq^1 v = u$. It follows that $\tilde{H}_\ast(P^n(2))$ has a basis $\{u, v\}$ with $|v| = n$ and $Sq^1 v = u$. Let $[A, A]$ be the two-sided ideal generated by the commutator $[\alpha, \beta] = \alpha \beta - \beta \alpha$ for $\alpha, \beta \in A$. Recall that $A/[A, A]$ is a divided polynomial algebra on one generator, that is

$$Sq^i Sq^j = \left( \begin{array}{c} i+j \\ i \end{array} \right) Sq^{i+j} \mod [A, A],$$

see [40, pp.25]. Let $\Gamma(x_1, \ldots, x_k)$ be the divided polynomial algebra generated one $x_1, \ldots, x_k$. We have $A/[A, A] \cong \Gamma(Sq^1)$.

**Lemma 2.1.** Let $X$ be a path-connected space. Suppose that the commutator $[Sq^a_\ast, Sq^b_\ast]$ acts trivially on $H_\ast(X)$ for $a, b \geq 1$. Then the action of $A$ on

(1) $H_\ast(\Omega \Sigma X)$ and

(2) $\tilde{H}_\ast((X)^{(n)})$ for $n \geq 1$

factors through the quotient $A/[A, A]$.

**Proof.** It suffices to show that $[Sq^a_\ast, Sq^b_\ast]$ acts trivially on $H_\ast(\Omega \Sigma X)$. Let $V = \tilde{H}_\ast(X)$. Recall that, as an algebra, $H_\ast(\Omega \Sigma X)$ is isomorphic to the tensor algebra $T(V)$. Thus the action of the Steenrod algebra on $H_\ast(\Omega \Sigma X)$ is uniquely determined by its action
on $V$ and the Cartan formula. By using the fact that $Sq^ksq^jx = Sq^jSq^kx$ for $x \in V$, we have 

$$Sq^b Sq^a(x_1x_2 \ldots x_k) = \sum_{i_1 + i_2 + \ldots + i_k = a \atop j_1 + j_2 + \ldots + j_k = b} Sq^{i_1} Sq^{i_2} x_1 Sq^{j_1} Sq^{j_2} x_2 \cdots Sq^{k_1} Sq^{k_2} x_k$$

for $x_i \in V$ and hence the result. 

\begin{proof}
Suppose that there exist positive integers $r$ and $k$ such that 

1. $H_j(B) = 0$ for $j \geq r$ and $H_r(C_f) \neq 0$. 
2. there is an element $v \in H_r(C_f)$ with the property that $Sq^k v \neq 0$ and $Sq^i v$ lies in the image of $i_* : H_*(B) \to H_*(C_f)$ for $j \geq k$. 

Then the map $f \wedge X : A \wedge X \to B \wedge X$ is essential for any finite dimensional complex $X$ with $\tilde{H}_*(X) \neq 0$. 

Suppose that $f \wedge X : A \wedge X \to B \wedge X$ were null homotopic. Then there is a map $\phi : C_f \wedge X \to B \wedge X$ such that $\phi \circ (i \wedge X)$ is homotopic to the identity map of $B \wedge X$. Let $y$ be a top cell of $X$, that is, $H_j(X) = 0$ for $j > |y|$ and $y$ is not zero in $H_{|y|}(X)$. There exists an integer $t \geq 0$ such that $Sq^j y = 0$ for $j > t$ and $Sq^t y \neq 0$. By the assumption (1), $H_{r+|y|}(B \wedge X) = 0$ and so $\phi_*(v \otimes y) = 0$. For each $j \geq k$, let $u_j$ be an element in $H_{r-J}(B)$ such that $Sq^j v = i_*(u_j)$. It follows that 

$$0 = Sq^k \phi_*(v \otimes y) = \sum_{j \geq k} \phi_*(Sq^j v \otimes Sq^{k-t-j})$$

$$= u_k \otimes Sq^j v + \sum_{j > k} u_j \otimes Sq^{k-t-j} y \neq 0$$

because $\tilde{H}_{k+t}(B \wedge X) = \bigoplus_j \tilde{H}_j(B) \otimes \tilde{H}_{k+t-j}$. We obtain a contradiction and hence the result. 

\end{proof}

\begin{corollary}
Let $X$ be any finite dimensional complex with $\tilde{H}_*(X) \neq 0$. Then the degree map $[2] : \Sigma X \to \Sigma X$ is essential.

\end{corollary}

\begin{proposition}
Let $n \geq 3$. Then the degree map $[2] : P^n(2) \to P^n(2)$ is homotopic to the composite 

$$P^n(2) \xrightarrow{\text{pinch}} S^n \xrightarrow{\eta} S^{n-1} \xrightarrow{c} P^n(2),$$

where $\eta$ is a suspension of the Hopf map.

\end{proposition}
Proof. It suffices to show that the statement holds for $n = 3$. Clearly the map $[2]$ restricted to the bottom cell $S^2$ is null homotopic and so there is a map $\alpha : S^3 \to P^3(2)$ such that the diagram

$$
\begin{array}{ccc}
P^3(2) & \xrightarrow{\text{pinch}} & S^3 \\
\downarrow & & \downarrow \alpha \\
P^3(2) & \xrightarrow{\text{pinch}} & S^3
\end{array}
$$

commutes up to homotopy. Since $P^3(2)$ is a torsion space, the composite

$$S^3 \xrightarrow{\alpha} P^3(2) \xrightarrow{\text{pinch}} S^3$$

is null homotopic and so $\alpha$ lifts to the homotopy fibre $F^3 \{2\}$ of the pinch map $P^3(2) \to S^3$. By computing the low homology of $F^3 \{2\}$, the bottom cell $S^2$ is the 3-skeleton of $F^3 \{2\}$ and so there is a map $\alpha' : S^3 \to S^2$ such that the map $\alpha$ is homotopic to the composite $S^3 \xrightarrow{\alpha'} S^2 \xleftarrow{\text{pinch}} P^3(2)$. By Corollary 2.4, the map $[2] : P^3(2) \to P^3(2)$ is essential and so is the composite $P^3(2) \xrightarrow{\text{pinch}} S^3 \xrightarrow{\alpha'} S^2$. It follows that $[\alpha']$ is a generator of $\pi_3(S^2) = \mathbb{Z}$ or $\alpha' \simeq \pm \eta$. The assertion follows. \qed

**Proposition 2.6.** Let $n \geq 3$. Then the group $[P^n(2), P^n(2)]$ is isomorphic to $\mathbb{Z}/4$.

**Proof.** We only need to compute $[P^3(2), P^3(2)]$. By the cofibre sequence

$$S^2 \xrightarrow{[2]} S^2 \xrightarrow{\text{pinch}} P^3(2),$$

there is an exact sequence

$$[S^3, P^3(2)] \xrightarrow{2} [S^3, P^3(2)] \to [P^3(2), P^3(2)] \to [S^2, P^3(2)] \xrightarrow{2} [S^2, P^3(2)].$$

Thus there is a short exact sequence

$$0 \to \pi_3(P^3(2))/2 \to [P^3(2), P^3(2)] \to \pi_2(P^3(2)) = \mathbb{Z}/2 \to 0.$$

By using the fact that $S^2$ is the 3-skeleton of $F^3 \{2\}$, the map $\pi_3(S^2) \to \pi_3(P^3(2))$ is an epimorphism and so $\pi_3(P^3(2))$ is a cyclic group $\mathbb{Z}/k$ for some integer $k$. It follows that $[P^3(2), P^3(2)]$ is isomorphic to $\mathbb{Z}/4$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/2$. By Corollary 2.4, we have

$$2 \cdot [P^3(2), P^3(2)] \neq 0$$

and hence $[P^3(2), P^3(2)] = \mathbb{Z}/4$, which is the result. \qed

**Corollary 2.7.** Let $f$ be a self homotopy equivalence of $P^n(2)$ with $n \geq 3$. Then $f$ is either homotopic to the identity map or the degree map $[-1] : P^n(2) \to P^n(2)$.

**Note.** Propositions 2.5 and 2.6 can be found in [43, Corollary, p.307] and [26, Lemma 1.1, p.271].
2.2. The James Construction and the Cohen Groups. Let $X$ be a pointed space with the base point $\ast$. Recall that the James construction $J(X)$ is the free monoid on $X$ subject to the single relation $\ast = 1$, see for instance [18, 44]. The James filtration $\{J_n(X)\}_{n \geq 0}$ with $J_0X = \ast$ and $J_1X = X$ is induced by the word-length filtration. Note that $J_n(X)/J_{n-1}(X) \cong X^{(n)}$. Let $q_n : X^n \to J_n(X)$ be the quotient map. The fundamental properties of $J(X)$ are as follows [5]:

1) If $X$ is path-connected, then $J(X)$ is (weak) homotopy equivalent to $\Omega X$.
2) The quotient $q_n : X^n \to J_n(X)$ has a functorial cross-section.
3) The inclusion $J_{n-1}(X) \to J_n(X)$ has a functorial retraction.
4) There is a functorial decomposition

$$J_n(X) \simeq \bigvee_{j=1}^n X^{(j)}$$

for $0 \leq n \leq \infty$.
5) The James-Hopf map $H_n : J(X) \to J(X^{(n)})$ is defined combinatorially by

$$H_n(x_1x_2 \cdots x_q) = \prod_{1 \leq i_1 < i_2 < \cdots < i_q \leq n} (x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_q}),$$

where the product of the right side is ordered lexicographically from right.
6) There is a fibre sequence, which is called EHP sequence,

$$S^n \to J(S^n) \simeq \Omega S^{n+1} \xrightarrow{H_2} J(S^{2n}) \simeq \Omega S^{2n+1}$$

localized at 2.

Below we go over some ideas and results from [7]. Let $f : X \to \Omega Y$ be a map. The group $K_n(f)$ is defined to be the subgroup of $[X^n, \Omega Y]$ generated by $x_i(f)$ represented by the composite

$$X^n \xrightarrow{\pi_i} X \xrightarrow{f} \Omega Y$$

for $1 \leq i \leq n$, where $\pi_i$ is the $i$-th coordinate projection. Let $F_n$ be the free group of rank $n$ generated by letters $x_1, \ldots, x_n$ and let

$$e_f : F_n \to [X^n, \Omega Y]$$

be the group homomorphism such that $e_f(x_i) = x_i(f)$ for $1 \leq i \leq n$. If $g : Y \to Y'$ and $h : X' \to X$ be maps, then there is a commutative diagram

$$\begin{array}{ccc}
F_n & \xrightarrow{e_f} & [X^n, \Omega Y] \\
\downarrow \updownarrow & & \downarrow \updownarrow \\
F_n & \xrightarrow{e_{\Omega g \circ f \circ h}} & [X'^n, \Omega Y']
\end{array}$$

Let $\mathcal{C}$ be a class (or set) of maps $f : X \to \Omega Y'$, that is, $X$ and $Y$ run over certain class of spaces and $f$ runs over certain maps from $X$ to $\Omega Y$. The Cohen group $K_n(\mathcal{C})$ is the quotient group of $F_n$ such that

1) If $f : X \to \Omega Y$ lies in $\mathcal{C}$, then the homomorphism $e_f : F_n \to [X^n, \Omega Y]$ factors through $K_n(\mathcal{C})$. 

2) There exists an object \( f: X \to \Omega Y \) in \( \mathcal{C} \) such that the resulting homomorphism \( e_f: K_n(\mathcal{C}) \to [X^n, \Omega Y] \) is a monomorphism.

**Note.**

1) If \( f: X \to \Omega Y \) is map between \( p \)-local (or \( p \)-complete) spaces, then the map \( e_f \) extends to the \( p \)-localization (or \( p \)-completion) of \( F_n \). Thus one can obtain the \( p \)-local (or \( p \)-complete) version of the Cohen groups along the same lines.

2) There is a short exact sequence

\[
1 \longrightarrow \bigcap_{f \in \mathcal{C}} \text{Ker}(e_f) \longrightarrow F_n \longrightarrow K_n(\mathcal{C}) \longrightarrow 1.
\]

Define the group homomorphisms \( d_i: F_n \to F_{n-1} \) and \( s_i: F_{n-1} \to F_n \) by

\[
d_i(x_j) = \begin{cases} 
  x_j & \text{for } j < i \\
  1 & \text{for } j = i \\
  x_{j-1} & \text{for } j > i
\end{cases}
\]

\[
s_i(x_j) = \begin{cases} 
  x_j & \text{for } j < i \\
  x_{j+1} & \text{for } j \geq i.
\end{cases}
\]

Let

\[
d^i: X^{n-1} \to X^n, \quad (z_1, z_2, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{i-1}, *, z_i, \ldots, z_{n-1})
\]

\[
\tilde{\pi}^i: X^n \to X^{n-1}, \quad (z_1, z_2, \ldots, z_n) \mapsto (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)
\]

be the maps for \( 1 \leq i \leq n \). Then there are commutative diagrams

\[
\begin{array}{ccc}
F_n & \xrightarrow{e_f} & [X^n, \Omega Y] \\
\downarrow d_i & & \downarrow \left\{ \begin{array}{c} e_f \\ \tilde{\pi}^i \end{array} \right. \\
F_{n-1} & \xrightarrow{e_f} & [X^{n-1}, \Omega Y]
\end{array}
\quad
\begin{array}{ccc}
F_{n-1} & \xrightarrow{e_f} & [X^{n-1}, \Omega Y] \\
\downarrow s_i & & \downarrow \left\{ \begin{array}{c} e_f \\ \tilde{\pi}^i \end{array} \right. \\
F_n & \xrightarrow{e_f} & [X^n, \Omega Y]
\end{array}
\]

for \( 1 \leq i \leq n \). It follows that the composite \( F_n \xrightarrow{d_i} F_{n-1} \xrightarrow{s_i} K_{n-1}(\mathcal{C}) \) factors through \( K_n(\mathcal{C}) \) and the composite \( F_{n-1} \xrightarrow{s_i} F_n \xrightarrow{\tilde{\pi}^i} K_{n-1}(\mathcal{C}) \) factors through \( K_{n-1}(\mathcal{C}) \). Let \( d_i: K_n(\mathcal{C}) \to K_{n-1}(\mathcal{C}) \) and \( s_i: K_{n-1}(\mathcal{C}) \to K_n(\mathcal{C}) \) denote the resulting homomorphisms. Then there are commutative diagrams

\[
\begin{array}{ccc}
K_n(\mathcal{C}) & \xrightarrow{e_f} & [X^n, \Omega Y] \\
\downarrow d_i & & \downarrow \left\{ \begin{array}{c} e_f \\ \tilde{\pi}^i \end{array} \right. \\
K_{n-1}(\mathcal{C}) & \xrightarrow{e_f} & [X^{n-1}, \Omega Y]
\end{array}
\quad
\begin{array}{ccc}
K_{n-1}(\mathcal{C}) & \xrightarrow{e_f} & [X^{n-1}, \Omega Y] \\
\downarrow s_i & & \downarrow \left\{ \begin{array}{c} e_f \\ \tilde{\pi}^i \end{array} \right. \\
K_n(\mathcal{C}) & \xrightarrow{e_f} & [X^n, \Omega Y]
\end{array}
\]

for \( f \in \mathcal{C} \) and \( 1 \leq i \leq n \).

By direct calculation, we have

**Lemma 2.8.** In the sequence of the groups \( \{K_n(\mathcal{C})\}_{n \geq 1} \), the following identities hold

1) \( d_j d_i = d_i d_{j+1} \) for \( j \geq i \).

2) \( s_j s_i = s_{i+1} s_j \) for \( j \leq i \).
3) \[
d_j s_i = \begin{cases} 
  s_{i-1} d_j & \text{for } j < i \\
  \text{id} & \text{for } j = i \\
  s_i d_{j-1} & \text{for } j > i.
\end{cases}
\]

**Note.** The third identity is different from the simplicial identity.

**Lemma 2.9.** The \([J_n(X), \Omega Y]\) is the equalizer of the homomorphisms
\[
d^i_*: [X^n, \Omega Y] \to [X^{n-1}, \Omega Y]
\]
for \(1 \leq i \leq n\).

**Proof.** The proof is given by induction on \(n\) and the assertion holds for \(n = 1\). Suppose that the assertion holds for \(n - 1\) with \(n > 1\). By the suspension splitting theorem, \(q_n^*: [J_n(X), \Omega Y] \to [X^n, \Omega Y]\) is a monomorphism and so \([J_n(X), \Omega Y]\) can be regarded as a subgroup of \([X^n, \Omega Y]\). Let \(G_n\) be the equalizer of the homomorphisms \(d_i: [X^n, \Omega Y] \to [X^{n-1}, \Omega Y]\) for \(1 \leq i \leq n\). By the definition of \(J_n(X)\), we have
\[
[J_n(X), \Omega Y] \subseteq G_n.
\]

Conversely let \(\alpha \in G_n\), that is, \(d_1 \alpha = d_2 \alpha = \ldots = d_n \alpha\). Then \(d_1 \alpha \in G_{n-1}\) and so \(d_1 \alpha \in [J_{n-1}(X), \Omega Y]\) by induction. Consider the commutative diagram
\[
\begin{array}{ccc}
[J_n(X), \Omega Y] & \longrightarrow & [X^n, \Omega Y] \\
\downarrow & & \downarrow d_1 \\
[J_{n-1}(X), \Omega Y] & \hookrightarrow & [X^{n-1}, \Omega Y].
\end{array}
\]

There exists an element \(\beta \in [J_n(X), \Omega Y]\) such that \(d_1 \beta = d_1 \alpha\). It follows that \(d_i(\alpha \beta^{-1}) = 1\) for all \(1 \leq i \leq n\). Let \(g: X^n \to \Omega Y\) be a map which represents \(\alpha \beta^{-1}\) and let \(g': \Sigma X^n \to Y\) be the adjoint of \(g\). Then the composite
\[
\Sigma \bigvee_{i=1}^{n} X^{n-1} \xrightarrow{\Sigma (d^i, \ldots, d^n)} \Sigma X^n \xrightarrow{g'} Y
\]
is null homotopic. Let
\[
(X[*])^n = \{(z_1, \ldots, z_n) \in X^n | z_i = * \text{ for some } i\}.
\]
Recall that the map
\[
\Sigma \bigvee_{i=1}^{n} X^{n-1} \xrightarrow{\Sigma (d^i, d^2, \ldots, d^n)} \Sigma (X[*])^n
\]
has a cross-section. Thus the composite
\[
\Sigma (X[*])^n \xrightarrow{\Sigma X^n} \Sigma X^n \xrightarrow{g'} Y
\]
is null homotopic and so \(g'\) factors through \(\Sigma X^{(n)}\) up to homotopy or
\[
\alpha \beta^{-1} \in [X^{(n)}, \Omega Y] \subseteq [J_n(X), \Omega Y] \subseteq [X^n, \Omega Y].
\]
Thus \(\alpha \in [J_n(X), \Omega Y]\) and hence the result. \(\square\)
The Cohen group $\mathcal{H}_n(C)$ is defined to be the equalizer of the group homomorphisms $d_i: K_n(C) \to K_{n-1}(C)$ for $1 \leq i \leq n$. By Lemma 2.9, there is a homomorphism $e_f: \mathcal{H}_n(C) \to [J_n(X), \Omega Y]$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{H}_n(C) & \xrightarrow{e_f} & [J_n(X), \Omega Y] \\
\downarrow & & \downarrow \\
K_n(C) & \xrightarrow{e_f} & [X^n, \Omega Y]
\end{array}
$$

commutes for $f \in C$. By the definition, there is a homomorphism $p_n: \mathcal{H}_n(C) \to \mathcal{H}_{n-1}(C)$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{H}_n(C) & \xrightarrow{p_n} & K_n(C) \\
\downarrow & & \downarrow \\
\mathcal{H}_{n-1}(C) & \xleftarrow{d_i} & K_{n-1}(C)
\end{array}
$$

commutes for each $1 \leq i \leq n$.

**Lemma 2.10.** The map $p_n: \mathcal{H}_n(C) \to \mathcal{H}_{n-1}(C)$ is an epimorphism for each $n$.

**Proof.** We show by induction that $p_{k,n} = p_{k+1} \circ \ldots \circ p_n: \mathcal{H}_n(C) \to \mathcal{H}_k(C)$ is an epimorphism for $k \leq n$. Clearly $p_{1,n}$ is an epimorphism. Suppose that $p_{k-1,n}$ is an epimorphism with $k > 1$ and let $\alpha \in \mathcal{H}_k(C)$. Since $p_{k-1,n}: \mathcal{H}_n(C) \to \mathcal{H}_{k-1}(C)$ is onto, we may assume that $\alpha$ lies in the kernel of $p_k: \mathcal{H}_k(C) \to \mathcal{H}_{k-1}(C)$. Let

$$
\alpha_{k,n} = \prod_{1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n} s_{i_{n-k}} s_{i_{n-k-1}} \ldots s_{i_1} \alpha \in K_n(C)
$$

with lexicographic order from right. Then it is a routine work to check that $\alpha_{k,n} \in \mathcal{H}_n(C)$ with $p_{k,n}(\alpha_{k,n}) = \alpha$ and hence the result. \qed

Thus one gets the progroup

$$
\mathcal{H}(C) = \lim_{\leftarrow p_n} \mathcal{H}_n(C)
$$

together with the representation $e_f: \mathcal{H}(C) \to [J(X), \Omega Y]$ for $f \in C$. The group $\mathcal{H}(C)$ catches most interesting elements in $[J(X), \Omega Y]$ which are related to the Hopf invariants and the Whitehead products.

Let $G$ be a group. We write $[a, b] = a^{-1}b^{-1}ab$ for $a, b \in G$. Let $\{\Gamma_n G\}$ be the descending central series of $G$ starting with $\Gamma_1 G = G$. Let $L_n(G) = \Gamma_n G/\Gamma_{n+1} G$.

**Example 2.11.** Let $C$ consist of the single object $f: S^{m-1} \xrightarrow{\epsilon} \Omega S^m$ localized at 2 with $m \geq 1$. We determine $\mathcal{H}(C)$.

**Solution.** Consider $e_f: F_2 \to [X^n, \Omega S^m]$, where $X = S^{m-1}$. Recall that the 3-fold Samelson product $W_3: S^{3(m-1)} \to \Omega S^m$ is null homotopic localized at 2, see for instance [5]. Thus

$$
e_f([x_i, x_j, x_k]) = 1$$
for any \( i, j, k \) and so \( e_f \) factors through \( F_n/\Gamma_3 F_n \). By direct calculation, we have \( \mathcal{H}_n(C) \cong \mathcal{H}_2(C) \) for \( n \geq 2 \) and so \( \mathcal{H}(C) \cong \mathcal{H}_2(C) \). Now we consider several cases:

**Case I.** The Whitehead square \( \omega_m = 0 \), that is, \( m = 1, 3 \) or 7. Then \( e_f([x_i, x_j]) = 1 \) for any \( i \) and \( j \). Thus \( e_f \) factors through \( F_n/\Gamma_2 F_n = L_1(F_n) \). It follows that \( K_n(C) \) is a free abelian group of rank \( n \) and so \( \mathcal{H}(C) \cong \mathcal{H}_1(C) \cong \mathbb{Z} \).

**Case II.** The Whitehead square \( \omega_m \neq 0 \) with \( m \) odd. Since \( 2\omega_m = 0 \), we have \( e_f([x_i, x_j]^2) = 1 \). It follows that there is a short exact sequence

\[
1 \longrightarrow L_2(F_n)/2L_2(F_n) \longrightarrow K_n(C) \longrightarrow L_1(F_n) \longrightarrow 1.
\]

By direct calculation, \( \mathcal{H}_1(C) \cong \mathbb{Z} \) and \( \mathcal{H}_n(C) \cong \mathcal{H}_2(C) \) for \( n \geq 2 \) with a short sequence

\[
1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathcal{H}_2(C) \longrightarrow \mathbb{Z} \longrightarrow 1.
\]

**Case III.** \( m \) is even. In this case, \( k\omega_m \neq 0 \) for any non-zero integer \( k \) and so \( e_f([x_i, x_j]^k) \neq 1 \) for \( k \neq 0 \). Then there is a short exact sequence

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_2(C) \longrightarrow \mathbb{Z} \longrightarrow 1.
\]

\[
\square
\]

The following theorem is a well-known result in classical homotopy theory [5].

**Theorem 2.12.** Let \( m > 1 \). Then \( \Omega[2] \simeq 2 \cdot (H_2 \circ \Omega\omega_m) : \Omega S^m \to \Omega S^m \).

**Proof.** Let \( C \) consist of the single object \( S^{m-1} \to \Omega S^m \). Observe that the maps \( 2|_{J_n(S^{m-1})} : J_n(S^{m-1}) \to \Omega S^m \) represent the element \( e_f(x_1 \cdots x_n)^2 \) and \( e_f(x_1^2 x_2^2 \cdots x_n^2) \), respectively. By commutator calculus,

\[
x_1^2 x_2^2 \cdots x_n^2 \equiv (x_1 x_2 \cdots x_n)^2 \prod_{1 \leq i < j \leq n} [x_i, x_j] \mod \Gamma_3(F_n).
\]

The assertion follows from Example 2.11. \( \square \)

A space \( X \) is called **conilpotent** of class \( \leq k \) if the \((k + 1)\)-fold reduced diagonal \( \bigwedge_{i=1}^{k+1} X \) is null homotopic. Clearly a space of \( LS \)-category \( \leq k \) is conilpotent of class \( \leq k \). A **coabelian** space means a conilpotent space of class \( \leq 1 \). For instance, a co-\( H \)-space is coabelian.

**Proposition 2.13.** Let \( X \) be a conilpotent space of class \( \leq k \) and let \( f : X \to \Omega Y \) be a map. Then the following relation holds in the group \( K_n(f) \):

The (iterated) commutator on generators

\[
[[x_{i_1}, x_{i_2}], \ldots, x_{i_t}] = 1
\]

if one of the generators \( x_1, \ldots, x_n \) occurs at least \( k + 1 \) times in the commutator bracket.

**Proof.** Let \( \{j_1, j_2, \ldots, j_s\} = \{i_1, i_2, \ldots, i_t\} \) with \( j_1 < j_2 < \ldots < j_s \). Let \( a_i \) be the occurrence of \( x_{j_i} \) in the commutator bracket. Then

\[
e_f([[x_{i_1}, x_{i_2}], \ldots, x_{i_t}]) \in [X^n, \Omega Y]
\]

is represented by the composite

\[
X^n \xrightarrow{\pi_{j_1,j_2,\ldots,j_s}} X^s \xrightarrow{\text{pinch}} X(s) \xrightarrow{\Delta} X^{(a_1)} \wedge \cdots \wedge X^{(a_s)} = X^{(t)} \xrightarrow{\sigma} X^{(t)} \xrightarrow{W_i} J(X) \xrightarrow{Jf} \Omega Y,
\]
where \( Jf \) is the loop map induced by \( f \), \( \sigma \) is a permutation of coordinates, and \( W_t \) is the \( t \)-fold Samelson product. By the assumption, one of \( a_i \) is \( \geq k+1 \) and hence the result.

\[ \text{Corollary 2.14.} \] Let \( f : X \to \Omega Y \). Suppose that \( X \) is a conilpotent space of class \( < \infty \). Then \( K_n(f) \) is a nilpotent group for each \( n \). In particular, if \( X \) is a path-connected finite complex, then \( K_n(f) \) is nilpotent for each \( n \).

Let \( R \) be a ring and let \( \tilde{V} \) be a free \( R \) module of rank \( n \) with a basis \( \{x_1, x_2, \ldots, x_n\} \). Recall that the module \( \text{Lie}(n) \) over \( R \) is the submodule of \( \tilde{V}^{\otimes n} \) spanned by the Lie elements \( \{[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)}]|\sigma \in S_n\} \). The symmetric group \( S_n \) acts on \( \text{Lie}(n) \) by permuting letters \( x_1, \ldots, x_n \). The following theorem is due to Cohen.

\[ \text{Theorem 2.15.} \] Let \( \mathcal{E} \) be the class of maps \( f : X \to \Omega Y \) with the condition that \( X \) is coabelian and let \( \mathcal{E}^{p^r} \) be the subclass of \( \mathcal{E} \) consisting of \( f : X \to \Omega Y \) with the additional condition that \( p^r | f = 1 \) in the group \([X, \Omega Y]\). Then

1) \( K_n(\mathcal{E}) \) is the quotient group of \( F_n = F(x_1, \ldots, x_n) \) subject to the relations for the iterated commutators on generators

\[ [[x_{i_1}, x_{i_2}], \ldots, x_{i_t}] = 1 \]

with \( i_a = i_b \) for some \( 1 \leq a < b \leq t \).

2) The kernel of the epimorphism \( \mathcal{H}_n(\mathcal{E}) \to \mathcal{H}_{n-1}(\mathcal{E}) \) is isomorphic to \( \text{Lie}(n) \) over \( \mathbb{Z} \).

3) \( K_n(\mathcal{E}^{p^r}) \) is the quotient group of \( K_n(\mathcal{E}) \) subject to the additional relations

\[ x_i^{p^r} = 1 \]

for \( 1 \leq i \leq n \).

4) The kernel of the epimorphism \( \mathcal{H}_n(\mathcal{E}^{p^r}) \to \mathcal{H}_{n-1}(\mathcal{E}^{p^r}) \) is isomorphic to \( \text{Lie}(n) \) over \( \mathbb{Z}/p^r \).

The following proposition is useful.

\[ \text{Proposition 2.16.} \] In the group \( K_n(\mathcal{E}) \), the following identity

\[ [[x_{i_1}^{m_1}, x_{i_2}^{m_2}], \ldots, x_{i_t}^{m_t}] = [[x_{i_1}, x_{i_2}], \ldots, x_{i_t}]^{n_1 n_2 \cdots n_t} \]

holds for the generators \( x_i \).

2.3. An Application to the Stiefel Manifolds. Recall that the Stiefel manifold \( V_{m+n,m} \) is the space of all \( m \)-frames in \( R^{m+n} \) or, equivalently,

\[ V_{m+n,m} = O(m+n)/O(n) = SO(m+n)/SO(n) \]

the quotient homogeneous space of \( O(m+n) \) by the sub Lie group \( O(n) \). In particular,

\[ V_{2n+1,2} = \{(x, y) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} | (x, y) = 0, \|x\| = 1, \|y\| = 1\} \]

is the spherical tangent bundle \( \tau(S^{2n}) \) of \( S^{2n} \). Let \( M \) be an orientable \( m \)-manifold and let \( f : M \to M \) be a map. The degree of \( f \), denoted by \( \text{deg}(f) \), is defined to the degree of the homomorphism \( f_* : H_m(M, \mathbb{Z}) = \mathbb{Z} \to H_m(M, \mathbb{Z}) = \mathbb{Z} \). A question arising from geometry is whether there is a self homomorphism \( f \) of \( M \) which changes the orientation, that is, \( \text{deg}(f) = -1 \). In the case where \( M = V_{2n+1,2} \) with \( n > 1 \), we will show that for any self homotopy equivalence of preserves the orientation and so the answer of the question above is negative when \( M = V_{2n+1,2} \) with \( n > 1 \).
Recall that $P^{2n}(2)$ is the $(4n - 2)$-skeleton of $V_{2n+1,2}$ and so there is a cofibre sequence

$$S^{4n-2} \xrightarrow{\lambda_{2n}} P^{2n}(2) \rightarrow V_{2n+1,2}.$$ 

The following lemma can be found in [31], [25, Lemma 12, p.185] or [26, Proposition 2.1, p.274].

**Lemma 2.17.** The order of the homotopy class $[\lambda_{2n}]$ in $\pi_{4n-2}(P^{2n}(2))$ is 4 if $n \equiv 0 \mod 2$ and 8 if $n \equiv 1 \mod 2$ and $n > 1$.

Let $\lambda'_{2n}: S^{4n-3} \rightarrow \Omega P^{2n}(2)$ be the adjoint map of $\lambda_{2n}$.

**Lemma 2.18.** Let $n > 1$. Then

$$[2] \circ \lambda_{2n} \simeq \ast \quad \text{and} \quad [-1] \circ \lambda_{2n} \simeq \lambda_{2n} \circ [-3]: S^{4n-2} \rightarrow P^{2n}(2)$$

for $n > 1$.

**Proof.** Observe that $J_2(P^{2n-1}(2))$ is the $(6n - 7)$-skeleton of $J(P^{2n-1}(2)) \simeq \Omega P^{2n}(2)$. It suffices to show that the homotopy class

$$[\Omega[-1]|_J]_{J_2(P^{2n-1}(2))} \circ \lambda'_{2n} = -3[\lambda'_{2n}]$$

in $\pi_{4n-3}(\Omega P^{2n}(2))$. Since the pinch map $P^{2n}(2) \rightarrow S^{2n}$ factors through $V_{2n+1,2}$, the composite $[2] \circ \lambda_{2n}$ is null homotopic by Proposition 2.5. Consider the Cohen representation

$$e_f: \mathcal{H}_2(\mathcal{E}) \rightarrow [J_2(P^{2n-1}(2)), \Omega P^{2n}(2)],$$

where $f$ is the inclusion $P^{2n-1} \subseteq \Omega P^{2n}(2)$. The maps

$$\Omega[-1]|_{J_2(P^{2n-1}(2))}, \quad \Omega[2]|_{J_2(P^{2n-1}(2))} \quad \text{and} \quad k|_{J_2(P^{2n-1}(2))}$$

represent elements $x_1^{-1}x_2^{-1}, x_1^2x_2^2$ and $(x_1x_2)^k$, respectively. Since

$$x_1^2x_2^2 = (x_1x_2)^2 \cdot [x_1, x_2] = (x_1x_2)^2 \cdot (x_1^{-1}x_2^{-1}) \cdot (x_1x_2)$$

in $\mathcal{H}_2(\mathcal{E})$, we have

$$0 = [\Omega[2]|_{J_2(P^{2n-1}(2))} \circ \lambda'_{2n}] = 2[\lambda'_{2n}] + [\Omega[-1]|_{J_2(P^{2n-1}(2))} \circ \lambda'_{2n}] + [\lambda'_{2n}]$$

in $\pi_{4n-3}(\Omega P^{2n}(2))$ and hence the result. \hfill $\Box$

**Theorem 2.19.** Let $n > 1$ and let $f: V_{2n+1,2} \rightarrow V_{2n+1,2}$ be a map. Let

$$g = f|_{P^{2n}(2)}: P^{2n}(2) \rightarrow P^{2n}(2)$$

be the restriction of $f$. Then

$$\deg(f) \equiv \begin{cases} 
1 \mod 4 & \text{if } g \simeq \text{id} \quad \text{and} \quad n \equiv 0 \mod 2 \\
1 \mod 8 & \text{if } g \simeq \text{id} \quad \text{and} \quad n \equiv 1 \mod 2 \\
1 \mod 4 & \text{if } g \simeq [-1] \quad \text{and} \quad n \equiv 0 \mod 2 \\
5 \mod 8 & \text{if } g \simeq [-1] \quad \text{and} \quad n \equiv 1 \mod 2 \\
0 \mod 4 & \text{if } g \simeq [2] \quad \text{and} \quad n \equiv 0 \mod 2 \\
0 \mod 8 & \text{if } g \simeq [2] \quad \text{and} \quad n \equiv 1 \mod 2 \\
0 \mod 4 & \text{if } g \simeq * \quad \text{and} \quad n \equiv 0 \mod 2 \\
0 \mod 8 & \text{if } g \simeq * \quad \text{and} \quad n \equiv 1 \mod 2 
\end{cases}$$
Proof. There is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccccccccc}
\Omega S^{4n-1} & \xrightarrow{\partial} & F_q & \xrightarrow{\Omega[\deg(f)]} & V_{2n+1,2} & \xrightarrow{q} & S^{4n-1} \\
\downarrow & & \downarrow \bar{f} & & \downarrow f & & \downarrow \deg(f) \\
\Omega S^{4n-1} & \xrightarrow{\partial} & F_q & \xrightarrow{\Omega[\deg(f)]} & V_{2n+1,2} & \xrightarrow{q} & S^{4n-1},
\end{array}
\]

where \( q \) is the pinch map. By computing low-dimensional homology of \( F_q \), \( P^{2n}(2) \) is the \((6n - 4)\)-skeleton of \( F_q \) and so there is a homotopy commutative diagram

\[
\begin{array}{cccccc}
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2) \\
\downarrow & & \downarrow \deg(f) \simeq g \\
S^{4n-2} & \xrightarrow{\lambda_{2n}} & P^{2n}(2).
\end{array}
\]

The assertion follows from Lemmas 2.17 and 2.18. \( \square \)

**Corollary 2.20.** Let \( n > 1 \). Then

1. Any self homotopy equivalence of \( V_{2n+1,2} \) preserves the orientation. In particular, there is no self homeomorphism of \( V_{2n+1,2} \) which changes the orientation.
2. If \( f \) is a 2-local homotopy equivalence of \( V_{2n+1,2} \), then \( \deg(f) \equiv 1 \mod 4 \).
3. If \( n \equiv 0 \mod 2 \), then there is no self map of \( V_{2n+1,2} \) of degree 2 or 3 \mod 4.
4. If \( n \equiv 1 \mod 2 \), then there is no self map of \( V_{2n+1,2} \) of degree 2, 3, 4, 6 or 7 \mod 8.
5. If \( n \equiv 1 \mod 2 \) and \( f \) is a self homotopy equivalence of \( V_{2n+1,2} \), then \( f \) restricted to \( P^{2n}(2) \) is homotopic to the identity map of \( P^{2n}(2) \).
6. If \( n \equiv 0 \mod 2 \), then there is a self homotopy equivalence \( f \) of \( V_{2n+1,2} \) such that \( f|_{P^{2n}(2)} : P^{2n}(2) \to P^{2n}(2) \) is homotopic to \([-1]\).

### 2.4. The Strong Form of the Kervaire Invariant One Problem.

Consider the Whitehead square \( \omega_n \) in \( \pi_{2n-1}(S^n) \). That \( \omega_n \) is zero precisely when \( n = 1, 3, 7 \) is equivalent to the classical problem of the existence of elements of Hopf invariant one which was solved completely in [1]. Restrict attention to odd integers \( n \) which are not equal to 1, 3 or 7. In these cases \( \omega_n \) is non-zero and is of order 2. There is a short exact sequence of groups localized at 2

\[
0 \to \mathbb{Z}/2 \to \pi_{2n-1}(S^n) \to \pi_{2n-1}(S^n) \to 0.
\]

Hence \( \omega_n \) is divisible by 2 in \( \pi_{2n-1}(S^n) \) if and only if this sequence fails to split. It is well known that \( \omega_n \) is not divisible by 2 if \( n \neq 2^k - 1 \) for some \( k \). A proof can be found in [5]. Of the remaining cases given by \( n = 2^k - 1 \), it is known that \( \omega_n \) is divisible by 2 if \( n = 1, 3, 7, 15, 31 \) or 63, see [3, 20, 34]. The cases for which \( n = 2^k - 1 > 63 \) remain open.

**Note.** For even integers \( n \), it is a routine exercise to show that \( \omega_n \) is not divisible by 2 if \( n \neq 2, 4 \) or 8. According to [42], \( \omega_2 \) is divisible by 2 but \( \omega_4 \) and \( \omega_8 \) are not.
The Strong form of the Kervaire invariant one conjecture is that $\omega_n$ is divisible by 2 when $n = 2^k - 1$. There are many reformulations of this conjecture, see for instance [5, 8, 34]. Several reformulations in terms of mod 2 Moore spaces and the Stiefel manifolds are discussed below.

2.4.1. Spherical Classes. Let $X$ be a space. An element $a \in H_\ast(X)$ is called spherical if there is an element $b \in \pi_\ast(X)$ with Hurewicz image in mod 2 homology given by $a$. Recall that $H_\ast(\Omega \mathbb{P}_n^{n+1}(2)) = T(u; v)$, where $|v| = n$ and $Sq^1 v = u$.

**Proposition 2.21.** The Whitehead square $\omega_{2n-1}$ is divisible by 2 in $\pi_{4n-3}S^{2n-1}$ if and only if $u^2$ is spherical in $H_\ast(\Omega \mathbb{P}_2^{2n+1}(2))$.

**Proof.** First suppose that $\omega_{2n-1} : S^{4n-3} \to S^{2n-1}$ is divisible by 2. Let $f : S^{4n-3} \to S^{2n-1}$ be any map and let $f' : S^{4n-4} \to \Omega S^{2n-1}$ be the adjoint map of $f$. Then the composite $S^{4n-4} \xrightarrow{f'} \Omega S^{2n-1} \xrightarrow{H_2} \Omega S^{4n-3}$ is null homotopic and so, by Theorem 2.12, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{[2]} & S^{2n-1} \\
\downarrow f & & \downarrow f \\
S^{4n-3} & \xrightarrow{[2]} & S^{4n-3}.
\end{array}
\]

Thus the composite $S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \xrightarrow{\omega_{2n-1}} P^{2n}(2)$ is null homotopic. Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
S^{4n-3} & \xrightarrow{\omega_{2n-1}} & S^{2n-1} \\
\downarrow & & \downarrow \\
\Sigma P^{2n-1}(2) \wedge P^{2n-1}(2) & \xrightarrow{W_2} & P^{2n}(2),
\end{array}
\]

where $W_2 : \Sigma X \wedge X \to \Sigma X$ is Whitehead product. It follows that there is a homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
S^{4n-3} & \to & * & \to & S^{4n-2} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma P^{2n-1}(2) \wedge P^{2n-1}(2) & \xrightarrow{W_2} & P^{2n}(2) & \xrightarrow{J_2(P^{2n}(2))} & P^{2n}(2) \wedge P^{2n}(2)
\end{array}
\]

and so $u^2$ is spherical.

Conversely suppose that $u^2$ is spherical in $H_\ast(\Omega \mathbb{P}_2^{2n+1}(2))$. We may assume that $n > 1$. Since $J_2(P^{2n}(2))$ is the $(6n-4)$-skeleton of $\Omega \mathbb{P}_2^{2n+1}(2) \simeq J(P^{2n}(2))$, the element $u^2$ is spherical in $H_\ast J_2(P^{2n}(2))$. Thus there is a homotopy commutative diagram of fibre
sequences
\[ \Omega(P^{2n}(2) \wedge P^{2n}(2)) \longrightarrow Fq \longrightarrow J_2(P^{2n}(2)) \xrightarrow{q} P^{2n}(2) \wedge P^{2n}(2) \]
\[ \Omega S^{4n-2} \longrightarrow * \longrightarrow S^{4n-2} \longrightarrow S^{4n-2} \]
where \( i \) is the canonical inclusion and \( p \) is the pinch map. By inspecting the Serre spectral sequence for the fibre sequence
\[ F \longrightarrow J_2(P^{2n}(2)) \longrightarrow P^{2n}(2) \wedge P^{2n}(2), \]
the space \( P^{2n}(2) \) is the \((6n - 5)\)-skeleton of \( F \). Thus the composite
\[ S^{4n-3} \longrightarrow \Sigma P^{2n-1}(2) \wedge P^{2n-1}(2) \longrightarrow P^{2n}(2) \]
is null homotopic or the composite
\[ S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \longrightarrow P^{2n}(2) \]
is null homotopic. It follows that \( \omega_{2n-1} : \Omega S^{4n-3} \rightarrow S^{2n-1} \) lifts to the homotopy fibre \( E \) of the injection \( S^{2n-1} \rightarrow P^{2n}(2) \).\]
Consider the fibre sequence
\[ \Omega S^{2n-1} \xrightarrow{\Omega j} \Omega P^{2n}(2) \longrightarrow E \xrightarrow{f} S^{2n-1} \xrightarrow{i} P^{2n}(2). \]
Since \( \Omega j_* : H_*(\Omega S^{2n-1}) \rightarrow H_*(\Omega P^{2n}(2)) \) is a monomorphism, the Serre spectral sequence for the fibre sequence \( \Omega S^{2n-1} \rightarrow \Omega P^{2n}(2) \rightarrow E \) collapses. By [12, Lemma 2.1], the class \([u, v]\) is spherical in \( H_*(\Omega P^{2n}(2)) \) and so is in \( H_*(E) \). Hence the \((4n - 3)\)-skeleton \( \text{sk}_{4n-3}(E) \) of \( E \) is homotopy equivalent to \( S^{2n-1} \vee S^{4n-3} \) and \( f|_{S^{4n-3}} \) is null. Now let \( \theta : S^{4n-3} \longrightarrow E \) be a lifting of \( \omega_{2n-1} \). Let \( \theta_1 \) and \( \theta_2 \) denote the composites
\[ S^{4n-3} \xrightarrow{\theta} \text{sk}_{4n-3}(E) \xrightarrow{\text{proj.}} S^{2n-1} \subset E \quad \text{and} \]
\[ S^{4n-3} \xrightarrow{\theta} \text{sk}_{4n-3}(E) \xrightarrow{\text{proj.}} S^{4n-3} \subset E, \]
respectively. Because \( f|_{S^{2n-1}} \) is of degree 2, we have
\[ \omega_{2n-1} \simeq f \circ \theta \simeq f \circ \theta_1 \simeq [2] \circ \theta_1. \]
The assertion follows from the fact that
\[ [2]_* : \pi_4n-3(S^{2n-1}) \rightarrow \pi_{4n-3}(S^{2n-1}). \]
\[ \square \]
Recall that \( H_*(S^n\{2\}) \cong P(u) \otimes E(v) \) as \( P(u) \)-module with \(|v| = n\) and \( Sq^1v = u \).

**Corollary 2.22.** The Whitehead square \( \omega_{2n-1} \) is divisible by 2 if and only if \( u^2 \) is spherical in \( H_*(S^n\{2\}) \).
Proof. The homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega S^{2n+1} & \longrightarrow & F^{2n+1}\{2\} \\
\downarrow & & \downarrow \text{pinch} \\
S^{2n} & \longrightarrow & S^{2n+1}
\end{array}
\]

induces a canonical inclusion \(j : S^{2n}\{2\} \rightarrow \Omega P^{2n+1}\{2\}\) so that

\[
j_* : H_r(S^{2n}\{2\}) \longrightarrow H_r(\Omega P^{2n+1}\{2\})
\]

is one to one for all \(r\) and onto for \(r \leq 4n-2\) and hence the result. \(\square\)

2.4.2. Exponents of Homotopy Groups.

Proposition 2.23. Let \(n \geq 2\). Suppose that \(u^2\) is spherical in \(H_{4n-2}(S^{2n}\{2\})\). Let \(x \in \pi_{4n-2}(S^{2n}\{2\})\) be any element with Hurewicz image in mod 2 homology given by \(u^2\). Then \(x\) generates a \(\mathbb{Z}/8\)-summand in \(\pi_{4n-2}(S^{2n}\{2\})\).

Proof. Since \(u^2\) is spherical, we have

\[
\text{sk}_{4n-2}(S^{2n}\{2\}) \simeq P^{2n}(2) \vee S^{4n-2}
\]

and so there is a cofibre sequence

\[
S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{j} \text{sk}_{4n-1}(S^{2n}\{2\}) \longrightarrow S^{4n-1}.
\]

Let \(\phi_1\) and \(\phi_2\) denote the composites

\[
S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{\text{proj}} P^{2n}(2) \quad \text{and} \quad S^{4n-2} \xrightarrow{\phi} P^{2n}(2) \vee S^{4n-2} \xrightarrow{\text{proj}} S^{4n-2},
\]

respectively. Since \(\tilde{H}_*(\text{sk}_{4n-1}(S^{2n}\{2\}))\) has a basis \(u, v, u^2\) and \(uv\) with \(Sq^1v = u\) and \(Sq^1(uv) = u^2\), the map \(\phi_2 : S^{4n-2} \rightarrow S^{4n-2}\) is of degree 2. Consider the homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
S^{4n-2} & \xrightarrow{\phi} & P^{2n}(2) \vee S^{4n-2} \xrightarrow{\text{sk}_{4n-1}(S^{2n}\{2\})} \xrightarrow{q} S^{4n-1} \\
\downarrow & & \downarrow \phi_1 \\
S^{4n-2} & \xrightarrow{\phi_1} & P^{2n}(2) \xrightarrow{f} X^{2n} \xrightarrow{q} S^{4n-1},
\end{array}
\]

where \(X^{2n} = C_{\phi_1}\). Since \(q_* : H_*(\text{sk}_{4n-1}(S^{2n}\{2\})) \rightarrow H_*(X^{2n})\) is onto, we have

\[
H_*(X^{2n}) \simeq E(\bar{u}) \otimes E(\bar{v})
\]

as coalgebras, where \(\bar{x} = q_*(x)\). By inspecting the Eilenberg-Moore spectral sequence for \(\Omega X^{2n} \rightarrow \ast \rightarrow X^{2n}\), we have

\[
H_*(\Omega X^{2n}) \simeq P(s^{-1}\bar{u}, s^{-1}\bar{v})
\]

the polynomial on \(s^{-1}\bar{u}\) and \(s^{-1}\bar{v}\). Thus

\[
\Omega f_* : H_*(\Omega P^{2n}(2)) \cong T(s^{-1}\bar{u}, s^{-1}\bar{v}) \rightarrow H_*(\Omega X^{2n}) \cong P(s^{-1}\bar{u}, s^{-1}\bar{v})
\]
is onto and so there is a short exact sequence of Hopf algebras
\[ H_*(\Omega F_f) \longrightarrow H_*(\Omega P^{2n}(2)) \longrightarrow H_*(\Omega X^{2n}). \]

Observe that \( \phi_1 : S^{4n-2} \to P^{2n}(2) \) lifts to \( F_f \) with \( \phi_1_* : H_{4n-2}(S^{4n-2}) \cong H_{4n-2}(F_f) \) and \( \tilde{H}_r(F_f) = 0 \) for \( r < 4n - 2 \). Let \( \phi'_1 : S^{4n-3} \to \Omega P^{2n}(2) \) be the adjoint map of \( \phi_1 \). Then \( \phi'_1(\ell_{4n-3}) = [s^{-1}u, s^{-1}v] \) which is the only primitive element in \( H_{4n-3}(\Omega P^{2n}(2)) \).

Since \( 2n \) should be a power of 2, the homotopy class \([\phi_1]\) of \( \phi_1 \) is of order 4 in \( \pi_{4n-2}(P^{2n}(2)) \) by [12, Theorem 2.2]. Now, by the cofibre sequence
\[ S^{4n-2} \overset{\phi}{\longrightarrow} P^{2n}(2) \vee S^{4n-2} \overset{j}{\longrightarrow} sk_{4n-1}(S^{2n}\{2\}) \longrightarrow S^{4n-1}, \]
there is a short exact sequence
\[ 0 \longrightarrow \mathbb{Z} \overset{\pi_{4n-2}(\phi_4)}{\longrightarrow} \pi_{4n-2}(P^{2n}(2)) \oplus \mathbb{Z} \overset{j_*}{\longrightarrow} \pi_{4n-2}(S^{2n}\{2\}) \longrightarrow 0 \]
with \( \phi_4(\ell_{4n-2}) = [\phi_1] + 2[\ell_{4n-2}] \). Thus
\[ 8x = 8j_*(\ell_{4n-2}) = 4j_*(2[\ell_{4n-2}]) = 4j_*(\ell_{4n-2}) = 0 \]
and
\[ 4x = 4j_*(\ell_{4n-2}) = j_*(2[\phi_1]) \neq 0 \]
since \( 2[\phi_1] \not\in \text{Im}(\phi_4) \). Observe that \( x \) is not divisible by 2. Thus \( x \) generates a \( \mathbb{Z}/8 \)-summand and hence the result. \( \square \)

**Proposition 2.24.** If \( \omega_{2n-1} \) is not divisible by 2, then \( 4\pi_r(S^{2n}\{2\}) = 0 \) for \( r \leq 4n - 2 \).

**Proof.** Consider the homotopy commutative diagram of fibre sequences
\[
\begin{array}{cccccccc}
\Omega^2 S^{4n-1} & \overset{\Omega^2[4]}{\longrightarrow} & \Omega^2 S^{4n-1} & \longrightarrow & \Omega(S^{4n-1}\{4\}) & \longrightarrow & \Omega S^{4n-1} & \overset{\Omega[4]}{\longrightarrow} & \Omega S^{4n-1} \\
& & H_2 & & H_2 & & H_2 & & H_2 \\
\Omega^2 S^{2n} & \overset{\Omega^2[2]}{\longrightarrow} & \Omega^2 S^{2n} & \longrightarrow & \Omega(S^{2n}\{2\}) & \longrightarrow & \Omega S^{2n} & \overset{\Omega[2]}{\longrightarrow} & \Omega S^{2n} \\
& & & & & & & & & \\
\Omega S^{2n-1} & \overset{\Omega[2]}{\longrightarrow} & \Omega S^{2n-1} & \longrightarrow & \Omega(S^{2n-1}\{2\}) & \longrightarrow & \Omega S^{2n-1} & \overset{\Omega[2]}{\longrightarrow} & \Omega S^{2n-1}. \\
\end{array}
\]

The map
\[ E_* : \pi_r(S^{2n-1}\{2\}) \longrightarrow \pi_r(\Omega S^{2n}\{2\}) \]
is onto for \( r \leq 4n - 4 \) and there is an exact sequence
\[ \pi_{4n-3}(S^{2n-1}\{2\}) \longrightarrow \pi_{4n-3}(\Omega(S^{2n}\{2\})) \longrightarrow \pi_{4n-3}(\Omega(S^{4n-1}\{4\})). \]

Since \( u^2 \) is not spherical, \( \pi_{4n-3}(\Omega(S^{2n}\{2\})) \to \pi_{4n-3}(\Omega(S^{4n-1}\{4\}))=\mathbb{Z}/4 \) is not an epimorphism. Recall that
\[ [\Omega H_2 \circ \omega_{2n-1}^*] = 2[\ell_{4n-3}] \]
in \( \pi_{4n-3}(\Omega^2 S^{4n-1}) \), where \( \omega_{2n} : S^{4n-3} \to \Omega^2 S^{2n} \) is the adjoint map of Whitehead square \( \omega_{2n} \).

Thus
\[ \text{Im}(\tilde{H}_2^* : \pi_{4n-3}(\Omega(S^{2n}\{2\})) \longrightarrow \pi_{4n-3}(\Omega(S^{4n-1}\{4\}))) = \mathbb{Z}/2 \]
which is generated by \([\tilde{H}_2 \circ \Omega f \circ \omega_{2n}']\). By Theorem 2.12, we have

\[ \Omega^2[2] \circ \omega_{2n}' = 2\omega_{2n}' + \Omega^2 \omega_{2n} \circ \Omega H_2 \circ \omega_{2n}' = 4\omega_{2n}'. \]

Thus \(4\Omega f_* (\omega_{2n}') = 0\) and so

\[ \pi_{4n-3}(S^{2n-1}\{2\}) \oplus \mathbb{Z}/4 \rightarrow \pi_{4n-3}(\Omega(S^{2n}\{2\})) \]

is onto. According to [12], \(4\pi_r(S^{2n-1}\{2\}) = 0\) for \(r \leq 4n-3\) and hence the result. \(\square\)

**Corollary 2.25.** Let \(n \geq 2\). Then the Whitehead square \(\omega_{2n-1}\) is not divisible by 2 if and only if \(4\pi_r(S^{2n}\{2\}) = 0\) for \(r \leq 4n-2\).

**Proposition 2.26.** Let \(n > 1\). Suppose that \(u^2\) is spherical in \(H_*(\Omega P^{2n+1}(2))\). Let \(x\) be any element in \(\pi_{4n-2}(\Omega P^{2n+1}(2))\) with the Hurewicz image in mod 2 homology given by \(u^2\). Then \(x\) is of order 4.

**Proof.** Suppose that \(2x = 0\). Then there is a map \(\phi : P^{4n+1}(2) \rightarrow \Omega P^{2n+1}(2)\) such that \(\phi|_{S^{4n-2}}\) is a representative of \(x\) in \(\pi_{4n-2}(\Omega P^{2n+1}(2))\). Let \(\{U, V\}\) be a basis for \(\tilde{H}_*(P^{4n-2}(2))\) with \(Sq_*^2 V = U\). Since \(\phi_* (U) = u^2\), we have \(Sq_*^2 \phi_* (V) = \phi_* (U) = u^2\) and so \(\phi_* (V) = uv\) or \(vu\). This is a contradiction because \(uv\) and \(vu\) are not primitive. Thus \(2x \neq 0\).

Now we show that \(4\pi_{4n-2}(\Omega P^{2n+1}(2)) = 0\) from which we obtain \(4x = 0\). In the Cohen group \(K_2(\mathbb{E}^{\mathbb{Z}/4})\), we have

\[ (x_1 x_2)^4 = [x_1, x_2]^2 \]

and so

\[ 4y = -2(\Omega W_2 \circ H_2)_*(y) \]

for any \(y \in \pi_r(\Omega P^{2n+1}(2))\) with \(r \leq 6n-4\), where \(W_2 : \Sigma(P^{2n}(2))^{(2)} \rightarrow P^{2n+1}(2)\) is the Whitehead product. Since \(2 \cdot \pi_{4n-2}(\Omega \Sigma(P^{2n}(2))^{(2)}) = 0\), we have \(4y = 0\) and hence the result. \(\square\)

**2.4.3. Partial H-spaces.** A map \(f : X \rightarrow Y\) is called a partial H-space if there is a map \(\tilde{f} : J_2(X) \rightarrow Y\) such that \(\tilde{f}|_X = f\), or equivalently there is a map \(\mu : X \times X \rightarrow Y\) such that \(\mu|_{XX} = \mu|_{XX} = f\). The identity map \(f = \text{id}_X : X \rightarrow X\) is a partial H-space if and only if \(X\) is an H-space. If \(X\) is an H-space, then any map from \(X \rightarrow Y\) is a partial H-space.

It is well known that \(S^{2n-1}\) is an H-space if and only if \(\omega_{2n-1}\) is null homotopic (and so if and only if \(2n-1 = 1, 3\) or 7). Consider the spherical fibration

\[ S^{2n-1} \xrightarrow{i} V_{2n+1,2} \rightarrow S^{2n}. \]

One may ask whether the inclusion \(i : S^{2n-1} \rightarrow V_{2n+1,2}\) is a partial H-space. This question is equivalent to the strong form of the Kervaire invariant one problem.

**Proposition 2.27.** The following are equivalent.

1. The Whitehead square \(\omega_{2n-1}\) is divisible by 2.
2. the inclusion \(i : S^{2n-1} \rightarrow P^{2n}(2)\) is a partial H-space.
3. the inclusion \(i : S^{2n-1} \rightarrow V_{2n+1,2}\) is a partial H-space.
Proof. (2) ⇒ (3) is obvious. (1) ⇒ (2). Since \( \omega_{2n-1} \) is divisible by 2, the composite
\[
S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \xrightarrow{i} P^{2n}(2)
\]
is null homotopic and so the inclusion \( i: S^{2n-1} \to P^{2n}(2) \) extends to \( J_2(S^{2n-1}) \) or \( i \) is a partial \( H \)-space.

(3) ⇒ (1). Since the inclusion \( i: S^{2n-1} \to V_{2n+1,2} \) extends to \( J_2(S^{2n-1}) \), the composite
\[
S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1} \xrightarrow{i} V_{2n+1,2}
\]
is null homotopic and so there is a homotopy commutative diagram
\[
\begin{array}{ccc}
S^{4n-3} & \xrightarrow{\omega} & S^{2n-1} \\
\downarrow g & & \downarrow \partial \\
\Omega S^{2n} & \xrightarrow{\partial} & S^{2n-1} \\
\downarrow E & & \downarrow i \\
S^{2n-1} & \xrightarrow{[2]} & S^{2n-1},
\end{array}
\]
where the middle row is the fibre sequence. Recall that \( E_\ast: \pi_\ast(S^{2n-1}) \to \pi_\ast(\Omega S^{2n}) \) is onto for \( r \leq 4n-3 \). Thus there is a map \( \tilde{g}: S^{4n-3} \to S^{2n-1} \) such that \( \omega_{2n-1} \simeq [2] \circ \tilde{g} \) and hence the result.

The question whether the inclusion \( i: S^{2n-1} \to V_{2n+1,2} \) is a partial \( H \)-space admits some geometric means, namely whether there is a map
\[
\phi: S^{2n-1} \times S^{2n-1} \xrightarrow{(\phi_0,\theta_0,\phi_1,\theta_1)} \mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}
\]
such that
\begin{enumerate}
  \item \( \phi_0(x,*) = \phi_0(*,x) = x \);
  \item \( \phi_1(x,*) = \phi_1(*,x) = 0 \);
  \item \( \theta_0(x,*) = \theta_0(*,x) = 0 \);
  \item \( \theta_1(x,*) = \theta_1(*,x) = 1 \);
  \item \( (\phi_0,\theta_0) \) and \( (\phi_1,\theta_1) \) are nowhere zero;
  \item \( (\phi_1,\theta_1) \) is orthogonal to \( (\phi_0,\theta_0) \).
\end{enumerate}
The map \( \phi_0 \) looks like a multiplication on \( \mathbb{R}^{2n} \) with possible zero divisors, where \( 2n \) is assumed to be a power of 2. The Cayley-Dickson multiplication might be a candidate for the map \( \phi_0 \). Condition (5) requires that \( \theta_0(x,y) \neq 0 \) whenever \( \phi_0(x,y) = 0 \) for \( x,y \in S^{2n-1} \). The map \( (\phi_1,\theta_1) \) looks like certain ‘derivative’ of \( (\phi_0,\theta_0) \).

2.4.4. Spherical Fibrations.

**Proposition 2.28.** Suppose that the Whitehead square \( \omega_{2n-1} \) is divisible by 2. Then there is a spherical fibration
\[
S^{2n-1} \longrightarrow E^{4n-1} \longrightarrow P^{2n}(2)
\]
such that the boundary map \( \partial: \Omega P^{2n}(2) \to S^{2n-1} \) is onto in mod 2 homology.
Proof. When \(2n - 1 = 1, 3 \text{ or } 7\), the space \(E^{4n-1}\) can be chosen as the homotopy pull back of the diagram \(S^{4n-1} \xrightarrow{q} S^{2n} \xleftarrow{\text{pinch}} P^{2n}(2)\) and so we may assume that \(n \neq 1, 3, 7\).

Let \(q: P^{2n-1}(2) \to S^{2n-1}\) be the pinch map. Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
S^{4n-5} & \xrightarrow{\omega_{2n-2}} & P^{2n-1}(2) \wedge P^{2n-2}(2) \\
\downarrow & & \downarrow \text{pinch} \\
S^{2n-2} & \xrightarrow{W_2} & P^{2n-1}(2) \\
\downarrow & & \downarrow q \\
J_2(S^{2n-2}) & \xrightarrow{\bar{q}} & J_2(P^{2n-1}(2)) \\
\end{array}
\]

where \(\bar{W}_2\) is the composite

\[
P^{2n-1}(2) \wedge P^{2n-2}(2)/S^{4n-5} \xrightarrow{\text{pinch}} P^{4n-3}(2) \xrightarrow{\text{pinch}} S^{4n-3} \xrightarrow{\omega_{2n-1}} S^{2n-1}.
\]

Since \(\omega_{2n-1}\) is divisible by 2, the map \(\bar{W}_2\) is null homotopic and so there is a retraction \(r: J_2(P^{2n-1}(2))/J_2(S^{2n-2}) \to S^{2n-1}\). Let \(\bar{q}: J_2(P^{2n-1}(2)) \to S^{2n-1}\) be the composite

\[
J_2(P^{2n-1}(2)) \xrightarrow{\text{pinch}} J_2(P^{2n-1}(2))/J_2(S^{2n-2}) \xrightarrow{r} S^{2n-1}.
\]

Then \(\bar{q}|_{P^{2n-1}(2)} = q\). Let \(\{u, v\}\) be a basis for \(\bar{H}_*(P^{2n-1}(2))\) with \(S\bar{q}^* u = u\). Consider the Serre spectral sequence for the fibre sequence

\[
\Omega S^{2n-1} \xrightarrow{\Omega F} F \xrightarrow{\bar{q}} J_2(P^{2n-1}(2)) \xrightarrow{\bar{q}} S^{2n-1}.
\]

Then

\[
H_r(F) \to H_r(J_2(P^{2n-1}(2)))
\]

is a monomorphism for \(r \leq 6n - 7\). A basis for \(H_r(F)\) for \(r \leq 6n - 7\) is given by \(\{u, u^2, [u, v], v^2\}\). Since \(\bar{q}|_{J_2(S^{2n-2})}\) is null homotopic. The inclusion

\[
J_2(S^{2n-2}) \to J_2(P^{2n-1}(2))
\]

lifts to \(F\) and so \(\text{sk}_{4n-4}(F) \simeq J_2(S^{2n-2})\) with a cofibre sequence

\[
P^{4n-3}(2) \to J_2(S^{2n-2}) \to \text{sk}_{4n-2}(F).
\]

Let \(s: \Sigma X^{(2)} \to \Sigma J_2(X)\) be a functorial cross-section of the pinch map \(\Sigma J_2(X) \to \Sigma X^{(2)}\) and let \(E^{4n-1}\) be the homotopy cofibre of the composite

\[
S^{4n-3} \xrightarrow{s} \Sigma J_2(S^{2n-2}) \subset \Sigma \text{sk}_{4n-2} F.
\]
Then there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^{4n-3} & \xrightarrow{s} & P^{2n-1} \wedge P^{2n}(2) \\
\downarrow & & \downarrow s \\
\Sigma \sk_{4n-2}(F_q) & \xleftarrow{\phi} & \Sigma J_2(P^{2n-1}(2)) \\
\downarrow & & \downarrow \\
E^{4n-1} & \xrightarrow{\theta} & P^{2n}(2).
\end{array}
\]

We show that the homotopy fibre of the map \( \theta \) is \( S^{2n-1} \). By considering the commutative diagram

\[
H_*(\Omega \Sigma \sk_{4n-2}(\tilde{F}_q)) \rightarrow H_*(\Omega \Sigma J_2(P^{2n-1}(2)))
\]

the elements \( u, [u, v] \) and \( v^2 \) of \( H_*(\Omega P^{2n}(2)) = T(u, v) \) lie in the image of \( \theta_* \) and so the Poincaré series

\[
\chi(H_*(\Omega E^{4n-1})) \geq \chi(T(u, [u, v], v^2)).
\]

By considering the Eilenberg-Moore spectral sequence for \( \Omega E^{4n-1} \to * \to E^{4n-1} \), we have

\[
\chi(H_*(\Omega E^{4n-1})) \leq \chi(T(u, [u, v], v^2)).
\]

Thus \( \chi(H_*(\Omega E^{4n-1})) = \chi(T(u, [u, v], v^2)) \) or \( H_*(\Omega E^{4n-1}) \cong T(u, [u, v], v^2) \) and therefore \( \theta_* : H_*(\Omega E^{4n-1}) \to H_*(\Omega P^{2n}(2)) \) is a monomorphism. Hence the Serre spectral sequence for the fibre sequence \( \Omega E^{4n-1} \to \Omega P^{2n}(2) \to F_\theta \) collapses and so

\[
\chi(H_*(F_\theta)) = \frac{\chi(T(u, v))}{\chi(T(u, [u, v], v^2))} = 1 + t^{2n-1}
\]

It follows that \( F_\theta \) is homotopy equivalent to \( S^{2n-1} \) and hence the result. \( \square \)

**Proposition 2.29.** Let \( n \geq 2 \). Suppose that there is a spherical fibration

\[
S^{2n-1} \longrightarrow E^{4n-1} \longrightarrow P^{2n}(2)
\]

such that the boundary map \( \partial : \Omega P^{2n}(2) \rightarrow S^{2n-1} \) is onto in mod 2 homology. Then

1) the homotopy cofibre of the map \( E^{4n-1} \to P^{2n}(2) \) has a non-trivial cup product.

2) The Whitehead square \( \omega_{2n-1} \) is divisible by 2.
Proof. Consider the Serre spectral sequence for $S^{2n-1} \rightarrow E^{4n-1} \rightarrow P^{2n}(2)$. The homology $\tilde{H}_*(E^{4n-1})$ has a basis $x_{2n-1}, x_{4n-2}$ and $x_{4n-1}$, where $|x_i| = i$. Since $\partial_* : H_*(\Omega P^{2n}(2)) \rightarrow S^{2n-1}$ is onto, the Serre spectral sequence for
$$\Omega E^{4n-1} \rightarrow \Omega P^{2n}(2) \rightarrow S^{2n-1}$$
collapses and so $H_*(\Omega E^{4n-1}) \cong T(u, [u, v], v^2)$. By expecting the Serre spectral sequence for $\Omega E^{4n-1} \rightarrow * \rightarrow E^{4n-1}$, the homology suspension
$$\sigma : Q(H_*(\Omega E^{4n-1})) \rightarrow \tilde{H}_*(E^{4n-1})$$
is an isomorphism, where $Q(A)$ is the set of indecomposable elements of an algebra $A$. It follows that $S_q^1 x_{4n-1} = x_{4n-2}$.

Let $X$ be the homotopy fibre of $E^{4n-1} \rightarrow P^{2n}(2)$. Then $\tilde{H}_*(X)$ has a basis $y_{2n}, y_{4n-1}$ and $y_{4n}$ with $S_q^1 y_{4n} = y_{4n-1}$. Consider the homotopy commutative diagram
$$\begin{CD}
\Omega P^{2n}(2) @>>> S^{2n-1} @>>> E^{4n-1} @>>> P^{2n}(2) \\
@VV\Omega f V @VV\Omega f V @VVf V @VVf V \\
\Omega X @>>> \Omega X @>>> * @>>> X.
\end{CD}$$

Let $\tau$ denote the transgression. We have
$$\tau(y_{2n})^2 = \Omega f_*(v^2) = 0.$$

By expecting the Serre spectral sequence for $\Omega X \rightarrow * \rightarrow X$, we have
$$y_{2n}^2 = y_{4n}^*$$
in the cohomology $H^*(X)$ and hence assertion (1). By [5, Theorem 9.1], assertion (2) follows. This finishes the proof. \qed

2.5. The Exponent Problem.

2.5.1. The Exponent Problems in the Homotopy Theory. Let $p$ be a prime integer and let $G$ be an abelian group. We write $\text{Tor}_p(G)$ for the $p$-torsion component of $G$. We call $p^r$ an exponent of $G$ if $p^r \cdot \text{Tor}_p(G) = 0$. Let $X$ be a space. The integer $p^r$ is called an exponent of $\pi_*(X)$ if $p^r \cdot \text{Tor}_p(\pi_*(X)) = 0$ for all $n \geq 2$. If $X$ is a simply connected CW of finite type, then each $\pi_n(X)$ has a bounded exponent which depends on $n$ in general. For instance $\pi_*(S^2 \vee S^2)$ does not have a bounded exponent. On the other hand, one can check that the homotopy groups of the mapping space from a simply connected finite torsion space to a space has a bounded exponent.

It was first known by James [19] that $\pi_*(S^{2n+1})$ has an exponent bounded by $2^{2n}$. The improvements given in [5, 35] state that the exponent of $\pi_*(S^{2n+1})$ is bounded by $2^{2n-[n/2]}$, where $[a]$ is the maximal integer $\leq a$. In the cases where $p > 2$, Toda [41] showed that $\pi_*(S^{2n+1})$ has an exponent bounded by $p^{2n}$. Selick [33] then showed that $\pi_*(S^3)$ has an exponent bounded by $p$ for $p > 2$. Later Cohen-Moore-Neisendorfer [9] proved that $\pi_*(S^{2n+1})$ has an exponent bounded by $p^n$ for $p > 2$. This is the best exponent because a theorem of Gray [15] gives that there are $\mathbb{Z}/p^n$-summands in $\pi_*(S^{2n+1})$ for $p > 2$. The best 2-exponent of $\pi_*(S^{2n+1})$ remains open.

The Moore conjecture says that the homotopy groups of a simply connected finite complex $X$ has a bounded $p$-exponent if the rational cohomology ring $H^*(\Omega X; \mathbb{Q})$ is
finitely generated. For any given simply connected finite complex \( X \), it was known by McGibbon and Wilkerson [21] that the Moore conjecture holds when the prime \( p \) is sufficiently large. By the arguments above, the Moore conjecture holds for spheres (for all primes \( p \)) and so for any space which admits a finite fibration resolution in terms of spheres, for instance the classical Lie groups.

The Barratt conjecture states that if the degree map \([p^r]: \Sigma X \to \Sigma X\) is null homotopic, then \( p^{r+1} \pi_*(\Sigma X) = 0 \). It was known by Neisendorfer [30] that the Barratt conjecture holds for the Moore spaces \( P^n(p^r) \) with \( p > 2 \). By [9], there is a family of \( \mathbb{Z}/p^{r+1} \)-summands in \( \pi_*(P^n(p^r)) \) and so \( p^{r+1} \) is the best exponent.

In [6], Cohen showed that \( \pi_*(P^n(2^r)) \) has a bounded exponent when \( r \geq 2 \) and so the Moore conjecture holds for these spaces. It is unknown whether the Barratt conjecture holds for \( P^n(2^r) \) with \( r \geq 2 \). But there is a family of \( \mathbb{Z}/2^{r+1} \)-summands in \( \pi_*(P^n(2^r)) \), see [4], and so \( 2^{r+1} \) will be the best exponent if the Barratt conjecture holds.

It is still unknown whether \( \pi_*(P^n(2)) \) has a bounded exponent. In other words, the conjectures of Moore and Barratt both remain open for mod 2 Moore spaces.

2.5.2. Exponents of \( \pi_*(S^n\{2\}) \). Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{c}
\Omega^2 S^{2n+1} & \longrightarrow & \Omega(S^{2n+1}\{4\}) & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{\Omega[4]} & \Omega S^{2n+1} \\
\downarrow & & \downarrow H_2 & & \downarrow H_2 & & \downarrow H_2 \\
\Omega^2 S^{n+1} & \longrightarrow & \Omega(S^{n+1}\{2\}) & \longrightarrow & \Omega S^{n+1} & \xrightarrow{\Omega[2]} & \Omega S^{n+1} \\
\downarrow & & \downarrow \bar{E} & & \downarrow E & & \downarrow E \\
\Omega S^n & \longrightarrow & S^n\{2\} & \longrightarrow & S^n & \xrightarrow{[2]} & S^n .
\end{array}
\]

We obtain the fibre sequence

\[ S^n\{2\} \xrightarrow{\bar{E}} \Omega(S^{n+1}\{2\}) \xrightarrow{\bar{H}_2} \Omega(S^{2n+1}\{4\}) . \]

Recall that \( S^n\{2\} \) is an \( \Omega S^n = J(S^{2n-1}) \)-space. By the above diagram, the map \( \bar{E} \) can be chosen as an \( \Omega S^n \)-map.

**Lemma 2.30.** Let \( n \geq 3 \). Then the map

\[ 4 \cdot \bar{E} : S^n\{2\} \longrightarrow \Omega(S^{n+1}\{2\}) \]

is null homotopic.

**Proof.** Filter \( S^n\{2\} \) by setting \( F_k S^n\{2\} \) to be the \( ((n - 1)k + 1) \)-skeleton. Let \( \mu : J(S^{n-1}) \times P^n(2) \to S^n\{2\} \) be the \( J(S^{n-1}) \)-action on \( S^n\{2\} \). Then \( \mu_k = \mu|_{J_k(S^{n-1}) \times P^n(2)} \)
lifts to $F_{k+1}S^n\{2\}$. Recall that

$$\Sigma J_k S^{n-1} \times P^n(2) \simeq \sum_{j=1}^{k} S^{(n-1)j} \wedge P^n(2) \lor \sum_{j=1}^{k} S^{(n-1)j} \lor P^n(2)$$

and

$$\Sigma F_{k+1} S^n\{2\} \simeq \sum_{j=0}^{k} S^{(n-1)j} \wedge P^n(2).$$

The map $\Sigma \mu_k$ is a retraction for $0 \leq k \leq +\infty$. Let

$$q_k : (S^{n-1})^k \times P^n(2) \rightarrow J_k(S^{n-1}) \times P^n(2)$$

denote the projection. Then $\Sigma q_k$ is a retraction and therefore

$$q_k^* \circ \mu_k^* : [F_{k+1}S^n\{2\}, \Omega Y] \rightarrow [(S^{n-1})^k \times P^n(2), \Omega Y]$$

is a monomorphism for any $Y$. Since $\tilde{E}$ is an $\Omega S^n$-map, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
(S^{n-1})^k \times P^n(2) & \xrightarrow{(\tilde{E}|_{S^{n-1}})^k \times \tilde{E}|P^n(2)} & (\Omega(S^{n+1}\{2\}))^{k+1} \\
\downarrow q_k & & \downarrow \mu \times \text{id} \\
J_k S^{n-1} \times P^n(2) & \xrightarrow{\tilde{E}|J_k(S^{n-1}) \times \tilde{E}|P^n(2)} & \Omega(S^{n+1}\{2\}) \times \Omega(S^{n+1}\{2\}) \\
\downarrow \mu_k & & \downarrow \mu \\
F_{k+1} S^n\{2\} & \xrightarrow{\tilde{E}} & \Omega(S^{n+1}\{2\}).
\end{array}
\]

Let $x_i$ and $y$ denote the homotopy classes of

$$(S^{n-1})^k \times P^n(2) \xrightarrow{\pi_i} S^{n-1} \xrightarrow{\tilde{E}|_{S^{n-1}}} \Omega(S^{n+1}\{2\})$$

and

$$(S^{n-1})^k \times P^n(2) \xrightarrow{\pi_{k+1}} P^n(2) \xrightarrow{\tilde{E}|_{P^n(2)}} \Omega(S^{n+1}\{2\}),$$

respectively, where $\pi_i$ the $i$-th coordinate projection. Then

$$q_k^* \circ \mu_k^*([\tilde{E}|_{F_{k+1}S^n\{2\}}]) = x_1 x_2 \cdots x_k y.$$

Now let $G_k$ be the subgroup of $[(S^{n-1})^k \times P^n(2), \Omega(S^{n+1}\{2\})]$ generated by $x_i$, $1 \leq i \leq k$, and $y$. Then the following identities hold in $G_k$:

1) $x_i^2 = 1$ for $1 \leq i \leq k$;
2) $y^k = 1$;
3) the commutators $[x_i, x_j] = 1$;
4) The (iterated) commutators

$$[[a_1, a_2], \ldots, a_s] = 1$$

if $a_i = a_j$ for some $i < j$, where $a_i = y$ or $x_j$ for some $j$. 

Identities (1) and (2) are obvious. Identity (4) follows from Theorem 2.15 and (3) follows from the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega^2 S^{n+1} & \rightarrow & \Omega(S^{2n+1}\{2\}) \\
\downarrow & & \downarrow \\
\Omega S^n & \rightarrow & S^n\{2\}.
\end{array}
\]

Now, by using identities (1) to (4),

\[(x_1 \cdots x_ky)^2 = x_1 \cdots x_ky \cdot x_1 \cdots x_ky = (x_1 \cdots x_k)^2 \cdot y^2 \cdot \delta = y^2 \cdot \delta\]

for some \(\delta \in \langle y \rangle\), where \(\langle y \rangle\) is the subgroup of \(G_k\) generated by \([y, \alpha]\) for \(\alpha \in G_k\). By identity (4), \(\langle y \rangle\) is abelian. Thus \((x_1 \cdots x_ky)^4 = y^4 \cdot \delta^2 = 1\) and so \(4 \cdot \tilde{E}|_{F_k+1S^n\{2\}}\) is null homotopic for each \(k\). Since

\[\left[ S^n\{2\}, \Omega Y \right] \cong \prod_{j=0}^{\infty} [P^{n+(n-1)j}(2), \Omega Y] \]

as sets for any \(Y\), the map \(4 \cdot \overline{E}\) is null homotopic and hence the result.

An \(H\)-space \(X\) is said to have an exponent \(p^r\) if the power map \(p^r : X \rightarrow X\) is null homotopic.

**Proposition 2.31.** Let \(n \geq 3\). Then the space \(\Omega^2(S^n\{2\})\) has an exponent 16.

**Proof.** By Lemma 2.30, the power map \(4 : \Omega(S^n\{2\}) \rightarrow \Omega(S^n\{2\})\) lifts to the fibre \(\Omega^3(S^{2n+1}\{4\})\). Since \(\Omega^2[4] \simeq 4 : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{2n+1}\), \(\Omega^3(S^{2n+1}\{4\})\) is homotopy equivalent to the homotopy fibre (\(\Omega^2 S^{2n+1}\{4\}\)) of the power map \(4 : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{2n+1}\). Thus \(\Omega^4(S^{2n+1}\{4\})\) has exponent 4 and so \(\Omega^2(S^n\{2\})\) has exponent 16, which is the assertion. \(\square\)

**Corollary 2.32.** Let \(n \geq 3\). Then \(16 \cdot \pi_*(S^n\{2\}) = 0\).

For the case \(n = 2\), we have

**Proposition 2.33.** There is a homotopy decomposition

\[\Omega^2(S^2\{2\}\langle 3 \rangle) \simeq \Omega^2(S^3\langle 3 \rangle) \times \Omega_0^3 S^3.\]

In particular, \(4 \cdot (\pi_*(S^2\{2\})) = 0\).

**Proof.** By the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega S^3 & \overset{\Omega[4]}{\rightarrow} & \Omega S^3 \\
\downarrow & & \downarrow \\
H_2 & & H_2 \\
\Omega S^2 & \overset{\Omega[2]}{\rightarrow} & \Omega S^2,
\end{array}
\]

we have \(\Omega_0(S^2\{2\}) \simeq (\Omega S^3)\{4\}\). According to [4], \(\Omega^2(S^3\langle 3 \rangle)\) has an exponent 4 and hence the result. \(\square\)
By Proposition 2.23, there are \( \mathbb{Z}/8 \)-summands in \( \pi_\ast(S^n\{2\}) \) for \( n = 4, 8, 16, 32 \) or 64. We do not know any element in \( \pi_\ast(S^n\{2\}) \) which order is 16 and so we wonder whether \( \pi_\ast(S^n\{2\}) \) has an exponent bounded by 8. This is true when \( n \) is odd.

**Proposition 2.34.** The space \( \Omega^4(S^{2n+1}\{2\}) \) has an exponent 8.

**Proof.** By using the fact that \( 2\Omega[2] \simeq 4 : \Omega^2S^{2n+1} \to \Omega^2S^{2n+1} \), there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccc}
(\Omega^3S^{2n+1}\{2\}) & \xrightarrow{\delta} & \Omega^3S^{2n+1} & \xrightarrow{2} & \Omega^3S^{2n+1} \\
| & & | & & | \\
(\Omega^3S^{2n+1}\{2\}) & \xrightarrow{g} & \Omega^2(S^{2n+1}\{2\}) & \xrightarrow{f} & (\Omega^2S^{2n+1}\{4\}) \\
| & & | & & | \\
(\Omega^2S^{2n+1}\{2\}) & \xrightarrow{} & \Omega^2S^{2n+1} & \xrightarrow{} & \Omega^2S^{2n+1} \\
| & & | & & | \\
(\Omega^2S^{2n+1}\{2\}) & \xrightarrow{} & \Omega^2S^{2n+1} & \xrightarrow{} & \Omega^2S^{2n+1} \\
\end{array}
\]

Observe that

\[ 2\Omega g \simeq 2\Omega k \circ \Omega \delta \simeq \Omega k \circ 2 \circ \Omega \delta \simeq \ast. \]

There is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccc}
\Omega^4(S^{2n+1}\{2\}) & \xrightarrow{} & (\Omega^4S^{2n+1}\{4\}) & \xrightarrow{\Omega g} & (\Omega^3S^{2n+1}\{2\}) \\
| & & | & & | \\
\Omega^4(S^{2n+1}\{2\}) & \xrightarrow{2} & \Omega^4(S^{2n+1}\{2\}) & \xrightarrow{2} & \Omega^3(S^{2n+1}\{2\}) \\
\end{array}
\]

and so a homotopy commutative diagram

\[
\begin{array}{cccc}
\Omega^4(S^{2n+1}\{2\}) & \xrightarrow{} & (\Omega^4S^{2n+1}\{4\}) & \xrightarrow{4 \simeq \ast} & (\Omega^4S^{2n+1}\{4\}) \\
| & & | & & | \\
\Omega^4(S^{2n+1}\{2\}) & \xrightarrow{2} & \Omega^4(S^{2n+1}\{2\}) & \xrightarrow{2} & \Omega^4(S^{2n+1}\{2\}) \\
\end{array}
\]
Thus \( \Omega^4(S^{2n+1}\{2\}) \to \Omega^4(S^{2n+1}\{2\}) \) is null homotopic and hence the result. \( \square \)

**Corollary 2.35.** The homotopy groups \( \pi_*(S^{2n+1}\{2\}) = 0 \) have an exponent bounded by 8.

So far we do not find any element in \( \pi_*(S^{2n+1}\{2\}) \) which order is 8. Below is a conjecture due to F. R. Cohen.

**Conjecture 2.36 (Cohen).** The homotopy groups \( \pi_*(S^{2n+1}\{2\}) \) have an exponent bounded by 4.

### 2.6. Some Lemmas on Fibrations

We refer the reader to [22] for basic properties of Hopf algebras. A *Hopf algebra* means an (graded) algebra \( A \) together with a comultiplication \( \psi: A \to A \otimes A \) such that \( A \) is a coalgebra under \( \psi \) and \( \psi \) is an algebraic map. Let \( M \) be an augmented module with the augmentation \( \epsilon: M \to k \), where \( k \) is the ground field. We write \( IM \) for the kernel of \( \epsilon \). Let \( A \) be a Hopf algebra. Let \( I^nA \) denote the \( n \)-fold self product of \( IA \) in \( A \) and let \( Q(A) = IA/I^2A \) be the set of indecomposable elements.

In the case where \( A \) is a tensor algebra (as an algebra), the indecomposable elements of subHopf algebras of \( A \) can be understood in the following sense.

**Lemma 2.37.** Let \( A \) be a connected Hopf algebra of finite type and let \( B \) be a subHopf algebra of \( A \). Suppose that \( A \) is a tensor algebra as an algebra. Then there is a short exact sequence

\[
0 \longrightarrow Q(B) \longrightarrow (k \otimes_B A) \otimes Q(A) \longrightarrow I(k \otimes_B A) \longrightarrow 0.
\]

**Proof.** Let \( C = k \otimes_B A \). Then \( C \) is a right \( A \)-module. Let \( V \) be a submodule of \( IA \) such that \( V \cong Q(A) \) and let \( J \) be the left ideal of \( A \) generated by \( IB \). Consider the commutative diagram

\[
\begin{array}{cccccc}
J \otimes V & \overset{\text{mult}}{\longrightarrow} & J \otimes V \\
\downarrow & & \downarrow \\
J \hookrightarrow A \otimes V \cong IA & \longrightarrow & IC \\
\downarrow & & \downarrow \\
J/(J \cdot V) \hookrightarrow C \otimes V & \longrightarrow & IC.
\end{array}
\]

we obtain the short exact sequence

\[
0 \longrightarrow J/(J \cdot V) \longrightarrow C \otimes V \longrightarrow IC.
\]

Now the composite

\[
IB \hookrightarrow J \longrightarrow J/(J \cdot V) = J/(J \cdot IA)
\]

is epimorphism and it factors through \( Q(B) = IB/I^2B \). Let \( \phi: Q(B) \to J/(J \cdot V) \) be the resulting map. Since \( \phi \) is an epimorphism, the Poincaré series

\[
\chi(Q(B)) \geq \chi(J/(J \cdot V)) = \chi(C \otimes V) - \chi(IC) = \chi(C)\chi(V) - \chi(C) + 1.
\]
Recall that any sub algebra of tensor algebra is free. By [22, Theorem 4.4], there is an isomorphism of $B$-modules
\[ A \cong B \otimes C. \]
In particular, we have
\[ \frac{1}{1 - \chi(V)} = \frac{1}{1 - \chi(Q(B))} \cdot \chi(C) \]
or $\chi(Q(B)) = \chi(C)\chi(V) - \chi(C) + 1$. Thus the map $\phi: Q(B) \to J/(J \cdot V)$ is an isomorphism and hence the result.

This lemma helps to understand the homology of certain spaces.

**Lemma 2.38.** Let $F \xrightarrow{i} E \xrightarrow{q} \Sigma X$ be a fibration. Suppose that
1) $X$ is path-connected;
2) $F$ is simply connected and
3) the boundary $H_*(\Omega X) \to H_*(F)$ is an epimorphism.

Then
1) There is a short exact sequence
\[
0 \longrightarrow Q(H_*(\Omega E)) \longrightarrow H_*(F) \otimes \tilde{H}_*(X) \longrightarrow \tilde{H}_*(F) \longrightarrow 0.
\]
2) The homology suspension
\[ \sigma: Q(H_*(\Omega E)) \to \tilde{H}_{*+1}(E) \]
is an isomorphism.

**Proof.** Since $H_*(\Omega X) \to H_*(F)$ is an epimorphism, the Serre spectral sequence for the fibre sequence $\Omega E \longrightarrow \Omega X \longrightarrow F$ collapses and so assertion (1) follows from Lemma 2.37.

Consider the cofibre sequence
\[ F \xrightarrow{i} E \xrightarrow{\theta} \Sigma X \times F/(\ast \times F). \]
Since $H_*(\Omega X) \to H_*(F)$ is onto, the map $i_*: \tilde{H}_*(F) \to \tilde{H}_*(E)$ is zero or
\[ \theta_*: \tilde{H}_*(E) \to \tilde{H}_*(\Sigma X \times F/(\ast \times F)) \]
is a monomorphism. It follows that
\[ \chi(s^{-1}\tilde{H}_*(E)) = \chi(Q(H_*(\Omega E))), \]
where $s^{-1}$ is the desuspension of graded modules. Since
\[ \chi(H_*(\Omega E)) = \frac{1}{1 - \chi(s^{-1}\tilde{H}_*(E))}, \]
the cobar spectral sequence for $E$ collapses and hence assertion (2). We finish the proof.

**Lemma 2.39.** Let $F \xrightarrow{i} E \xrightarrow{q} \Sigma X$ be a fibration and let $f: X \to F$ be the composite $X \xhookrightarrow{} \Omega X \xrightarrow{\partial} F$. Then there is a cofibre sequence
\[ C_f \longrightarrow E \longrightarrow \Sigma X \wedge F. \]
Proof. By [44, Theorem 1.1, chapter VII], there is a cofibre sequence

\[ F \xrightarrow{i} E \xrightarrow{} (\Sigma X \times F)/(\ast \times F). \]

Since \( i \circ f : X \to E \) is null homotopic, there is a homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
F & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
(\Sigma X \times F)/(\ast \times F) & \xrightarrow{} & \Sigma X \\
\downarrow & & \downarrow \\
* & \xrightarrow{} & \Sigma X \wedge F
\end{array}
\]

and hence the result. \( \square \)

3. Decompositions of Self Smash Products

In this chapter, we investigate splittings of self smash products of the projective plane \( \mathbb{R}P^2 \). An instructional example is the 3-fold self smash \((\mathbb{R}P^2)^{(3)}\). First we can decompose this space as wedge of 3 factors

\[ (\mathbb{R}P^2)^{(3)} \simeq X_1 \lor X_2 \lor X_3, \]

where \( X_i \) is an indecomposable space. Since there is only one bottom cell in \((\mathbb{R}P^2)^{(3)}\), we assume that \( X_1 \) contains the bottom cell, that is, \( H_3(X_1) = \mathbb{Z}/2 \). Then we show that \( X_2 \simeq X_3 \simeq P^5(2) \). By analysis on the cell structure, we show that \( X_1 \simeq CP^2 \lor \mathbb{R}P^2 \). In other words, the atomic space \( X_1 \) admits a smash product decomposition. This gives an interesting relation between different projective planes. We will largely investigate this kind of decompositions in this chapter.

There is a connection between decompositions of self smashes of \( \mathbb{R}P^2 \) and the modular representation theory. The homology of the (functorial) indecomposable factors of self smashs of a two-cell suspension has been determined recently in [38]. We will go over these results and connections in the first section. In the second section, we focus on decompositions of self smash products of \( \mathbb{R}P^2 \). Since \( \mathbb{R}P^2 \) is not a suspension, we first desuspend the decomposition formulas. Then we study the smash product decompositions of indecomposable factors.

3.1. Functorial Decompositions of Self Smashes of Two-cell Complexes.

Let \( X \) be a path-connected \( p \)-local \( CW \)-complex. We start with some observations. Consider the \( n \)-fold self smash product \( X^{(n)} \). Let the symmetric group \( S_n \) act on \( X^{(n)} \) by permuting positions. Thus, for each \( \sigma \in S_n \), we have a map \( \sigma : X^{(n)} \to X^{(n)} \). Let \( Z_p \) denote \( p \)-local integers and let \( \alpha = \sum_{\sigma \in S_n} k_\sigma \sigma \) be an element in the group algebra
$\mathbb{Z}_p(S_n)$. By using the group structure in $[\Sigma X^{(n)}, \Sigma X^{(n)}]$, we obtain a map

$$\alpha = \sum_{\sigma \in S_n} k_\sigma \sigma : \Sigma X^{(n)} \to \Sigma X^{(n)}.$$  

(Note. $[\Sigma X^{(n)}, \Sigma X^{(n)}]$ is not abelian in general and the product above is given by a fixed choice of order.) The map $\alpha$ has the following properties:

1) The map $\alpha$ is functorial with respect to $X$.
2) Let $S_*(Y)$ be the singular chain complex of $Y$ with coefficients in $\mathbb{Z}_p$. Then the induced chain map

$$\alpha_* : S_*(\Sigma X^{(n)}) \simeq \Sigma S_*(X)^{\otimes n} \to S_*(\Sigma X^{(n)}) \simeq \Sigma S_*(X)^{\otimes n}$$

is homotopic to the linear combination $\sum_{\sigma \in S_n} k_\sigma \Sigma \sigma$, where $S_n$ acts on $S_*(X)^{\otimes n}$ by permuting factors in graded sense. In other words, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}_p(S_n) & \xrightarrow{\theta_1} & [\Sigma S_*(X)^{\otimes n}, \Sigma S_*(X)^{\otimes n}] \\
\| & & \| \\
\mathbb{Z}_p(S_n) & \xrightarrow{\theta_2} & [\Sigma X^{(n)}, \Sigma X^{(n)}],
\end{array}$$

where $\theta_1$ is a map of rings and $\theta_2$ is only a function.

For each $\alpha \in \mathbb{Z}_p(S_n)$, let $\alpha(X) = \text{hocolim}_\alpha \Sigma X^{(n)}$ be the homotopy colimit of the sequence

$$\Sigma X^{(n)} \xrightarrow{\alpha} \Sigma X^{(n)} \xrightarrow{\alpha} \Sigma X^{(n)} \xrightarrow{\alpha} \ldots.$$  

Now let

$$1 = \sum_\alpha e_\alpha$$

be an orthogonal decomposition of the identity in $\mathbb{Z}_p(S_n)$ in terms of primitive idempotents. For each $e_\alpha$, we have a $e_\alpha(X) = \text{hocolim}_{e_\alpha} \Sigma X^{(n)}$. The composite

$$\Sigma X^{(n)} \xrightarrow{\text{comult}} \bigvee_{\alpha} \Sigma X^{(n)} \longrightarrow \bigvee_{\alpha} e_\alpha(X)$$

is a homotopy equivalence because its induced map on the singular chains over $p$-local integers is a homotopy equivalence. Furthermore, this decomposition is functorial with respect to $X$. In other words, each $e_\alpha$ is a functor from path-connected spaces to co-H-spaces. (Note: for each $\alpha$, we can fix a choice of the representative map $e_\alpha$. This gives a particular choice of space $e_\alpha(X)$ such that $e_\alpha(X)$ is strictly functorial with respect to $Y$. Furthermore, there is a choice of the map

$$\phi : \Sigma X^{(n)} \to \bigvee_{\alpha} e_\alpha(X)$$

such that $\phi$ is strictly functorial on $X$ and is a homotopy equivalence for each $X$.)
Recall that primitive idempotents, up to conjugates, in \( \mathbb{Z}_{(p)}(S_n) \) are one-to-one corresponds to \( p \)-regular partitions of \( n \). A \textit{(proper)} partition of \( n \) is a sequence of positive integers \( \lambda = (\lambda_1, \cdots, \lambda_s) \) such that

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \quad \text{and} \quad \sum_{j=1}^s \lambda_j = n.
\]

\( s \) is called the length of \( \lambda \), which is denoted by \( \text{len}(\lambda) \). A partition \( \lambda = (\lambda_1, \cdots, \lambda_s) \) is called \textit{\( p \)-regular} if there is NO subscript \( i \) with \( 1 \leq i \leq s \) such that \( \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1} \). For each \( p \)-regular partition \( \lambda \), there is an idempotent \( Q^\lambda \in \mathbb{Z}_{(p)}(S_n) \) such that the (left) ideal generated by \( Q^\lambda \) is a projective cover of the Specht module corresponding to \( \lambda \) (See [18] for details). Any primitive idempotent in \( \mathbb{Z}_{(p)}(S_n) \) is conjugate to one and only one \( Q^\lambda \).

Let \( d_\lambda \) be the number of idempotents in decomposition 1 which are conjugate to \( Q^\lambda \). We called \( d_\lambda \) the \textit{multiplicity} of primitive idempotent \( Q^\lambda \) in \( \mathbb{Z}_{(p)}(S_n) \). Let \( Q^\lambda(X) \) be the homotopy colimit

\[
Q^\lambda(X) = \text{hocolim}_\lambda \Sigma X^{(n)}.
\]

Then decomposition 2 can be rewritten as

\[
(3) \quad \Sigma X^{(n)} \cong \bigvee_{\lambda}^d Q^\lambda(X),
\]

where \( \lambda \) runs over all \( p \)-regular partitions of \( n \).

Let \( H_*(X) \) denote the mod \( p \) homology of \( X \).

**Theorem 3.1.** [38, Theorem 1.1] Let \( X = \Sigma Y \) be a \( p \)-local suspension. Suppose that \( p = 2 \) or \( \tilde{H}_*(X)^\text{odd} = 0 \). Then \( Q^\lambda(X) \) is contractible if and only if \( \text{len}(\lambda) > \dim \tilde{H}_*(X) \).

**Note:** If \( p > 2 \) and \( \tilde{H}_*(X)^\text{odd} \neq 0 \), then \( Q^\lambda(X) \) may not be contractible even if \( \text{len}(\lambda) > \dim \tilde{H}_*(X) \). For instance, let \( X \) be an odd sphere, then \( Q^{(1,1)}(X) \cong \Sigma X^{(2)} \).

Let \( X \) be a path-connected two-cell complex localized at 2. By Theorem 3.1, we have

\[
(4) \quad \Sigma X^{(n)} \cong \bigvee_{n/2 < a \leq n}^{c(a,n-a)} Q^{(a,n-a)}(X).
\]

The homology of \( Q^{(a,n-a)}(Y) \) for each \( n/2 < a \leq n \) and the multiplicity \( d(a,n-a) \) can be determined.

Let \( u, v \) be a basis for \( V = \tilde{H}_*(X) \) with \( |u| \leq |v| \). Let \( S(V) \) be the free commutative algebra generated by \( V \) and let \( P(V) \) be the set of primitive elements in \( S(V) \). Then

\[
P(V) = \bigoplus_{k=0}^\infty P_{2^k}(V)
\]

and \( P_{2^k}(V) \) has a basis \( u^{2^k}, v^{2^k} \). Since \( V \) is a module over the Steenrod algebra, \( P_{2^k}(V) \) is a module over the Steenrod algebra in the sense that \( P_{2^k}(V) \) is a submodule of
Let $s(n)$ be the integer such that $2^{s(n)} - 1 \leq n \leq 2^{s(n)+1} - 2$ and let
\[
n - 2^{s(n)} + 1 = 2^{a(1;n)} + 2^{a(2;n)} + \cdots + 2^{a(l(n);n)}
\]
with $0 \leq a(1;n) < a(2;n) < \cdots < a(l(n);n)$.

**Theorem 3.2.** [38, Theorem 1.2] Let $X$ be a path connected two-cell complex localized at 2 and let $V = H_*(X)$ with a basis $|u|, |v|$. Then
1) there is an isomorphism of modules over the Steenrod algebra
\[
\Sigma^{-1}H_*(Q^{(n)}(X)) \cong \bigotimes_{j=0}^{s(n)-1} P_{2^j}(V) \otimes \bigotimes_{j=1}^{l(n)} P_{2^{s(j);n}}(V);
\]
2) there is a homotopy equivalence
\[
Q^{(a,n-a)}(X) \simeq \Sigma^{(|u|+|v|)(n-a)} Q^{(n-2(n-a))}(X)
\]
for $n/2 < a \leq n$.
3) the (mod 2) homology of the decomposition 4 is a complete decomposition over the Steenrod algebra if $H_*(X)$ has a non-trivial Steenrod operation.

Assertion 3 says that any idempotent $\phi: H_*(X^{(n)}) \to H_*(X^{(n)})$ over the Steenrod algebra can be geometrically realizable by an idempotent $f: \Sigma X^{(n)} \to \Sigma X^{(n)}$, that is $f_* = \phi$. Furthermore the map $f$ can be chosen to be a functorial map
\[
f = \sum_{\sigma \in S_n} k_\sigma \sigma: \Sigma X^{(n)} \to \Sigma X^{(n)}
\]
for some $k_\sigma \in \mathbb{Z}_2$.

For each $a$ with $n/2 < a \leq n$, we have
\[
2^{s(2a-n)} - 1 \leq 2a - n \leq 2^{s(2a-n)+1} - 2.
\]
Let
\[
2a - n - 2^{s(2a-n)} + 1 = \eta_0 + \eta_1 2 + \cdots + \eta_{s(2a-n)-1} 2^{s(2a-n)-1}
\]
be the 2-adic resolution, where $\eta_i = 0, 1$. Let
\[
\omega(a) = (\eta_0 + 1, \eta_1 + 1, \ldots, \eta_{s(2a-n)-1} + 1).
\]
Let $m$ be a non-negative integer. Define
\[
\epsilon(m) = \begin{cases} 
2 & \text{if } m > 0 \text{ and } m \text{ is even}, \\
1 & \text{if } m \text{ is odd}, \\
0 & \text{if } m = 0.
\end{cases}
\]
Let $I = (i_1, \ldots, i_s)$ be a sequence of non-negative integers. Let
\[
\epsilon(I) = (\epsilon(i_1), \ldots, \epsilon(i_s)).
\]
Theorem 3.3. [38, Theorem 1.3] For each \( n/2 < a \leq n \), the multiplicity \( d_{(a,n-a)} \) of \( Q^{(a,n-a)} \) is given by the formula
\[
d_{(a,n-a)} = \sum_{I = (i_0, \ldots, i_s), i_0 = n} k_I,
\]
where
\[
I = (i_0, \ldots, i_s), i_0 = n \quad \epsilon(I) = \omega(a) \quad 0 \leq i_l \leq \frac{n-1-\epsilon(i_{l-1})}{2} \quad i_{l-1} > \epsilon(i_l) \text{ for } 0 \leq l \leq s \quad 0 \leq i_s \leq \epsilon(i_s)
\]

\[
k_I = \left( \frac{i_0-\epsilon(i_0)}{i_1} \right) \left( \frac{i_1-\epsilon(i_1)}{i_2} \right) \cdots \left( \frac{i_{s-1}-\epsilon(i_{s-1})}{i_s} \right) 2^{(n-(\epsilon(i_0)+\epsilon(i_1)+\cdots+i_{s-1}+\epsilon(i_{s-1})+i_s))/2}.
\]

3.2. Decompositions of Self Smashes of \( \mathbb{R}P^2 \). When \( X \) is a projective plane, the decompositions in Theorem 3.2 are desuspensionable.

Proposition 3.4. Let \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( K \) and let \( X = \mathbb{F}P^2 \). Then there is a sequence of spaces \( FQ^k \) for \( k \geq 1 \) such that
\[
1) X^{(n)} \simeq FQ^n \lor \bigvee_{i=1}^{[\frac{n}{2}]} \bigvee_{j=1}^{\Sigma^{(|u|+|v|)i}} FQ^{n-2i}.
\]
\[
2) \Sigma FQ^n \simeq Q^{(n)}(X).
\]
\[
3) FQ^{2n} \simeq FQ^{2n-1} \lor \mathbb{F}P^2.
\]

Proof. Suppose that there is a sequence of space \( FQ^n \) such that assertions (1) and (2) hold. Then the composite
\[
\mathbb{F}Q^{2n-1} \land \mathbb{F}P^2 \xhookrightarrow{\partial} (\mathbb{F}P^2)^{(2n)} \xrightarrow{\partial} \mathbb{F}Q^{2n}
\]
is a homotopy equivalence by checking the homology. Now we show that there exists a sequence of spaces \( FQ^n \) such that assertions (1) and (2) hold by induction on \( n \). Let \( FQ^1 = \mathbb{F}P^2 \). Suppose that the spaces \( FQ^k \) defined for \( k < n \) with \( n > 1 \) such that assertions (1) and (2) holds for all \( k < n \). Let \( d = \dim F \) over \( \mathbb{R} \). Then \( Q^{(n)}(X) \) is \((dn)\)-connected \((2dn+1)\)-dimensional complex and so there is a space \( Y \) such that \( \Sigma Y \simeq Q^{(n)}(X) \). Since \( X^{(n)} \) is \((dn-1)\)-connected \((2dn)\)-dimensional complex, there is a decomposition
\[
\text{sk}_{2dn-1}(X^{(n)}) \simeq \text{sk}_{2dn-1}(Y) \lor B,
\]
where
\[
B = \bigvee_{i=1}^{[\frac{n}{2}]} \bigvee_{j=1}^{\Sigma^{(|u|+|v|)i}} FQ^{n-2i}(X).
\]
Consider the cofibre sequence
\[
S^{2dn-1} \xrightarrow{\partial} \text{sk}_{2dn-1}(Y) \lor B \xrightarrow{\partial} X^{(n)}.
\]
Since \( \Sigma X^{(n)} \simeq \Sigma Y \lor \Sigma B \) and \( B \) is \((dn)\)-connected, the composite
\[
S^{2dn-1} \xrightarrow{\partial} \text{sk}_{2dn-1}(Y) \lor B \xrightarrow{\text{proj}} B
\]
is null homotopic. Observe that \( \text{sk}_{2dn-1}(Y) \) is the \((2dn-1)\)-skeleton of the homotopy fibre of the projection map \( \text{sk}_{2dn-1}(Y) \vee B \to B \). There is a homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
S^{2dn-1} & \rightarrow & \text{sk}_{2dn-1}(Y) \\
\downarrow & & \downarrow \\
S^{2dn-1} & \rightarrow & \text{sk}_{2dn-1}(Y) \vee B \\
\end{array}
\]

By expecting the homology, the map \( \mathbb{F}Q^n \vee B \to X^{(n)} \) is a homotopy equivalence, which is assertion (1). By assertion 3 of Theorem 3.2, \( \Sigma \mathbb{F}Q^n \simeq Q^{(n)}(X) \). The induction is finished and hence the result. \( \square \)

Now we investigate the atomic spaces \( \mathbb{R}Q^n \) for low \( n \). By expecting the homology, \( \mathbb{R}Q^1 = \mathbb{R}P^2 \) and \( \mathbb{R}Q^2 = \mathbb{R}P^2 \wedge \mathbb{F}P^2 \). The first proper factor is \( \mathbb{R}Q^3 \). By expecting the homology, this is a four cell complex. This factor of \( X^{(3)} \) can obtained in an explicit way after suspension.

**Lemma 3.5.** Let \( X \) be a 2-local path-connected space and let \( \sigma = (123) \) be the cyclic element in \( S_3 \) of order 3. Then

\[
Q^{(3)}(X) = \text{hocolim}_{\hat{X}} \hat{X}(\sigma + \sigma^2 + 1) \Sigma X^{(3)}.
\]

The proof follows immediately by observing that \( \frac{1}{3}(\sigma + \sigma^2 + 1) \) is a primitive idempotent in \( \mathbb{Z}/2(S_3) \) which acts on \( x \otimes x \otimes x \) trivially.

**Proposition 3.6.** The space \( \mathbb{R}Q^3 \) is homotopy equivalent to \( \mathbb{R}P^2 \wedge \mathbb{C}P^2 \).

**Proof.** By assertion 1 of Theorem 3.2, \( \bar{H}_*(\mathbb{R}Q^3) \) has a basis \( x_3, x_4, x_5 \) and \( x_6 \) with the nontrivial Steenrod operations \( S^1 q_1^1 x_6 = x_5, S^1 q_2^2 x_6 = x_4, S^1 q_4^4 x_6 = x_3, S^1 q_5^5 x_5 = x_3 \) and \( S^1 q_6^6 x_1 = x_3 \). Clearly \( \text{sk}_4(\mathbb{R}Q^3) \simeq \mathbb{R}P^4 \simeq P^4(2) \). Since \( \pi_4(P^4(2)) = \mathbb{Z}/2 \), we have \( \text{sk}_5(\mathbb{R}Q^3) \simeq \mathbb{R}P^5_3 \). Now we compute \( \pi_5(\mathbb{R}P^5_3) \). Since \( P^4(2) \) is the 6-skeleton of the homotopy fibre of the pinch map \( q: \mathbb{R}P^5_3 \to S^6 \), there is an exact sequence

\[
\pi_5(S^4) = \mathbb{Z}/2 \rightarrow \pi_5(P^4(2)) = \mathbb{Z}/4 \rightarrow \pi_5(\mathbb{R}P^5_3) \rightarrow \pi_5(S^5) \rightarrow \pi_4(P^4(2)) \rightarrow 0
\]

and so a short exact sequence

\[
0 \rightarrow \pi_5(P^4(2))/2 \rightarrow \pi_5(\mathbb{R}P^5_3) \rightarrow q_* 2 \cdot \pi_5(S^5).
\]

Let \( \alpha \in \pi_5(\mathbb{R}P^5_3) \) such that \( q_*(\alpha) = 2 \cdot \pi_5(S^6) \) and let \( \delta \) be the nontrivial element of \( \text{Im}(\pi_5(P^4(2)) \to \pi_5(\mathbb{R}P^5_3)) = \mathbb{Z}/2 \). Let \( \beta \) be any element in \( \pi_5(\mathbb{R}P^5_3) \) such that \( q_*(\beta) = \pm 2 \cdot \pi_5(S^5) \). Then \( \beta \) is one of the four elements \( \{ \pm \alpha, \pm (\alpha + \delta) \} \). It follows that there are exactly two different, up to homotopy, 4-cell complexes \( Y \) such that \( \text{sk}_5(Y) \simeq \mathbb{R}P^5_3 \) and \( S^1 q_6^6: H_6(Y) \to H_5(Y) \) is an isomorphism. Observe that \( \mathbb{R}P^6_5 \) and \( \mathbb{R}P^2 \wedge \mathbb{C}P^2 \) are the complexes which satisfy these conditions with different Steenrod operations. By expecting the Steenrod operations, \( \mathbb{R}Q^3 \) is homotopy equivalent to \( \mathbb{R}P^2 \wedge \mathbb{C}P^2 \) and hence the result. \( \square \)

**Corollary 3.7.** There is homotopy decomposition

\[
(\mathbb{R}P^2)^{(3)} \simeq \mathbb{R}P^2 \wedge \mathbb{C}P^2 \vee P^5(2) \vee P^5(2).
\]
By using this corollary, we can directly work out decompositions of \((\mathbb{R}P)^{(n)}\) for \(n < 6\).

**Example 3.8.** When \(n = 4\), then
\[
(\mathbb{R}P^2)^{(4)} \simeq (\mathbb{R}P^2 \wedge \mathbb{C}P^2 \vee P^5(2) \vee P^5(2)) \wedge \mathbb{R}P^2 \simeq (\mathbb{R}P^2)^{(2)} \wedge \mathbb{R}P^2 \rightarrow P^5(2) \wedge \mathbb{R}P^2
\]
is a complete decomposition by considering the homology and so \(\mathbb{R}Q^4 \simeq (\mathbb{R}P^2)^{(2)} \wedge \mathbb{C}P^2\).

When \(n = 5\), then
\[
(\mathbb{R}P^2)^{(5)} \simeq (\mathbb{R}P^2)^{(3)} \wedge \mathbb{C}P^2 \vee \Sigma^3(\mathbb{R}P^2)^{(3)} \vee \Sigma^3(\mathbb{R}P^2)^{(3)}
\]
\[
\simeq \mathbb{R}P^2 \wedge (\mathbb{C}P^2)^{(2)} \vee \bigvee_{j=1}^{4} P^5(2) \wedge \mathbb{C}P^2 \vee \bigvee_{j=1}^{4} P^8(2).
\]
Thus \(\mathbb{R}Q^5 \simeq \mathbb{R}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2\).

The next example is \(\mathbb{R}Q^7\). Observe that \(\mathbb{R}Q^7\) is a retract of \(\mathbb{R}Q^6 \wedge \mathbb{R}P^2\) and so, by Corollary 3.7, \(\mathbb{R}Q^7\) is a retract of \(\mathbb{R}P^2 \wedge (\mathbb{C}P^2)^{(3)}\). The space \((\mathbb{C}P^2)^{(3)}\) has further decomposition by Proposition 3.4.

Let \(f, g \in \pi_n(Y)\). We write \(f \sim g\) if there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{f} & Y \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
S^n & \xrightarrow{g} & Y
\end{array}
\]

for some self homotopy equivalences \(\phi_1\) and \(\phi_2\). Clearly, if \(f \sim g\), then the homotopy cofibre \(C_f \simeq C_g\).

**Lemma 3.9.** Let \(Z = S^4 \cup e^8\) be a complex such that \(Sq^4: H^4(Z) \to H^8(Z)\) is an isomorphism. Then \(Z\) is homotopy equivalent to \(\mathbb{H}P^2\) localized at 2.

**Proof.** Let \(f: S^7 \to S^4\) be the attaching map for \(Z\). Recall that \(\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/4\) with generators \(\nu_4\) and \(E(\nu')\), see [42]. Since \(Sq^4: H^4(Z) \to H^8(Z)\) is an isomorphism, we may assume that \(f = \nu_4 + aE(\nu')\), where \(a = 0, 1\) or 2. Now we apply the Cohen group to show that \(\nu_4 + aE(\nu') \sim \nu_4\) for each \(a\). The result will follow from this statement.

Observe that \(\Omega[k]: \Omega S^4 \to \Omega S^4\) restricted to \(J_2(S^3)\) is represented by the element \(x^1_1x^k_2\) in the Cohen group for the inclusion \(E: S^3 \hookrightarrow J(S^3)\). Let \(\alpha_k\) denote the element \(x^1_1x^k_2(x_1x_2)^{-k}\). Then \(\alpha_k \in \Gamma_3K_2(E)\). By Example 2.11, we have \(\Gamma_3K_2(E) = 1\) and so

\[
\alpha_{k+1} = x^1_1x^{k+1}_2(x_1x_2)^{-k-1} = x_1 \cdot x^k_1 \cdot x_2 \cdot x^k_2 \cdot (x_1x_2)^k \cdot (x_1x_2)^{-1} \cdot (x_1x_2)^{-1} = [x_1, x_2]^k \alpha_k \cdot x_1 \cdot x_2 \cdot (x_1x_2)^{-1} = [x_1, x_2]^k \alpha_k.
\]

Since \(\alpha_1 = 1\), we have

\[
\alpha_k = [x_1, x_2]^{(k-1)+(k-2)+\cdots+1} = [x_1, x_2]^{k(k-1)/2}.
\]

Note that \([x_1, x_2]\) is represented by the composite

\[
J_2(S^3) \xrightarrow{\text{pinch}} S^6 \xrightarrow{\omega_4} J(S^3) \simeq \Omega S^4.
\]
Corollary 3.11. There is a homotopy decomposition
\[ \Omega([k])_* \nu_4 = k \nu_4 + [x_1, x_2]_*^{k(k-1)/2} \] and hence the result.

Lemma 3.10. There is a homotopy decomposition
\[ (\mathbb{C}P^2)^{(3)} \simeq \mathbb{H}^2 \wedge \mathbb{C}P^2 \vee \Sigma^6 \mathbb{C}P^2 \vee \Sigma^6 \mathbb{C}P^2 \]
localized at 2 and so \( \mathbb{C}Q^5 \simeq \mathbb{H}^2 \wedge \mathbb{C}P^2 \).

Proof. Observe that the homology class \([u, v] = u \otimes v - v \otimes u\) is spherical in \( \tilde{H}_*(\mathbb{C}P^2 \wedge \mathbb{C}P^2) \). There is a cofibre sequence
\[ S^6 \xrightarrow{f} \mathbb{C}P^2 \wedge \mathbb{C}P^2 \longrightarrow C_f, \]
where \( f_* (x_6) = [u, v] \). Now \( \tilde{H}_*(C_f) \) has a basis \( \{x_4, x_6, x_8\} \) with the nontrivial Steenrod operations \( Sq^2 x_6 = x_4 \) and \( Sq^4 x_8 = x_4 \). Since \( Sq^2 x_8 = 0 \), the homology class \( x_6 \) is cospherical. Let \( g : C_f \rightarrow S^6 \) such that \( g_* (x_6) = x_6 \). Let \( Z \) be the 8-skeleton of the homotopy fibre of \( g : C_f \rightarrow S^6 \). By expecting the Serre spectral sequence for the fibre sequence \( F_g \rightarrow C_f \rightarrow S^6 \), the map \( Z \rightarrow C_f \) induces a monomorphism in homology and so there is a cofibre sequence
\[ Z \longrightarrow C_f \longrightarrow S^6. \]
Since \( Z \) is a two-cell complex with nontrivial \( Sq^4 \), we have \( Z \simeq \mathbb{H}^2 \) by Lemma 3.9. By Proposition 3.4, we have
\[ (\mathbb{C}P^2)^{(3)} \simeq S^6 \wedge \mathbb{C}P^2 \vee C_f \wedge \mathbb{C}P^2 \simeq S^6 \wedge \mathbb{C}P^2 \vee S^6 \wedge \mathbb{C}P^6 \vee \mathbb{H}^2 \wedge \mathbb{C}P^2 \]
and hence the result. \( \Box \)

Corollary 3.11. There is a homotopy decomposition
\[ (\mathbb{R}P^2)^{(7)} \simeq \mathbb{H}^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2 \vee \bigvee_{j=1}^{6} \Sigma^3 (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2 \vee \bigvee_{j=1}^{14} \Sigma^6 \mathbb{C}P^2 \wedge \mathbb{R}P^2 \vee \bigvee_{j=1}^{8} P^{11}(2). \]
and so \( \mathbb{R}Q^7 \simeq \mathbb{H}^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2 \).

In general, we have the following theorem for \( \mathbb{R}Q^n \) with \( n \geq 7 \).

Theorem 3.12. Let \( k \geq 1 \). Then
Let $X$ excluded. As a stable complex, $X$ is torsion free in these cases. We write $Q^n(X)$ for the integral homology functorially. Thus $Q^n(X)$ is the functorially atomic factor of $X^{(n)}$ which contains the bottom cell. Let $\{u, v\}$ be a basis for $H_*(X)$. We assume that $|u| = 0$ and $|v| > 1$. (So mod 2 Moore space is excluded.) As a stable complex, $X = S^0 \cup_f e^{[v]}$, where $f$ runs over all elements in the 2-torsion of the stable homotopy groups $\pi_*(S^0)$. Observe that the integral homology of $X$ is torsion free in these cases. We write $ab$ for $a \otimes b$ in a tensor product $V \otimes W$ and write $s^i x$ for the $n$-fold suspension of an element $x$ in a graded module.

**Example 3.13.** The spaces $\mathbb{R}Q^{2n+1}$ for $9 \leq n \leq 23$ admit the following smash product decompositions:

1) $\mathbb{R}Q^{2k-1} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$;
2) $\mathbb{R}Q^{2k} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{C}P^2 \wedge (\mathbb{R}P^2)^{(2)}$;
3) $\mathbb{R}Q^{2k+1} \cong \mathbb{H}Q^{2k-1} \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2$;
4) $\mathbb{R}Q^{2k+2} \cong \mathbb{H}Q^{2k-1} \wedge (\mathbb{C}P^2)^{(2)} \wedge (\mathbb{R}P^2)^{(2)}$;
5) $\mathbb{R}Q^{2k+3} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$;
6) $\mathbb{R}Q^{2k+4} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge (\mathbb{R}P^2)^{(2)}$;
7) $\mathbb{R}Q^{2k+5} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge \mathbb{R}P^2$;
8) $\mathbb{R}Q^{2k+6} \cong \mathbb{H}Q^{2k-1} \wedge \mathbb{H}P^2 \wedge (\mathbb{C}P^2)^{(2)} \wedge (\mathbb{R}P^2)^{(2)}$;

The proof follows by induction and by using Theorem 3.2, Propositions 3.4 and 3.6 and Lemma 3.9.

**Lemma 3.14.** Let $X = S^0 \cup_f e^{[v]}$ be a 2-cell complex with $|v| > 1$. Suppose that $|v| \equiv 0 \mod 2$. Then there exists a map $\phi_n: \Sigma^{(n-1)[v]} X \to X^{(n)}/\text{sk}_{(n-2)[v]}(X^{(n)})$ such that in the integral homology $\phi_n, s^{(n-1)[v]} v = v^n$ and $\phi_n(s^{(n-1)[v]} u) = \sum_{i+j=n-1} v^i u^j$.

**Proof.** The proof is given by induction on $n$. The statement is obvious for $n = 1$. Now we show that the statement holds for $n = 2$. Consider the cofibre sequence

$$
\Sigma^{-1} S^2[v] \xrightarrow{\partial} \text{sk}_{[v]}(X^{(2)}) \longrightarrow X^{(2)} \xrightarrow{\text{pinch}} S^2[v] \xrightarrow{-\partial} \Sigma \text{sk}_{[v]}(X^{(2)}).
$$

Let $g_1, g_2: S^{|v|} \to \text{sk}_{[v]}(X^{(2)})/S^0$ be the maps which sends $e^{[v]}$ to $uv$ and $vu$, respectively. By using the cofibre sequence

$$
X \wedge S^0 \longrightarrow X^{(2)} \longrightarrow X \wedge S^{|v|} \xrightarrow{-\text{id}_X \wedge (\Sigma f)} X \wedge S^1,
$$
we obtain the commutative diagram
\[
\begin{array}{ccc}
\Sigma^{-1} S^2[v] & \xrightarrow{\partial} & \text{sk}_v(X^{(2)})/S^0 \\
\downarrow & & \downarrow \\
\Sigma^{-1} S^2[v] & \xrightarrow{\Sigma^{-1}(\Sigma f \wedge \text{id}_{S[v]})} & S^0 \wedge S^v \\
\end{array}
\]
\[
\xrightarrow{P_1} \\
\Sigma^{-1} S^2[v] & \xrightarrow{\partial} & \text{sk}_v(X^{(2)})/S^0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & X \wedge S^v.
\]

By computing the homology, we have
\[
(p_1 \circ g_1)_*(\iota_{[v]}) = \iota_{[v]} \quad \text{and} \quad (p_1 \circ g_2)_*(\iota_{[v]}) = 0.
\]

Similarly, we have a map \(p_2: \text{sk}_v(X^{(2)})/S^0 \to S^v\) such that
\[
p_2 \circ \partial = \Sigma^{-1}(\text{id}_{S[v]} \wedge \Sigma f): \Sigma^{-1} S^2[v] \to S^v,
\]
\[
(p_2 \circ g_1)_*(\iota_{[v]}) = 0, \quad \text{and} \quad (p_2 \circ g_2)_*(\iota_{[v]}) = \iota_{[v]}.
\]

Since \(\Sigma^{-1}(\text{id}_{S[v]} \wedge \Sigma f) \simeq (\Sigma f \wedge \text{id}_{S[v]}): S^2[v] \to S^v\), there is a homotopy commutative diagram
\[
\begin{array}{ccc}
\Sigma^{-1} S^2[v] & \xrightarrow{\Sigma^{-1}(\Sigma f \wedge \text{id}_{S[v]})} & S^v \\
\downarrow & & \downarrow \\
\Sigma^{-1} S^2[v] & \xrightarrow{\partial} & \text{sk}_v(X^{(2)})/S^0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & X \wedge S^v
\end{array}
\]
\[
\xrightarrow{\phi_2} \\
\Sigma^{-1} S^2[v] & \xrightarrow{\partial} & \text{sk}_v(X^{(2)})/S^0 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi} & X \wedge S^v.
\]

where in the integral homology \(g_*\) sends \(\iota_{[v]}\) to \(uv + (\Sigma^1/\Sigma^1)uv = uv + vu\). Thus the statement holds for \(n = 2\).

Suppose that the statement holds for \(n - 1\) with \(n > 2\). There is a homotopy commutative diagram
\[
\begin{array}{ccc}
\Sigma^{(n-2)[v]} X \wedge X & \xrightarrow{\phi_{n-1} \wedge \text{id}_X} & (X^{(n-1)}/\text{sk}_{(n-3)[v]}(X^{(n-1)})) \wedge X \\
\text{pinch} & & \text{pinch} \\
\Sigma^{(n-2)[v]} X^{(2)}/S^0 & \xrightarrow{\partial} & X^{(n)}/\text{sk}_{(n-2)[v]}(X^{(n)})
\end{array}
\]

Let \(\phi_n: \Sigma^{(n-1)[v]} X \to X^{(n)}/\text{sk}_{(n-2)[v]}(X^{(n)})\) be the map defined by the composite
\[
\Sigma^{(n-1)[v]} X \xrightarrow{\Sigma^{(n-2)[v]} \phi_2} \Sigma^{(n-2)[v]} X^{(2)}/S^0 \xrightarrow{\partial} X^{(n)}/\text{sk}_{(n-2)[v]}(X^{(n)}).
\]

By induction, we have \(\phi_{n*}(s^{(n-1)[v]} u) = v^n\) and
\[
\phi_{n*}(s^{(n-1)[v]} u) = \sum_{i+j=n-2} v^i uv^j + v^{n-1} u = \sum_{i+j=n-1} v^i uv^j.
\]

Hence the result. \(\Box\)
Theorem 3.15. Let \( X = S^0 \cup_f e^{|v|} \) be a stable 2-local two-cell complex with \(|v| > 0 \) and \(|v| \equiv 0 \mod 2 \). Then the cell structure of \( \bar{Q}^{2k-1}(X) \) is given by
\[
\bar{Q}^{2k-1}(X) = S^0 \cup_f e^{|v|} \cup 2f e^{2|v|} \cup 3f e^{3|v|} \cup \cdots \cup (2k-1)f e^{(2k-1)|v|}
\]
for each \( k \geq 0 \).

Proof. Let \( V = \tilde{H}_s(X) \) and let \( q: X^{(2^k-1)} \to \bar{Q}^{2k-1}(X) \) be the projection map. By Theorem 3.2, \( \tilde{H}_s(Q^{2k-1}(X)) \) is the symmetric quotient \( S_{2k-1}(V) \) of \( V \otimes 2^{k-1} \). Let
\[
j_n: X^{(n)} = (S^0)^{(2k-1-n)} \wedge X^{(n)} \to X^{(2k-1)}
\]
be the injection for \( n \leq 2^k - 1 \). Let \( \theta \) be the composite
\[
\Sigma^{(n-1)|v|} X \xrightarrow{\phi_n} X^{(n)}/\sk_{(n-2)|v|}(X^{(n)}) \xrightarrow{j_n} \sk_{|v|}(\bar{Q}^{2k-1}(X))/\sk_{(n-2)|v|}(\bar{Q}^{2k-1}(X)),
\]
where \( \phi_n \) is given in Lemma 3.14. Then
\[
\theta_n: H_j(\Sigma^{(n-1)|v|} X) \xrightarrow{\phi_n} H_j(\sk_{|v|}(\bar{Q}^{2k-1}(X))/\sk_{(n-2)|v|}(X))
\]
is an isomorphism for \( j \neq (n-1)|v| \) and of degree \( n \) for \( j = (n-1)|v| \). Since \( \Sigma^{(n-1)|v|} X \) is attached by \( f \), the attaching map for the two-complex
\[
\sk_{|v|}(\bar{Q}^{2k-1}(X))/\sk_{(n-2)|v|}(\bar{Q}^{2k-1}(X))
\]
is \( nf \) and hence the result. \( \square \)

Corollary 3.16. The space \( \mathbb{H}Q^3 \) has the stable cell structure
\[
S^{12} \cup_v e^{16} \cup_{2v} e^{20} \cup_{3v} e^{24}.
\]
Thus \( \mathbb{H}Q^3 \) is not homotopy equivalent to \( \mathbb{K}P^2 \wedge \mathbb{H}P^2 \).

Proposition 3.17. Let \( X = S^0 \cup_f e^{|v|} \) be a stable two-cell complex with \(|v| > 0 \) and \(|v| \equiv 0 \mod 2 \). Then
\[
\sk_n(\bar{Q}^{2k-1}(X)) \simeq \sk_n(Q^{2k-1}(X))
\]
for \( n \leq \min\{2^k-1, 2^t-1\} \).

Proof. We may assume that \( 2^k - 1 \leq 2^t - 1 \). Then the composite
\[
Q^{2k-1}(X) \xrightarrow{\jmath} X^{(2^k-1)} \xrightarrow{\jmath} X^{(2^t-1)} \xrightarrow{\text{pinch}} \bar{Q}^{2k-1}(X)
\]
induces a homotopy equivalence from \( Q^{2k-1}(X) \) to \( \sk_{2k-1}(\bar{Q}^{2t-1}(X)) \). \( \square \)

Let \( J_n(X) \) denote the \( n|v| \)-skeleton of \( \bar{Q}^{2k-1}(X) \) for some \( k \) such that \( n \leq 2^k - 1 \). By the above proposition, the homotopy type of \( J_n(X) \) is independent of the choice of \( \bar{Q}^{2k-1}(X) \). By Theorem 3.15, the cell structure of \( J_n(X) \) is given by
\[
J_n(X) = S^0 \cup_f e^{|v|} \cup 2f e^{2|v|} \cup 3f e^{3|v|} \cup \cdots \cup n_f e^{n|v|}.
\]
The projection \( q: X^{(2^k-1)} \to \bar{Q}^{2k-1}(X) \) induces a map
\[
g_n: X^{(n)} = \sk_n(X^{(2k-1)}) \xrightarrow{q|X^{(n)}} J_n(X) = \sk_n(\bar{Q}^{2k-1}(X)).
\]
Let \( D \) denote the Spanier-Whitehead dual and let \( c_n \) be the composite
\[
\Sigma^{2n|v|} D(J_n(X)) \xrightarrow{D} \Sigma^{2n|v|} D(X^{(n)}) \simeq X^{(n)} \xrightarrow{q_n} J_n(X).
\]
The homology information on the map \( c_n \) is given as follows.
Proposition 3.18. Let $X = S^0 \cup_f e^{[v]}$ with $|v| > 0$ and $|v| \equiv 0 \mod 2$. Then

$$c_{n*}: \tilde{H}_j|v|(\Sigma^{2n}|v|) \rightarrow \tilde{H}_j|v|(J_n(X)); \mathbb{Z}(2) = \mathbb{Z}(2),$$

is of degree $\binom{n}{j}$.

Proof. Let $V = \tilde{H}_*(X)$. Observe that

$$q_{n*}: \tilde{H}_*(X^{(n)}) = V^\otimes n \longrightarrow \tilde{H}_*(J_n(X)) = S_n(V)$$

is the canonical quotient from the self tensor product to symmetric product. Thus

$$D q_{n*}: \tilde{H}_*(\Sigma^{2n}|v|) \rightarrow \tilde{H}_*(X^{(n)}) = V^\otimes n$$

is the canonical inclusion of the $n$-th homogeneous of the commutative divided algebra into $n$-fold self tensor product. It follows that

$$c_{n*}: \tilde{H}_j|v|(\Sigma^{2n}|v|) \rightarrow \tilde{H}_j|v|(J_n(X))$$

is of degree $\dim(V_j^\otimes n) = \binom{n}{j}$, which is the assertion. \hfill \Box

Corollary 3.19. Let $X = S^0 \cup_f e^{[v]}$ with $|v| > 0$ and $|v| \equiv 0 \mod 2$. Then $\tilde{Q}^{2^k-1}(X)$ is self-dual for each $k \geq 0$.

As an application, we give a new proof of the classical result that the Whitehead square $\omega_{15}$ is divisible by 2.

Theorem 3.20 (Toda). The Whitehead square $\omega_{15} \in \pi_{29}(S^{15})$ is divisible by 2.

Proof. Let $X = \Sigma^{-8}K\mathbb{P}^2$ and let $f$ be the composite

$$K\mathbb{P}^2 \simeq \Sigma^{32}D(J_2(X))/S^0 \xrightarrow{c_2} J_2(X)/S^0 \xrightarrow{i} J_3(X)/S^0.$$ 

Then $f_*: H_*(K\mathbb{P}^2; \mathbb{Z}(2)) \rightarrow H_*(J_3(X)/S^0; \mathbb{Z}(2))$ is of degree 2 and 1 in dimensions 8 and 16, respectively. Thus $H_*(C_f)$ has a basis $\{x_8, x_9, x_{24}\}$ with $Sq^1 x_9 = x_8$ and $Sq^{16} x_{24} = x_8$. It is well-known that the existence of such a complex if and only if the Whitehead square $\omega_{15}$ is divisible by 2 and hence the result. \hfill \Box

Example. Let $X = \Sigma^{-8}K\mathbb{P}^2$ and let $g$ be the composite

$$\Sigma^{16}J_5(X) \simeq (\Sigma^{80}D(J_5(X)))/sk_8(\Sigma^{80}D(J_5(X))) \xrightarrow{c_5} J_5(X)/X \xrightarrow{\iota} J_6(X)/X.$$ 

Then in $\mathbb{Z}(2)$-homology, $g_*$ is of degree 1, 5, 10 and 10 in dimensions 40, 32, 24 and 16, respectively. Thus $H_*(C_g)$ has a basis $\{x_{16}, x_{17}, x_{24}, x_{25}, x_{48}\}$ with $Sq^1 x_{17} = x_{16}$, $Sq^4 x_{25} = x_{24, g}, Sq^8 x_{24} = x_{16}, Sq^8 x_{25} = x_{17}$ and $Sq^{32} x_{48} = x_{16}$. Clearly the attaching map between the 48-cell and 25-cell is trivial and so we obtain a stable complex $Y$ such that $\tilde{H}_*(Y)$ has a basis $\{x_{16}, x_{17}, x_{24}, x_{48}\}$ with $Sq^1 x_{17} = x_{16}, Sq^8 x_{24} = x_{16}$ and $Sq^{32} x_{48} = x_{16}$. 


This chapter is the preliminary for computing the homotopy groups of mod 2 Moore spaces. In the first section, we briefly go over some results in [12, 36, 37, 39]. In section 2, we will apply these general results to our special space and determine the factors of $\Omega P^n(2)$ up to certain dimension. This will help computing the homotopy groups up to certain range. We give some properties on the spaces $F^n \{2\}$ in section 3. In section 4, we study the $EHP$ sequences for mod 2 Moore spaces.

4.1. Functorial Decompositions of Loop Suspensions. We start with some algebraic results. Let $V$ be a vector space over a field $k$ and let $T(V)$ be the tensor algebra generated by $V$. $T(V)$ is a Hopf algebra by saying $V$ primitive. Consider $T$ as a functor from (ungraded) modules to coalgebras. Let $A_{\text{min}}$ is a smallest retract of $T$ which contains $V$ in the following sense

1) $A_{\text{min}}(V)$ is a functorial coalgebra retract of $T(V)$;
2) $V \subseteq A_{\text{min}}(V)$;
3) If $A(V)$ is any functorial coalgebra retract of $T(V)$ with $V \subseteq A(V)$, then $A_{\text{min}}(V)$ is a functorial coalgebra retract of $A(V)$.

It was shown in [36] that the functor $A_{\text{min}}$ exists and unique up to natural equivalence. The coalgebra $A_{\text{min}}(V)$ can described as follows.

Recall that $\text{Lie}(n)$ is module over the symmetric group $k(S_n)$, see 2.15. Let $\text{Lie}_{\text{max}}(n)$ be the maximal projective $k(S_n)$-submodule of $\text{Lie}(n)$. (If $\text{Lie}(n)$ itself is projective, then $\text{Lie}_{\text{max}}(n) = \text{Lie}(n)$.) For any $k$-module $V$, define the functor

$$L_{\text{max}}^n(V) = V^{\otimes n} \otimes_{k(S_n)} \text{Lie}_{\text{max}}(n).$$

Then $L_{\text{max}}^n(V)$ is a functorial submodule of $L_n(V)$. Let $B_{\text{max}}^n(V)$ be the subHopf algebra of $T(V)$ generated by $L_{\text{max}}^n(V)$ for $n \geq 2$. Let $Q_{\text{max}}^n(V)$ denote the set of indecomposable elements of $B_{\text{max}}^n(V)$ of tensor length $n$.

**Theorem 4.1.** [36, Theorem 6.5, Corollary 8.9] There are functorial coalgebra decompositions

$$T(V) \cong B_{\text{max}}^n(V) \otimes A_{\text{min}}(V) \cong T\left(\bigoplus_{n=2}^{\infty} Q_{\text{max}}^n(V)\right) \otimes A_{\text{min}}(V).$$

Let $p$ be the characteristic of $k$.

**Theorem 4.2.** [36, Theorem 1.5] If $n$ is not a power of $p$, then $L_n(V) \subseteq B_{\text{max}}^n(V)$.

Let $A_{\text{min}}^n(V)$ be the submodule of $A_{\text{min}}(V)$ consisting of homogeneous elements of tensor length $n$.

**Proposition 4.3.** [36, Proposition 8.10] There is a functorial isomorphism of $k$-modules

$$A_{\text{min}}^n(V) \otimes V \cong Q_{\text{max}}^n(V) \oplus A_{\text{min}}^n(V)$$

for each $n \geq 1$.

These algebraic results admit geometric realization in the following sense. A *homotopy functor* from $CW$-complexes to spaces means a functor from the homotopy category of $CW$-complexes to the homotopy category of spaces.
Theorem 4.4. [37, Theorem 1.2] Let $X$ be any path-connected $p$-local CW complex. Then there are homotopy functors $Q_n^{\max}$, $n \geq 2$, and $A_n^{\min}$ from path-connected $p$-local CW complexes to spaces with the following properties

1) $Q_n^{\max}(X)$ is a functorial retract of $\Sigma X(n)$;
2) there is a functorial fibre sequence

$$A_n^{\min}(X) \xrightarrow{j_X} \bigvee_{n=2}^{\infty} Q_n^{\max}(X) \xrightarrow{\pi_X} \Sigma X$$

with $j_X \simeq *$ functorially;
3) there is a functorial decomposition

$$\Omega \Sigma X \simeq A_n^{\min}(X) \times \Omega(\bigvee_{n=2}^{\infty} Q_n^{\max}(X))$$

4) there is a functorial coalgebra filtration on the mod $p$ homology $H_\ast(A_n^{\min}(X))$ such that there is a functorial isomorphism of coalgebras

$$\text{Gr}.H_\ast(A_n^{\min}(X)) \cong A_n^{\min}(V),$$

where $V = \tilde{H}_\ast(X)$ by forgetting the comultiplication.

The following lemmas will be useful.

Lemma 4.5. [12, Lemma 3.3] The symmetric group module $\text{Lie}(n)$ is projective if and only if $n \not\equiv 0 \mod p$. Thus $\text{Lie}^{\max}(n) = \text{Lie}(n)$ if and only if $n \not\equiv 0 \mod p$.

Recall that the set of finitely generated indecomposable projective modules of the symmetric group $k(S_n)$ is one-to-one correspondent to the set of $p$-regular partitions of $n$. The indecomposable projective $k(S_n)$-module corresponding to $\lambda$ is denoted by $Q^{\lambda}$.

Lemma 4.6. [39] The symmetric group module $\text{Lie}^{\max}(8)$ has the decomposition

$$\text{Lie}^{\max}(8) \cong [L_3, L_5] + Q^{(6,2)} + 4Q^{(5,2,1)} + Q^{(4,3,1)} \cong 2Q^{(6,2)} + Q^{(5,3)} + 8Q^{(5,2,1)} + 4Q^{(4,3,1)},$$

where $kM$ is the $k$-fold self direct sum of a module $M$ and $M + N = M \oplus N$.

The following lemma is well-known.

Lemma 4.7. Let $G$ be a finite group and let $\text{Syl}_p(G)$ be the Sylow $p$-subgroup. Suppose that $P$ is a finitely generated projective $k(G)$-module. Then $P$ is a free $k(\text{Syl}_p(G))$-module, where $p$ is the characteristic of $k$. In particular, the dimension of $P$ is divisible by the order of the Sylow $p$-subgroup of $G$.

4.2. Determination of $Q_k^{\max}(P^n(2))$ for $k \leq 9$. In this section, we determine $Q_k^{\max}(P^n(2))$ for low $k$. Since $Q_k^{\max}(P^n(2))$ is a retract of $\Sigma(P^n(2))^{(k)}$, there is a corresponding retract $Q_k^{\max}(P^n(2))$ of $(P^n(2))^{(k)}$ such that $Q_k^{\max}(P^n(2)) \simeq \Sigma Q_k^{\max}(P^n(2))$ by Proposition 3.4. We will determine $Q_k^{\max}(P^n(2))$ for $n \geq 2$ and $k \leq 9$. We assume that the ground field $k$ is of characteristic 2. Let $V$ be a two-dimensional module with a basis $\{u, v\}$ and $|v| = |u| + 1 = n$. Let $\text{ad}(v)(u) = [u, v]$ and $\text{ad}^n(v)(u) = [\text{ad}^{n-1}(v)(u), v]$ for $n \geq 2$. For a 2-regular partition $\lambda$ of $k$. Let $Q^{\lambda}(V) = V^{\otimes k} \otimes_{k(S_k)} Q^{\lambda}$. By Theorem 3.1, we have $Q^{\lambda}(V) = 0$ if $\text{len}(\lambda) > 2$. By Proposition 3.4, each module $Q^{\lambda}(V)$ is geometric realizable by a space $Q^{\lambda}(P^n(2))$.
which is a retract of \((P^n(2))^{(k)}\). For simplicity, we write \(kX\) for the \(k\)-fold self wedge of \(X\).

Our algorithm for determining decompositions of \(Q_k^{\max}(P^n(2))\) for low \(k\) is as follows:

1) Determine the module \(Q_k^{\max}(V)\).

2) Count the dimension \(t_1 = \dim Q_k^{\max}(V)_l\), where \(Q_k^{\max}(V)_{j} = 0\) for \(j > l\) and \(Q_k^{\max}(V)_l \neq 0\). In other words, count the dimension of the top degree submodule of \(Q_k^{\max}(V)\). By Theorem 3.2, there is one and only one indecomposable factor \(Q_k^{(a_1,k-a_1)}(P^n(2))\) of \((P^n(2))^{(k)}\) such that (1) \(Q_k^{(a_1,k-a_1)}(P^n(2))\) has exactly one top cell in dimension \(l\) and (2) the \(t_1\)-fold self wedge of \(Q_k^{(a_1,k-a_1)}(P^n(2))\) is a retract \(Q_k^{\max}(P^n(2))\). In this step, we obtain the factor \(Q_k^{(a_1,k-a_1)}(P^n(2))\) with the multiplicity \(t_1\).

3) By subtracting the factor \(t_1Q_k^{(a_1,k-a_1)}(V)\) from \(Q_k^{\max}(V)\), we obtain the proper summand \(Q_k^{\max}(V)/(t_1Q_k^{(a_1,k-a_1)}(V))\) of \(Q_k^{\max}(V)\). By repeating step 2, we obtain a factor \(Q_k^{(a_2,k-a_2)}(P^n(2))\) of \(Q_k^{\max}(P^n(2))\) with a multiplicity \(t_2\).

4) After a finite steps, we obtain a (finite) sequence of indecomposable factors of \(Q_k^{\max}(P^n(2))\) together with their multiplicities. This will give a complete decomposition of \(Q_k^{\max}(P^n(2))\) in terms of indecomposable factors.

4.2.1. \(Q_k^{\max}(P^n(2))\) for \(k \leq 4\). Since \(\dim \text{Lie}(2) = 1\) and \(\dim \text{Lie}(4) = 6\), we obtain that \(\text{Lie}^{\max}(2) = \text{Lie}(4) = 0\) by Lemma 4.7. It follows that

\[
Q_k^{\max}(V) = L_k^{\max}(V) = Q_4^{\max}(V) = L_4^{\max}(V) = 0
\]

and so

\[
Q_2^{\max}(P^n(2)) \cong Q_4^{\max}(P^n(2)) \cong *.
\]

By Proposition 4.5, \(\text{Lie}_2^{\max}(3) = \text{Lie}(3)\) and so

\[
Q_3^{\max}(V) = L_3^{\max}(V) = L_3(V).
\]

This is a two-dimensional module with a basis \([[u, v], [u, v], u]]\) and so

**Proposition 4.8.** There is a homotopy equivalence

\[
Q_3^{\max}(P^n(2)) \cong P^{3n-1}(2).
\]

4.2.2. \(Q_5^{\max}(P^n(2))\). By Proposition 4.5, \(\text{Lie}_5^{\max}(5) = \text{Lie}(5)\) and so \(L_5^{\max}(V) = L_5(V)\). Observe that \(L_5^{\max}(V) \to Q(B^{\max}(V))\) is a monomorphism. We have

\[
Q_5^{\max}(V) = L_5(V).
\]

This is 6-dimensional module.

Observe that the only top dimensional element in \(L_5(V)\) is given by \([[u, v], v, v, v]]\). Thus \(Q^{(4,1)}(V)\) is a summand of \(L_5(V)\) with the multiplicity one. By Example 3.8, \(Q^{(4,1)}(P^n(2)) \cong \mathbb{C}P^2 \wedge P^{5n-5}(2)\).

Now \(L_5(V)/Q^{(4,1)}(V)\) is a two-dimensional module. Thus we have

**Proposition 4.9.** There is a homotopy decomposition

\[
Q_5^{\max}(P^n(2)) \cong \mathbb{C}P^2 \wedge P^{5n-5}(2) \vee P^{5n-2}(2).
\]
4.2.3. \( Q_6^{\text{max}}(P^n(2)) \). By Theorem 4.2, we have
\[
L_6(V) \subseteq B_6^{\text{max}}(V) \cong T(Q_3^{\text{max}}(V) \oplus Q_5^{\text{max}}(V) \oplus Q_6^{\text{max}}(V) \oplus \ldots)
\]
and so \( Q_6^{\text{max}}(V) = L_6(V)/[L_3(V), L_5(V)] \). Observe that the only top dimensional element in \( L_6(V)/[L_3(V), L_5(V)] \) is represented by \( \text{ad}^5(v)(u) \). Thus \( Q_{(5,1)}^{(5,1)}(V) \) is a summand of \( Q_6^{\text{max}}(V) \). Since
\[
\dim L_6(V)/[L_3(V), L_5(V)] = \frac{1}{6}(2^6 - 2^3 - 2^2 + 2) - 1 = 8 = \dim Q_{(5,1)}^{(5,1)}(V)
\]
by the Witt formula and Example 3.8, we have

**Proposition 4.10.** There is a homotopy equivalence
\[
Q_6^{\text{max}}(P^n(2)) \cong \mathbb{CP}^2 \wedge \mathbb{RP}^2 \wedge P^{6n-7}(2).
\]

4.2.4. \( Q_7^{\text{max}}(P^n(2)) \). By Theorem 4.2, we have \( Q_7^{\text{max}}(V) = L_7(V) \). Since the top dimensional element of \( L_7(V) \) is \( \text{ad}^6(v)(u) \), \( Q_{(6,1)}^{(6,1)}(V) \) is a summand of \( L_7(V) \). By expecting the next top dimensional elements in \( L_7(V) \) and \( Q_{(6,1)}^{(6,1)}(V) \), \( Q_{(5,2)}^{(5,2)}(V) \) is a summand of \( V \) with multiplicity 2. By extracting these three factors from \( L_7(V) \), we obtain a 2-dimensional factor which is \( Q^{(4,3)}(V) \). By Corollary 3.11, we have

**Proposition 4.11.** There is a homotopy decomposition
\[
Q_7^{\text{max}}(P^n(2)) \cong (\mathbb{CP}^2)^{(2)} \wedge P^{7n-9}(2) \vee 2(\mathbb{CP}^2 \wedge P^{7n-6}(2)) \vee P^{7n-3}(2).
\]

4.2.5. \( Q_8^{\text{max}}(V) \). Since \( \dim V = 2 \), we have
\[
L_8^{\text{max}}(V) \cong [L_3(V), L_5(V)] \oplus Q_{(6,2)}^{(6,2)}(V) \cong Q_{(6,2)}^{(6,2)}(V) \oplus Q_{(5,3)}^{(5,3)}(V)
\]
by Lemma 4.6. Observe that the factor \([L_3(V), L_5(V)]\) lies in the set of decomposable elements of \( B_8^{\text{max}}(V) \) and in dimension 8, there are no more possible decomposable elements. We obtain
\[
Q_8^{\text{max}}(V) \cong Q_{(6,2)}^{(6,2)}(V).
\]
Thus we have

**Proposition 4.12.** There is a homotopy equivalence
\[
Q_8^{\text{max}}(P^n(2)) \cong \mathbb{CP}^2 \wedge \mathbb{RP}^2 \wedge P^{8n-8}(2).
\]

This proposition also gives information for primitive elements of tensor length 8 in \( A_{\text{min}}^{\text{min}}(V) \). Let \( L_8^{\text{min}}(V) = L_8(V) \cap A_{\text{min}}^{\text{min}}(V) \). Observe that there is a decomposition
\[
L_8(V) \cong [L_3(V), L_5(V)] \oplus Q_8^{\text{max}}(V) \oplus L_8^{\text{min}}(V).
\]
Thus
\[
\dim L_8^{\text{min}}(V) = \dim L_8(V) - \dim [L_3(V), L_5(V)] - \dim Q_8^{\text{max}}(V) = \frac{1}{8}(2^8 - 2^4) - 2 \times 6 - 8 = 10.
\]

Let \( L \) be the sub Lie algebra of \( L(V) \) generated by \( L_2(V), L_3(V), L_4(V), L_5(V) \) and \( L_6(V) \). Then \( L \) is a free Lie algebra generated by
\[
\{L_2(V), L_3(V), L_4(V)/L_2(L_2(V)) = L_4(V), L_5(V), L_6(V)\},
\]
where \( L_6(V) \cong L_6(V)/[L_3(L_2(V)) \oplus L_2(L_3(V))] \cong L_6(V)/[L_3(V), L_3(V)] \). This gives another (non-functorial) decomposition of \( L_8(V) \):
\[
L_8(V) \cong L_4(L_2(V)) \oplus [L_4(V), L_4(V)] \oplus [L_2(V), L_6(V)] \oplus [L_3(V), L_5(V)] \oplus M_8,
\]
where \( \dim M_8 = 7 \). According to [39], \( Q_8^{\max}(V) \) is a summand of \([L_2(V), L_6(V)]\) and so \( Q_8^{\max}(V) \cong [L_2(V), L_6(V)] \). Observe that \( L_4(L_2(V)) = 0 \). Thus

\[ L_8^{\min}(V) \cong [L_4(V), L_4(V)] \oplus M_8 \]

as modules over the Steenrod algebra. A basis for \( M_8 \) is given by \( \{Sq^i \circ \delta^k \mid 1 \leq i \leq 7\} \) and \( \dim[L_4(V), L_4(V)] = 3 \).

**Note.** If \( \dim W \geq 3 \), then \([L_4(W), L_4(W)]\) contains factors which lie in \( B^{\max}(V) \). These factors come from some of the factors \( Q^\lambda \) with \( \text{len}(\lambda) \geq 3 \) in Lemma 4.6.

4.2.6. \( Q_9^{\max}(P^n(2)) \). By expecting the Lie elements of tensor length 9 in the free Lie algebra

\[ L(L_3(V), L_5(V), Q_6^{\max}(V), L_7(V), Q_8^{\max}(V)) \]

we have the short exact sequence

\[ 0 \rightarrow L_3(L_3(V)) \oplus [L_3(V), Q_6^{\max}(V)] \rightarrow L_9(V) \rightarrow Q_9^{\max}(V) \rightarrow 0. \]

Thus

\[ \dim Q_9^{\max}(V) = \frac{1}{9}(2^9 - 3) - 2 - 2 \times 8 = 38. \]

The top element in \( Q_9^{\max}(V) \) is \( \text{ad}^8(v)(u) \) which occurs once. Thus

\[ Q^{(8,1)}(V) \cong L_2(V) \otimes Q^{(7)}(V) \]

is a summand of \( Q_9^{\max}(V) \) with multiplicity one. Now consider the top cells, that is, in the submodule of \( V^\otimes 9 \) spanned by monomials in which \( u \) occurs twice and \( v \) occurs 7 times. The modules \( L_9(V), L_3(L_3(V)) \) and \([L_3, Q_6^{\max}(V)]\) restricted to this submodule are of dimension 4, 0 and 1, respectively, by the Witt formula. Thus \( Q_9^{\max}(V) \) restricted to this submodule is of dimension 3. By Theorem 3.2, \( Q^{(8,1)}(V) \) restricted to this submodule is of dimension 1. Thus \( Q^{(7,2)}(V) \) is a summand of \( Q_9^{\max}(V) \) with multiplicity 2. Similarly, by expecting the submodule of \( V^\otimes 9 \) spanned by monomials in which \( u \) occurs 3 times and \( v \) occurs 6 times, \( Q^{(6,3)}(V) \) is a summand of \( Q_9^{\max}(V) \) with multiplicity 3. By subtracting the factors obtained, there is a two-dimensional submodule of \( Q_9^{\max}(V) \) left. Thus \( Q^{(5,4)}(V) \) is a summand of \( Q_9^{\max}(V) \) with multiplicity 1 and so we have

**Proposition 4.13.** There is a homotopy decomposition

\[ Q_9^{\max}(P^n(2)) \simeq \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge P^{9n-13}(2) \vee 2(\mathbb{C}P^2)^{(2)} \wedge P^{9n-10}(2) \vee 3 \mathbb{C}P^2 \wedge P^{9n-7}(2) \vee P^{9n-4}(2). \]

**Note.** By using this algorithm, one may determine \( Q_k^{\max}(V) \) for \( k \leq 15 \) with the help of computer program.

4.3. The Fibre of the Pinch Map. Let \( F^{n+1}\{2\} \) be the homotopy fibre of the pinch map \( P^{n+1}(2) \longrightarrow S^{n+1} \). There is a principal fibre sequence

\[ \Omega S^{n+1} \overset{s_{n+1}}{\longrightarrow} F^{n+1}\{2\} \longrightarrow P^{n+1}\{2\} \]

and so \( \Omega S^{n+1} \) acts on \( F^{n+1}\{2\} \).

**Lemma 4.14.** [9, Proposition 8.1, Corollary 8.2] If \( n \geq 2 \), then

1. the reduced integral homology \( \tilde{H}_k(F^{n+1}\{2\}; \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module on generators \( y_k \) with \( k \geq 1 \) and degree \( |y_k| = kn \);
2. The map \( s_{(n+1)*} : H_{kn}(\Omega S^{n+1}; \mathbb{Z}) \longrightarrow H_{kn}(F^{n+1}\{2\}; \mathbb{Z}) = \mathbb{Z} \) is of degree 2;
3) $H_*(F^{n+1}\{2\})$ is a free $H_*(\Omega S^{n+1})$ module on a single generator $y_1$.

The following proposition can be found in [16, Corollary 5.8. p.513] and [27, Lemma 1.4].

**Proposition 4.15.** The $(2n)$-skeleton $\text{sk}_{2n}(F^{n+1}\{2\})$ of $F^{n+1}\{2\}$ is the homotopy co-fibre of $2\omega_n: S^{2n-1} \to S^n$. Thus $\text{sk}_{4n-2}(F^{2n}\{2\}) \simeq S^{2n-1} \vee S^{4n-2}$.

**Proof.** Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccc}
\Omega S^{n+1} & \xrightarrow{\Omega[2]} & \Omega S^{n+1} & \xrightarrow{g} & S^{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega S^{n+1} & \xrightarrow{s_{n+1}} & F^{n+1}\{2\} & \xrightarrow{P^{n+1}(2)} & S^{n+1}.
\end{array}
\]

By assertion 2 of Lemma 4.14, we have

\[2^{k-1}t_n^k = \Omega[2]_*((i_n^k) = g_*((s_{n+1})_*(i_n^k)) = 2g_*(y_k)\]

and so $g_*(y_k) = 2^{k-1}t_n^k$

\[g_*: H_{kn}(F^{n+1}\{2\}; \mathbb{Z}) \xrightarrow{} H_{kn}(\Omega S^{n+1}; \mathbb{Z})\]

is of degree $2^{k-1}$ for $k \geq 1$. Thus there is a homotopy commutative diagram of co-fibre sequences

\[
\begin{array}{cccc}
S^{2n-1} & \xrightarrow{\partial} & S^n & \xrightarrow{\text{sk}_{2n}(F^{n+1}\{2\})} & \text{pinch} & S^{2n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{2n-1} & \xrightarrow{\omega_n} & S^n & \xrightarrow{J_2(S^n)} & \text{pinch} & S^{2n}.
\end{array}
\]

and hence the result. \hfill \square

By assertion 2 of Theorem 4.4, there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccc}
A^{\text{min}}(P^n(2)) & \xrightarrow{j_{P^n(2)}} & \bigvee_{k=2}^\infty \tilde{Q}_k^{\text{max}}(P^n(2)) & \xrightarrow{\pi P^n(2)} & P^{n+1}(2) \\
\downarrow & & \downarrow & & \downarrow \\
A^{\text{min}}(S^n) & \xrightarrow{j_{S^n}} & \bigvee_{k=2}^\infty \tilde{Q}_k^{\text{max}}(S^n) & \simeq & \pi S^n & \rightarrow & S^{n+1}.
\end{array}
\]
Thus the map \( \bigvee_{k=1}^{\infty} Q_n^{\text{max}}(P^n(2)) \to P^{n+1}(2) \) lifts to \( F^{n+1}(2) \) and so there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
\tilde{A}^{n+1} & \xrightarrow{=} & \tilde{A}^{n+1} \\
\downarrow & & \downarrow \tilde{j} \\
A^{\text{min}}(P^n(2)) & \xrightarrow{j_{P^n(2)}} & \Sigma \bigvee_{k=2}^{\infty} Q_k^{\text{max}}(P^n(2)) \\
\downarrow & & \downarrow \pi_{P^n(2)} \\
\Omega S^{n+1} & \xrightarrow{=} & F^{n+1}(2) \\
\end{array}
\]

Since \( j_{P^n(2)} \) is null homotopic, so is \( \tilde{j} \). Thus we have

**Proposition 4.16.** If \( n \geq 2 \), then there is a homotopy decomposition

\[
\Omega F^{n+1}(2) \simeq \tilde{A}^{n+1} \times \Omega \Sigma \bigvee_{k=2}^{\infty} Q_k^{\text{max}}(P^n(2))
\]

with a fibre sequence \( \tilde{A}^{n+1} \to A^{\text{min}}(P^n(2)) \to \Omega S^{n+1} \).

Let \( \tilde{B}^{n+1} \) be the pull back of the diagram \( S^n \to F^{n+1}(2) \leftarrow \Sigma \bigvee_{k=2}^{\infty} Q_k^{\text{max}}(P^n(2)) \).

There is a principal fibre sequence \( \Omega S^n \to \tilde{A}^{n+1} \to \tilde{B}^{n+1} \). When \( n \) is odd, this fibre sequence splits off.

**Lemma 4.17.** There is a homotopy decomposition

\[
\tilde{A}^{2n} \simeq \Omega S^{2n-1} \times \tilde{B}^{2n}.
\]

for \( n > 1 \).

**Proof.** Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
F & \to & F^{2n}(2) \to \tau(S^{2n}) \\
\downarrow & & \downarrow \text{pinch} \\
F & \to & P^{2n}(2) \to S^{2n-1} \\
\downarrow & & \downarrow \text{pinch} \\
* & \to & S^{2n} = S^{2n}.
\end{array}
\]

The bottom cell \( S^{2n-1} \) is a retract of \( F^{2n}(2) \) and so \( \Omega S^{2n-1} \) is a retract \( \Omega F^{2n}(2) \). It follows that \( \Omega S^{2n-1} \) is a retract of \( \tilde{A}^{2n} \) and hence the result. \( \square \)
4.4. The EHP Sequences for mod 2 Moore Spaces. Let $\tilde{H}_2$ denote the composite

$$\tilde{H}_2: \Omega P^{n+1} \simeq J(P^n(2)) \xrightarrow{H_2} J(P^n(2) \wedge P^n(2)) \xrightarrow{J(q)} J(P^{2n}(2)) \simeq \Omega D^{2n+1}(2),$$

where $q$ is the pinch map $(P^n(2))^{(2)} \to (P^n(2))^{(2)}/(S^{n-1} \wedge P^n(2)) = P^{2n}(2)$. There is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccccc}
\Omega^2 S^{2n+1} & \xrightarrow{P} & S^n & \xrightarrow{E} & \Omega S^{n+1} & \xrightarrow{H_2} & \Omega S^{2n+1} \\
\Omega^2 P^{2n+1}(2) & \xrightarrow{P} & T^{n+1} & \xrightarrow{E} & \Omega P^{n+1}(2) & \xrightarrow{\tilde{H}_2} & \Omega P^{2n+1}(2). \\
\end{array}
\]

We will determine the atomic factor of $T^{n+1}$ which contains the bottom cell. We need some terminologies on simplicial sets. Let $X$ and $Y$ be pointed simplicial sets and let $f: X \to Y$ be a simplicial map. Let $F(Y)$ be the Milnor construction on $Y$. Define the simplicial set $E^f$ to be the pull-back of the diagram

$$J(X) \xrightarrow{J(f)} J(Y) \subseteq F(Y) \xleftarrow{PF} Y,$$

where $PZ$ is the Moore path-space of a fibrant simplicial set $Z$. As a space, $E^f$ is homotopy equivalent to the loop space of the homotopy fibre of $\Sigma f: \Sigma X \to \Sigma Y$.

Consider the strictly commutative diagram

\[
\begin{array}{cccccc}
J(X) & \xrightarrow{J(f)} & J(Y) & \subseteq & F(Y) & \xleftarrow{PF} Y \\
H_2 & \downarrow & H_2 & \downarrow & \tilde{H}_2 & \downarrow \bar{H}_2 \\
J(X \wedge X) & \xrightarrow{J(f \wedge f)} & J(Y \wedge Y) & \subseteq & F(Y \wedge Y) & \xleftarrow{PF(Y \wedge Y)}, \\
\end{array}
\]

where $\tilde{H}_2: F(Y) \to F(Y \wedge Y)$ is a functorial extension of the map $H_2$. There is a unique simplicial map

$$H_2: E^f \rightarrow E^{f \wedge f}$$

such that the diagram

\[
\begin{array}{cccc}
E^f & \subseteq & J(X) \times PF(Y) \\
H_2 & & H_2 \times \bar{H}_2 \\
E^{f \wedge f} & \subseteq & J(X \wedge X) \times PF(Y \wedge Y) \\
\end{array}
\]

commutes strictly.

**Lemma 4.18.** Let $f: X \to Y$ be a simplicial map and let $Z$ be a simplicial subset of $X$ such that $f|_Z: Z \to Y$ is the trivial map. Then there exist simplicial homomorphisms $\phi_1: J(Z) \to E^f$ and $\phi_2: J(Z \wedge X) \to E^{f \wedge f}$ such that

1) $\phi_1$ is a lifting of $J(Z) \subseteq J(X)$;
2) \( \phi_2 \) is a lifting of \( J(X \wedge Z) \hookrightarrow J(X \wedge X) \) and

3) the diagram

\[
\begin{array}{ccc}
J(Z) & \xrightarrow{\phi_1} & E^f \\
\downarrow H_2 & & \downarrow H_2 \\
J(Z \wedge Z) & \xrightarrow{\phi_2|_{J(Z \wedge Z)}} & E^{f \wedge f}
\end{array}
\]

commutes strictly.

**Proof.** Since \( J(f)|_{J(Z)} \) and \( J(f \wedge f)|_{J(Z \wedge X)} \) are trivial maps, the simplicial submonoids \( J(Z) \) of \( J(X) \) and \( J(X \wedge Z) \) of \( J(X \wedge X) \) admits the cross-sections, \( x \mapsto (x, 1) \), into the pull-back \( E^f \) and \( E^{f \wedge f} \), respectively. The assertions follow immediately. \( \square \)

**Lemma 4.19.** There exists a map \( \phi: T^{n+1} \to S^n\{2\} \) such that

\[ \phi_*: H_{n-1}(T^{n+1}) \longrightarrow H_{n-1}(S^n\{2\}) \]

is an isomorphism.

**Proof.** Let \( f: X \to Y \) be a fibration with the fibre \( F \) such that

1) \( X \simeq Y \simeq S^n \);
2) \( f \) is a map of degree 2.

Then there is a map \( g: P^n(2) \to F \) such that the composite \( P^n(2) \to F \to X \) is homotopic to the pinch map \( P^n(2) \to S^n \). By Lemma 4.18, there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
T^{n+1} & \xrightarrow{\phi} & \Omega P^{n+1}(2) \\
\downarrow \phi & & \downarrow \phi_1 \\
S^n\{2\} & \xrightarrow{H_2} & \Omega S^{n+1}\{2\}
\end{array}
\]

\[
\begin{array}{ccc}
\Omega P^{n+1}(2) & \xrightarrow{\phi_2$ & \Omega \Sigma(P^n(2) \wedge S^n) \\
\downarrow \phi & & \downarrow \phi_2 \\
\Omega S^{n+1}\{2\} & \xrightarrow{H_2} & \Omega S^{2n+1}\{4\}.
\end{array}
\]

Clearly \( \phi \) induces an isomorphism on \( H_{n-1} \) and hence the result. \( \square \)

Our next step is to construct a cross-section map for \( T^{n+1} \to S^n\{2\} \). Let

\[
A \xleftarrow{i} X \longrightarrow X/A
\]

be a cofibration such that \( A \simeq X \simeq S^{n-1} \) and \( i \) is of degree 2. Then \( \Omega P^{n+1}(2) \simeq F(X/A) \) is the quotient simplicial group of \( F(X) \) by the normal subgroup generated by \( F(A) \). Clearly the simplicial coset of \( F(X) \) by the action of the simplicial subgroup \( F(A) \) is homotopy equivalent to \( S^n\{2\} \). Thus there is a map of \( F(X) \)-spaces

\[ s: S^n\{2\} \to \Omega P^{n+1}(2) \]
such that the diagram

\[
\begin{array}{c}
F(X) \\
\downarrow \\
S^n\{2\} \\
\downarrow \\
\Omega P^{n+1}(2)
\end{array}
\]

commutes. Let \( c: J(X) \land J(Y) \to J(X \land Y) \) be the map defined by

\[
c((x_1 x_2 \cdots x_n) \land (y_1 y_2 \cdots y_m)) = (x_1 \land y_1) \cdot (x_2 \land y_1) \cdots (x_n \land y_1) \cdot (x_1 \land y_2) \cdots (x_n \land y_2) \cdots (x_1 \land y_m) \cdots (x_n \land y_m)
\]

for \( x_i \in X \) and \( y_j \in Y \), see [47]. Let \( \tilde{K}_{k,l} \) be the subgroup of

\[
[(S^{n-1})^k \times (P^n(2))^l, \Omega P^{n+1}(2)]
\]

generated by the elements \( x_i \), \( 1 \leq i \leq k \) and \( y_j \), \( 1 \leq j \leq l \), represented by the maps:

\[
x_i: (S^{n-1})^k \times (P^n(2))^l \xrightarrow{\pi_i} S^{n-1} \xrightarrow{c} \Omega P^{n+1}(2),
\]

\[
y_j: (S^{n-1})^k \times (P^n(2))^l \xrightarrow{\pi_{k+j}} P^n(2) \xrightarrow{c} \Omega P^{n+1}(2),
\]

respectively, where \( \pi_i \) the \( i \)-th coordinate projection. By the proof of Lemma 2.30, we have

**Lemma 4.20.** If \( n \geq 3 \), then the following identities hold in the group \( \tilde{K}_{k,l} \):

1) \( x_i^2 = 1 \) for \( 1 \leq i \leq k \);
2) \( y_j^2 = 1 \);
3) The (iterated) commutators

\[
[[a_1, a_2], \ldots, a_s] = 1
\]

if \( a_i = a_j \) for some \( i < j \), where \( a_i = y_s \) or \( x_t \) for some \( s, t \).

**Theorem 4.21.** If \( n \geq 3 \), then the map \( \phi: T^{n+1} \to S^n\{2\} \) has a cross-section and so \( S^n\{2\} \) is a retract of \( T^{n+1} \).

**Proof.** Since \( S^n\{2\} \) is atomic, it suffices to show that the composite

\[
S^n\{2\} \xrightarrow{s} \Omega P^{n+1}(2) \xrightarrow{H_2} \Omega P^{2n+1}(2)
\]

is null homotopic. We use the filtration \( \{F_k S^n\{2\}\} \) is the proof of Lemma 2.30. Observe that the product element \( x_1 x_2 \cdots x_k y_1 \in \tilde{K}_{k,1} \) is represented by the composite

\[
(S^{n-1})^k \times P^n(2) \xrightarrow{c} F_{k+1} S^n\{2\} \xrightarrow{s} \Omega P^{n+1}(2).
\]

Consider the function

\[
H_{2*}: [(S^{n-1})^k \times P^n(2), \Omega P^{n+1}(2)] \to [(S^{n-1})^k \times P^n(2), \Omega \Sigma(P^n(2))^{(2)}].
\]

According to [47, Lemma 3.4], The element \( H_{2*} (x_1 x_2 \cdots x_k y) \) is represented by the product of the maps \( \alpha \) and \( \beta \) given by

\[
\alpha: (S^{n-1})^k \times P^n(2) \xrightarrow{\text{proj}} (S^{n-1})^k \xrightarrow{\text{proj}} J_k(S^{n-1}) \xrightarrow{H_2} J(S^{2n-2}) \xrightarrow{c} J(P^n(2) \land P^n(2)),
\]

\[
\beta: (S^{n-1})^k \times P^n(2) \xrightarrow{\text{proj}} J_k(S^{n-1}) \land P^n(2) \xrightarrow{c} J(S^{n-1} \land P^n(2)) \xrightarrow{c} J(P^n(2) \land P^n(2)).
\]
Since both $\alpha$ and $\beta$ are null homotopic after composing with the map

$$ J((P^n(2))^{(2)}) \xrightarrow{\alpha} J((P^n(2))^{(2)}/(S^{n-1} \land P^n(2))) = J(P^{2n}(2)), $$

the element

$$ \tilde{H}_2(x_1 \cdots x_k y) = 1 $$

in $[(S^{n-1})^k \otimes P^n(2), \Omega P^{2n+1}(2)]$ and hence the result. \hfill $\square$

The homology of the space $T^{n+1}$ can be described as follows.

**Proposition 4.22.** If $n \geq 3$, then the Serre spectral sequence for the fibre sequence

$$ T^{n+1} \xrightarrow{\tilde{H}_2} \Omega P^{n+1}(2) \xrightarrow{\tilde{H}_2} \Omega P^{2n+1}(2) $$

collapses. Furthermore, there is a submodule $W$ of the free Lie algebra $L(u, v)$ such that $H_*(T^{n+1})$ is isomorphic to $T(W) \otimes H_*(S^n\{2\})$ as coalgebras.

**Proof.** Let $\tilde{V} = \tilde{H}_*(P^{2n}(2))$ with a basis $\{\tilde{u}, \tilde{v}\}$ and $Sq^1 \tilde{v} = \tilde{u}$. Consider

$$ \tilde{H}_2: H_*(\Omega P^{n+1}(2)) = T(V) \xrightarrow{\mu} H_*(\Omega P^{2n+1}(2)) = T(\tilde{V}). $$

Let $A$ be the cotensor product $k \square_{T(\tilde{V})} T(u, v)$. Then $A$ is a subHopf algebra of $T(u, v)$ generated by $\{u^2, [u, v], v^2, [[u, v], u], [[u, v], v]\}$. Let $j: A \to T(u, v)$ be the inclusion map and let $s: E(u, v) \to T(u, v)$ be a $k$-linear map defined by $s(u) = u, s(v) = v$ and $s(uv) = uv$. Then $s$ is a coalgebra map and the composite

$$ A \otimes E(u, v) \xrightarrow{j \otimes s} T(u, v) \otimes T(u, v) \xrightarrow{\mu} T(u, v) $$

is an isomorphism of coalgebras over $A$. According to [11, Proposition 5.1] and [47, Proposition 3.12], the map

$$ H_2: T(V) \to T(V \otimes V) $$

is a morphism of coalgebras over $A$ and so is $\tilde{H}_2$, where the map $\tilde{H}_2$ restricted to $A$ is a morphism of Hopf algebras given by $\tilde{H}_2([u, v]) = \tilde{u}, \tilde{H}_2(v^2) = \tilde{v}$ and

$$ \tilde{H}_2(u^2) = \tilde{H}_2([[u, v], u]) = \tilde{H}_2([[u, v], v]) = 0. $$

Let $B = k \square_{T(\tilde{V})} A$. By [22, Theorem 4.4], there is an isomorphism of coalgebras

$$ A \cong B \otimes T(\tilde{V}), $$

where the coaction of $T(\tilde{V})$ on $B$ is trivial. Observe that the composite

$$ E(u, v) \xrightarrow{s} T(u, v) \xrightarrow{H_2} T(\tilde{V}) $$

is trivial map. Thus

$$ T(V) \cong E(u, v) \otimes B \otimes T(\tilde{V}) $$

as comodules over $T(\tilde{V})$, where the coactions of $T(\tilde{V})$ on the factors $E(u, v)$ and $B$ are trivial. Thus $H^*(\Omega P^{n+1}(2))$ is a free $H^*(\Omega P^{2n+1}(2))$-module. By expecting the Eilenberg-Moore spectral sequence for the fibre sequence

$$ T^{n+1} \xrightarrow{\tilde{H}_2} \Omega P^{n+1}(2) \xrightarrow{\tilde{H}_2} \Omega P^{2n+1}(2), $$

$H_*(T^{n+1})$ is a sub-quotient of $E(u, v) \otimes B$. In particular, the Poincaré series

$$ \chi(H_*(T^{n+1})) \leq \chi(E(u, v))\chi(B). $$
By expecting the Serre spectral sequence for the fibre sequence above, we have
\[
\chi(E(u, v))\chi(B)\chi(T(\tilde{V})) = \chi(T(V)) \leq \chi(H_*(T^{n+1}))\chi(T(\tilde{V})).
\]
Thus
\[
\chi(H_*(T^{n+1})) = \chi(E(u, v))\chi(B)
\]
and so the Serre spectral sequence collapses. Furthermore, there is an isomorphism of coalgebra
\[
H_*(T^{n+1}) \cong E(u, v) \otimes B.
\]
Consider the commutative diagrams of short exact sequences of Hopf algebras
\[
\begin{array}{ccc}
B \Box P(u^2) & \subset & B \\
\downarrow & & \downarrow \\
A \Box P(u^2) & \subset & A \\
\downarrow & & \downarrow \\
T(\tilde{V}) & = & T(\tilde{V}).
\end{array}
\]
There is a submodule \(W\) of \(L(V)\) such that \(B \Box P(u^2) \cong T(W)\). It follows that
\[
H_*(T^{n+1}) \cong E(u, v) \otimes P(u^2) \otimes T(W) \cong H_*(S^n\{2\}) \otimes T(W)
\]
as coalgebras. The proof is finished.

Now we are going to give a product decomposition of \(T^{n+1}\) up to certain dimension. We need to do some lemmas on the James-Hopf maps. Let \(\mathcal{E}\) be the collection of spaces consisting of all suspensions. For each suspension \(X\), let \(K_n(k)(X)\) be the subgroup of \([X^n, J(X^{(k)})]\) generated by the elements labelled by \(\{x_{i_1} | x_{i_2} | \cdots | x_{i_k}\}\), \(1 \leq i_1, i_2, \ldots, i_s \leq n\), represented by the map
\[
X^n \overset{\pi_{i_1, i_2, \ldots, i_k}}{\longrightarrow} X^k \overset{\text{pinch}}{\longrightarrow} X^{(k)} \overset{\subseteq}{\longrightarrow} J(X^{(k)}),
\]
where \(\pi_f\) is the coordinate projection. By letting \(X\) run over all objects in \(\mathcal{E}\), one gets a universal group \(K_n(k)\). The generators for \(K_n(k)\) are labelled by \(\{x_{i_1} | x_{i_2} | \cdots | x_{i_k}\}\) subject to the relations given in [47, Lemma 2.2]. Observe that \(K_n(1) = K_n(\mathcal{E})\) is the Cohen group given in Section 2.2. The James-Hopf map \(H_k : J(X) \rightarrow J(X^{(k)})\) induces a function
\[
H_{k*} : [X^n, J(X)] \longrightarrow [X^n, J(X^{(k)})].
\]
When $X$ runs over all objects in $\mathcal{E}$, this function induces a unique function $H_k : K_n(1) \rightarrow K_n(k)$ such that the diagram

$$
\begin{array}{ccc}
K_n(1) & \xrightarrow{e_X} & [X^n, J(X)] \\
\downarrow H_k & & \downarrow H_k \\
K_n(k) & \xrightarrow{e_X} & [X^n, J(X^{(k)})]
\end{array}
$$

commutes for all $X \in \mathcal{E}$. Let $\beta_n : X^{(n)} \rightarrow J(X)$ be the $n$-fold Samelson product. Observe that the element $e_X([x_1, x_2, \cdots, x_n])$ in $[X^n, J(X)]$ is represented by the composite $X^n \xrightarrow{\text{pinch}} X^{(n)} \xrightarrow{W_n} J(X)$. Thus the element

$e_X(H_k([x_1, x_2, \cdots, x_n]))$

in $[X^n, J(X^{(k)})]$ is represented by composite

$$X^n \xrightarrow{\text{pinch}} X^{(n)} \xrightarrow{\beta_n} J(X) \xrightarrow{H_k} J(X^{(k)}).$$

**Lemma 4.23.** If $n$ is odd, then the composite

$$X^{(n)} \xrightarrow{\beta_n} J(X) \xrightarrow{H_2} J(X^{(2)})$$

is null homotopic for any suspension $X$.

*Proof.* The homotopy commutative diagram

$$
\begin{array}{ccc}
J(X) & \xrightarrow{J([-1])} & J(X) \\
\downarrow H_2 & & \downarrow H_2 \\
J(X^{(2)}) & \xrightarrow{\text{id}} & J(X^{(2)})
\end{array}
$$

induces a group homomorphism $J([-1]) : K_n(1) \rightarrow K_n(1)$ such that the diagram

$$
\begin{array}{ccc}
K_n(1) & \xrightarrow{J([-1])} & K_n(1) \\
\downarrow H_2 & & \downarrow H_2 \\
K_n(2) & & K_n(2)
\end{array}
$$

commutes. Observe that $J([-1])(x_i) = -x_i$. Thus we have

$H_2(\{[x_1, x_2], \cdots, x_n\}) = H_2 \circ J([-1])(\{[x_1, x_2], \cdots, x_n\}) = H_2(\{[x_1, x_2], \cdots, x_n\}^{(-1)^n})$.

By [47, Theorem 3.8], $H_2$ restricted to the commutator subgroup $\Gamma^2 K_n(2)$ is a group homomorphism. Thus the element

$(H_2(\{[x_1, x_2], \cdots, x_n\}))^2 = 1$
in the group $K_n(2)$. Since $K_n(2)$ is a torsion free group, the element
\[ H_2([[x_1, x_2], \cdots, x_n]) = 1 \]
and so the composite
\[ X^n \xrightarrow{q} X^{(n)} \xrightarrow{\beta_n} J(X) \xrightarrow{H_2} J(X^{(2)}) \]
is null homotopic. The assertion follows from the fact that $q^*: [X^{(n)}, \Omega Y] \to [X^n, \Omega Y]$ is a monomorphism for any $Y$.

**Note.** This lemma can be generalized as follows: The composite
\[ X^{(l)} \xrightarrow{W_l} J(X) \xrightarrow{H_k} J(X^{(k)}) \]
is null homotopic if $l \not\equiv 0 \mod k$.

**Corollary 4.24.** If $n \geq 3$ and $k$ is odd, then the composite
\[ (P^n(2))^{(k)} \xrightarrow{\beta_k} \Omega P^{n+1} \xrightarrow{H_2} \Omega P^{2n+1}(2) \]
is null homotopic.

**Lemma 4.25.** In the group $K_4(2)$, the element $H_2([[x_1, x_2], x_3, x_4])$ is equal to the product
\[ \{x_4|x_3\}, \{x_1|x_2\} \cdot \{x_4|x_1\}, \{x_3|x_2\} \cdot \{x_3|x_1\}, \{x_4|x_2\} \cdot \{x_2|x_1\}, \{x_4|x_3\}. \]

The proof follows from a direct calculation by using the methods in [47].

**Lemma 4.26.** If $n \geq 3$, then the composite
\[ (P^n(2))^{(3)} \wedge S^{n-1} \xrightarrow{\sigma} (P^n(2))^{(4)} \xrightarrow{\beta_4} \Omega P^{n+1}(2) \xrightarrow{H_2} \Omega P^{2n+1}(2) \]
is null homotopic.

**Proof.** Let
\[ q: (P^n(2))^{(2)} \longrightarrow P^{2n}(2) = (P^n(2))^{(2)} / S^{n-1} \wedge P^n(2) \]
be the pinch map. Observe that the element $\{x_4|x_i\}$ restricted to $(P^n(2))^3 \times S^{n-1}$ is represented by the composite
\[ (P^n(2))^3 \times S^{n-1} \xrightarrow{\pi_{4,i}} S^{n-1} \wedge P^n(2) \xrightarrow{\sigma} J((P^n(2))^{(2)}). \]
Thus $J(q)_i(\{x_4|x_i\}) = 1$ for $1 \leq i \leq 3$. The assertion follows from Lemma 4.25. \( \square \)

Let $f: X \to J(Y)$ be a map. We write $J(f): J(X) \to J(Y)$ be the homomorphism of monoids induced by $f$.

**Lemma 4.27.** Let $\phi: Y \to (P^n(2))^{(k)}$ be a map such that the composite
\[ Y \xrightarrow{\phi} (P^n(2))^{(k)} \xrightarrow{\beta_k} \Omega P^{n+1}(2) \xrightarrow{H_2} \Omega P^{2n+1}(2) \]
is null homotopic. Suppose that $n \geq 3$ and $k \geq 2$. Then there is a map
\[ \mu_\phi: \Omega \Sigma Y \times T^{n+1} \to T^{n+1} \]
such that the diagram

\[
\begin{array}{ccc}
\Omega \Sigma Y \times T^{n+1} & \xrightarrow{\mu_\phi} & T^{n+1} \\
\downarrow \quad \Omega \Sigma \phi \times E & & \downarrow E \\
\Omega \Sigma (P^n(2))^{(k)} \times \Omega P^{n+1} & \xrightarrow{J(\beta_k) \cdot \text{id}_{\Omega P^{n+1}(2)}} & \Omega P^{n+1}(2)
\end{array}
\]

commutes up to homotopy.

**Proof.** By Theorem [47, Theorem 3.10], there is a map \(\beta_{k,2}: \Sigma X^{(k)} \to \Sigma X^{(2)}\) such that \(\Omega \beta_{k,2}\) is homotopic to the composite

\[
\Omega \Sigma X^{(k)} \xrightarrow{J(\beta_k)} \Omega \Sigma X \xrightarrow{H_2} \Omega \Sigma X^{(2)}
\]

for any suspension \(X\) and any integer \(k \geq 2\). According to [47, Proposition 3.12], there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega \Sigma X^{(k)} \times \Omega \Sigma X & \xrightarrow{J(\beta_k) \cdot \text{id}_{\Omega \Sigma X}} & \Omega \Sigma X \\
\downarrow \quad \Omega \beta_{k,2} \times H_2 & & \downarrow H_2 \\
\Omega \Sigma X^{(2)} \times \Omega \Sigma X^{(2)} & \xrightarrow{\mu} & \Omega \Sigma X^{(2)}
\end{array}
\]

for any suspension \(X\) and any integer \(k \geq 3\). It follows that the composite

\[
\Omega \Sigma Y \times T^{n+1} \xrightarrow{\Omega \Sigma \phi \times E} \Omega \Sigma (P^n(2))^{(k)} \times \Omega P^{n+1}(2) \xrightarrow{J(\beta_k) \cdot \text{id}_{\Omega P^{n+1}(2)}} \Omega P^{n+1}(2)
\]

lifts to the fibre \(T^{n+1}\) and hence the result. \(\square\)

**Theorem 4.28.** If \(n \geq 3\), there is a map \(\theta: S^n\{2\} \times \Omega P^3(2) \times \Omega P^{4n-1}(2) \times \Omega (\mathbb{CP}^2 \wedge P^{5n-4}(2) \vee P^{5n-1}(2)) \to T^{n+1}\) such that \(H_j(\theta)\) is an isomorphism for \(j \leq 6n - 6\).

**Proof.** Let \(s: S^n\{2\} \to T^{n+1}\) be a cross-section of the map \(\phi\) defined in Lemma 4.19. By Lemma 4.26 and Corollary 4.24, the maps

\[
\begin{align*}
\beta_3 &: Q_3^{\max}(P^n(2)) = P^{3n-1}(2) \to \Omega P^{n+1}(2), \\
\beta_5 &: Q_5^{\max}(P^n(2)) = \mathbb{CP}^2 \wedge P^{5n-5}(2) \vee P^{5n-2}(2) \to \Omega P^{n+1}(2), \\
\tilde{\beta}_4 &: Q_3^{\max}(P^n(2)) \wedge S^{n-1} = P^{4n-2}(2) \to Q_3^{\max}(P^n(2)) \wedge P^n(2) \xrightarrow{\beta_4} \Omega P^{n+1}(2)
\end{align*}
\]

are null homotopic after composing with \(H_2\). The assertion follows from Lemma 4.27 and by expecting the homology of \(T^{n+1}\) described in Proposition 4.22. \(\square\)

5. **The Homotopy Groups** \(\pi_{n+r}(\Sigma^n \mathbb{R}P^2)\) for \(n \geq 2\) and \(r \leq 8\)

In this chapter, we compute specific homotopy groups. Thus this chapter will be divided into parts A: \(\pi_* (P^4(2))\), B: \(\pi_* (P^5(2))\), C: \(\pi_* (P^6(2))\), D: \(\pi_* (P^7(2))\), E: \(\pi_* (P^8(2))\), F: \(\pi_* (P^9(2))\) and G: \(\pi_* (P^n(2))\) for \(n \geq 10\).
5.1. Preliminary and Notations.

**Proposition 5.1.** The first three homotopy groups are as follows.

1. $\pi_{n-1}(P^n(2)) = \mathbb{Z}/2 \cong \pi^s_1(\mathbb{R}P^2)$ for $n \geq 2$;
2. $\pi_n(P^n(2)) = \mathbb{Z}/2 \cong \pi^s_2(\mathbb{R}P^2)$ for $n \geq 4$;
3. $\pi_{n+1}(P^n(2)) = \mathbb{Z}/4 \cong \pi^s_3(\mathbb{R}P^2)$ for $n \geq 4$.

The proof is immediate.

Recall that there is a decomposition

$$\Omega P^{n+1}(2) \cong \Omega \Sigma \left( \bigvee_{k=3}^{\infty} Q^\max_k(P^n(2)) \right) \times A^\min(P^n(2)).$$

By applying the Hilton-Milnor theorem, we have a decomposition

$$\Omega \Sigma \left( \bigvee_{k=3}^{\infty} Q^\max_k(P^n(2)) \right) \cong \prod_{k=3}^{\infty} \Omega \Sigma L_k.$$

Let $V = \tilde{H}_*(P^n(2))$ with a basis $\{u, v\}$ with $S_q^1v = u$. When $k$ is odd, We write $L_k$ for $L_k(P^n(2))$ as a geometric realization of module $L_k(V)$. When $k = 6$, then $L_k$ is a geometric realization of $L_6(V)/[L_3(V), L_3(V)] = Q^\max_6(V)$. Observe that $L_k$ is $(kn - k)$-connected. For computing $\pi_{n+r}(\Omega P^{n+1}(2))$ with $n \geq 3$ and $r \leq 8$, we only need factors $L_3 \cong P^{3n-1}(2)$ and $L_5 \cong \mathbb{C}P^2 \wedge P^{5n-5} \vee P^{5n-2}(2)$. We rewrite the decomposition

$$\Omega P^{n+1}(2) \cong \Omega L_3 \times \Omega L_5 \times Y_{n+1},$$

where $\pi_{n+r}(Y_{n+1}) = \pi_{n+r}(A^\min(P^n(2)))$ for $n \geq 3$ and $r \leq 8$. The decomposition in Proposition 4.16 is rewritten as

$$\Omega F^{n+1}\{2\} \cong \Omega L_3 \times \Omega L_5 \times A_{n+1},$$

where $\pi_{n+r}(A_{n+1}) = \pi_{n+r}(\tilde{A}^{n+1})$ for $n \geq 3$ and $r \leq 8$. Recall that there is a fibre sequence

$$A_{n+1} \longrightarrow Y_{n+1} \longrightarrow \Omega S^{n+1}.$$

By Lemma 4.17, $A_{2n}$ has a further decomposition

$$A_{2n} \cong \Omega S^{2n-1} \times B_{2n}.$$

We will need some information on the stable homotopy groups of $\mathbb{R}P^2$.

**Lemma 5.2.** The stable homotopy groups $\pi^s_r(\mathbb{R}P^2)$ for $4 \leq r \leq 10$ are as follows

1. $\pi^s_4(\mathbb{R}P^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
2. $\pi^s_5(\mathbb{R}P^2) = \mathbb{Z}/2$,
3. $\pi^s_6(\mathbb{R}P^2) = 0$,
4. $\pi^s_7(\mathbb{R}P^2) = \mathbb{Z}/2$,
5. $\pi^s_8(\mathbb{R}P^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
6. $\pi^s_9(\mathbb{R}P^2) = \mathbb{Z}/2^{\oplus 3}$,
7. $\pi^s_{10}(\mathbb{R}P^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

**Proof.** The proof follows from the long exact sequence of stable homotopy groups associated with the cofibre sequence

$$S^1 \xrightarrow{[2]} S^1 \longrightarrow \mathbb{R}P^2.$$
together with calculations from [42] for the homotopy groups of spheres and the fact that the degree 2 map \[ [2]: \mathbb{R}P^2 \to \mathbb{R}P^2 \], in the stable category, is the composite
\[
\mathbb{R}P^2 \xrightarrow{\text{pinch}} S^2 \xrightarrow{\eta} S^1 \xrightarrow{\pi} \mathbb{R}P^2.
\]

5.2. The Homotopy Groups \( \pi_*(P^4(2)) \). Now we start to compute \( \pi_*P^4(2) \) for \( * \leq 12 \).

5.2.1. The Space \( B_4 \).

**Lemma 5.3.** There exists a map \( \phi: \Omega S^6 \times P^{10}(2) \to B_4 \) so that \( \phi_*: H_*(\Omega S^6 \times P^{10}(2)) \to H_*B_4 \) is an isomorphism for \( * \leq 10 \).

**Proof.** The primitive elements \( PH_r(B_4) \) for \( r \leq 10 \) has a basis
\[
\{[u, v], [u, v]^2, [[[u, v], u], u], [[[u, v], v], u]\}.
\]
Let \( \phi_1: \Omega S^6 \to \Omega F^4\{2\} \) be a map so that \( (\phi_1)_*(\iota_5) = [u, v] \). Since
\[
L_3(P^n(2)) = P^{3n-1}(2),
\]
there exists a map \( \psi: P^8(2) \to \Omega F^4\{2\} \) such that
\[
\psi_*(\iota_8) = [[[u, v], v], u] \quad \text{and} \quad \psi_*(\iota_7) = [[[u, v], v], u].
\]
Let \( j: S^2 \to \Omega F^4\{2\} \) be the injection so that \( j_*(\iota_2) = u \). Define \( \phi_2 \) to be Samelson product \( [\psi, j]: P^{10}(2) = P^8(2) \wedge S^2 \to \Omega F^4\{2\} \). Let \( \phi \) be the composite
\[
\Omega S^6 \times P^{10}(2) \xrightarrow{\phi_1 \times \phi_2} \Omega F^4\{2\} \times \Omega F^4\{2\} \xrightarrow{\text{prod}} \Omega F^4\{2\} \to B_4.
\]
Then the map \( \phi \) has the desired property and hence the result.

**Lemma 5.4.** There is a cofibre sequence
\[
S^{10} \xrightarrow{\psi} \text{sk}_{15}(\Omega S^6 \times P^{10}(2)) \xrightarrow{} \text{sk}_{15}(B_4),
\]
which satisfies:
1) the composite
\[
S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \xrightarrow{\text{proj}} P^{10}(2)
\]
is \( \bar{\eta} \) which is a representative of the generator for \( \pi_{10}P^{10}(2) = \mathbb{Z}/2 \) and
2) the composite
\[
S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \xrightarrow{} \Omega S^6
\]
is \( 2\omega_6 \).

**Proof.** Note that \( PH_r(B_4) \) for \( r \leq 16 \) has a basis
\[
[u, v], [u, v, u^2], [u, v]^2, [[[u, v], v], u], [[[u, v], v], v]
\]
with \( Sq^1[u, v, v, u] = [u, v, u^2] \) and \( Sq^2[u, v, v, v] = [u, v, u^2] \). By [9, Lemma 10.4], \( \beta_2[u, v, v, v] = [u, v]^2 \). Thus \( \text{sk}_{15}(B_4)/\text{sk}_{15}(\Omega S^6 \times P^{10}(2)) = S^{11} \) and there is a cofibre sequence
\[
S^{10} \xrightarrow{\psi} \text{sk}_{15}(\Omega S^6 \times P^{10}(2)) \xrightarrow{} \text{sk}_{15}(B_4).
\]
Since $S^2[[u,v,v,v]] = [[u,v,u^2]] \neq 0$, $S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \xrightarrow{\psi} P^{10}(2)$ is essential and hence condition (1). Let $\psi_2$ be the composite $S^{10} \xrightarrow{\psi} \Omega S^6 \times P^{10}(2) \xrightarrow{\psi} \Omega S^6$. There is a homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
S^{10} & \xrightarrow{\psi} & \sk_{15}(\Omega S^6 \times P^{10}(2)) \\
\downarrow & & \downarrow \\
S^{10} & \xrightarrow{\psi_2} & J_2(S^5) \\
\downarrow & & \downarrow \\
\phantom{S^{10}} & & X \\
\downarrow & & \downarrow \\
S^{11} & \xrightarrow{} & S^{11}
\end{array}
\]

Since $H_*(\sk_{15}(B_4)) \to H_*X$ is onto, the Bockstein $\beta_2 : H_{11}X \to H_{10}X$ is an isomorphism. Observe that $\phi_2 : S^{10} \to \Omega S^6$ maps into the 10-skeleton $J_2(S^5)$ up to homotopy. The composite

\[
S^{10} \xrightarrow{\psi_2} J_2(S^5) \xrightarrow{\text{pinch}} S^{10},
\]

is of degree 4. Since $\pi_{11}(S^6) = \mathbb{Z}$ which is generated by $\omega_6$, we have $\psi_2 = 2\omega_6$ and hence the result.

5.2.2. The Homotopy Groups $\pi_*(F^4\{2\})$.

**Lemma 5.5.** The homotopy groups $\pi_k(B_4)$ for $5 \leq k \leq 12$ are as follows:

\[
\pi_k(B_4) = \begin{cases} 
\mathbb{Z} & \text{for } k = 5 \\
\mathbb{Z}/2 & \text{for } k = 6 \\
\mathbb{Z}/2 & \text{for } k = 7 \\
\mathbb{Z}/8 & \text{for } k = 8 \\
\mathbb{Z}/2 & \text{for } k = 9 \\
\mathbb{Z}/4 & \text{for } k = 10 \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } k = 11 \\
\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } k = 12.
\end{cases}
\]

**Proof.** By Lemma 5.4, $\pi_r(B_4) \cong \pi_r(\Omega S^6)$ for $r \leq 8$ and $\pi_9(B_4) = \pi_9(\Omega S^6) \oplus \pi_9(P^{10}(2)) = \mathbb{Z}/2$.

1. $\pi_{10}(B_4)$: There is an exact sequence

\[
\pi_{10}(S^{10}) \xrightarrow{\psi_*} \pi_{10}(\Omega S^6) \oplus \pi_{10}(P^{10}(2)) \xrightarrow{} \pi_{10}B_4 \xrightarrow{} 0 \quad \text{or}
\]

\[
\mathbb{Z} \xrightarrow{(2,\eta_2)} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{} \pi_{10}B_4 \xrightarrow{} 0.
\]

Thus $\pi_{10}(\Omega S^6) \xrightarrow{} \pi_{10}(B_4)$ is onto and $\pi_{10}(B_4) = \mathbb{Z}/4$.

2. $\pi_{11}(B_4)$: There is an exact sequence

\[
\pi_{11}(S^{10}) \xrightarrow{\psi_*} \pi_{11}(\Omega S^6) \oplus \pi_{11}(P^{10}(2)) \xrightarrow{} \pi_{11}B_4 \xrightarrow{} 0 \quad \text{or}
\]

\[
\mathbb{Z}/2 \xrightarrow{\psi_*} \mathbb{Z}/2 \oplus \mathbb{Z}/4 \xrightarrow{} \pi_{11}B_4 \xrightarrow{} 0.
\]

Observe that

\[
\psi_*(\eta) = 2\omega_6 \circ \eta + \bar{\eta}^2 = \bar{\eta} \circ \eta = 2\alpha,
\]

where $\alpha$ is the generator of $\pi_{11}(P^{10}(2)) = \mathbb{Z}/4$. Thus

\[
\pi_{11}(B_4) \cong \pi_{11}(\Omega S^6) \oplus \pi_{11}(P^{10}(2))/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]
3. $\pi_{12}(B_4)$: Consider the exact sequence

$$\pi_{12}(S^{10}) \xrightarrow{\psi_*} \pi_{12}(\Omega S^6) \oplus \pi_{12}(P^{10}(2)) \longrightarrow \pi_{12}(B_4) \longrightarrow 0.$$ 

Note that

$$\psi_*(\eta^2) = 2\omega_4 \circ \eta^2 + \bar{\eta} \circ \eta^2 = 0.$$ 

Thus

$$\pi_{12}(B_4) \cong \pi_{12}(\Omega S^6) \oplus \pi_{12}(P^{10}(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$ 

The calculation is finished now. 

By Lemmas 5.2 and 5.5, we have

**Proposition 5.6.** The homotopy groups $\pi_r = \pi_r(F^4\{2\})$ for $r \leq 13$ are as follows.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\pi_r(S^3)\oplus$</th>
<th>$\pi_{r-1}(B_4)\oplus$</th>
<th>$\pi_r(P^9(2))\oplus$</th>
<th>$\pi_r(X)\oplus$</th>
<th>$\pi_r(P^{14}(2))\oplus^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}/4$</td>
<td>$\mathbb{Z}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>$\mathbb{Z}/8$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\mathbb{Z}/2^{\oplus^2}$</td>
<td>$\mathbb{Z}/2^{\oplus^2}$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$(\mathbb{Z}/2)^{\oplus^2}$</td>
</tr>
<tr>
<td>13</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td>$\mathbb{Z}/2^{\oplus^2} \oplus \mathbb{Z}/4$</td>
<td>0</td>
<td>0</td>
<td>$(\mathbb{Z}/2)^{\oplus^2}$</td>
</tr>
</tbody>
</table>

where $X = \mathbb{C}P^2 \wedge P^{11}(2)$, one copy of $P^{14}(2)$ comes from $\Sigma L_5$ and another copy of $P^{14}(2)$ is the bottom two-cells for $\Sigma L_6$.

5.2.3. The Fibre of $P^4(2) \to BS^3$. Now consider the homotopy commutative diagram of fibre sequences

$$
\begin{array}{cccccc}
S^3 & \longrightarrow & S^7 & \nu_4 & S^4 & \longrightarrow & BS^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^3 & \longrightarrow & X^4 & \delta & BS^3 \\
\downarrow i & & \downarrow q & & \downarrow j \\
S^3 & \longrightarrow & P^4(2) & \delta & BS^3 \\
\downarrow j & & \downarrow \delta & & \downarrow j \\
F^4\{2\} & \longrightarrow & F^4\{2\} & & \\
\end{array}
$$

where $\nu_4 : S^7 \to S^4$ is the Hopf fibration.

**Lemma 5.7.** There is a cofibre sequence

$$S^6 \xrightarrow{(\nu',2)} S^3 \vee S^6 \longrightarrow X^4$$
**Proof.** By Lemma 2.38, the mod 2 homology $H_r(X^4) = \mathbb{Z}/2$ for $r = 3, 6, 7$ and 0 otherwise.

Consider the Serre spectral sequence for the fibre sequence $F^4\{2\} \to X^4 \to S^7$. There is a cofibre sequence

$$\text{sk}_6(F^4\{2\}) \to X^4 \to S^7.$$  

Since $F^4\{2\}(6) = S^3 \vee S^6$, there is a cofibre sequence

$$S^6 \to S^3 \vee S^6 \xrightarrow{f} X^4.$$  

Now consider the homotopy commutative diagram of fibre sequences

$$\begin{array}{ccc}
\Omega S^7 & \to & F^4\{2\} \to X^4 \\
\downarrow & & \downarrow \\
\Omega S^4 & \to & F^4\{2\} \to P^4(2).
\end{array}$$

Observe that $H_6(\Omega S^4; \mathbb{Z}) \to H_6(F^4\{2\}; \mathbb{Z})$ is of degree 2. The composite

$$S^6 \to S^3 \vee S^6 \xrightarrow{\text{proj}} S^6$$

is of degree 2. Recall that $\pi_6(S^3) = \mathbb{Z}/4$ which is generated by $\nu'$. The composite $S^6 \to S^3 \vee S^6 \xrightarrow{\text{proj}} S^6$ is $k\nu'$. We claim that $k = \pm 1$.

Suppose that $k \equiv 0 \mod 2$. Let $\lambda_4$ denote the composite

$$S^6 \to S^3 \vee S^6 = \text{sk}_6(F^4\{2\}) \to X^4.$$  

Then, in $\pi_6(X^4)$, $2[\lambda_4] = -kf_*[\nu']$. Since $i : S^3 \to X^4$ is of degree 2 and $S^3$ is an H-space, we have

$$i_*([k/2\nu']) = f_*([2\nu'] = f_*(-k\nu') = 2[\lambda_4].$$

Hence $2q_*[\lambda_4] = 0$ in $\pi_6(P^4(2))$. Let $\lambda_4 : S^5 \to \Omega P^4(2)$ denote the adjoint map of $q \circ \lambda_4 : S^6 \to P^4(2)$. Then $(\lambda_4)_*(i_5) = [u, v]$ in $H_5(\Omega P^4(2))$. By [12, Theorem 2.2],

$$2q_*[\lambda_4] = 2[\lambda_4] \neq 0.$$  

This is a contradiction. Thus $k = \pm 1$ and hence the result.  

\[ \square \]

5.2.4. **The Homotopy Groups $\pi_*(P^4(2))$.**

**Theorem 5.8.** The homotopy groups $\pi_r(P^4(2))$ for $6 \leq r \leq 12$ are as follows.

<table>
<thead>
<tr>
<th>$\pi_r(P^4(2))$</th>
<th>$\pi_{r-1}(Y_4)\oplus$</th>
<th>$\pi_r(P^3(2))\oplus$</th>
<th>$\pi_r(\mathbb{C}P^2 \wedge P^{11}(2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 6$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$7$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
</tr>
<tr>
<td>$9$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4$</td>
<td></td>
</tr>
<tr>
<td>$11$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td></td>
</tr>
<tr>
<td>$12$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
</tbody>
</table>
Proof. 1. \(\pi_6(P^4(2))\): By expecting the exact sequence

\[
\pi_6(S^6) \xrightarrow{(\nu',2)} \pi_6(S^3) \oplus \pi_6(S^6) \xrightarrow{} \pi_6(X^4) \xrightarrow{} 0,
\]
we have \(\pi_6(X^4) = \mathbb{Z}/8\) generated by \([\lambda_4]\). Since
\[i_*(\nu') = 2\nu' = 4[\lambda_4],\]
there is a short exact sequence
\[
0 \xrightarrow{} \mathbb{Z}/4 \xrightarrow{} \pi_6(P^4(2)) \xrightarrow{} \pi_5(S^3) \xrightarrow{} 0.
\]

Since \(q_*([\lambda_4]) = [\lambda_4]\) in \(\pi_6(P^4(2)) \cong \pi_5(\Omega P^4(2))\) and \(\text{h}([\lambda_4]) = [u,v]\) in \(H_*\Omega P^4(2)\), the element \([\lambda_4]\) is not divisible by 2. Thus the above short exact sequence splits and so \(\pi_6(P^4(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2\).

2. \(\pi_7(P^4(2))\): Consider the fibre sequence
\[
\Omega^2S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \xrightarrow{} \Omega X^4 \rightarrow \Omega S^7,
\]
where \(Q = \prod_{k \geq 3} \Omega \Sigma L_k\). Since \(\Omega P^4(2) \cong Y_4 \times Q\), the composite
\[
\Omega^2S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \xrightarrow{\text{proj.}} Q
\]
is null homotopic. By Lemma 5.7, the diagram
\[
\begin{array}{ccc}
\Omega S^6 & \xrightarrow{E} & \Omega^2S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \\
\downarrow \Omega[2] & & \downarrow \text{proj.} \\
\Omega S^6 & \xrightarrow{\phi_1} & B_4
\end{array}
\]
commutes up to homotopy commutes and the composite
\[
\Omega S^6 \xrightarrow{E} \Omega^2S^7 \xrightarrow{\delta} \Omega S^3 \times B_4 \times Q \xrightarrow{\text{proj.}} \Omega S^3
\]
is \(\Omega \nu'\). Consider the exact sequence
\[
\pi_6(\Omega^2S^7) \xrightarrow{\delta_*} \pi_6(\Omega S^3) \oplus \pi_6(B_4) \oplus \pi_6(Q) \xrightarrow{} \pi_6(\Omega X^4) \xrightarrow{} 0.
\]
Observe that
\[\delta_*(\eta) = \nu' \circ \eta + (\phi_1)_* \circ \Omega[2]*(\eta) = \nu' \circ \eta.\]
Thus \(\pi_6(\Omega X^4) \cong \pi_6(B_4) \oplus \pi_6(Q) = \mathbb{Z}/2\) and \(\delta_* : \pi_6(\Omega^2S^7) \rightarrow \pi_6(\Omega F^4\{2\})\) is 1-1. Since \(i_*(\nu' \circ \eta) = f_*(2\nu' \circ \eta) = f_*(0) = 0\), there is a short exact sequence
\[
0 \xrightarrow{} \pi_7(X^4) \xrightarrow{q} \pi_7(P^4(2)) \xrightarrow{} 2 \cdot \pi_6(S^3) \xrightarrow{} 0 \quad \text{or}
\]
\[
0 \xrightarrow{} \pi_7(P^4(2)) \xrightarrow{} \mathbb{Z}/2 \xrightarrow{} \mathbb{Z}/2 \xrightarrow{} 0
\]
and therefore \((\lambda_4)_* : \pi_6(S^5) \rightarrow \pi_7(P^4(2))\) is a monomorphism. Consider the second reduced James-Hopf map \(\tilde{H}_2 : \Omega P^4(2) \rightarrow \Omega P^7(2)\). Then
\[
(\tilde{H}_2 \circ (\lambda_4)_*)(t_5) = (\tilde{H}_2)_*([u,v]) \neq 0
\]
in $H_5(\Omega P^7(2))$ and so $\tilde{H}_2 \circ \lambda_4 : S^5 \to \Omega P^7(2)$ is the inclusion of the bottom cell. Since $\tilde{H}_2 \circ \lambda_4$ induces an isomorphism on $\pi_6$, the subgroup $\pi_6(S^5) = \mathbb{Z}/2$ is a summand of $\pi_6(\Omega P^4(2))$ or the above short exact sequence for $\pi_7(P^4(2))$ splits. Thus

$$\pi_7(P^4(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

generated by $\lambda_4 \circ \eta$ and $2\nu'$.

3. $\pi_8(P^4(2))$: By the commutative diagram

$$\pi_8(S^3) \xrightarrow{i_*} \pi_8(X^4) \xrightarrow{q_*} \pi_8(P^4(2)) \xrightarrow{\delta_*} \pi_7(S^3)$$

$$\pi_7(\Omega S^3) \oplus \pi_7(B_4) \oplus \pi_7(Q) \cong \pi_8(\mathbb{F}^4\{2\})$$

we have

$$\delta_*(\eta^2) = \nu' \circ \eta^2 + (\phi_1)_* \circ [2]_*(\eta^2) = \nu' \circ \eta^2 \neq 0.$$}

Thus

$$\pi_8(X^4) \cong \pi_7(B_4) \oplus \pi_7(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and there is an exact sequence

$$\pi_8(S^3) \xrightarrow{i_*} \pi_7(B_4) \xrightarrow{q_*} \pi_7(Y_4) \xrightarrow{\delta_*} \pi_7(S^3) \to 0.$$}

Since $i_*(\nu' \circ \eta^2) = f_*(2(\nu' \circ \eta^2)) = 0$, there is a short exact sequence

$$0 \to \pi_7(B_4) \xrightarrow{q_*} \pi_7(Y_4) \xrightarrow{\delta_*} \pi_7(S^3) \to 0.$$}

Recall that $(\lambda_4)_*(\iota_5) = [u,v]$. There is a commutative diagram

$$\pi_7(B_4) \xrightarrow{q_*} \pi_7(Y_4) \xrightarrow{\delta_*} \pi_8(P^4(2))$$

$$\mathbb{Z}/2 = \pi_7(S^5) \xrightarrow{\mu_*} \pi_7(\Omega(P^6(2) \vee S^6)) \xrightarrow{\Omega g_*} \pi_7(\Omega\Sigma(P^3(2) \wedge P^3(2)),$$

where $\mu : S^5 \to \Omega(P^6(2) \vee S^6)$ is the composite $S^5 \xrightarrow{E} \Omega S^6 \xrightarrow{} \Omega(P^6(2) \vee S^6)$ and $\pi_7(P^6(2) \vee S^6) \xrightarrow{g} \Sigma P^3(2) \wedge P^3(2) \to S^7$.
is a cofibre sequence. Observe that
\[ g_* : \pi_8(P^6(2) \vee S^6) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \rightarrow \pi_8(P^3(2) \wedge P^4(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \]
is an isomorphism. There exists a homomorphism \( \psi : \pi_7(\Omega\Sigma(P^3(2) \wedge P^3(2))) \rightarrow \mathbb{Z}/2 \) so that
\[ \pi_7(Y_4) \cong \pi_7(B_4) \oplus \pi_7(S^3) \]
generated by \( \lambda_4 \circ \eta^2 \) and \( \nu' \circ \eta \). Thus
\[ \pi_8(P^4(2)) = \pi_7(Y_4) \oplus \pi_7(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]

4. \( \pi_9(P^4(2)) \): Consider the commutative diagram

Since \( \pi_8(B_4) \cong \pi_8(\Omega S^6) = \mathbb{Z}/8 \), we have \( \delta_*(\nu_7) = 2\nu_6 \) and so
\[ \pi_9(X^4) \cong \pi_8(B_4)/2 \oplus \pi_8(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]
with a short exact sequence
\[ 0 \rightarrow \pi_8(\Omega S^6)/2 \rightarrow \pi_8(Y_4) \rightarrow \pi_8(S^3) \rightarrow 0. \]

Now consider the commutative diagram

Since \( \Omega g_* (\nu_6) \neq 0 \), the above short exact sequence for \( \pi_8(Y_4) \) splits and therefore
\[ \pi_8(Y_4) \cong \pi_8(\Omega S^6)/2 \oplus \pi_8 S^3. \]
Thus \( \pi_9(P^4(2)) \cong \pi_8(\Omega S^6)/2 \oplus \pi_8(S^3) \oplus \pi_8(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)
5. $\pi_{10}(P^4(2))$: Consider the commutative diagram of exact sequences

\[
\begin{array}{ccc}
\pi_{10}(S^7) & \longrightarrow & \pi_{10}(S^4) \\
\pi_{10}(S^3) = 0 & \xrightarrow{i_*} & \pi_{10}(X^4) & \xrightarrow{q_*} & \pi_{10}(P^4(2)) & \xrightarrow{\delta_*} & \pi_9(S^3) = 0 \\
 & & \pi_9(\Omega S^3) \oplus \pi_9(B_4) \oplus \pi_9(Q) \equiv \pi_{10}(F^4\{2\}) \\
\pi_{11}(S^7) = 0 & \longrightarrow & \pi_{11}(S^4).
\end{array}
\]

Since $\text{Ker}(\delta_*: \pi_{10}(S^7) \to \pi_9(F^4\{2\})) = 4 \cdot \pi_{10}(S^7)$, we have $\pi_{10}(X^4) \cong \pi_{10}(P^4(2))$ with short exact sequences

\[
0 \longrightarrow \pi_9(B_4) \oplus \pi_9(Q) \longrightarrow \pi_{10}(P^4(2)) \longrightarrow 4 \cdot \pi_{10}(S^7) \longrightarrow 0 \quad \text{and}
\]

\[
0 \longrightarrow \pi_9(B_4) \longrightarrow \pi_9(Y_4) \longrightarrow 4 \cdot \pi_{10}(S^7) \longrightarrow 0.
\]

Recall that $\pi_9(B_4) \cong \pi_9(P^{10}(2)) = \mathbb{Z}/2$. Let $\alpha$ be the non-zero element in $\pi_9(B_4)$. Then the Hurewicz image $H(\alpha) = [u, v, u]$ in $H_*(B_4)$ and so in $H_\ast \Omega P^4(2)$. Thus $\alpha$ is not divisible by 2 in $\pi_{10}(P^4(2))$ and

\[
\pi_9(Y_4) \cong \pi_9(B_4) \oplus 4 \cdot \pi_{10}(S^7)
\]

generated by Samelson product $\langle (\lambda_4, u), u \rangle$ and $4\nu_4^2$. Hence

\[
\pi_{10}(P^4(2)) \cong \pi_9(B_4) \oplus 4 \cdot \pi_{10}(S^7) \oplus \pi_9(Q) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.
\]
6. \( \pi_{11}(P^4(2)) \): Consider the commutative diagram of exact sequences

\[
\begin{array}{ccc}
\pi_{11}(S^7) = 0 & \rightarrow & \pi_{11}(S^4) \\
\pi_{11}(S^3) = \mathbb{Z}/2 & \rightarrow & \pi_{11}(X^4) \\
\pi_{10}(\Omega S^3) \oplus \pi_{10}(B_4) \oplus \pi_{10}(Q) & \rightarrow & \pi_{11}(F^4\{2\}) \\
\delta_* & \rightarrow & \pi_{12}(S^7) = 0 \\
& \rightarrow & \pi_{12}(S^4).
\end{array}
\]

Recall that \( \pi_{11}(S^3) = \mathbb{Z}/2 \) generated by \( \epsilon_3 \) and \( i_*(\epsilon_3) = f_* \circ [2]*_*(\epsilon_3) = f_*(2\epsilon_3) = 0 \). Thus

\[
\pi_{11}(P^4(2)) \cong \pi_{11}(X^4) \cong \pi_{11}(F^4\{2\}) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

7. \( \pi_{12}(P^4(2)) \): Consider the commutative diagram of exact sequences

\[
\begin{array}{ccc}
\pi_{12}(S^7) = 0 & \rightarrow & \pi_{12}(S^4) \cong \pi_{11}(S^3) \\
\pi_{12}(S^3) & \rightarrow & \pi_{12}(X^4) \\
\pi_{11}(\Omega S^3) \oplus \pi_{11}(B_4) \oplus \pi_{11}(Q) & \rightarrow & \pi_{12}(F^4\{2\}) \\
\delta_* & \rightarrow & \pi_{13}(S^7) = \mathbb{Z}/2 \\
& \rightarrow & \pi_{13}(S^4).
\end{array}
\]

Observe that

\[
\delta_*(\nu^2) = \nu' \circ \nu_6^2 + (\phi_1)_* \Omega[2]*_*(\nu_6^2) = \nu' \circ \nu_6^2 = 0
\]

and \( 2\pi_{12}S^3 = 0 \). Thus

\[
\pi_{12}(X^4) \cong \pi_{11}(\Omega S^3) \oplus \pi_{11}(B_4) \oplus \pi_{11}(Q)
\]
with a short exact sequence

\[ 0 \longrightarrow \pi_{11}(\Omega S^3) \oplus \pi_{11}(B_4) \longrightarrow \pi_{11}(Y_4) \xrightarrow{\delta_*} \pi_{11}(S^3) = \mathbb{Z}/2 \longrightarrow 0. \]

Now let \( \alpha \in \pi_{11}(Y_4) \subseteq \pi_{11}(\Omega P^4(2)) \) so that \( \delta'_*(\alpha) = \epsilon_3 \). Then \( p_*(\alpha) = \epsilon_4 \in \pi_{11}(\Omega S^4) \).

Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{11}(\Omega F^4\{2\}) & \longrightarrow & \pi_{11}(\Omega P^4(2)) \xrightarrow{p_*} \pi_{11}(\Omega S^4) \\
\downarrow & & \downarrow \\
\pi_{11}(Q(S^2)) & \longrightarrow & \pi_{11}(Q(P^3(2))) \xrightarrow{p_*} \pi_{11}(Q(S^3))
\end{array}
\]

Then \( \delta'_*(E^\infty \alpha) = E^\infty \epsilon_3 = \epsilon \) and \( E^\infty(2\alpha) = 2E^\infty \alpha = \epsilon \circ \eta \neq 0 \) in \( \pi_{11}(Q(P^3(2))) \). It follows that \( 2\alpha \neq 0 \). Since \( 2\pi_{11}\Omega S^3 = 2\pi_{11}B_4 = 0 \), we have

\[ \pi_{11}(Y_4) \cong \frac{1}{2} \pi_{11}(\Omega S^3) \oplus \pi_{11}(B_4) \oplus \pi_{11}(\Omega S^3)/(\epsilon_3 \circ \eta) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

generated by \( 1/2\epsilon_3, \mu_3, \lambda_4 \circ r_0^2, \beta \), where \( \beta \) is the image of the generator for \( \pi_{11}(P^{10}(2)) \) with \( 2\beta = 0 \) in \( \pi_{11}(\Omega P^4(2)) \). Hence

\[ \pi_{12}(P^4(2)) \cong \pi_{11}(Y_4) \oplus \pi_{12}(P^9(2)) \oplus \pi_{12}(\mathbb{C}P^2 \wedge P^{11}(2)) \cong (\mathbb{Z}/2)^{\oplus 5} \oplus \mathbb{Z}/4. \]

We finish the calculation. \( \square \)

5.3. The Homotopy Groups \( \pi_*(P^5(2)) \). Now we compute \( \pi_*P^5(2) \). By Proposition 4.15, there is a cofibre sequence

\[ S^7 \xrightarrow{2\omega_4} S^4 \longrightarrow \text{sk}_8(F^5\{2\}) \]

and so there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
\Omega F^5\{2\} & \longrightarrow & \Omega F^5\{2\} \\
\Omega j & \searrow & j \\
\Omega S^4 \xrightarrow{\phi_1} X^5 & \longrightarrow & S^4 \\
\downarrow & & \downarrow \\
X^5 & \xrightarrow{2\omega_4} & S^4
\end{array}
\]

Let \( \phi_2 = [S_3, j] : L_3(P^4(2)) \wedge S^3 = P^{14}(2) \to \Omega F^5\{2\} \) be Samelson product and let \( \phi \) denote the composite

\[ X^5 \times P^{14}(2) \xrightarrow{\phi_1 \times \phi_2} \Omega F^5\{2\} \times \Omega F^5\{2\} \xrightarrow{\text{multi}} \Omega F^5\{2\} \longrightarrow A_5. \]

**Lemma 5.9.** The map \( \phi_r : H_r(X^5 \times P^{14}(2)) \to H_r(A_5) \) is an isomorphism for \( r \leq 13 \) and therefore \( \phi_r : \pi_r(X^5 \times P^{14}(2)) \to \pi_r(A_5) \) is an isomorphism for \( r \leq 12 \).

**Proof.** The proof follows by observing that the primitive elements \( P\mathcal{H}_r(A_5) \) for \( r \leq 14 \) has a basis \( u, u^2, [u, v], u^4, [[u, v], u], u, [u, v]^2, [[u, v],], u \) with \( \beta_2[u, v] = u^2 \) and \( Sq^1[[u, v], v], u = [[[u, v], u], u] \). \( \square \)

There is an overlap between the following theorem and [28, Theorem 1.2, p.64].
Theorem 5.10. The homotopy groups $\pi_r(P^5(2))$ for $7 \leq r \leq 13$ are as follows.

<table>
<thead>
<tr>
<th>$\pi_r(P^5(2))$</th>
<th>$\pi_{r-1}(Y_5)\oplus$</th>
<th>$\pi_r(P^{12}(2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 7$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
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<tr>
<td>9</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
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<td>10</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
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<td>11</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
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<td>12</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
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<tr>
<td>13</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$</td>
<td>$\mathbb{Z}/4$</td>
</tr>
</tbody>
</table>

Proof. 1. $\pi_7(P^5(2))$: Consider the exact sequence

$$\pi_7(S^7) = \mathbb{Z} \xrightarrow{(2\omega_4)_*} \pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{j_*} \pi_0(X^5) \longrightarrow 0.$$

Since

$$(2\omega_4)_*(\nu_7) = \omega_4 \circ (2\nu_7) = 2(2\nu_4 - E\nu') = 4\nu_4 - 2E\nu',$$

we have

$$\pi_6(X^5) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$$

generated by $j_*(\nu_4)$ and $j_*(\omega_4)$. Now consider the exact sequence

$$\pi_8(S^5) \xrightarrow{\delta_*} \pi_7(F^5\{2\}) \cong \pi_6(X^5) \xrightarrow{i_*} \pi_7(P^5(2)) \xrightarrow{p_*} \pi_7(S^5) \longrightarrow 0$$

and the commutative diagram

$$\pi_8(S^5) \xrightarrow{\delta_*} \pi_7(F^5\{2\})$$

\[\begin{array}{c}
\uparrow E \\
\downarrow i_* \\
\pi_7(S^4) \xrightarrow{[2]_*} \pi_7(S^4).
\end{array}\]

Since $[2]_*(\nu_4) = 2\nu_4 + \omega_4$, we have

$$\delta_*(\nu_5) = \delta_* \circ E(\nu_4) = 2j_*(\nu_4) + j_*(\omega_4).$$

Thus there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4(i_*j_*(\nu_4)) \longrightarrow \pi_7(P^5(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

The homotopy commutative diagram of fibre sequences

$$F^5\{2\} \xrightarrow{} P^5(2) \xrightarrow{} S^5$$

$$Q(S^4) \xrightarrow{} Q(P^5(2)) \xrightarrow{} Q(S^5)$$
induces a commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}/4 & \rightarrow & \pi_7(P^5(2)) & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0 \\
      &      & \downarrow E^\infty &      &        &      & \downarrow &      &      \\
0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & \pi_7(Q(P^5(2))) & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0.
\end{array}
\]

Since \(E^\infty(i_*,j_*(\nu_4)) \neq 0\) and \(\pi_7(Q(P^5(2))) = \mathbb{Z}/2 \oplus \mathbb{Z}/2\), the element \(i_*,j_*(\nu_4)\) is not divisible by 2 and so

\[
\pi_7(P^5(2)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2
\]

with the epimorphism \(E^\infty: \pi_7(P^5(2)) \rightarrow \pi_7(P^5(2))\).

2. \(\pi_8(P^5(2))\): By expecting the exact sequence

\[
\pi_8(S^7) \twoheadrightarrow \pi_8(S^4) \twoheadrightarrow \pi_7(X^5) \twoheadrightarrow 0,
\]

we have

\[
\pi_8(S^4) \cong \pi_7(X^5) \cong \pi_7(A_5) \cong \pi_8(F^5\{2\}).
\]

Now consider the commutative diagram

\[
\begin{array}{ccc}
\pi_9(S^5) & \overset{\delta_*}{\longrightarrow} & \pi_8(F^5\{2\}) \\
\uparrow E & \cong & \downarrow \pi_8(P^5(2)) \\
\pi_8(S^4) & \overset{[2]_*}{\longrightarrow} & \pi_8(S^4),
\end{array}
\]

where the top arrow is exact. Since \([2]_*(\nu_4 \circ \eta) = 2\nu_4 \circ \eta + \omega_4 \circ \eta = -E\nu' \circ \eta\) and \(\pi_8(S^4) = \mathbb{Z}/2(\nu_4 \circ \eta) \oplus \mathbb{Z}/2(E\nu' \circ \eta)\), there is a short exact sequence

\[
\mathbb{Z}/2(\nu_4 \circ \eta) \rightarrow \pi_8(P^5(2)) \rightarrow 4 \cdot \pi_8(S^5) \rightarrow 0
\]

with the monomorphism \(\delta_*: \pi_9(S^5) \rightarrow \pi_8(F^5\{2\})\). By the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
F^5\{2\} & \rightarrow & P^5(2) \\
\downarrow E & & \downarrow E \\
\Omega F^6\{2\} & \rightarrow & \Omega P^6(2) \\
\end{array}
\]

\[
\begin{array}{cc}
\rightarrow & \rightarrow \\
\Omega S^6,
\end{array}
\]

there is a commutative diagram of exact sequences

\[
\begin{array}{cccccccc}
\pi_9(S^5) & \overset{\delta_*}{\longrightarrow} & \pi_8(F^5\{2\}) & \overset{\cong}{\longrightarrow} & \pi_8(P^5(2)) & \overset{\cong}{\longrightarrow} & \pi_7(F^5\{2\}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = \pi_{10}(S^6) & \overset{\delta_*}{\longrightarrow} & \pi_9(F^6\{2\}) & \overset{\cong}{\longrightarrow} & \pi_9(P^6(2)) & \overset{\cong}{\longrightarrow} & \pi_8(S^6) & \overset{\cong}{\longrightarrow} & \pi_8(F^6\{2\}).
\end{array}
\]
By Proposition 4.16 and Lemma 4.17, there is decomposition

$$\Omega F^6 \{2\} \simeq \Omega S^5 \times B_6 \times Q_6,$$

where both $B_6$ and $Q_6$ are 8-connected. Thus $\pi_r(S^5) \cong \pi_r(F^6 \{2\})$ for $r \leq 9$ and so there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2(\nu_4 \circ \eta) & \longrightarrow & \pi_8(P^5(2)) & \longrightarrow & 4 \cdot \pi_8(S^5) & \longrightarrow & 0 \\
& & \cong & & E & & \cong & & \\
0 & \longrightarrow & \mathbb{Z}/2(\nu_5 \circ \eta) & \longrightarrow & \pi_9(P^6(2)) & \longrightarrow & 4 \cdot \pi_9(S^6) & \longrightarrow & 0.
\end{array}$$

Hence $\pi_8(P^5(2)) \cong \pi_8(P^6(2))$. We show that $2 \cdot \pi_9(P^6(2)) = 0$. Recall that the degree 2 map $[2] : P^6(2) \to P^6(2)$ is the composite

$$P^6(2) \xrightarrow{\text{pinch}} S^6 \xrightarrow{\eta} S^5 \xrightarrow{\pi} P^6(2)$$

and $\eta_5 \circ \nu_6 = 0$ (see [42, pp.44]). Thus $[2]_* : \pi_9(P^6(2)) \to \pi_9(P^6(2))$ is zero. By the distributivity law [2],

$$\Omega[2]_* = 2 + (\Omega W_2 \circ H_2)_* : \pi_8(\Omega P^6(2)) \longrightarrow \pi_8(\Omega P^6(2)).$$

(Note the dimension restrictions in this last equation.) Since the composite

$$P^5(2) \xrightarrow{E} \Omega P^6(2) \xrightarrow{H_2} \Omega \Sigma(P^5(2) \land P^5(2)) \xrightarrow{\Omega W_2} \Omega P^6(2)$$

is null homotopic, the composite

$$\pi_8(P^5(2)) \xrightarrow{E} \pi_8(\Omega P^6(2)) \xrightarrow{(\Omega W_2 \circ H_2)_*} \pi_8(\Omega P^6(2))$$

is zero and so is $(\Omega W_2 \circ H_2)_* : \pi_8(\Omega P^6(2)) \to \pi_8(\Omega P^6(2))$. Thus

$$2 \cdot \pi_8(\Omega P^6(2)) = 0$$

and hence $\pi_8(P^5(2)) \cong \pi_9(P^6(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

3. $\pi_9(P^5(2))$: From the exact sequence

$$\begin{array}{cccccc}
\pi_9(S^7) & \xrightarrow{(2\omega_4)_*} & \pi_9(S^4) & \longrightarrow & \pi_8(X^5) & \longrightarrow & \pi_8(S^7) & \longrightarrow & 0,
\end{array}$$

we obtain a short exact sequence

$$\begin{array}{cccccc}
0 & \longrightarrow & \pi_9(S^4) & \longrightarrow & \pi_8(X^5) & \longrightarrow & \pi_8(S^7) & \longrightarrow & 0.
\end{array}$$

Consider the commutative diagram

$$\begin{array}{cccccc}
\pi_10(S^5) & \xrightarrow{\delta_*} & \pi_9(F^5 \{2\}) & \longrightarrow & \pi_9(P^5(2)) & \longrightarrow & 0 \\
\pi_9(S^4) & \xrightarrow{[2]_*} & \pi_9(S^4) & \longrightarrow & \pi_8(X^5) & & \\
E & & & & \cong & &
\end{array}$$
Since \([2]_*(\nu_4 \circ \eta^2) = 2\nu_4 \circ \eta^2 + \omega_4 \circ \eta^2\), we have \(\delta_*(\nu_5 \circ \eta^2) = j_*(\omega_4 \circ \eta^2) \neq 0\) and so there is a short exact sequence

\[
0 \rightarrow \pi_9(S^4)/(\omega_4 \circ \eta^2) \rightarrow \pi_8(X^5)/j_*(\omega_4 \circ \eta^2) \rightarrow \pi_8(S^7) \rightarrow 0
\]

The commutative diagram of fibre sequences

\[
\begin{array}{ccc}
\Omega S^5 & \rightarrow & S^5 \{2\} \\
\downarrow & & \downarrow \\
F^5 \{2\} & \rightarrow & P^5(2)
\end{array}
\]

induces a commutative diagram

\[
\begin{array}{ccc}
\pi_9(S^4) & \rightarrow & \pi_9(F^5\{2\}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 = \pi_9(\Omega S^5) & \rightarrow & \pi_9(\Omega S^5),
\end{array}
\]

where \(E\) is onto. Thus \(j_*(\nu_4 \circ \eta^2)\) is not divisible by 2 in \(\pi_9(F^5\{2\})\) and therefore in \(\pi_9(P^5(2))\). Hence \(\pi_9(P^5(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\).

4. \(\pi_{10}(P^5(2))\): Consider the exact sequence

\[
\pi_{10}(S^7) \xrightarrow{(2\omega_4)_*} \pi_{10}(S^4) \rightarrow \pi_9(X^5) \rightarrow \pi_9(S^7) \rightarrow 0.
\]

By the EHP sequence \(\pi_{12}(S^5) \xrightarrow{H} \pi_{12}(S^9) \xrightarrow{P} \pi_{10}(S^4) \xrightarrow{E} \pi_{11}(S^5) \rightarrow 0\), we have \(P(\nu_9) = 2\nu_4^2\) and so \((2\omega_4)_*(\nu_7) = 4\nu_4^2\). By the commutative diagram

\[
\begin{array}{ccc}
\pi_{10}(S^4) & \rightarrow & \pi_9(X^5) \cong \pi_9(F^5\{2\}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 = \pi_{11}(S^5) & \rightarrow & \pi_{11}(S^5),
\end{array}
\]

the element \(\nu_4^2\) is not divisible by 2 in \(\pi_9(X^5)\) and so

\[
\pi_9(X^5) \cong \pi_{10}(S^4)/4 \oplus \pi_9(S^7) = \mathbb{Z}/4 \oplus \mathbb{Z}/2.
\]
Consider the exact sequence

\[
\pi_{11}(S^5) \xrightarrow{\delta_*} \pi_{10}(F^5\{2\}) \xrightarrow{\pi_{10}(P^5(2))} \pi_{10}(S^4) \xrightarrow{[2]_*} \pi_{10}(S^4).
\]

Since \([2]_*(\nu^2_1) = 2\nu^2_1 + \omega_4 \circ \nu_7 = 4\nu^2_1 \equiv 0\) in \(\pi_{10}(F^5\{2\})\), we have \(\delta_*(\nu^2_1) = 0\) and so \(\pi_{10}(F^5\{2\}) \cong \pi_{10}(P^5(2))\). Thus \(\pi_{10}(P^5(2)) \cong \pi_{10}(S^4)/4 \oplus \pi_9(S^7) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2\).

5 \(\pi_{11}(P^5(2))\): Consider the exact sequences

\[
\pi_{12}(S^5) \xrightarrow{\delta_*} \pi_{11}(F^5\{2\}) \xrightarrow{\pi_{11}(P^5(2))} \pi_{11}(S^5) \xrightarrow{\pi_{10}(F^5\{2\})} \pi_{10}(S^4) = 0 \xrightarrow{\pi_{10}(X^5)} \pi_{10}(S^7) \xrightarrow{\pi_{10}(S^4)} = 0.
\]

Since \((2\omega_4)_*(\nu_7) = 4\nu^2_1\), we have \(\pi_{10}(X^5) \cong 2 \cdot \pi_{10}(S^7) \cong \mathbb{Z}/4\). By Lemma 5.9, \(\pi_{10}(X^5) \cong \pi_{10}(A_5)\) and

\[
\pi_{11}(F^5\{2\}) \cong \pi_{10}(A_5) \oplus \pi_{11}(P^{12}(2)) \cong \pi_{10}(X^5) \oplus \pi_{11}(P^{12}(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2,
\]

where \(\Omega P^{12}(2) = \Omega \Sigma L_3\) is a common factor of \(\Omega F^5\{2\}\) and \(\Omega P^5(2)\). We need to determine the boundary \(\delta_* : \pi_{12}(S^5) \rightarrow \pi_{11}(F^5\{2\})\). Consider the homotopy commutative diagram of \(EHP\) sequences

\[
\begin{array}{cccccc}
\Omega^2 S^9 & P & S^4 & E & \Omega S^5 & H & \Omega S^9 \\
\Omega^2 P^9(2) & P & T^5 & E & \Omega P^5(2) & \bar{H}_2 & \Omega P^9(2) \\
\Omega^2 F^9\{2\} & P & E_5 & \bar{E} & \Omega F^5\{2\} & \bar{H}_2 & \Omega F^9\{2\}.
\end{array}
\]

Observe that the inclusion \(\Omega S^4 \hookrightarrow \Omega F^5\{2\}\) lifts to \(E_5\) and the composite

\[
L_3(P^4(2)) \rightarrow \Omega F^5\{2\} \rightarrow \Omega F^9\{2\}
\]

is null homotopic. Thus \(sk_{12}(E_5) \simeq sk_{12}(\Omega S^4 \times \Omega P^{12}(2))\) and so

\[
\pi_{10}(E_5) = \pi_{11}(S^4) \oplus \pi_{11}(P^{12}(2)) = 0 \oplus \mathbb{Z}/2 = \mathbb{Z}/2.
\]

It follows that the composite

\[
\mathbb{Z}/4 = \pi_{10}(X^5) \rightarrow \pi_{10}(\Omega F^5\{2\}) \rightarrow \pi_{10}(\Omega F^9\{2\}) \simeq \pi_{11}(S^8) = \mathbb{Z}/8
\]
is a monomorphism. Now consider the commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
\pi_{12}(S^5) & \xrightarrow{\delta_*} & \pi_{10}(\Omega F^5 \{2\}) & \to & \pi_{10}(\Omega P^5(2)) \\
| & & | & & | \\
H & \downarrow & \tilde{H}_2* & & \tilde{H}_2* \\
\pi_{12}(S^9) & \xrightarrow{\delta'} & \pi_{10}(\Omega F^9 \{2\}) & \to & \pi_{10}(\Omega P^9(2)).
\end{array}
\]

Recall that \(\pi_{12}(S^5)\) is generated by \(\sigma''\) with \(H(\sigma'') = 4\nu_9\), see [42, Lemma 5.13]. We obtain

\[
\tilde{H}_2* \circ \delta_* (\sigma'') = \delta'_* \circ H(\sigma'') = \delta'_*(4\nu_9) = 8\nu_7 = 0
\]

in \(\pi_{10}(S^7) \cong \pi_{10}(\Omega F^9 \{2\})\). Thus \(\delta_* (\sigma'') = 0\) and so there are short exact sequences

\[
0 \to \pi_{10}(\Omega F^5 \{2\}) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \to \pi_{10}(\Omega P^5(2)) \to \pi_{11}(S^5) = \mathbb{Z}/2 \to 0
\]

for some \(n_\sigma \in \mathbb{Z}\). Note that the diagram

\[
\begin{array}{ccccccc}
\Omega \Sigma (P^4(2))^{(3)} & \to & \Omega \Sigma L_3 = \Omega P^{12}(2) \\
| & & | & & | \\
\Omega W_3 & & & & \Omega P^5(2)
\end{array}
\]

commutes up to homotopy and \(\pi_{10}(\Omega P^{12}(2)) = \mathbb{Z}/2\). We have

\[
2\Omega W_{3*} \circ \Omega \sigma_* \circ H_{3*}(\alpha) = 0
\]

for any \(\alpha \in \pi_{10}(\Omega P^5(2))\). Recall that \(\Omega[2]_* : \Omega P^5(2) \to \Omega P^5(2)\) is the composite

\[
\Omega P^5(2) \to \Omega S^5 \to \Omega S^4 \to \Omega P^5(2).
\]

Since \(\pi_{10}(\Omega S^4) = 0\), \(\Omega[2]_* : \pi_{10}(\Omega P^5(2)) \to \pi_{10}(\Omega P^5(2))\) is zero. Since

\[
\pi_{11}(P^5(2) \land P^4(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,
\]

we have \(2(\Omega W_{2} \circ H_{2})_* (\alpha) = 0\) for any \(\alpha \in \pi_{11}(P^5(2))\). Thus

\[
4\alpha = 0
\]

for all \(\alpha \in \pi_{11} P^5(2)\) and so \(\pi_{10}(Y_5) = \mathbb{Z}/4 \oplus \mathbb{Z}/2\). Hence

\[
\pi_{11}(P^5(2)) = \pi_{10}(Y_5) \oplus \mathbb{Z}/2 = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

6. \(\pi_{12}(P^5(2))\): Consider the exact sequence

\[
\pi_{12}(S^7) = 0 \to \pi_{12}(S^4) \to \pi_{11}(X^5) \to 0.
\]
We have $\pi_{12}(S^4) \cong \pi_{11}(X^5) \cong \pi_{11}(A_5)$. Now consider the commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
\pi_{13}(S^5) & \xrightarrow{\delta_*} & \pi_{11}(A_5) & \xrightarrow{j_*} & \pi_{11}(Y_5) & \xrightarrow{} & \pi_{12}(S^5) & \xrightarrow{} & 0 \\
& & \uparrow E^\infty & & \uparrow E^\infty & & \uparrow E^\infty & & \\
\pi_{13}^s(S^5) & \xrightarrow{2} & \pi_{12}^s(S^4) & \xrightarrow{j_*^s} & \pi_{12}^s(P^5(2)) & \xrightarrow{} & \pi_{12}^s(S^5), & & \\
\end{array}
$$

where

$E^\infty: \pi_{11}(A_5) \cong \pi_{12}(S^4) \rightarrow \pi_{13}^s(S^5)$ and $E^\infty: \pi_{12}(S^5) \rightarrow \pi_{12}^s(S^5)$

are monomorphisms, see [42, Proposition 5.15, Theorem 7.1]. Observe that

$E^\infty j_*(\epsilon_4) = j_*(\epsilon) \neq 0$

in $\pi_{12}^s(P^5(2)) = \mathbb{Z}/2^{\mathbb{N}}$. Thus there is a short exact sequence

$$0 \rightarrow \pi_{11}(A_5) \rightarrow \pi_{11}(Y_5) \rightarrow \pi_{12}(S^5) \rightarrow 0.$$

Since $E^\infty \pi_{11}(A_5)$ is a summand of $\pi_{12}^s(P^5(2))$, we have

$\pi_{11}(Y_5) = \pi_{11}(A_5) \oplus \pi_{12}(S^5)$

and $E^\infty: \pi_{12}(Y_5) \rightarrow \pi_{12}^s(P^5(2))$ is a monomorphism. Thus $\pi_{12}(P^5(2)) = \pi_{11}(Y_5) \oplus \pi_{12}(P^5(2)) = \mathbb{Z}/2^{\mathbb{N}}$.

7. $\pi_{13}(P^5(2))$: Consider the exact sequence

$$\pi_{13}(S^7) \xrightarrow{(2\omega_1)} \pi_{13}(S^4) \rightarrow \pi_{12}(X^5) \rightarrow \pi_{12}(S^7) = 0.$$

Since $2 \cdot \pi_{13}(S^7) = 0$, $(2\omega_2)_* : \pi_{13}(S^7) \rightarrow \pi_{13}(S^4)$ is zero and so

$\pi_{13}(S^4) \cong \pi_{12}(X^5) \cong \pi_{12}(A_5)$.

Consider the commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
\pi_{14}(S^5) & \xrightarrow{\delta_*} & \pi_{12}(A_5) & \xrightarrow{} & \pi_{12}(Y_5) & \xrightarrow{} & \pi_{12}(S^5) & \xrightarrow{} & 0 \\
& & & \uparrow & & \uparrow & & \uparrow & & \\
\pi_{14}^s(S^5) & \xrightarrow{2} & \pi_{13}^s(S^4) & \xrightarrow{\pi_{13}^s(P^5(2))} & \pi_{13}^s(S^5) & \xrightarrow{} & 0 \\
& & & & \uparrow & & \uparrow & & \\
& & & & \pi_{13}(S^4) & \xrightarrow{\pi_{13}^s(P^5(2))} & \pi_{13}^s(S^5) & \xrightarrow{} & 0 \\
\end{array}
$$

By Theorem [42, Theorem 7.2], $\pi_{13}(S^4) \cong \pi_{13}^s(S^4) \cong \mathbb{Z}/2^{\mathbb{N}}$ generated by $\nu_4^3$, $\mu_4$ and $\eta_4 \circ \epsilon_5$. There is a commutative diagram of short exact sequences

$$0 \rightarrow \pi_{13}(S^4) \rightarrow \pi_{12}(Y_5) \rightarrow \pi_{13}(S^5) \cong \mathbb{Z}/2 \rightarrow 0.$$
Thus \( \pi_{12}(Y_5) \to \pi_{13}^s(P^5(2)) \) is a monomorphism and
\[
\pi_{12}(Y_5) \cong \frac{1}{2} \pi_{13}(S^5) \oplus \pi_{13}(S^4)/ (\eta_4 \circ \epsilon_5) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]
Thus \( \pi_{13}(P^5(2)) = \pi_{12}(Y_5) \oplus \pi_{13}(P^{12}(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \). This completes the calculation. \(\square\)

5.4. The Homotopy Groups \( \pi_*(P^6(2)) \).

**Theorem 5.11.** The homotopy groups \( \pi_r(P^6(2)) \) for \( r \leq 14 \) are as follows.

1. \( \pi_r(P^6(2)) \cong \pi_r^s(P^6(2)) \) for \( r \leq 8 \).
2. \( \pi_9(P^6(2)) \cong \pi_8(P^5(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) maps onto \( \pi_8^s(P^6(2)) \).
3. \( \pi_{10}(P^6(2)) = \mathbb{Z}/8 \) maps onto \( \pi^s_8(P^6(2)) = 0 \).
4. \( \pi_{11}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) maps onto \( \pi_{11}^s(P^6(2)) = \mathbb{Z}/2 \).
5. \( \pi_{12}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) with the stable image \( \mathbb{Z}/2 \) in \( \pi_{12}^s(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).
6. \( \pi_{13}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) with the stable image \( \mathbb{Z}/2^{\oplus 2} \) in \( \pi_{13}^s(P^6(2)) = \mathbb{Z}/2^{\oplus 3} \).
7. \( \pi_{14}(P^6(2)) = \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4 \) with the stable image \( \mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/4 \) in \( \pi_{14}^s(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).

An application is as follows.

**Proposition 5.12.** Let \( X \) be a simply connected CW-complex. Suppose that the integral homotopy \( H_5(X; \mathbb{Z}) = \mathbb{Z}/2 \) and the mod 2 cohomology \( H^*(X) \) is an exterior algebra \( E(u, v) \) with \( |u| = 5 \) and \( |v| = 6 \). Then \( X \) is homotopy equivalent to \( \tau(S^6) \). In other words, the Stiefel manifold \( \tau(S^6) \) is uniquely determined, up to homotopy, by its fundamental group and its cohomology ring.

**Proof.** By the assumption, \( X = P^6(2) \cup_f e^{11} \). Since \( H^*(X) \cong E(u, v) \), the homology \( H_*(\Omega X) \) is isomorphic to the polynomial algebra \( S(u, v) \). Let \( f^*: S^9 \to \Omega P^6(2) \) be the adjoint of the attaching map \( f: S^{10} \to P^6(2) \). Then \( f_*(\alpha) = [u_4, v_5] \) and the homotopy class \([f]\) is not divisible by 2 in \( \pi_{10}(P^6(2)) \). Let \( \lambda_6: S^{10} \to P^6(2) \) be the attaching map for \( \tau(S^9) \). By assertion 3, \( f \simeq k\lambda_6 \) for \( k = \pm 1 \) or \( \pm 3 \). Clearly \( X \simeq \tau(S^6) \) if \( k = \pm 1 \). By Lemma 2.18, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^{10} & \xrightarrow{\lambda_6} & P^6(2) \\
\downarrow[-3] & & \downarrow[-1] \\
S^{10} & \xrightarrow{\lambda_6} & P^6(2).
\end{array}
\]

Thus \( X \simeq \tau(P^6(2)) \) if \( k = -3 \). If \( k = 3 \), then \( X \simeq P^6(2) \cup_f e^{11} \simeq \tau(S^6) \) and hence the result. \(\square\)

**Proof of Theorem 5.11.**

1. \( \pi_r(P^6(2)) \) for \( r \leq 9 \): Since \( S^5 \) is the 9-skeleton of \( F^6\{2\} \), assertion (1) follows. Assertion (2) was proved in the Step 2 of Theorem 5.10.

2. \( \pi_{10}(P^6(2)) \): By Lemma 4.17, \( A_6 \simeq \Omega S^5 \times B_6 \). Since \( S^9 \) is the 16-skeleton of \( B_6 \), \( \pi_r(\Omega S^5 \times S^9) \cong \pi_r(A_6) \) for \( r \leq 15 \). Observe that \( \Sigma L_3(P^5(2)) = P^{15}(2) \). The
homotopy groups \( \pi_r(P^6(2)) \cong \pi_{r-1}(Y_6) \) for \( r \leq 13 \) and \( \pi_{14}(P^6(2)) = \pi_{13}(Y_4) \oplus \mathbb{Z}/2 \).

Now consider the commutative diagram of exact sequences

\[
\begin{array}{cccccc}
\pi_{11}(S^6) & \xrightarrow{\delta_*} & \pi_{10}(F^6\{2\}) & \xrightarrow{H} & \pi_{10}(P^6(2)) & \xrightarrow{H_2^*} & \pi_{10}(S^6) = 0 \\
\pi_{11}(S^{11}) & \xrightarrow{2} & \pi_{10}(F^{11}\{2\}) & \xrightarrow{H_2^*} & \pi_{10}(P^{11}(2)) & \xrightarrow{H} & \pi_{10}(S^{11}) = 0.
\end{array}
\]

Since \( H : \pi_{11}(S^6) \to \pi_{11}(S^{11}) \) is of degree 2, the composite

\[
\pi_{11}(S^6) \xrightarrow{\delta_*} \pi_{10}(F^6\{2\}) \cong \pi_9(\Omega S^5 \times S^9) \xrightarrow{\text{proj}} \pi_9(S^9)
\]

is of degree 4. Consider the exact sequence

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_*} \mathbb{Z}/2 \oplus \mathbb{Z} \longrightarrow \pi_{10}(P^6(2)) \longrightarrow 0,
\]

where \( \delta_*(\omega_6) = k\nu_5 \circ \eta^2 + 4\iota_9 \) for some \( k = 0, 1 \). By [12, Theorem 2.2], there is a \( \mathbb{Z}/8 \)-summand in \( \pi_{10}(P^6(2)) \). Thus \( k = 1 \) and \( \pi_{10}(P^6(2)) = \mathbb{Z}/8 \).

3. \( \pi_{11}(P^6(2)) \): Consider the commutative diagram

\[
\begin{array}{cccccc}
\pi_{12}(S^6) & \xrightarrow{} & \pi_{11}(F^6\{2\}) & \xrightarrow{} & \pi_{11}(P^6(2)) & \xrightarrow{} & 0 \\
\pi_{11}(S^5) & \xrightarrow{[2]^*} & \pi_{11}(S^5)
\end{array}
\]

Recall that \( \Omega[2] \cong 2 + \Omega \omega_5 \circ H_2 : \Omega S^5 \to \Omega S^5 \). Thus \( [2]^*(\nu_5^2) = 2\nu_5^2 = 0 \) and

\[
\pi_{11}(P^6(2)) \cong \pi_{11}(F^6\{2\}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

Since \( \pi_{11}(S^5) \cong \pi_{11}^+(S^5) \), the map \( \pi_{11}(P^6(2)) \to \pi_{11}^+(P^6(2)) \) is onto.

4. \( \pi_{12}(P^6(2)) \): Consider the commutative diagram

\[
\begin{array}{cccccc}
\mathbb{Z}/4 = \pi_{13}(S^6) & \xrightarrow{\delta_*} & \pi_{12}(F^6\{2\}) & = \pi_{12}(S^5) \oplus \pi_{11}(S^9) & \xrightarrow{} & \pi_{12}(P^6(2)) & \xrightarrow{} & \pi_{12}(S^6) & \xrightarrow{} & 0 \\
\pi_{12}^+ S^5 = \mathbb{Z}/16 & \xrightarrow{2} & \pi_{12}^+ S^5 = \mathbb{Z}/16 & \xrightarrow{} & \pi_{12}^+ (P^6(2)) & \xrightarrow{} & \pi_{12}^+ (S^6) & \xrightarrow{} & 0
\end{array}
\]

where \( \pi_{13}(S^6) \twoheadrightarrow \pi_{12}^+(S^5) \) is a monomorphism by [42, Proposition 5.15]. Thus

\[
E^\infty \circ \delta_*(\sigma''') = 8\sigma \neq 0
\]

and so there is a short exact sequence

\[
0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{12}(P^6(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.
\]
Consider the commutative diagram of exact sequences
\[
\begin{array}{ccccccccc}
\mathbb{Z}/2 & \rightarrow & \pi_{12}(F^6\{2\}) & \rightarrow & \pi_{12}(P^6(2)) & \rightarrow & \pi_{12}(S^6) & \rightarrow & 0 \\
\downarrow H_{2*} & & \downarrow H_{2*} & & \downarrow H & & \downarrow \text{null homotopic} & & \\
0 & \rightarrow & \pi_{12}(F^{11}\{2\}) & \rightarrow & \pi_{12}(P^{11}(2)) & \rightarrow & \pi_{12}(S^{11}) & \rightarrow & 0.
\end{array}
\]
We have \(\pi_{12}(P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2\). By [12, Lemma 2.1], the composite
\[S^9 \xrightarrow{\lambda_6} \Omega P^6(2) \rightarrow Q(P^6(2))\]
is null homotopic and so there is an exact sequence
\[0 \rightarrow \mathbb{Z}/2 \rightarrow \pi_{12}(P^6(2)) \xrightarrow{E^\infty} \pi_{12}(P^6(2)) \rightarrow \mathbb{Z}/2 \rightarrow 0.\]
5. \(\pi_{13}(P^6(2))\): Consider the commutative diagram
\[
\begin{array}{ccccccccc}
\mathbb{Z}/2 = \pi_{13}(S^5) & \xrightarrow{[2]_*} & \pi_{13}(S^5) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\mathbb{Z}/2 \oplus \mathbb{Z}/8 = \pi_{14}(S^6) & \xrightarrow{\delta_*} & \pi_{13}(F^6\{2\}) & \rightarrow & \pi_{13}(P^6(2)) & \rightarrow & 2 \cdot \pi_{13}(S^6) & \rightarrow & 0 \\
\downarrow & & \downarrow H_{2*} & & \downarrow & & \downarrow & & \\
\mathbb{Z}/8 = \pi_{14}(S^{11}) & \xrightarrow{2} & \mathbb{Z}/8 = \pi_{13}(F^{11}\{2\}) & \rightarrow & \pi_{13}(P^{11}(2)) & \rightarrow & \pi_{13}(S^{11}) & \rightarrow & 0,
\end{array}
\]
where \(\pi_{14}(S^6) \rightarrow \pi_{14}(S^{11})\) and \(\pi_{13}(F^6\{2\}) \rightarrow \pi_{13}(F^{11}\{2\})\) are epimorphisms and \(E: \pi_{13}(S^5) \rightarrow \pi_{13}(S^6)\) is a monomorphism. Observe that
\[\pi_{13}(F^6\{2\}) \cong \pi_{13}(\Omega S^5 \times S^9) = \mathbb{Z}/2 \oplus \mathbb{Z}/8\]
and \([2]_* (\epsilon_5) = 2\epsilon_5 + (\Omega \omega_5 \circ H_2)_* (\epsilon_5) = 0\).
Thus \(\delta_* (\tilde{\nu}_6) = 2\lambda_6 \circ \nu_9 + k\epsilon_5\) for some \(k = 0, 1\) and \(\delta(\epsilon_6) = 0\). Consider the commutative diagram
\[
\begin{array}{ccccccccc}
\pi_{14}(S^6) & \xrightarrow{\delta_*} & \pi_{13}(F^6\{2\}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\pi_{13}(S^5) & \xrightarrow{2} & \pi_{13}(S^5) & & & & & & \\
\downarrow E^\infty & & \downarrow & & & & & & \\
\pi_{13}(S^5) & \xrightarrow{E^\infty} & \pi_{13}(S^5) & & & & & & \\
\end{array}
\]
Since \(E^\infty(\lambda_6) = 0\), we have \(E^\infty(\lambda_6 \circ \nu_9) = 0\) and so
\[E^\infty \circ \delta_* (\tilde{\nu}_6) = E^\infty (k\epsilon_5) = k\epsilon = 2E^\infty (\nu_6) = 2\tilde{\nu} = 0\]
Thus \(k = 0\) and \(\delta_* (\tilde{\nu}_6) = 2\lambda_6 \circ \nu_9\) with a short exact sequence
\[0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \pi_{13}(P^6(2)) \rightarrow 2 \cdot \pi_{13}S^6 \rightarrow 0.\]
Now consider the commutative diagram
\[
\begin{array}{ccc}
\pi_{13}(P^6(2)) & \twoheadrightarrow & 2 \cdot \pi_{13}S^6 \\
\downarrow & & \downarrow \cong \\
\pi_{12}(P^5(2)) & \twoheadrightarrow & \pi_{12}(S^5)
\end{array}
\]

By Step 6 of Theorem 5.10, \(\pi_{12}(P^5(2)) \to \pi_{12}S^5\) is split onto and so
\[
\pi_{13}(P^6(2)) \cong 2 \cdot \pi_{13}(S^6) \oplus \pi_{13}(S^5) \oplus \pi_{12}(S^5)/2 \cong \mathbb{Z}/2^{\oplus 3}
\]
with an exact sequence
\[
0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{13}(P^6(2)) \longrightarrow \pi_{13}(P^6(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.
\]

6. \(\pi_{14}(P^6(2))\): Consider the exact sequence
\[
\pi_{15}(S^6) \xrightarrow{\delta_*} \pi_{13}(A_6) \longrightarrow \pi_{13}(Y_6) \overset{p_*}{\longrightarrow} \pi_{14}(S^6) \xrightarrow{\delta_*} \pi_{13}(P^6\{2\}),
\]
where
\[
\text{Ker}(\delta_* : \pi_{14}(S^6) \to \pi_{13}(P^6\{2\})) = \{4\nu_6\} \oplus \{\epsilon_5\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]
Since \(\pi_{13}(S^9) = 0\), we have \(\pi_{13}(A_6) \cong \pi_{14}(S^5)\). Consider the commutative diagram
\[
\begin{array}{ccc}
\pi_{14}(S^5) & = \mathbb{Z}/2^{\oplus 3} & \xrightarrow{j_*} \pi_{13}(Y_6) \overset{p_*}{\longrightarrow} \{4\nu_6\} \oplus \{\epsilon_6\} \longrightarrow 0 \\
\cong & E^{\infty} & \overset{E^{\infty}}{\longrightarrow} \pi_{14}(P^6\{2\}) \longrightarrow \pi_{14}(S^6) \longrightarrow 0.
\end{array}
\]
Thus \(\pi_{14}(S^5) \longrightarrow \pi_{13}(Y_6)\) is a monomorphism. Let \(\alpha_1 \in \pi_{13}(Y_6)\) so that \(p_*(\alpha_1) = \epsilon_6\). Then \(E^{\infty}(2\alpha_1) = \eta \circ \epsilon_6 = j_* \circ E^{\infty}(\eta_5 \circ \epsilon_6) \neq 0\) and so \(\alpha_1\) is of order 4. Now let \(\alpha_2 \in \pi_{13}(Y_6)\) so that \(p_*(\alpha_2) = 4\nu_6\). Then there exists \(\beta \in \pi_{14}(S^5)\) so that \(j_*(\beta) = 2\alpha_2\). Since \(p_* \circ E^{\infty}(\alpha_2) = E^{\infty}4\nu_6 = 0\), we have \(E^{\infty}(\alpha_2) \in \pi_{14}^{1}(S^5)\) and so
\[
2E^{\infty}(\alpha_2) = E^{\infty}(2\alpha_2) = E^{\infty}j_*(\beta) = j_*E^{\infty}(\beta) = 0.
\]
Thus \(\beta = 0\) or \(2\alpha_2 = 0\). Hence
\[
\pi_{13}(Y_6) \cong \{4\nu_6\} \oplus 1/2\{\epsilon_5\} \oplus \pi_{14}(S^5)/\{\eta_5 \circ \epsilon_6\} \cong \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4 \quad \text{and}
\]
\[
\pi_{14}(P^6(2)) \cong \pi_{13}(Y_6) \oplus \pi_{14}(P^{15}(2)) \cong \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4
\]
with an exact sequence
\[
0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \pi_{14}(P^6(2)) \longrightarrow \pi_{14}(P^6(2)) \longrightarrow 0.
\]
We finish the proof. \(\square\)
5.5. The Homotopy Groups $\pi_*(P^7(2))$.

**Theorem 5.13.** The homotopy groups $\pi_*(P^7(2))$ for $r \leq 15$ are as follows.

1. $\pi_r(P^7(2)) \cong \pi_r^sP^7(2)$ for $r \leq 10$.
2. $\pi_{11}(P^7(2)) = \mathbb{Z}/2$.
3. $\pi_{12}(P^7(2)) = \mathbb{Z}/2$.
4. $\pi_{13}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
5. $\pi_{14}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
6. $\pi_{15}(P^7(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$.

**Proof.**

1. $\pi_r(P^7(2))$ for $r \leq 10$: Since $S^6$ is the 11-skeleton of $F^7\{2\}$, assertion (1) follows.

2. $\pi_{11}(P^7(2))$: By Proposition 4.15, the 17-skeleton $sk_{17}(F^7\{2\})$ of $F^7\{2\}$ is the homotopy cofibre of $2\omega_6: S^{11} \to S^6$. Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccccc}
\Omega F^7\{2\} & \longrightarrow & \Omega F^7\{2\} & \longrightarrow & F^7\{2\} & \\
\downarrow \Omega j & & \downarrow \phi & & \downarrow j & \\
\Omega S^6 & \longrightarrow & X^7 & \longrightarrow & S^{11} & \longrightarrow & 2\omega_6 & \longrightarrow & S^6.
\end{array}
\]

Let $\psi$ be the composite

\[
X^7 \times \Omega \Sigma L_3(P^6(2)) \xrightarrow{\phi \times J(\beta_1)} \Omega F^7\{2\} \times \Omega F^7\{2\} \xrightarrow{\text{multi.}} \Omega F^7\{2\}.
\]

Then

\[
\psi_* : H_r(X^7 \times \Omega \Sigma L_3(P^6(2)) \to H_r(\Omega F^7\{2\})
\]

is an isomorphism for $r \leq 19$. Since $\Sigma L_3(P^6(2)) = P^{18}(2)$, the map

\[
\psi_* : \pi_r(X^7) \longrightarrow \pi_r(\Omega F^7\{2\})
\]

is an isomorphism for $r \leq 15$. By expecting the exact sequence

\[
\pi_{11}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{11}(S^6) \longrightarrow \pi_{10}(X^7) \longrightarrow 0,
\]

we have $\pi_{10}(X^7) = \mathbb{Z}/2$. From the exact sequence

\[
\pi_{12}(S^7) = 0 \longrightarrow \pi_{10}(X^7) \longrightarrow \pi_{11}(P^7(2)) \longrightarrow \pi_{11}(S^7) = 0,
\]

we have $\pi_{11}(P^7(2)) \cong \pi_{10}(X^7) = \mathbb{Z}/2$.

3. $\pi_{12}(P^7(2))$: Consider the exact sequence

\[
\pi_{12}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{12}(S^6) \longrightarrow \pi_{11}(X^7) \longrightarrow 0.
\]

Since $\pi_{12}(S^{11}) = \mathbb{Z}/2$, $(2\omega_6)_* : \pi_{12}(S^{11}) \to \pi_{12}(S^6)$ is zero and so $\pi_{12}(S^6) \cong \pi_{11}(X^7)$.

Now consider the commutative diagram

\[
\begin{array}{cccccc}
\pi_{13}(S^7) & \delta_* & \pi_{11}(X^7) & \longrightarrow & \pi_{12}(P^7(2)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \cong & & \\
\pi_{12}(S^6) & \xrightarrow{[2]_*} & \pi_{12}(S^6).
\end{array}
\]
Recall that $\Omega[2] = 2 + (\Omega_6 \circ H_2)$, and $(H_2)_*(\nu^7_2) = 0$. Thus $[2]_*(\nu^7_6) = 2\nu^7_6 = 0$ and so $\delta_*(\nu^7_2) = 0$. Hence $\pi_{12}(P^7(2)) \cong \pi_{11}(X^7) \cong \mathbb{Z}/2$.

4. $\pi_{13}(P^7(2))$: Consider the exact sequence

$$
\pi_{13}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{13}(S^6) \xrightarrow{j_*} \pi_{12}(X^7) \xrightarrow{} \pi_{12}(S^{11}) \xrightarrow{} 0.
$$

Since $(2\omega_6)_* : \pi_{13}(S^{11}) \to \pi_{13}(S^6)$ is zero, there is a short exact sequence

$$
0 \to \pi_{13}(S^6) \to \pi_{12}(X^7) \to \pi_{12}(S^{11}) \to 0.
$$

From the commutative diagram of exact sequences

$$
\begin{array}{ccc}
\pi_{13}(\Omega S^7) & \xrightarrow{\Omega[2]_*} & \pi_{13}(\Omega S^7) \\
\downarrow & & \downarrow \\
\pi_{13}(\Omega S^7) & \xrightarrow{\delta_*} & \pi_{12}(X^7) \\
\downarrow & & \downarrow \\
\pi_{13}(\Omega S^7) & \xrightarrow{\delta'_*} & \pi_{13}(P^7(2)) \\
\downarrow & & \downarrow \\
\pi_{14}(S^{13}) & \xrightarrow{\delta'_*} & \pi_{13}(F^{13}\{2\}) \cong \pi_{13}(S^{12}) \to \pi_{13}(P^{13}(2)) \to \pi_{13}(S^{13})
\end{array}
$$

we have $f_{1*} \circ \delta_*(\sigma') = 2\sigma'$. Consider the commutative diagram of exact sequences

$$
\begin{array}{ccc}
\pi_{14}(S^7) & \xrightarrow{\delta_*} & \pi_{12}(X^7) \\
\downarrow & & \downarrow \\
\pi_{13}(P^7(2)) \to \pi_{13}(S^7)
\end{array}
$$

Since $\delta'_* : \pi_{14}(S^{13}) \to \pi_{13}(F^{13}\{2\})$ is zero, the composite

$$
\pi_{14}(S^7) \xrightarrow{\delta_*} \pi_{12}(X^7) \xrightarrow{} \pi_{12}(S^{11})
$$

is zero and so $\delta_*(\sigma') = j_*(k\sigma'')$ for some integer $k$. Now

$$
2\sigma' = (f_{1*} \circ \delta_*(\sigma') = (f_{1*} \circ j_*(k\sigma'') = kE(\sigma'') = 2k\sigma'.
$$

Thus $k \equiv 1(4)$ and so $\delta_*(\sigma') = \pm j_*(\sigma'')$. Hence there is a short exact sequence

$$
0 \to \mathbb{Z}/2 \to \pi_{13}(P^7(2)) \to \pi_{13}(S^7) \to 0.
$$

Consider the commutative diagram

$$
\begin{array}{ccc}
\pi_{13}(P^7(2)) & \xrightarrow{} & \pi_{13}(S^7) \\
\downarrow & & \downarrow \\
\pi_{12}(P^6(2)) & \to & \pi_{12}(S^6)
\end{array}
$$

By Step 4 of Theorem 5.11, we have $\pi_{12}(P^6(2)) \cong \pi_{12}(S^6) \oplus \mathbb{Z}/2$ and so

$$
\pi_{13}(P^7(2)) \cong \pi_{12}(X^7)/(\pi_{13}(S^6)) \oplus \pi_{13}(S^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.
$$

5. $\pi_{14}(P^7(2))$: Consider the exact sequence

$$
\pi_{14}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{14}(S^6) \xrightarrow{} \pi_{13}(X^7) \xrightarrow{} \pi_{13}(S^{11}) \xrightarrow{} 0.
$$
By [42, Lemma 6.2], $P(\nu_{13}) = \pm 2\nu_6$ in the EHP sequence for $S^6 \xrightarrow{E} \Omega S^7 \xrightarrow{H} \Omega S^{13}$. Thus $(2\omega_6)_*(\nu_{11}) = \pm 4\nu_6$ and so there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/4 \longrightarrow \pi_{13}(X^7) \xrightarrow{p_*} \pi_{13}(S^{11}) \longrightarrow 0.$$ 

Consider the commutative diagram

$$\begin{array}{ccc}
\pi_{14}(S^6) & \xrightarrow{j_2} & \pi_{13}(X^7) \\
\downarrow{E} & & \downarrow{f_1}\pi_{15}(S^7) \\
\pi_{15}(S^7) & \xrightarrow{p_*} & \pi_{13}(S^{11}) \longrightarrow 0
\end{array}$$

Let $\alpha \in \pi_{13}(X^7)$ so that $p_*(\alpha) = \eta^2$. Then $2\alpha = j_*(k\nu_6 + l\epsilon_6)$ for some integers $k, l$. Then

$$E(k\nu_6 + l\epsilon_6) = (f_1)_* \circ j_*(k\nu_6 + l\epsilon_6) = (f_1)_*(2\alpha) = 0 = k\nu_7 + l\epsilon_7.$$ 

It follows that $l = 0$ and $k \equiv 0(2)$. Now $p_*(\alpha - j_*(k/2\nu_6)) = \eta^2$ and $2(\alpha - j_*(k/2\nu_6)) = 0$. Thus

$$\pi_{13}(X^7) \cong \pi_{13}(S^{11}) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$ 

Now consider the exact sequence

$$\pi_{15}(S^7) \xrightarrow{\delta_*} \pi_{13}(X^7) \longrightarrow \pi_{14}(P^7(2)) \longrightarrow \pi_{14}(S^7) \xrightarrow{\delta_*} \pi_{12}(X^7).$$

By Step 4, $\delta_*(\sigma') = \pm j_*\sigma''$ and so Ker$(\delta_* : \pi_{14}(S^7) \rightarrow \pi_{12}(X^7)) = 4 \cdot \pi_{14}(S^7)$. By [42, pp. 64 and Theorem 7.1], we have

$$\pi_{15}(S^7) = \{\sigma' \circ \eta_{14}\} \oplus \{\nu_7\} \oplus \{\epsilon_7\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{and} \quad \sigma'' \circ \eta_{13} = 4\nu_6.$$ 

Hence

$$\delta_*(\sigma' \circ \eta_{14}) = j_*(\sigma'' \circ \eta_{13}) = j_*(4\nu_6) = 0$$

in $\pi_{13}(X^7)$. Now

$$\delta_*(\nu_7) = \delta_*E(\nu_6) = j_* [2]_*(\nu_6) = j_* (2\nu_6 + (\Omega\omega_6 \circ H_2)_*(\nu_6))$$

$$= j_* (2\nu_6 + (\Omega\omega_6)_*(\nu_{11})) = j_* (2\nu_6 + P(\nu_{13})) = j_*(4\nu_6) = 0.$$ 

and

$$\delta_*(\epsilon_7) = j_* [2]_*(\epsilon_6) = j_* (2\epsilon_6 + (\Omega\omega_6 \circ H_2)_*(\epsilon_6)) = 0$$

This shows that the boundary map $\delta_* : \pi_{15}(S^7) \longrightarrow \pi_{13}(X^7)$ is zero and so there is a short exact sequence

$$0 \longrightarrow \pi_{13}(X^7) \longrightarrow \pi_{14}(P^7(2)) \longrightarrow 4 \cdot \pi_{14}(S^7) \longrightarrow 0.$$
Consider the commutative diagram

\[
\begin{array}{cccc}
\pi_{14}(P^7(2)) & \rightarrow & 4 \cdot \pi_{14}(S^7) & \rightarrow & 0 \\
\downarrow E & & \downarrow \cong & & \downarrow E \\
\pi_{13}(P^6(2)) & \rightarrow & 2 \cdot \pi_{13}(S^6) & \rightarrow & 0 \\
\downarrow \cong & & \downarrow E & & \\
\pi_{12}(P^5(2)) & \rightarrow & \pi_{12}(S^5) & \rightarrow & 0
\end{array}
\]

By Step 6 of Theorem 5.10, \(\pi_{12}P^5(2) \rightarrow \pi_{12}S^5\) is split onto. Thus

\[
\pi_{14}(P^7(2)) \rightarrow 4 \cdot \pi_{14}(S^7)
\]

is split onto and so

\[
\pi_{14}(P^7(2)) \cong 4 \cdot \pi_{14}(S^7) \oplus \pi_{13}(X^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4.
\]

6. \(\pi_{15}(P^7(2))\): Consider the exact sequence

\[
\pi_{15}(S^{11}) = 0 \xrightarrow{(2\omega_6)_*} \pi_{15}(S^6) \rightarrow \pi_{14}(X^7) \rightarrow \pi_{14}(S^{11}) \xrightarrow{(2\omega_6)_*} \pi_{14}(S^6),
\]

where \(\text{Ker}((2\omega_6)_* : \pi_{14}(S^{11}) \rightarrow \pi_{14}(S^6)) = 2 \cdot \pi_{14}(S^{11})\). There is a short exact sequence

\[
0 \rightarrow \pi_{15}(S^6) \rightarrow \pi_{14}(X^7) \rightarrow 2 \cdot \pi_{14}(S^{11}) \rightarrow 0.
\]

Now consider the commutative diagram

\[
\begin{array}{cccc}
\pi_{15}(S^6) & \rightarrow & \pi_{14}(X^7) & \rightarrow & \pi_{15}(S^{11}) \\
\downarrow E & & \downarrow E & & \downarrow E \\
\pi_{16}(S^7) & \rightarrow & \pi_{16}(S^7) & \rightarrow & \pi_{16}(S^7)
\end{array}
\]

By [42, Theorem 7.2], we have \(\pi_{16}(S^7) \cong \pi_{15}(S^6) \oplus \mathbb{Z}/2\) and so \(\pi_{15}(S^6)\) is a summand of \(\pi_{14}(X^7)\). Thus

\[
\pi_{14}(X^7) \cong \pi_{15}(S^6) \oplus 2 \cdot \pi_{14}(S^{11}).
\]

Consider the exact sequence

\[
\pi_{16}(S^7) \xrightarrow{\delta_*} \pi_{14}(X^7) \rightarrow \pi_{15}(P^7(2)) \rightarrow \pi_{15}(S^7) \rightarrow 0.
\]

According to [42, Theorem 7.2],

\[
\pi_{16}S^7 = \{\sigma' \circ \eta_{14}^2\} \oplus \{\nu_2^3\} \oplus \{\mu_7\} \oplus \{\eta_7 \circ \epsilon_8\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

Observe that

\[
\begin{align*}
\delta_*(\sigma' \circ \eta_{14}^2) = \delta_*(\sigma' \circ \eta_{14}) \circ \eta_{15} = 0 \\
\delta_*(\nu_2^3) = \delta_*E(\nu_0^3) = j_*[2]_*(\nu_0^3) = j_*[2]_(2\nu_6^3 + (\Omega\omega_6 \circ H_2)_*(\nu_0^3)) = 0 \\
\delta_*(\mu_7) = \delta_*E\mu_6 = j_*[2]_*(\mu_6) = j_*[2]_(2\mu_6 + (\Omega\omega_6 \circ H_2)_*(\mu_6)) = 0
\end{align*}
\]
\[ \delta_*(\eta_7 \circ \epsilon_8) = \delta_*E(\eta_6 \circ \epsilon_7) = j_*[2]_*\eta_6 \circ \epsilon_7 = j_*[2\eta_6 \circ \epsilon_7 + (\Omega \omega_6 \circ H_2)*\eta_6 \circ \epsilon_7] = 0. \]

Thus \( \delta_* : \pi_{16}(S^7) \longrightarrow \pi_{14}(X^7) \) is zero and so there is a short exact sequence
\[ 0 \longrightarrow \pi_{14}(X^7) \longrightarrow \pi_{15}(P^7(2)) \longrightarrow \pi_{15}(S^7) \longrightarrow 0. \]

Consider the commutative diagram of exact sequences
\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_{14}(X^7) \\
\downarrow & & \downarrow \\
E^{\infty} & \longrightarrow & \pi_{15}(P^7(2)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_{15}^s(S^6) \\
\end{array}
\]
\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_{15}^s(P^7(2)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_{15}^s(S^7) \\
\end{array}
\]
where \( E^{\infty} : \pi_{14}(X^7) \longrightarrow \pi_{15}^s(S^6) \) and \( E^{\infty} : \pi_{15}(S^7) \longrightarrow \pi_{15}^s(S^7) \) are epimorphisms. It follows that
\[ E^{\infty} : \pi_{15}(P^7(2)) \longrightarrow \pi_{15}^s(P^7(2)) \]
is an epimorphism and so there is a commutative diagram of short exact sequences
\[
\begin{array}{ccc}
\mathbb{Z}/4 & \longrightarrow & G \\
\downarrow & & \downarrow p_* \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 & \longrightarrow & \pi_{15}(P^7(2)) \\
\downarrow & & \downarrow p_* \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\end{array}
\]
where the rows and columns are short exact sequence. To solve the group extension problem, we need to compute \( \pi_{15}(P^7(2) \wedge P^6(2)) \). According to [27, Lemma 1.6(ii), p.518], \( \pi_{2n+3}(\Sigma P^n(2) \wedge P^n(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) for \( n \geq 4 \). In particular,
\[ \pi_{15}(P^7(2) \wedge P^6(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2. \]

Now let \( \alpha \in G \) so that \( p_*(\alpha) = \sigma' \circ \eta_{14} \) in \( \pi_{16}(S^7) \). From the commutative diagram
\[
\begin{array}{ccc}
\pi_{15}(P^7(2)) & \longrightarrow & \pi_{15}(S^7) \\
\downarrow H_{2*} & & \downarrow H \\
\pi_{15}(P^7(2) \wedge P^6(2)) & \longrightarrow & \pi_{15}(S^{13}) \\
\end{array}
\]
we have
\[ p_* \circ H_{2*}(\alpha) = H \circ p_*(\alpha) = H(\sigma' \circ \eta_{14}) = \eta_{13}^2. \]

Observe that there is a short exact sequence
\[ 0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \pi_{15}(P^{12}(2)) \oplus \pi_{15}(S^{12})/2 \longrightarrow \pi_{15}(P^7(2) \wedge P^6(2)) \longrightarrow \pi_{15}(S^{13}) \longrightarrow 0, \]
Thus $2H_{2n}(\alpha) \neq 0$ and so $2\alpha \neq 0$. It follows that the short exact sequence

$$0 \rightarrow \mathbb{Z}/4 \rightarrow G \rightarrow \mathbb{Z}/2 \rightarrow 0$$

does not split and so $G = \mathbb{Z}/8$. By Barratt’s exponent theorem [2, 7], we have $8\pi_{15}(P^7(2)) = 0$ and so

$$\pi_{15}(P^7(2)) = \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$ 

This completes the calculation.

**Note.** It was known in [12] that there is a family of $\mathbb{Z}/8$-summands in $\pi_*(P^n(2))$ for $n \geq 3$ by expecting certain spherical classes in $H_*(\Omega P^n(2))$. The $\mathbb{Z}/8$-summand of $\pi_{15}(P^7(2)) = \pi_{14}(\Omega P^7(2))$ does not come from a spherical class because $H_{14}(\Omega P^7(2)) = 0$. It might be interesting to know whether this summand is detected by other homology theory. According to [27, Proposition 4.2], a generator of $\mathbb{Z}/8$-summand of $\pi_{15}(P^7(2))$ is given as $\tilde{\sigma}$, where $\tilde{\sigma}$ is a generator for $\pi_7(P^7(2)) = \mathbb{Z}/4$. According to the referee’s comments, the other generators for $\pi_{15}(P^7(2))$ are the following: $[\tilde{r}_7]$, $[\epsilon_7]$ and $i_7\nu_6$, where $[\alpha]$ stands for a lifting of $\alpha$ and $[\epsilon_7] = \tilde{\epsilon}_6$ (a coextension of $\epsilon_6$).

5.6. The Homotopy Groups $\pi_*(P^8(2))$. Consider the homotopy commutative diagram of fibre sequences

$$
\begin{array}{ccc}
S^7 & \longrightarrow & S^{15} \\
\downarrow i & & \downarrow q \\
S^7 & \longrightarrow & X^8 \\
\downarrow p & & \downarrow f \\
& & F^8\{2\} \\
& & \longrightarrow F^8\{2\}
\end{array}
$$

where $S^7 \rightarrow S^{15} \rightarrow S^8$ is the Hopf fibration.

**Lemma 5.14.** The space $X^8$ is the homotopy cofibre of $S^{14} \xrightarrow{(\sigma',[2])} S^7 \vee S^{14}$ and therefore $X^8$ is the 20-skeleton of $\Omega^2 S^9$.

**Proof.** By Lemma 2.38, $\tilde{H}_*(X^8)$ has a basis $\{x_7, x_{14}x_{15}\}$ with $Sq^1 x_{15} = x_7$. Thus $X^8$ is the homotopy cofibre of $\delta|_{S^{14}} : S^{14} \longrightarrow \text{sk}_{14}(F^8\{2\})$, where $\delta : \Omega S^{15} \rightarrow F^8\{2\}$ is the boundary. Since $\text{sk}_{14}(F^8\{2\}) \simeq S^7 \vee S^{14}$, there is a cofibre sequence

$$S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{f} X^8,$$

with the composite $S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{\text{proj.}} S^{14}$ of degree 2. Let $\phi_1$ be the composite $S^{14} \xrightarrow{\psi} S^7 \vee S^{14} \xrightarrow{\text{proj.}} S^7$. Then $\phi_1 \simeq k\sigma'$ for some $k$. It suffices to show that $k \neq 0$ mod 2. Suppose that $k$ were even. Let $\lambda_8 : S^{14} \rightarrow F^8\{2\}$ be the composite

$$S^{14} \xrightarrow{\rho} S^7 \vee S^{14} \simeq \text{sk}_{14}(F^8\{2\}) \xrightarrow{\rho} F^8\{2\}.$$
Then $\bar{\lambda}_8(\iota_{13}) = [u, v]$ in $H_*\Omega F^8\{2\}$ and therefore in $H_*\Omega P^8(2)$, where $\bar{\lambda}'$ is the adjoint map of $\lambda_8$. By [12, Theorem 2.2], the homotopy class
\[ 2[q \circ f \circ \bar{\lambda}_8] \neq 0 \quad \text{and} \quad 4[q \circ f \circ \bar{\lambda}_8] = 0 \]
in $\pi_{14}(P^8(2))$. Since $i: S^7 \to X^8$ is of degree 2 on the bottom cell, we have
\[ 0 = q_! \circ i_!(\frac{k}{2}\sigma') = q_! \circ f_!(k\sigma') = q_!(2f_!(\bar{\lambda}_8)) = -2[q \circ f \circ \bar{\lambda}] \neq 0 \]
This is a contradiction and hence the result. \hfill \square

**Corollary 5.15.** There is an exact sequence
\[ \pi_r(S^7) \xrightarrow{2E^2} \pi_r(\Omega^2 S^9) \to \pi_r(P^8(2)) \to \pi_{r-1}(S^7) \xrightarrow{2E^2} \pi_{r-1}(\Omega^2 S^9) \]
for $r \leq 19$.

**Theorem 5.16.** The homotopy groups $\pi_r(P^8(2))$ for $r \leq 16$ are as follows.
1. $\pi_r(P^8(2)) \cong \pi_r^*(P^8(2))$ for $r \leq 13$.
2. $\pi_{14}(P^8(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$.
3. $\pi_{15}(P^8(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.
4. $\pi_{16}(P^8(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

**Proof.**
1. $\pi_r(P^8(2))$ for $r \leq 13$: Note that $F^8\{2\} \simeq S^7 \vee S^{14}$ and $\pi_r(S^7) \cong \pi_r^*(S^7)$ for $r \leq 13$. The assertion (1) follows.
2. $\pi_{14}(P^8(2))$: By Corollary 5.15, there is a short exact sequence
\[ 0 \to \mathbb{Z}/4 = \pi_{14}(\Omega^2 S^9)/4 \to \pi_{14}(P^8(2)) \to \pi_{13}(S^7) = \mathbb{Z}/2 \to 0. \]
By [12, Theorem 2.2], there is a $\mathbb{Z}/4$-summand in $\pi_{14}(P^8(2))$ and so
\[ \pi_{14}(P^8(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2. \]
3. $\pi_{15}(P^8(2))$: Since $2\pi_{15}(S^7) = 0$ and the kernel of
\[ 2E^2: \pi_{14}(S^7) = \mathbb{Z}/8 \to \pi_{14}(\Omega^2 S^9) = \mathbb{Z}/16 \]
is $4\pi_{14}(S^7) \cong \mathbb{Z}/2$. There is a short exact sequence
\[ 0 \to \pi_{17}(S^9) \to \pi_{14}(P^8(2)) \to 4\pi_{14}(S^7) \to 0. \]
Consider the commutative diagram
\[
\begin{array}{ccc}
\pi_{12}(P^5(2)) & \xrightarrow{E^2} & \pi_{14}(P^7(2)) \\
\downarrow & & \downarrow \\
\pi_{12}(S^5) & \xrightarrow{E^2} & 4 \cdot \pi_{14}(S^7) \\
\end{array}
\]
By Step 6 of Theorem 5.10, the homomorphism $\pi_{12}(P^5(2)) \to \pi_{12}(S^5)$ is split onto and so is $\pi_{14}(P^8(2)) \to 4 \cdot \pi_{14}(S^7)$. Thus
\[ \pi_{15}(P^8(2)) \cong \pi_{17}(S^9) \oplus 4 \cdot \pi_{14}(S^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]
4. $\pi_{16}(P^8(2))$: Recall that $\pi_{15}(S^7) = \mathbb{Z}/2^3$ is generated by $\{\sigma' \circ \eta_{14}, \nu_7, \epsilon_7\}$ and $\pi_{16}(S^7) = \mathbb{Z}/2^4$ is generated by $\{\sigma' \circ \eta_{14}^2, \nu_7^2, \mu_7, \eta_7 \circ \epsilon_8\}$, see [42, Theorems 7.1 and 7.2]. Since $2\pi_{16}(S^7) = 0$, there is a short exact sequence

$$0 \longrightarrow \pi_{16}(\Omega^2 S^9) \longrightarrow \pi_{16}(P^8(2)) \longrightarrow \pi_{15}(S^7) \longrightarrow 0.$$ Consider the commutative diagram of fibre sequences

\[
\begin{array}{cccccc}
\Omega X^8 & \xrightarrow{\Omega q} & \Omega P^8(2) & \xrightarrow{\delta} & S^7 & \xrightarrow{g} X^8 \\
\downarrow{\Omega j} & & \downarrow{\phi} & & \downarrow{p} & \downarrow{[2]} \\
\Omega S^7 & \xrightarrow{g} S^7 \{2\} & \xrightarrow{p} S^7 & \xrightarrow{[2]} & S^7 \\
\end{array}
\]

Since $S^7$ is an $H$-space, we have $\Omega[2] \simeq 2: \Omega S^7 \to \Omega S^7$ and so

$$\Omega^3(S^7 \{2\}) \simeq (\Omega^3 S^7) \{2\} = \text{Map}_*(P^4(2); S^7).$$

Thus the power map $2 : \Omega^3(S^7 \{2\}) \to \Omega^3(S^7 \{2\})$ is homotopic to the composite

$$\Omega^3(S^7 \{2\}) \simeq \text{Map}_*(P^4(2); S^7) \xrightarrow{id^P} \Omega^3 S^7 \xrightarrow{id^g} \Omega^4 S^7 \longrightarrow \Omega^3(S^7 \{2\}),$$

where $s : S^3 \to P^4(2)$ is the canonical inclusion. Since $2\pi_{15}(S^7) = 0$, the map $p_* : \pi_{15}(S^7 \{2\}) \to \pi_{15}(S^7)$ is onto. Let $\alpha \in \pi_{15}(S^7 \{2\})$ so that $p_*(\alpha) = \sigma' \circ \eta_{14}$. Then $2\alpha = g_* (\sigma' \circ \eta_{14}^2)$ and so

$$2\phi_*(\alpha) = \phi_*(2\alpha) = \phi_* g_*(\sigma' \circ \eta_{14}^2) = \Omega q_* \circ \Omega j_* (\sigma' \circ \eta_{14}^2).$$

Consider the cofibre sequence

$$S^{14} \xrightarrow{(\sigma', [2])} S^7 \vee S^{14} \xrightarrow{f} X^8.$$

Then

$$j \circ \sigma' \circ \eta_{14}^2 = f|_{S^7} \circ \sigma' \circ \eta_{14}^2 \simeq f|_{S^{14}} \circ [-2] \circ \eta_{14}^2 \simeq \ast.$$ Thus $\Omega j_* (\sigma' \circ \eta_{14}^2) = 0$ and $2\phi_*(\alpha) = 0$, that is, there exists $\alpha_1 \in \pi_{16}(P^8(2))$ so that

(1) $\delta_*(\alpha_1) = \sigma' \circ \eta_{14}$ and

(2) $2\alpha_1 = 0$.

Now consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{16}(P^8(2)) & \longrightarrow & \pi_{16}(P^8(2)) \\
\downarrow{E} & & \downarrow{\delta_*} \\
\pi_{15}(P^7(2)) & \xrightarrow{p'_*} & \pi_{15}(S^7)
\end{array}
\]

By Step 6 of Theorem 5.13, the $p'_* : \pi_{15}(P^7(2)) \to \pi_{15}(S^7)$ is onto. Let $\alpha_2, \alpha_3$ be elements in $\pi_{15}(P^7(2))$ so that $p'_*(\alpha_2) = \nu_7$ and $p'_*(\alpha_3) = \epsilon_7$. Then

$$\{\alpha_2 \oplus \alpha_3\} \cong E^\infty \{\alpha_2\} \oplus E^\infty \{\alpha_3\}$$
is a summand of $\pi_{15}^r(P^7(2))$ and so $E\{\alpha_2\} \oplus E\{\alpha_3\}$ is a summand of $\pi_{16}(P^8(2))$. Thus there is a commutative diagram of short exact sequences

$$
\begin{array}{c}
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 = \{\alpha_1\} \oplus \{\alpha_2\} \oplus \{\alpha_3\} \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.
\end{array}
$$

and so $\pi_{10}(P^8(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. We finish the calculation.

5.7. The Homotopy Groups $\pi_*(P^9(2))$.

Theorem 5.17. The homotopy groups $\pi_r(P^9(2))$ for $r \leq 17$ are as follows.

1) $\pi_r(P^9(2)) \cong \pi_r^s(P^9(2))$ for $r \leq 14$.
2) $\pi_{15}(P^9(2)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$.
3) $\pi_{16}(P^9(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.
4) $\pi_{17}(P^9(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4$.

Proof. 1. $\pi_r(P^9(2))$ for $r \leq 14$: Since $S^8$ is the 15-skeleton of $F^9\{2\}$. The assertion (1) follows.

2. $\pi_{15}(P^9(2))$: Consider the cofibre sequence $S^{15} \xrightarrow{2\omega_8} S^8 \xrightarrow{j} \text{sk}_{23}(F^9\{2\})$. There is an exact sequence

$$
\pi_{15}(S^{15}) \xrightarrow{(2\omega_8)_*} \pi_{15}(S^8) \xrightarrow{j_*} \pi_{15}(F^9\{2\}) \longrightarrow 0.
$$

Since $(2\omega_8)_*(\iota_{15}) = 2 \cdot (2\sigma_8 - E\sigma') = 4\sigma_8 - 2E\sigma'$, we have

$$
\pi_{15}(F^9\{2\}) = \mathbb{Z}/16 \oplus \mathbb{Z}/2
$$
generated by $j_*(\sigma_8)$ and $j_*(\omega_8)$. Now consider the commutative diagram

$$
\begin{array}{c}
\pi_{16}(S^9) \xrightarrow{\delta_*} \pi_{15}(F^9\{2\}) \xrightarrow{i_*} \pi_{15}(P^9(2)) \xrightarrow{p_*} \pi_{15}(S^9) \longrightarrow 0 \\
\pi_{15}(S^8) \xrightarrow{[2]*} \pi_{15}(S^8),
\end{array}
$$

where $E$ is onto. Since $[2]*(\sigma_8) = 2\sigma_8 + \omega_8$, we obtain

$$
\delta_*(\sigma_9) = \delta_*E\sigma_8 = j_*(2\sigma_8 + \omega_8) = 2j_*(\sigma_8) \oplus j_*(\omega_8),
$$

and so there is a short exact sequence

$$
0 \longrightarrow \mathbb{Z}/4 \longrightarrow \pi_{15}(P^9(2)) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.
$$
Notice that \( h(\sigma_8) = i_2^2 \) in \( H_{14}(\Omega S^8) \). Thus \( h([1 \circ j \circ \sigma_8]) = u^2 \) in \( H_{14}(\Omega P^9(2)) \) and \([i \circ j \circ \sigma_8]\) is not divisible by 2. Hence \( \pi_{15}(P^9(2)) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \).

3. \( \pi_{16}(P^9(2)) \): Consider the exact sequence

\[
\pi_{16}(S^{15}) \xrightarrow{(2\omega_8)_*} \pi_{16}(S^8) \xrightarrow{j_*} \pi_{16}(F^9(2)) \xrightarrow{} 0.
\]

Since \( 2\pi_{16}(S^{15}) = 0 \), \((2\omega_8)_*: \pi_{16}(S^{15}) \rightarrow \pi_{16}(S^8)\) is zero and \( \pi_{16}(S^8) \cong \pi_{16}(F^9(2)) \). Now consider the exact sequence

\[
\pi_{17}(S^9) \xrightarrow{\delta_*} \pi_{16}(F^9(2)) \rightarrow \pi_{16}(P^9(2)) \rightarrow 8 \cdot \pi_{16}(S^9) \rightarrow 0.
\]

Observe that

\[
\delta_*(\sigma_9 \circ \eta) = \delta_*(\sigma_9) \circ \eta = j_*(\omega_8 \circ \eta) = j_*(E\sigma' \circ \eta)
\]
\[
\delta_*(\nu_9) = \delta_* E^2(\nu_7) = j_* E(2\nu_7) = 0
\]
\[
\delta_*(\epsilon_9) = \delta_* E^2(\epsilon_7) = j_* E(2\epsilon_7) = 0
\]

Thus there is a short exact sequence

\[
0 \rightarrow \pi_{16}(F^9(2))/\{E\sigma' \circ \eta\} \rightarrow \pi_{16}(P^9(2)) \rightarrow 8 \cdot \pi_{16}(S^9) \rightarrow 0 \quad \text{or}
\]
\[
0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow \pi_{16}(P^9(2)) \rightarrow \mathbb{Z}/2 \rightarrow 0.
\]

Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{16}(P^9(2)) & \xrightarrow{8} & 8\pi_{16}S^9 \\
\downarrow & & \downarrow \cong \\
\pi_{12}(P^5(2)) & \rightarrow & \pi_{12}(S^5).
\end{array}
\]

Then \( \pi_{16}(P^9(2)) \rightarrow 8 \cdot \pi_{16}(S^9) \) is split onto and so \( \pi_{16}(P^9(2)) = \mathbb{Z}/2^{\oplus 4} \).

4. \( \pi_{17}(P^9(2)) \): Consider the exact sequence

\[
\pi_{17}(S^{15}) \xrightarrow{(2\omega_8)_*} \pi_{17}(S^8) \xrightarrow{j_*} \pi_{17}(F^9(2)) \rightarrow \pi_{16}(S^{15}) \rightarrow 0.
\]

Since \( 2 \cdot \pi_{17}(S^{15}) = 0 \), \((2\omega_8)_*: \pi_{17}(S^{15}) \rightarrow \pi_{17}(S^8)\) is zero and so there is a short exact sequence

\[
0 \rightarrow \pi_{17}(S^8) \xrightarrow{j_*} \pi_{17}(F^9(2)) \rightarrow \pi_{16}(S^{15}) \rightarrow 0.
\]

By [42, Theorem 7.2], \( \pi_{17}(S^8)/\{E\sigma' \circ \eta^2_{15}\} \cong \pi_{18}(S^9) \). Recall that the suspension \( E: S^8 \rightarrow \Omega S^9 \) factors through \( F^9(2) \). Thus \( \pi_{17}(S^8)/\{E\sigma' \circ \eta^2_{15}\} \) is a summand of \( \pi_{17}(F^9(2))/j_*\{E\sigma' \circ \eta^2_{15}\} \) and so

\[
\pi_{17}(F^9(2))/j_*\{E\sigma' \circ \eta^2_{15}\} \cong \pi_{17}(S^8)/\{E\sigma' \circ \eta^2_{15}\} \oplus \pi_{16}(S^{15}).
\]

Now consider the exact sequence

\[
\pi_{18}(S^9) \xrightarrow{\delta_*} \pi_{17}(F^9(2)) \rightarrow \pi_{17}(P^9(2)) \rightarrow \pi_{17}(S^9) \xrightarrow{\delta_*} \pi_{16}(F^9(2)).
\]

By Step 3, \( \text{Ker}(\delta_*: \pi_{17}(S^9) \rightarrow \pi_{16}(F^9(2))) = \{\widetilde{v}\} \oplus \{\epsilon_9\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). Note that

\[
\delta_*(\nu_9 \circ \eta^2_{16}) = \delta_*(\nu_9 \circ \eta_{16}) \circ \eta_{17} = j_*(E\sigma' \circ \eta^2_{15})
\]
\[
\delta_*(\nu_9^3) = \delta_* E^2\nu_7 = j_* E(2\nu_7^2) = 0
\]
\[
\delta_*(\mu_9) = \delta_* E^2\mu_7 = j_* E(2\mu_7) = 0
\]
\[
\delta_*(\epsilon_9) = \delta_* E^2\epsilon_7 = j_* E(2\epsilon_7) = 0.
\]
Thus there is a short exact sequence

\[ 0 \to \pi_{17}(S^8)/\{E\sigma' \circ \eta^2_{13}\} \oplus \pi_{16}(S^{15}) \to \pi_{17}(P^9(2)) \to \{\bar{v}_9\} \oplus \{\epsilon_9\} \to 0 \]

or

\[ 0 \to \mathbb{Z}/2^{\oplus 5} \to \pi_{17}(P^9(2)) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0. \]

Consider the commutative diagram

\[
\begin{array}{cccccc}
\pi_{17}(P^9(2)) & \to & \{\bar{v}\} \oplus \{\epsilon_9\} & \to & 0 \\
\downarrow & & \downarrow \cong & & \\
\pi_{17}^s(P^9(2)) & \to & \pi_{17}^s(S^9) & \to & 0 \\
p & & & & \\
\mathbb{Z}/4 \oplus \mathbb{Z}/4 & \to & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \to & 0,
\end{array}
\]

where \( p \) is the projection. Thus \( \pi_{17}(P^9(2)) = \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4^{\oplus 2} \). This completes the calculation. \( \square \)

5.8. The Homotopy Groups \( \pi_*(P^n(2)) \) for \( n \geq 10 \).

**Theorem 5.18.** The homotopy groups \( \pi_*(P^n(2)) \) for \( n \geq 10 \) and \( r \leq n + 8 \) are as follows.

1) \( \pi_r(P^{10}(2)) \cong \pi_r^s(P^{10}(2)) \) for \( r \leq 16 \).
2) \( \pi_r(P^{11}(2)) \cong \pi_r^s(P^{11}(2)) \) for \( r \leq 18 \).
3) \( \pi_r(P^n(2)) \cong \pi_r^s(P^n(2)) \) for \( r \leq n + 8 \) and \( n \geq 12 \).
4) \( \pi_{17}(P^{10}(2)) = \mathbb{Z}/2^{\oplus 4} \).
5) \( \pi_{18}(P^{10}(2)) = \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).
6) \( \pi_{19}(P^{11}(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).

**Proof.** Assertions (1), (2) and (3) are obvious.

1. \( \pi_{17}(P^{10}(2)) \): Consider the exact sequence

\[
\pi_{18}(S^{10}) \xrightarrow{\delta_s} \pi_{17}(F^{10}\{2\}) \to \pi_{17}(P^{10}(2)) \to 8 \cdot \pi_{17}(S^{10}) \to 0.
\]

Since \( F^{10}\{2\} \simeq S^9 \vee S^{18} \), we have \( \pi_{17}(S^9) \cong \pi_{17}(F^{10}\{2\}) \). Note that

\[
\delta_s(\bar{v}_{10}) = \delta_s E^3(\bar{v}_7) = j_* E^2(2\bar{v}_7) = 0
\]

\[
\delta_s(\epsilon_{10}) = \delta_s E^3(\epsilon_7) = j_* E^2(2\epsilon_7) = 0
\]

Thus there is a short exact sequence

\[ 0 \to \pi_{17}(S^9) \to \pi_{17}(P^{10}(2)) \to 8 \cdot \pi_{17}(S^{10}) \to 0. \]
Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{17}(P^{10}(2)) & \rightarrow & 8 \cdot \pi_{17}S^{10} \\
\downarrow & & \downarrow \cong \\
\pi_{12}(P^{5}(2)) & \rightarrow & \pi_{12}(S^{5}).
\end{array}
\]

By Step 6 of Theorem 5.10, the map \( \pi_{12}(P^{5}(2)) \rightarrow \pi_{12}(S^{5}) \) is split onto. Thus \( \pi_{17}(P^{10}(2)) \cong \mathbb{Z}/2^{\oplus 4} \).

2. \( \pi_{18}(P^{10}(2)) \): Consider the commutative diagram of exact sequences

\[
\begin{array}{ccc}
\pi_{19}(S^{10}) & \xrightarrow{\delta_*} & \pi_{18}(F^{10}\{2\}) & \xrightarrow{j_*} & \pi_{18}(P^{10}(2)) & \xrightarrow{p_*} & \pi_{18}(S^{10}) & \rightarrow & 0 \\
\downarrow H & & \downarrow \tilde{H}_{2*} & & \downarrow H_{2*} \\
\pi_{19}(S^{19}) & \xrightarrow{\delta'_*} & \pi_{18}(F^{19}\{2\}) & \rightarrow & \pi_{18}(P^{19}(2)).
\end{array}
\]

By [42, Theorem 7.2], \( \pi_{19}(S^{10}) = \{\Delta(\tau_{21})\} \oplus \{\nu_{10}^3\} \oplus \{\mu_{10}\} \oplus \{\epsilon_{10}\} \cong \mathbb{Z} \oplus \mathbb{Z}/2^{\oplus 3} \), where \( \Delta = P: \Omega^2S^{21} \rightarrow S^{10} \). Note that

\[
\delta_*(\nu_{10}^3) = \delta_*(\mu_{10}) = \delta_*(\epsilon_{10}) = 0
\]

\[
\tilde{H}_{2*} \circ \delta_* \Delta(\tau_{21}) = \delta'_* \circ H \Delta(\tau_{21}) = \delta'_*(2\tau_{19}) = 4\tau_{18}.
\]

Let \( \tilde{\lambda}_{10}: S^{18} \rightarrow F^{10}\{2\} \) be the composite

\[
\begin{array}{ccc}
S^{18} & \rightarrow & S^9 \vee S^{18} \cong F^{10}\{2\}_{(18)} \rightarrow F^{10}\{2\}.
\end{array}
\]

By [12, Theorem 2.2], \( 4[j \circ \tilde{\lambda}_{10}] \neq 0 \) and \( 8[j \circ \tilde{\lambda}_{10}] = 0 \). Thus

\[
\delta_* \Delta(\tau_{21}) = 4[\tilde{\lambda}_{10}] + \alpha
\]

for some \( \alpha \neq 0 \) in \( \pi_{18}(S^9) \) and so there is a short exact sequence

\[
0 \rightarrow \pi_{18}(S^9)/\{\alpha\} \oplus \mathbb{Z}/8 \rightarrow \pi_{18}(P^{10}(2)) \rightarrow \pi_{18}(S^{10}) \rightarrow 0.
\]
From the commutative diagram
\[
\begin{array}{ccccccc}
\mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & 0 \\
\pi_{18}(P^{10}(2)) & \longrightarrow & \pi_{18}(S^{10}) & \longrightarrow & 0 \\
\pi_{18}(P^{10}(2)) & \longrightarrow & \pi_{18}(S^{10}) & \longrightarrow & 0 \\
\mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/4 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \longrightarrow & 0,
\end{array}
\]

we obtain \( \pi_{18}(P^{10}(2)) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).

3. \( \pi_{19}(P^{11}(2)) \): From the exact sequence
\[
\pi_{19}(S^{19}) \overset{(2\omega_{10})}{\longrightarrow} \pi_{19}(S^{10}) \longrightarrow \pi_{19}(F^{11}\{2\}) \longrightarrow 0,
\]

we have \( \pi_{19}(F^{11}\{2\}) = \pi_{19}(S^{10})/\{2\Delta(\omega_{21})\} \). Now consider the exact sequence
\[
\pi_{20}(S^{11}) \overset{\delta_*}{\longrightarrow} \pi_{19}(F^{11}\{2\}) \longrightarrow \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11}) \longrightarrow 0.
\]

Since \( E^4 : \pi_{16}(S^7) \to \pi_{20}(S^{11}) \) is onto, \( \delta_* : \pi_{20}(S^{11}) \longrightarrow \pi_{19}(F^{11}\{2\}) \) is zero and so there is a short exact sequence
\[
0 \longrightarrow \pi_{19}(F^{11}\{2\}) \longrightarrow \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11}) \longrightarrow 0.
\]

By [42, Theorem 7.2], we have \( \pi_{19}(S^{11}) \cong \pi_{19}(S^{11}) \). By Lemma 5.2, the group extension for the short exact sequence above is essential and
\[
\pi_{19}(P^{11}(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4.
\]

This completes the calculation. \( \square \)

5.9. Remarks on the generators for the homotopy groups. In this subsection, we give some remarks along the suggestions given by the referee. Although our paper essentially contains the classical methods completely, in the metastable range the classical method is very useful to settle generators of the group. The classical method is to use the homotopy exact sequence of a pair \((P^n(2), S^{n-1})\) and to determine the relative homotopy group \( \pi_*(P^n(2), S^{n-1}) \) by using the James exact sequence [17, Theorem 2.1]. By using this method, it seems not hard to determine the group structures and the generators of \( \pi_{n+r}(P^n(2)) \) for \( r \leq 2n-5 \) if \( \pi_{n+r}(S^m) \) \((m = n-1, n)\) is given by Toda’s composition methods. (Note. When \( n \) is small, for instance \( n = 3 \) or \( 4 \), the range up to \( 2n-5 \) is not so good.) The following example was given by the referee.
Example 5.19. \( \pi_{19}(P^{11}(2)) \cong (\mathbb{Z}/4)^{\oplus 2} \oplus \mathbb{Z}/2^{\oplus 2} \), which is generated by the elements \( \tilde{\epsilon}_{10}, \tilde{\nu}_{10}, i_{11}\mu_{10} \) and \( i_{11} \circ [\iota_{10}, \iota_{10}] \).

**Proof.** We consider the homotopy exact sequence of a pair \((P^{11}(2), S^{10})\):

\[
\pi_{20}(P^{11}(2), S^{10}) \xrightarrow{\partial} \pi_{19}(S^{10}) \xrightarrow{i_{11}^*} \pi_{19}(P^{11}(2)) \xrightarrow{j_*} \pi_{19}(P^{11}(2), S^{10}).
\]

By Blakers-Massey’s theorem,

\[
\pi_{19}(P^{11}(2), S^{10}) \cong \pi_{19}(S^{11}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

So \( j_* \) is equivalent to the homomorphism \( p_{11*} : \pi_{19}(P^{11}(2)) \longrightarrow \pi_{19}(S^{11}) \). Since \( \pi_{19}(S^{11}) = \Sigma \pi_{18}(S^{10}) \) and \( \pi_{18}(S^{10}) = \Sigma^4 \pi_{14}(S^6) \), there exist coextensions \( \tilde{\epsilon}_{10} \) and \( \tilde{\nu}_{10} \) of \( \epsilon_{10} \) and \( \nu_{10} \) satisfying \( p_{11*}(\tilde{\epsilon}_{10}) = \epsilon_{11} \) and \( p_{11*}(\tilde{\nu}_{10}) = \nu_{11} \) respectively. By [17, Theorem 2.1], we obtain

\[
\pi_{20}(P^{11}(2), S^{10}) = \{[\omega, \iota_{10}]\} \cong \mathbb{Z},
\]

where \( \omega \in \pi_{11}(P^{11}(2), S^{10}) \) is the characteristic map of the 11-cell of \( P^{11}(2) \) and \([, ,]\) stands for the relative Whitehead product. We obtain

\[
\partial[\omega, \iota_{10}] = -[2\iota_{10}, \iota_{10}].
\]

Thus, by the fact that

\[
\pi_{19}(S^{10}) \cong \mathbb{Z}\{[\iota, \iota]\} \oplus \mathbb{Z}/2\{\mu_{10}\} \oplus \mathbb{Z}/2\{\nu_{10}\} \oplus \mathbb{Z}/2\{\eta_{10}\epsilon_{11}\},
\]

we conclude that

\[
\pi_{19}(P^{11}(2)) \cong (\mathbb{Z}/4)^{\oplus 2} \oplus \mathbb{Z}/2^{\oplus 2},
\]

which is generated by elements \( \tilde{\epsilon}_{10}, \tilde{\nu}_{10}, i_{11}\mu_{10} \) and \( i_{11} \circ [\iota_{10}, \iota_{10}] \). The relations are the following:

\[
2\tilde{\epsilon}_{10} = i_{11}\eta_{10}\epsilon_{11} \quad \text{and} \quad 2\tilde{\nu}_{10} = i_{11}\nu_{10}^3.
\]

Here we note that \( \tilde{\epsilon}_{10} \) and \( \tilde{\nu}_{10} \) are chosen as suspended, so, by the fact \([2] = i_{11}\eta_{10}\), we obtain

\[
2\tilde{\epsilon}_{10} = [2_*]\tilde{\epsilon}_{10} = i_{11}\eta_{10}\epsilon_{11} \circ \tilde{\epsilon}_{10} = i_{11}\eta_{10}\epsilon_{11} \quad \text{and} \quad 2\tilde{\nu}_{10} = [2_*]\tilde{\nu}_{10} = i_{11}\eta_{10}\nu_{11}\tilde{\nu}_{10} = i_{11}\nu_{10}\nu_{11} = i_{11}\nu_{10}^3.
\]

\[ \square \]

**Note.** \( \tilde{\epsilon}_{10} = \Sigma^7 \tilde{\epsilon}_3 \) and \( \tilde{\nu}_{10} = \Sigma^4 [\tilde{\nu}_7] \).

6. **The Homotopy Theory of \( \Sigma \mathbb{R}P^2 \)**

In this chapter, we study the homotopy theory of the special space \( P^3(2) = \Sigma \mathbb{R}P^2 \). This chapter will be divided into six sections. In the first section, we consider certain canonical fibrations over \( \Sigma \mathbb{R}P^n \). In section 2, we consider the special case where \( n = 2 \). In this case, we obtain a product decomposition of the triple loop space of \( \Sigma \mathbb{R}P^2 \). This decomposition theorem will help us to compute the homotopy groups of \( \Sigma \mathbb{R}P^2 \). In section 3, we prove that the suspension \( E : \pi_*(\mathbb{R}P^2) \to \pi_{*+1}(\Sigma \mathbb{R}P^2) \) is the trivial homomorphism for \( * \geq 3 \). This information has been to help us to find certain non-suspension co-H-spaces \( X \) which admits the form \( P^3(2) \cup_f e^n \). An application of our results to the homogeneous space \( SU(3)/SO(3) \) is given in section 4. We will start to compute the homotopy groups from section 5. From our decomposition theorem, the homotopy groups of \( \pi_*(\Sigma \mathbb{R}P^4_2) \) for \( * \geq 4 \) is a summand of \( \pi_*(P^3(2)) \) and so we first compute \( \pi_*(\Sigma \mathbb{R}P^4_2) \) in section 5 and then do \( \pi_*(P^3(2)) \) in section 6.
6.1. **Fibrations over $\Sigma\mathbb{R}P^n$**. Let $\gamma_n \colon \Sigma\mathbb{R}P^n \to BSO(n + 1)$ be the adjoint map of the canonical inclusion $\mathbb{R}P^n \to SO(n + 1) = \Omega BSO(n + 1)$, where $BSO(n + 1)$ is the classifying space of $SO(n + 1)$. Let the space $Z_n$ be the pull-back of the diagram

\[ Z_n \longrightarrow ESO(n + 1) \]

\[ \text{pull} \quad q \]

\[ \Sigma\mathbb{R}P^n \xrightarrow{\gamma_n} BSO(n + 1), \]

where $SO(n + 1) \to ESO(n + 1) \to BSO(n + 1)$ is the universal principle $SO(n + 1)$-bundle. We obtain a principal $SO(n + 1)$-bundle over $\Sigma\mathbb{R}P^n$. For instance,

\[ Z_1 \longrightarrow \Sigma\mathbb{R}P^1 = S^2 \]

is the classical Hopf fibration. From now on, we assume that $n \geq 2$.

Let $F$ be any space with $SO(n + 1)$-action. We have the induced fibre bundle

\[ F \longrightarrow Z_n \times_{SO(n+1)} F \longrightarrow \Sigma\mathbb{R}P^n. \]

Let $Y_n = Z_n \times_{SO(n+1)} S^n$. Then there is a homotopy commutative diagram of fibre sequences

\[ SO(n + 1) \longrightarrow S^n \longrightarrow BSO(n) \longrightarrow BSO(n + 1) \]

\[ \Omega\Sigma\mathbb{R}P^n \xrightarrow{\partial} S^n \longrightarrow Y_n \longrightarrow \Sigma\mathbb{R}P^n \]

Observe that the composite $\mathbb{R}P^n \xhookrightarrow{\Omega\Sigma\mathbb{R}P^n} S^n$ is the pinch map and so its homotopy cofibre is $\Sigma\mathbb{R}P^{n-1}$. By Lemma 2.39, there are cofibre sequences

\[ SO(n + 1)/\mathbb{R}P^n \longrightarrow Z_n \xrightarrow{\theta_n} \Sigma\mathbb{R}P^n \wedge SO(n + 1) \quad \text{and} \quad \Sigma\mathbb{R}P^{n-1} \xhookrightarrow{Y_n} \Sigma\mathbb{R}P^n \wedge S^n = \Sigma^{n+1}\mathbb{R}P^n, \]

where $\theta_n$ is a monomorphism in mod 2 homology and $\tilde{\theta}_n$ is an epimorphism in mod 2 homology. Furthermore the map $S^n \to Y_n$ is the composite

\[ S^n \longrightarrow \Sigma\mathbb{R}P^{n-1} \longrightarrow Y_n, \]

where $q \colon S^{n-1} \to \mathbb{R}P^{n-1}$ is the quotient map. Observe that the composite

\[ \Sigma\mathbb{R}P^{n-1} \xhookrightarrow{Y_n} \Sigma\mathbb{R}P^n \]

is the canonical inclusion. Recall that the Thom space $C_n$ of the vector bundle

\[ Z_n \times_{SO(n+1)} \mathbb{R}P^n \to \Sigma\mathbb{R}P^n \]
is the homotopy cofibre of the map \( p: Y_n \to \Sigma \mathbb{R}P^n \). We have the homotopy commutative diagram of cofibre sequences

\[
\begin{array}{cccccc}
\Sigma \mathbb{R}P^{n-1} & \longrightarrow & Y_n & \longrightarrow & \Sigma^{n+1} \mathbb{R}P^n \\
\downarrow & & \downarrow p_n \downarrow & & \downarrow \delta_n \\
\Sigma \mathbb{R}P^{n-1} & \longrightarrow & \Sigma \mathbb{R}P^n & \longrightarrow & S^{n+1}
\end{array}
\]

We will give some information on \( H^*(C_n) \) and some properties of the map \( \delta_n: \Sigma^{n+1} \mathbb{R}P^n \to S^{n+1} \).

We need a lemma. A space \( X \) is called stably atomic if \( X \) does not have a nontrivial stable decomposition. The following lemma may be well-known.

**Lemma 6.1.** If \( n \neq 3 \) or \( 7 \), then \( \mathbb{R}P^n \) is stably atomic.

**Proof.** It is easy to check that \( \mathbb{R}P^{2n} \) is stably atomic by induction and by considering the Steenrod operations. Suppose that \( \mathbb{R}D^{2n+1} \) is not stably atomic. Let \( q: S^{2n} \to \mathbb{R}P^{2n} \) be the quotient map. Since \( \pi_{4n+1}(\Sigma^{2n+1} \mathbb{R}P^{2n}) \cong \pi_{4n+1}(\Sigma^{2n+1} \mathbb{R}P^{2n}) \), the map \( \Sigma^{2n+1}q: S^{4n+1} \to \Sigma^{2n+1} \mathbb{R}P^{2n} \) is null homotopic. By [12], there is a homotopy commutative diagram

\[
\begin{array}{cccccc}
\Omega^2 S^{4n+3} & \longrightarrow & S^{2n+1} & \longrightarrow & \Omega S^{2n+2} & \longrightarrow & \Omega S^{4n+3} \\
\downarrow P & \downarrow \phi_{2n} & \downarrow E & \downarrow \phi_{2n+1} & \downarrow H & \downarrow \text{pinch} & \downarrow \Omega S^{4n+2},
\end{array}
\]

where the top row is the \( EHP \)-sequence, the bottom row is the cofibre sequence and the map \( \phi_n: \Sigma^{n+1} \mathbb{R}P^n \to S^{n+1} \) is the adjoint map of the canonical map

\[
\mathbb{R}P^n \longrightarrow O(n+1) \longrightarrow \Omega^{n+1} S^{n+1}.
\]

It follows that the Whitehead square \( \omega_{2n+1} \) is null homotopic. Thus \( 2n+1 = 3 \) or \( 7 \) and hence the result. \( \Box \)

**Proposition 6.2.** Let \( n \geq 2 \). Then

1) \( Sq^{n+1}: H^{n+1}(C_n) \to H^{2n+2}(C_n) \) is an isomorphism.

2) The composite

\[
\Sigma \mathbb{R}P^n \longrightarrow \Omega \Sigma^{n+1} \mathbb{R}P^n \longrightarrow \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}
\]

is non-trivial mod 2 homology.

3) If \( n \neq 3 \) or \( 7 \), then \( \delta_n: \pi_{n+2}(\Sigma^{n+1} \mathbb{R}P^n) \to \pi_{n+2}(S^{n+1}) \) is onto.
**Proof.** Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega \Sigma \mathbb{RP}^n & \longrightarrow & S^n & \overset{i_n}{\longrightarrow} & Y_n & \overset{p_n}{\longrightarrow} & \Sigma \mathbb{RP}^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega C_n & = & \Omega C_n & = & * & = & C_n.
\end{array}
\]

Let \(x_{n+1}\) be a generator for \(H_{n+1}(C_n) = \mathbb{Z}/2\). Then

\[\tau(x_{n+1})^2 = 0\]

because \(H_{n+1}(\Sigma \mathbb{RP}^n) \cong H_{n+1}(C_n)\). Thus the dual \(x_{n+1}^*\) has a nontrivial self cup product and hence assertion (1).

(2) Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega Y_n & \longrightarrow & \Omega \Sigma \mathbb{RP}^n & \longrightarrow & S^n & \longrightarrow & \Sigma \mathbb{RP}^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega \Sigma^{n+1} \mathbb{RP}^n & \overset{\Omega \delta_n}{\longrightarrow} & \Omega S^{n+1} & \overset{\Omega \phi}{\longrightarrow} & F_n & \longrightarrow & \Sigma^{n+1} \mathbb{RP}^n \\
\end{array}
\]

Let \(\gamma_i(u)\) be a basis for \(H_i(\mathbb{RP}^n)\). Then \(H_*(\Omega \Sigma \mathbb{RP}^n)\) is the tensor algebra generated by \(\gamma_i(u)\) for \(1 \leq i \leq n\). By the decomposition

\[H_*(\Omega \Sigma \mathbb{RP}^n) = H_*(\Omega Y_n) \oplus H_*(\Omega Y_n) \cdot \gamma_n(u),\]

we have \(\gamma_n(u)^2 = \alpha + \beta \cdot \gamma_n(u)\) for some \(\alpha \in H_2n(\Omega Y_n)\) and \(\beta \in H_1(\Omega Y_n)\). Note that \(\Omega \phi_*\) is an algebraic map with \(\Omega \phi_*(\gamma_i(u)) = 0\) for \(i < n\) and \(Q(H_n(\Omega Y_n)) = 0\). We have \(\Omega \phi_*(\beta \cdot \gamma_n(u)) = 0\) and so

\[\Omega \phi_*(\gamma_n(u)^2) = \Omega \phi_*(\alpha).\]

It follows that the composite

\[H_{2n}(\Sigma \mathbb{RP}^n) \cong H_{2n}(\Omega \Sigma^{n+1} \mathbb{RP}^n) \overset{\Omega \delta_n}{\longrightarrow} H_{2n}(\Omega S^{n+1})\]

is onto and hence assertion (2).

(3) Let \(f\) be the composite

\[
\begin{array}{cccc}
\Sigma^\infty \mathbb{RP}^n & \longrightarrow & \Sigma^\infty \Omega^{n+1} \Sigma^{n+1} \mathbb{RP}^n & \overset{\Omega^{n+1} \delta_n}{\longrightarrow} & \Sigma^\infty \Omega^{n+1} S^{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^\infty Q_0(S^0) & \overset{H_2}{\longrightarrow} & \Sigma^\infty Q(\mathbb{RP}^\infty) & \longrightarrow & \Sigma^\infty \mathbb{RP}^\infty
\end{array}
\]

in the stable category. Then there is a self map \(g\) of \(\Sigma^\infty \mathbb{RP}^n\) such that \(f = j \circ g\), where \(j: \mathbb{RP}^n \to \mathbb{RP}^\infty\) is the inclusion. By assertion (2), \(f_*: H_n(\mathbb{RP}^n) \to H_n(\mathbb{RP}^\infty)\) is an isomorphism and so is \(g_*: H_n(\mathbb{RP}^n) \to H_n(\mathbb{RP}^n)\). By Lemma 6.1, the map \(g\) is a homotopy equivalence and hence the result. \(\square\)

**Question 6.3.** Is the map \(\delta_n: \Sigma^{n+1} \mathbb{RP}^n \to S^{n+1}\) homotopic to the adjoint map of the canonical map \(\mathbb{RP}^n \longrightarrow SO(n+1) \longrightarrow \Omega^{n+1} S^{n+1}\)?
6.2. A Decomposition of $\Omega_5^3 P^3(2)$. Now we study the spaces $Z_n$ and $Y_n$ in the special case where $n = 2$. By Lemma 2.38, $Q(H_*(Y_2))$ and $Q(H_*(Z_2))$ have bases \{u, [u, v], [v^2]\} and \{u^2, [u, v], v^2, [[u, v], u], [[u, v], v]\}, respectively, where $Sq_i v^2 = [u, v]$, $Sq_i v^2 = u^2$ and $Sq_i v = [[u, v], u]$. Thus $H_*(Y_2)$ and $H_*(Z_2)$ have bases \{x_2, x_4, x_5\} and \{y_3, y_4, y_5, z_5, z_6\}, respectively, with the Steenrod operations $Sq_i x_5 = x_4$, $Sq_i y_5 = y_4$, $Sq_i z_6 = y_3$. The fibre sequence

$$Z_2 \longrightarrow Y_2 \longrightarrow BSO(2) = \mathbb{CP}^\infty$$

shows that $Z_2$ is the 2-connected cover of $Y_2$ with $j_*(y_3) = x_4$ and $j_*(y_4) = x_5$.

**Proposition 6.4.** There is a cofibre sequence $S^4 \to S^2 \vee S^4 \to Y_2$.

**Proof.** Consider the fibre sequence $S^2 \longrightarrow Y_2 \longrightarrow P^3(2)$. The 4-skeleton $sk_4(Y_2) \cong S^2 \cup_k e^4$ for some $k \in \mathbb{Z}$. If $k \neq 0$, then $\pi_3(Y_2) = \mathbb{Z}/k$ and so the boundary map $\pi_4(P^3(2)) \to k \cdot \pi_3(S^2)$ is onto. This is a contradiction because $\pi_4(P^3(2))$ is torsion group. Thus $k = 0$ and so $sk_4(Y_2) \cong S^2 \vee S^4$. By the cofibre sequence $S^2 \longrightarrow Y_2 \longrightarrow P^5(2)$, we have $Y_2 = (S^2 \vee S^4) \cup_{(a, [2])} e^5$ for some $a : S^4 \to S^2$. Since $\pi_4(S^2) = \mathbb{Z}/2$, it suffices to show that $a$ is essential. Suppose that $a$ is null homotopic. Then $Y \cong S^2 \vee P^5(2)$ and so the bottom cell $S^3$ is a retract of $Y_2$. It follows that the bottom cell $S^3$ is a retract of $Z_2$. This is impossible because $Sq^2 y_5 = y_3$. Thus $a$ is essential and hence the result.

We compute the first three homotopy groups of $P^3(2)$.

**Proposition 6.5.** The homotopy groups $\pi_n(P^3(2))$ for $n \leq 4$ are as follows.

1) $\pi_2(P^3(2)) \cong \pi_2^s(P^3(2)) = \mathbb{Z}/2$.

2) $\pi_3(P^3(2)) = \mathbb{Z}/4$ with a generator represented by the composite

$$\tilde{\eta} : S^3 \longrightarrow S^2 \hookrightarrow P^3(2).$$

3) $\pi_4(P^3(2)) = \mathbb{Z}/4$ with a generator represented by the composite

$$\lambda_3 : S^4 \hookrightarrow S^2 \vee S^4 = sk_4(Y_2) \hookrightarrow Y_2 \longrightarrow P^3(2).$$

**Proof.** Assertion (1) is obvious. Consider the fibre sequence $S^2 \longrightarrow Y_2 \longrightarrow P^3(2)$. We have the exact sequence

$$0 \longrightarrow \pi_4(S^2) \longrightarrow \pi_4(Y_2) \longrightarrow \pi_4(P^3(2)) \longrightarrow \pi_3(S^2) \longrightarrow \pi_3(Y_2) \longrightarrow \pi_3(P^3(2)) \longrightarrow \pi_2(S^2).$$

Since $i : S^2 \to Y_2$ is of degree 2 on the bottom cell of $Y_2$, we have

$i_* : \pi_3(S^2) = \mathbb{Z} \to \pi_3(Y_2)$

is of degree 4 and $i_* : \pi_4(S^2) \to \pi_4(Y_2)$ is zero. Thus

$$\pi_3(P^3(2)) = \mathbb{Z}/4 \quad \text{and} \quad \pi_4(Y_2) \cong \pi_4(P^3(2)).$$

By the cofibre sequence

$$S^4 \longrightarrow S^2 \vee S^4 \longrightarrow Y_2,$$

we have $\pi_4(Y_2) = \mathbb{Z}/4$ and hence the result. \qed
By expecting the homology of $Z_2$, the classes $z_5$ and $z_6$ come from a basis for $L_3(V)$ by homology suspension. This information suggests a splitting of $Z_2$.

**Proposition 6.6.** There is a splitting $Z_2 \simeq \Sigma \mathbb{R}P^4 \vee P^6(2)$.

**Proof.** Since $\pi_3(Z_2) \cong \pi_3(Y_2) \cong \pi_3(\text{sk}_4(Y_2)) = \pi_3(S^2 \vee S^4) = \mathbb{Z}$, we have

\[
\text{sk}_4(Z_2) \simeq S^3 \vee S^4.
\]

Observe that $Sq^1 y_5 = y_4$ and $Sq^2 y_5 = y_3$. We obtain that $\text{sk}_3(Z_2) \simeq \Sigma \mathbb{R}P^4 \vee S^5$ and so there is a map $\phi: \Sigma \mathbb{R}P^4 \to Z_2$ such that the image of $\phi_*$ has a basis $\{y_3, y_4, y_5\}$. Let $g: S^4 \to Z_2$ be a map such that $g_*(\iota_4) = y_4$ and let $g': S^3 \to \Omega Z_2$ be the adjoint map of $g$. Let $\psi = [g, \text{id}_{\mathbb{R}P^4}]: S^3 \wedge \mathbb{R}P^2 \to \Omega Z_2$ be the relative Samelson product on the fibre sequence $\Omega Z_2 \to \Omega P^3(2) \to SO(3)$. Then $\psi_*: H_*(P^3(2)) \to Q(H_*(\Omega Z_2))$ has the image with a $\{[[u, v], u], [[u, v], v]\}$. Let $\psi': P^6(2) \to Z_2$ be the adjoint map of $\psi$. Then the map

\[
\Sigma \mathbb{R}P^2 \vee P^6(2) \xrightarrow{\phi \vee \psi'} Z_2 \vee Z_2 \xrightarrow{\text{fold}} Z_2
\]

is a homotopy equivalence and hence the result. \(\square\)

**Theorem 6.7.** [46, Theorem 1.1] There are homotopy decompositions localized at 2.

\[
\Omega^3_0(P^3(2)) \simeq \Omega^2(S^3(3)) \times \Omega^3(\Sigma \mathbb{R}P^4(3) \vee P^6(2)).
\]

**Proof.** The assertion follows from the facts that

1) The map $i: S^2 \to Y_2$ is of degree 2 on the bottom cell.
2) $\Omega^2([2]): \Omega^2(S^2(3)) = \Omega^2(S^3(3)) \to \Omega^2(S^2(3)) = \Omega^2(S^3(3))$ is null homotopic.
3) The 2-connected cover $Z_2 \cong \Sigma \mathbb{R}P^4 \vee P^6(2)$ by Proposition 6.6.

\(\square\)

There are some applications of this theorem.

**Proposition 6.8.** Let $X$ be a simply connected CW-complex and let $E \to X$ be a principle $SO(3)$-bundle. Suppose that the second and the third Whitney classes $w_2$ and $w_3$ are not zero in $H^*(X)$. Then

\[
\Omega^3_0 X \cong \Omega^3_0 E \times \Omega^2(S^3(3)).
\]

In particular, the higher homotopy groups of $S^3$ are summands of $\pi_*(X)$.

**Proof.** There is a map $f: X \to BSO(3)$ such that $E$ is homotopy equivalent to the homotopy fibre of $f$. By the assumption, the inclusion $P^3(2) \longrightarrow BSO(3)$ lifts to $X$ and hence the result. \(\square\)

**Corollary 6.9.** Let $G$ be any 2-connected topological group. Suppose that $G$ contains $SO(3)$ as a subgroup. Let $G/SO(3)$ be the obtained homogeneous space. Then $\Omega^3_0(G/SO(3))$ is (weak) homotopy equivalent to the product $\Omega^3_0 G \times \Omega^2(S^3(3))$.

**Proof.** Consider the fibration $SO(3) \to G \to G/SO(3)$. By the assumption, we have $\pi_2(G/SO(3)) \cong \pi_2(BSO(3))$ and hence the result. \(\square\)

As an example, consider that $SO(3)$ is a canonical subgroup of $SU(3)$ by complexification. We have the decomposition $\Omega^3_0(SU(3)/SO(3)) \simeq \Omega^3_0 SU(3) \times \Omega^2(S^3(3))$. By using the language in (unstable) $K$-theory, we have
Corollary 6.10. Let $X$ be a path-connected space. Then the complexification of any orientable 3-dimensional vector bundle over $\Sigma^3 X$ is a trivial bundle.

Consider the inclusion $E : \mathbb{R}P^2 \to \Omega P^3(2)$. Let $q : S^2 \to \mathbb{R}P^2$ be the quotient map. Then there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^2 & \xrightarrow{q} & \mathbb{R}P^2 \\
| & & | \\
S^2 & \xrightarrow{\bar{q}} & \Omega P^3(2).
\end{array}
\]

Observe that

1) $[2]_* : \pi_r(S^2) \to \pi_r(S^2)$ is 0 for $r \geq 4$ and of degree 4 for $r = 3$;
2) $q_* : \pi_r(S^2) \to \pi_r(\mathbb{R}P^2)$ is isomorphic and
3) $\pi_4(P^3(2)) = \mathbb{Z}/4$.

We have the following suspension theorem.

Proposition 6.11. The suspension $E_* : \pi_r(\mathbb{R}P^2) \to \pi_{r+1}(P^3(2))$ is zero for $r \geq 3$.

Corollary 6.12. Let $X$ be a complex which admits the form of $P^3(2) \cup e^n$ with $n \geq 5$. If the attaching map $f$ is essential, then $X$ is not homotopy equivalent to a suspension.

This result has been used for finding a family of non-suspension co-$H$-spaces $X$ which admits the form of $P^3(2) \cup e^n$ in [48].

The map $\lambda_3 : S^4 \to P^3(2)$, which represents a generator for $\pi_4(P^3(2))$, has some special properties. First the adjoint map $\lambda'_3 : S^3 \to \Omega P^3(2)$ has the property that $\lambda'_{3*}(t_3) = [u, v]$, that is, $\lambda_3$ is from the spherical class $[u, v]$. The homotopy cofibre of $\lambda_3$ is actually the homogeneous space $SU(3)/SO(3)$.

Proposition 6.13. There is a cofibre sequence $S^4 \xrightarrow{\lambda_3} P^3(2) \xrightarrow{\beta} SU(3)/SO(3)$.

Proof. By [24, Theorem 6.7], the cohomology $H^*(SU(3)/SO(3))$ is the exterior algebra $E(w_2, w_3)$, where $w_i$ is the $i$-th Whitney class. Thus there is a cofibre sequence $S^4 \xrightarrow{f} P^3(2) \xrightarrow{\beta} SU(3)/SO(3)$.

Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
S^4 & \xrightarrow{f} & P^3(2) \xleftarrow{c} SU(3)/SO(3) \\
| & & | \\
Z_2 & \xrightarrow{g} & P^3(2) \xrightarrow{\beta} BSO(3),
\end{array}
\]

where the top row is a cofibre sequence and the bottom row is a fibre sequence. The map $g : S^4 \to Z_2$ is nontrivial in mod 2 homology. Thus $g$ represents a generator for $\pi_4(Z_2)$ and hence the result. \qed
The next proposition shows that \( \lambda_3 \) represents a generator for the stable homotopy group \( \pi_*^s(P^3(2)) \).

**Proposition 6.14.** There is an isomorphism \( \pi_4(P^3(2)) \cong \pi_4^s(P^3(2)) \).

**Proof.** By Proposition 6.2, \( \delta_{2*}: \pi_4(P^3(2)) \to \pi_4(S^3) \) is an isomorphism and so the composite \( S^4 \xrightarrow{\lambda_3} P^3(2) \xrightarrow{\text{pinch}} S^3 \) is homotopic to \( \eta \). It follows that the homotopy class \([\lambda_3]\) is a generator for \( \pi_4^s(P^3(2)) = \mathbb{Z}/4 \) or \( \pi_4(P^3(2)) = \mathbb{Z}/4 \to \pi_4^s(P^3(2)) \) is an isomorphism and hence the result. \( \square \)

**Example 6.15.** Since \( \pi_4(P^3(2)) = \mathbb{Z}/4 \), there are only three different complexes \( X \) which admits the form \( X = P^3(2) \cup e^5 \) given by \( P^3(2) \cup_{\lambda_3} e^5 \cong SU(3)/SO(3), P^3(2) \cup_{2\lambda_3} e^5 \) and \( P^3(2) \cup e^5 \). The space \( P^3(2) \cup_{2\lambda_3} e^5 \) has the cell structure \( S^2 \cup_{\eta^2} e^4 \cup e^5 \). This shows that the space \( SU(3)/SO(3) \) is uniquely determined, up to homotopy, by its fundamental group and its cohomology ring. More precisely, if \( X \) is a simply connected complex such that the cohomology ring \( H^*(X) \) is isomorphic to \( H^*(SU(3)/SO(3)) \), then \( X \) is homotopy equivalent to \( SU(3)/SO(3) \). This information might be useful in the theory of 5-dimensional manifolds.

Unlike Theorem 2.19 on the Stiefel manifolds, there is a self homotopy equivalence of \( SU(3)/SO(3) \) which changes the orientation.

**Proposition 6.16.** There is a self homotopy equivalence of \( SU(3)/SO(3) \) which changes the orientation.

**Proof.** From the commutative diagram

\[
\begin{array}{ccc}
\pi_4(P^3(2)) & \cong & \pi_4^s(P^3(2)) \\
E_\infty \downarrow & & \downarrow E_\infty \\
[-1], & & -1 \\
\pi_4(P^3(2)) & \cong & \pi_4^s(P^3(2)),
\end{array}
\]

we have \([−1]*(\lambda_3) = −\lambda_3\) and so there is a homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
S^4 & \xrightarrow{\lambda_3} & P^3(2) \xrightarrow{\text{pinch}} S^5 \\
\downarrow [−1] & & \downarrow \phi \\
S^4 & \xrightarrow{\lambda_3} & P^3(2) \xrightarrow{\text{pinch}} S^5.
\end{array}
\]

By the 5-lemma, \( \phi \) is a homotopy equivalence and hence the result. \( \square \)

6.3. The Homotopy Groups \( \pi_*(\Sigma\mathbb{R}P^2) \). Now we start to compute \( \pi_r(P^3(2)) \) for \( r \leq 11 \). The first three homotopy groups have been computed in the previous section. The higher homotopy groups will be computed using Theorem 6.7. Observe that there are two factors in this decomposition theorem. The homotopy groups of the factor \( \Omega^2(S^3(3)) \), up to this range, has been computed in [42]. Thus our work is to compute
\[\pi_*(\Sigma\mathbb{R}P^4_2 \vee P^6(2))\]. We first compute \(\pi_*(\Sigma\mathbb{R}P^4_2)\), which is the most difficult part, and then, by applying the Hilton-Milnor theorem, determine the homotopy groups of \(\Sigma\mathbb{R}P^4_2 \vee P^6(2)\).

### 6.3.1. The Homotopy Groups \(\pi_*(\Sigma\mathbb{R}P^4_2)\)

We need some preliminary lemmas.

**Notation 6.17.** Let \(F\) and \(B\) denote the spaces in the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
S^3 & \xrightarrow{i} & SU(3) \\
\downarrow & & \downarrow \text{q} \\
\Sigma\mathbb{R}P^4_2 & \xrightarrow{p} & S^5
\end{array}
\]

\[
\begin{array}{ccc}
F & \xrightarrow{} & \Sigma\mathbb{R}P^4_2 \\
\downarrow & & \downarrow \text{p} \\
B & \xrightarrow{} & B
\end{array}
\]

where \(p\) is the pinch map, \(S^3 \rightarrow SU(3) \rightarrow S^5\) is the canonical fibration and \(q\) is the composite \(q: \Sigma\mathbb{R}P^4_2 \rightarrow \Sigma\mathbb{C}P^2 \hookrightarrow SU(3)\).

**Lemma 6.18.** There is a homotopy equivalence

\[\Omega F \simeq \Omega S^3 \times \Omega B\]

**Proof.** Since \(\pi_3(S^3) \cong \pi_3(SU(3))\), \(\pi_3(F) \cong \pi_3(\Sigma\mathbb{R}P^4_2)\). Thus \(\pi_3(\Sigma\mathbb{R}P^4_2) \cong \pi_3(SU(3))\) and hence the result. \(\square\)

Recall that \(H_*(\Omega\mathbb{R}P^4_2) \cong T(\bar{H}_*(\mathbb{R}P^4)) \cong T(x_2, u_3, v_4)\) as algebras with \(Sq^1_3 v_4 = u_3\) and \(Sq^2_3 v_4 = x_2\). Although \(H_*(\Omega\Sigma\mathbb{R}P^4_2)\) is not primitively generated, we will show that both \(H_*(\Omega F)\) and \(H_*(\Omega B)\) are primitively generated.

**Lemma 6.19.** Let \(\alpha \in PH_*(\Omega\Sigma\mathbb{R}P^4_2)\). Then \([\alpha, v_4] + x_2[\alpha, x_2]\) is primitive.

**Proof.** Observe that \(\psi(v_4) = v_4 \otimes 1 + 1 \otimes v_4 + x_2 \otimes x_2\) and \(\psi(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha\). We have

\[
\psi([\alpha, v_4]) = [\alpha, v_4] \otimes 1 + 1 \otimes [\alpha, v_4] + x_2 \otimes [x_2, \alpha] + [x_2, \alpha] \otimes x_2,
\]

\[
\psi(x_2[\alpha, x_2]) = x_2[\alpha, x_2] \otimes 1 + 1 \otimes x_2[\alpha, x_2] + x_2 \otimes [x_2, \alpha] + [x_2, \alpha] \otimes x_2
\]

and hence the result. \(\square\)

**Notation 6.20.** For \(\alpha \in H_*(\Omega\Sigma\mathbb{R}P^4_2) \cong T(x_2, u_3, v_4)\), we define \(\overrightarrow{ad}^0(v_4)(\alpha) = \alpha\), \(\overrightarrow{ad}^1 (v_4)(\alpha) = [\alpha, v_4] + x_2[\alpha, x_2]\) and \(\overrightarrow{ad}^k(v_4)(\alpha) = \overrightarrow{ad}^{k-1}(v_4)(\alpha)\) for \(k \geq 2\).

By Lemma 6.19, there are operations \(\overrightarrow{ad}^k(v_4) : PH_*(\Omega\Sigma\mathbb{R}P^4_2) \rightarrow PH_*(\Omega\Sigma\mathbb{R}P^4_2)\).

**Proposition 6.21.** \(H_*(\Omega F)\) is primitively generated and \(H_*(\Omega F) \cong T(V_F)\) as sub-Hopf algebra of \(H_*(\Omega\Sigma\mathbb{R}P^4_2) \cong T(x_2, u_3, v_4)\), where \(V_F \subseteq PH_*(\Omega\Sigma\mathbb{R}P^4_2)\) has a basis \(\overrightarrow{ad}^k(v_4)(x_2)\) and \(\overrightarrow{ad}^k(v_4)(u_3)\) for \(k \geq 0\).
Proof. Observe that \( \Omega p_* : H_*(\Omega \Sigma \mathbb{R}P^4) \to H_*(\Omega S^5) \) is an epimorphism. The Serre spectral sequence for the fibre sequence \( \Omega F \to \Omega \Sigma \mathbb{R}P^4 \to \Omega S^5 \) collapses and so there is a short exact sequence of Hopf algebras

\[
H_*(\Omega F) \longrightarrow H_*(\Omega \Sigma \mathbb{R}P^4) \to H_*(\Omega S^5).
\]

Thus there is an exact sequence

\[
0 \longrightarrow PH_*(\Omega F) \longrightarrow PH_*(\Omega \Sigma \mathbb{R}P^4) \longrightarrow PH_*(\Omega S^5).
\]

Since both \( x_2 \) and \( u_3 \) are primitive, the elements \( \text{ad}^k(v_4)(x_2), \text{ad}^k(v_4)(u_3) \in PH_*(\Omega F) \) for \( k \geq 0 \) by Lemma 6.19. Let \( A \) be the subalgebra of \( T(x_2, u_3, v_4) \) generated by \( \text{ad}^k(v_4)(x_2) \) and \( \text{ad}^k(v_4)(u_3) \) for \( k \geq 0 \). We claim that \( \mathbb{Z}/2 \otimes_A T(x_2, u_3, v_4) = T(v_4) \).

Observe that \( \mathbb{Z}/2 \otimes_A T(x_2, u_3, v_4) = T(x_2u_3v_4)/IA \cdot T(x_2, u_3, v_4) \). Since \( x_2, u_3 \in IA \), it suffices to show that \( v_4^k \cdot y \in IA \cdot T(x_2, u_3, v_4) \) for each \( k \geq 0 \), where \( y = x_2 \) or \( u_3 \).

This is shown by induction on \( k \). If \( k = 0 \), \( v_4^0 y = y \in IA \cdot T(x_2, u_3, v_4) \). Suppose that \( v_4^k y \in IA \cdot T(x_2, u_3, v_4) \). Then

\[
v_4^{k+1} y = \sum_i v_4^{i} \text{ad}^{i}(v_4)(y_i) \cdot \alpha_i
\]

for some \( t_i \geq 0 \), \( y_i = x_2 \) or \( u_3 \) and \( \alpha_i \in T(x_2, u_3, v_4) \) and so

\[
v_4^{k+1} y = \sum_i v_4^{i} \text{ad}^{i}(v_4)(y_i) \cdot \alpha_i.
\]

It suffice to show that \( v_4^{i} \text{ad}^{i}(v_4)(y_i) \in IA \cdot T(x_2, u_3, v_4) \) for each \( i \). Now

\[
\text{ad}^{i+1}(v_4)(y_i) = [\text{ad}^{i}(v_4)(y_i), v_4] + x_2 \cdot \text{ad}^{i}(v_4)(y_i), x_2
\]

Thus

\[
v_4^{i} \text{ad}^{i}(v_4)(y_i) = \text{ad}^{i+1}(v_4)(y_i) + \text{ad}^{i}(v_4)(y_i) \cdot v_4 + x_2 [\text{ad}^{i}(v_4)(y_i), x_2] \in IA \cdot T(x_2, u_3, v_4).
\]

The induction is completed.

Similarly, \( T(x_2, u_3, v_4) \otimes_A \mathbb{Z}/2 = T(v_4) \). Thus \( A \) is a normal subalgebra of \( T(x_2, u_3, v_4) \). Observe that

\[
\chi(A) = (1 - t^2 - t^3 - t^4)^{-1} \cdot (1 - t^4) = (1 - \sum_{k=0}^{\infty} (t^{4k+2} + t^{4k+3})^{-1} = \chi(H_*(\Omega F)) = \chi(T(V_F)).
\]

Since \( A \subseteq H_*(\Omega F) \) and \( T(V_F) \to A \) is onto, \( A = H_*(\Omega F) \) and so

\[
T(V_F) \cong A = H_*(\Omega F).
\]

Note that \( V_F \subseteq PH_*(\Omega \Sigma \mathbb{R}P^4) \). Thus \( H_*(\Omega F) \) is primitively generated and hence the result.

\[\square\]

Corollary 6.22. The Hopf algebra \( H_*(\Omega B) \) is primitively generated and \( H_*(\Omega B) \cong T(V_B) \) as a subalgebra of \( H_*(\Omega \Sigma \mathbb{R}P^4) \cong T(x_2, u_3, v_4) \), where \( V_B \subseteq PH_*(\Omega \Sigma \mathbb{R}P^4) \) has a basis \( \text{ad}^{j_2}(x_2) \text{ad}^{j_1}(v_4)(x_2) \) for \( j_2 \geq 0, j_1 \geq 1 \) and \( \text{ad}^{j_2}(x_2) \text{ad}^{j_1}(v_4)(u_3) \) for \( j_2, j_1 \geq 0 \).
Proof. There is a short exact sequence of Hopf algebras
\[ H_*(\Omega B) \hookrightarrow H_*(\Omega F) \rightarrow H_*(\Omega S^3). \]
Note that \( H_*(\Omega F) \cong T(V_F) \) is primitively generated. The assertion follows from [9, Lemma 3.13].

Now we are going to decompose \( \Omega B \) up to dimension 11.

**Lemma 6.23.** There exists a map \( \theta: B \rightarrow \Omega S^5 \) so that the composite
\[ S^4 \xrightarrow{j} B \xrightarrow{\theta} \Omega S^5 \]
is of degree 1.

**Proof.** Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
SU(3) & \xrightarrow{2} & SU(3) \\
\uparrow \Sigma \mathbb{R}P^4_2 & \xrightarrow{[2]} & \Sigma \mathbb{R}P^4_2 \\
\Sigma \mathbb{R}P^4_2 & \xrightarrow{p} & S^5 \\
\downarrow p & & \downarrow p \\
\Sigma \mathbb{R}P^4_2 & \xrightarrow{[2]} & \Sigma \mathbb{R}P^4_2 \\
\downarrow \text{pinch} & & \downarrow \text{pinch} \\
P^5(2) & \xrightarrow{[2]} & P^5(2) \\
\end{array}
\]

The composite \( \Sigma \mathbb{R}P^4_2 \rightarrow SU(3) \xrightarrow{2} SU(3) \rightarrow S^5 \) is null homotopic and so there is a homotopy commutative diagram
\[
\begin{array}{ccc}
\Omega SU(3) & \xrightarrow{\theta} & \Sigma \mathbb{R}P^4_2 \\
\downarrow 2 & & \downarrow 2 \\
\Omega SU(3) & \xrightarrow{\theta_*} & SU(3) \\
\end{array}
\]
Recall that \( \pi_4(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/4 \). From the commutative diagram of exact sequences
\[
\begin{array}{ccc}
0 & \rightarrow & \pi_4(\Omega SU(3)) \\
\downarrow 2 & & \downarrow \theta_* \\
0 & \rightarrow & \pi_4(\Omega S^5) \\
\end{array}
\]
the map \( \theta_* : \pi_4(B) \rightarrow \pi_4(\Omega S^5) \) is of degree 1 and hence the result.

**Corollary 6.24.** The bottom cell \( S^3 \) is a stable retract of \( \Omega B \)

**Proof.** There is a retraction: \( \Sigma^\infty \Omega^2 S^5 \rightarrow \Sigma^\infty S^3 \) and hence the result.
Lemma 6.25. The 7-skeleton $\text{sk}_7(\Omega B)$ of $\Omega B$ is homotopy equivalent to $S^3 \vee S^7 \vee Y^7$, where $\tilde{H}_*(Y^7)$ has a basis $[x_2, u_3], [x_2, v_4], \overline{\text{ad}}(v_4)(u_4)$ with $Sq^4_1[x_2, v_4] = [x_2, u_3]$, $Sq^2_1 \overline{\text{ad}}(v_4)(u_3) = [x_2, u_3]$ and the second Bockstein $\beta_2(\overline{\text{ad}}(v_4)(u_3)) = u_3^2$.

Proof. By the Cartan formula, we have

$$
Sq^4_1[x_2, v_4] = [x_2, u_3] \quad \text{and} \quad Sq^2_1 \overline{\text{ad}}(v_4)(u_3) = [x_2, u_3].
$$

We first show that $\beta_2(\overline{\text{ad}}(v_4)(u_3)) = u_3^2$. Consider the homotopy commutative diagram of fibre sequences

$$
\begin{array}{ccc}
\Omega B & \longrightarrow & \Omega S^3 \\
\Omega \phi & & \Omega SU(3) \\
\Omega F^5\{2\} & \longrightarrow & \Omega P^5(2) \longrightarrow \Omega S^5
\end{array}
$$

We have $\Omega \phi_*(\overline{\text{ad}}(v_4)(u_3)) = [u_3, v_4]$ and $\Omega \phi_*(u_3^2) = u_3^2$ in $H_*\Omega F^5\{2\}$. Since $\beta_1(\overline{\text{ad}}(v_4)(u_3)) = Sq^4_1(\overline{\text{ad}}(v_4)(u_3)) = 0$, we have $\Omega \phi_*(\beta_2(\overline{\text{ad}}(v_4)(u_3))) = u_3^2$ and so $\beta_2(\overline{\text{ad}}(v_4)(u_3)) = u_3^2 + \varepsilon[x_2, v_4] \equiv u_3^2$ in Bockstein spectral sequence for $\tilde{H}_*(\Omega B)$. Now $\tilde{H}_*(\text{sk}_7(\Omega B))$ has a basis

$$
\{u_3, [x_2, u_3], u_3^2, [x_2, v_4], \overline{\text{ad}}(v_4)(u_3), [[x_2, u_3], x_2].
$$

Since $x_2$ and $u_3$ are spherical in $H_*(\Omega S^3)$, the elements $u_3, [x_2, u_3], u_3^2, [[x_2, u_3], x_2]$ are spherical in $\Omega(\Sigma \mathbb{RP}^4_2)$ by Samelson products and Hopf Invariants 1. Observe that $\Omega(\Sigma \mathbb{RP}^4_2)$ is a commutative H-space. All of Samelson products lift to the fibre $\Omega B$. It follows that $u_3, [x_2, u_3], u_3^2$ and $[[x_2, u_3], u_3]$ are spherical in $H_*(\Omega B)$. Thus the 5-skeleton $\text{sk}_5(\Omega B) \simeq S^3 \vee S^5$ and the 6-skeleton $\text{sk}_6(\Omega B) \simeq S^6 \vee X$, where $\tilde{H}_*(X)$ has a basis $\{u_3, [x_2, u_3], [x_2, v_4]\}$. The space $X$ is the homotopy cofibre of the attaching map $S^5 \xrightarrow{f} S^3 \vee S^5$. Since $Sq^4_1[x_2, v_4] = [x_2, u_3]$, the composite

$$
S^5 \xrightarrow{f} S^3 \vee S^5 \xrightarrow{\text{proj}} S^5
$$

is of degree 2. Let $\phi$ be the composite $S^5 \xrightarrow{f} S^3 \vee S^5 \xrightarrow{\text{proj}} S^3$. Then $\phi = k\eta^2$ for some $k = 0, 1$. If $k = 1$, then $\pi_5(X) = \mathbb{Z}/4$ and $\pi_5(S^3)$ is not a summand of $\pi_5(X)$, which contradicts the facts that

i) $\pi_5(S^3) \cong \pi_5(\Omega B) = \mathbb{Z}/2$;

ii) $S^5$ is a stable retract of $\Omega B$ by Corollary 6.24.

iii) $\pi_5(\Omega B) \cong \pi_5(\Omega B)$.

Thus $k = 0$ and so $X \simeq S^3 \vee P^6(2)$ or $\text{sk}_6(\Omega B) \simeq S^3 \vee S^6 \vee P^6(2)$. Now we have $\text{sk}_7(\Omega B) \simeq S^7 \vee Y$, where $\tilde{H}_*(Y)$ has a basis $u_3, [x_2, u_3], u_3^2, [x_2, v_4]$ and $\overline{\text{ad}}(v_4)(u_3)$. There is a cofibre sequence

$$
S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \longrightarrow Y.
$$

Since $Sq^2_1 \overline{\text{ad}}(v_4)(u_3) = [x_2, u_3]$, the composite $S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \xrightarrow{\text{proj}} P^6(2)$ is $\tilde{\eta}$ which is a generator for $\pi_6(P^6(2)) = \mathbb{Z}/2$. Note that $\beta_2(\overline{\text{ad}}(v_4)(u_3)) = u_3^2$. The composite $S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \xrightarrow{\text{proj}} S^6$ is of degree 4. Observe that the composite
Y \longrightarrow \Omega B \longrightarrow \Omega^2 S^5 \text{ maps into the 7-skeleton } \text{sk}_7(\Omega^2 S^5) \text{ up to homotopy. There is a commutative diagram of cofibre sequences}

\[
\begin{array}{c}
S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \xrightarrow{t} Y \xrightarrow{t} S^7 \\
\downarrow \text{proj.} \quad \downarrow \quad \downarrow \quad \downarrow \\
S^6 \xrightarrow{(\nu',[2])} S^3 \vee S^6 \xrightarrow{\text{sk}_7(\Omega^2 S^5)} Y \xrightarrow{t} S^7
\end{array}
\]

Thus the map \( t : S^6 \to S^6 \) is of degree 2 and there is a projection \( p_1 : S^3 \vee S^6 \vee P^6(2) \to S^3 \) such that the composite \( S^6 \xrightarrow{\phi} S^3 \vee S^6 \vee P^6(2) \xrightarrow{p_1} S^3 \) is \( 2\nu' = \eta_3^3 = E\eta_3^2 \). Hence \( \phi = (\eta_3^2, 4, \bar{\eta}) = E(\eta_2^3, 4, \bar{\eta}) : S^6 \to S^3 \vee S^6 \vee P^6(2) \).

It follows that \( Y \simeq \Sigma Y' \) for some \( Y' \). Now let \( f : S^3 \vee S^6 \vee P^6(2) \to S^3 \) be the composite \( S^3 \vee S^6 \vee P^6(2) \xrightarrow{p} P^6(2) \xrightarrow{\eta_3^2} S^3 \), where \( p \) is the projection and \( \eta_3^2 \) is the extension of \( \eta_3^2 : S^5 \to S^3 \). Then \( f \circ \phi = \eta_3^3 \). Define \( g : S^3 \vee S^6 \vee P^6(2) \to S^3 \) by \( g = p_1 + f \). Then \( [g] \circ [\phi] = [p_1 \circ \phi] + [f \circ \phi] = 2\nu' + \eta_3^3 = 4\nu' = 0 \) and so \( g \) extends to \( \tilde{g} : Y \to S^3 \). Note that the composite \( S^3 \longrightarrow S^3 \vee S^6 \vee P^6(2) \xrightarrow{g} S^3 \) is a homotopy equivalence and \( Y \) is a suspension. The space \( S^3 \) is a retract of \( Y \) and hence the result. \( \square \)

**Notation 6.26.** Let \( \bar{L}_3 \) denote \( \text{hocolim}_{\beta_3} \Sigma(\mathbb{R}P^4_2)^{(3)} \), where \( \beta_3 \) is the composite

\[
\Sigma X(3) \xrightarrow{\text{id} \circ \Sigma T_{12}} \Sigma X(3) \xrightarrow{\text{id} \circ \Sigma \sigma} \Sigma X(3)
\]

with \( T_{12}(x_1 \wedge x_2 \wedge x_3) = (x_2 \wedge x_1 \wedge x_3) \) and \( \sigma(x_1 \wedge x_2 \wedge x_3) = x_3 \wedge x_1 \wedge x_2 \). Then \( \Sigma(\mathbb{R}P^4_2)^{(3)} \simeq \bar{L}_3 \vee M_3 \), see Chapter 3. Let \( L_3 \) denote \( \text{sk}_{11}(\Omega(\bar{L}_3)) \).

**Lemma 6.27.** With the notations above, we have

1) \( \Sigma L_3 \simeq \bar{L}_3 \).
2) \( L_3 \) is a retract of \( (\mathbb{R}P^4_2)^{(3)} \).

**Proof.** The map \( L_3 \longrightarrow \Omega \bar{L}_3 \) induces an \( H \)-map \( \Omega \Sigma L_3 \to \Omega \bar{L}_3 \). Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{c}
\Omega \Sigma L_3 \longrightarrow \ast \longrightarrow \Sigma L_3 \\
\Omega \bar{L}_3 \longrightarrow \ast \longrightarrow \bar{L}_3.
\end{array}
\]

Since \( \bar{L}_3 \) is 8-connected and \( \text{dim} \bar{L}_3 = 12 \), the homology suspension \( \sigma|_{H_*(L_3)} : \tilde{H}_*(L_3) \longrightarrow \tilde{H}_{*+1}(\bar{L}_3) \) is an isomorphism and hence assertion 1.

2) Note that \( J((\mathbb{R}P^4_2)^{(3)}) / (\mathbb{R}P^4_2)^{(3)} \) is 11-connected. The composite

\[
L_3 \longrightarrow \Omega \bar{L}_3 \longrightarrow J((\mathbb{R}P^4_2)^{(3)})
\]
maps into the subcomplex \((\mathbb{R}P^4_2)^{(3)}\) up to homotopy. Since \(\Omega L_3/L_3 \simeq \Omega \Sigma L_3/L_3\) is 13-connected, \((\mathbb{R}P^4_2)^{(3)} \to \Omega L_3\) maps into \(L_3\) up to homotopy. Thus there is a commutative diagram

\[
\begin{array}{ccc}
\Omega L_3 & \xrightarrow{\Omega \Sigma(\mathbb{R}P^4_2)^{(3)}} & \Omega \text{hocolim} \beta_3 \\
\downarrow & & \downarrow \\
L_3 & \xrightarrow{\text{(}\mathbb{R}P^4_2)^{(3)}\text{)}}, & L_3
\end{array}
\]

and hence the result. \(\square\)

**Proposition 6.28.** There is a map which is a homotopy equivalence through 11-skeleton

\[\phi: S^3 \times \Omega \Sigma Y^7 \times \Omega \Sigma L_3 \times P^{11}(2) \times \Sigma^6 \mathbb{R}P^5_3 \times S^{11} \to \Omega B.\]

**Proof.** The idea is to construct a map \(\phi\) and show that \(\phi\) induces an isomorphism in homology through dimension 11. The construction of the map \(\phi\) is given by several steps.

**Step 1.** Let \(\phi_1: \Omega \Sigma L_3 \to \Omega B\) be the composite

\[
\begin{array}{ccc}
\Omega \Sigma L_3 & \xrightarrow{\Omega \Sigma(\mathbb{R}P^4_2)^{(3)}} & \Omega \tilde{S}_3 \\
\downarrow & & \downarrow \\
\Omega L_3 & \xrightarrow{\phi} & \Omega B
\end{array}
\]

where \(\tilde{S}_3\) is the \(H\)-map induced by the Samelson product \(S_3: (\mathbb{R}P^4_2)^{(3)} \to \Omega B\). We check the homology information. Observe that \(\tilde{H}_*(L_3) = PH_*(L_3)\) has a basis \(\alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \beta_8, \beta_9, \beta_{10}\) and \(\beta_{11}\) with the Steenrod operations given as follows:

\[
S_3^4 \beta_{11} = \alpha_7, S_3^3 \beta_{11} = \beta_8, S_3^2 \beta_{11} = \beta_9, S_3 \beta_{11} = \beta_{10}
\]

\[
S_3^3 \alpha_{10} = \alpha_7, S_3^2 \alpha_{10} = \alpha_8, S_3 \alpha_{10} = \alpha_9, S_3^2 \alpha_9 = \alpha_8 \quad \text{and} \quad S_3 \alpha_8 = \alpha_7.
\]

Note that \(H_7(\Omega(\mathbb{R}P^4_2)^{(3)})) \cong H_7(\mathbb{R}P^4_2)^{(3)}\) has a basis

\[
\{x_2 \otimes x_2 \otimes u_3, x_2 \otimes u_3 \otimes x_2, u_3 \otimes x_2 \otimes x_2\},
\]

where both \(x_2\) and \(u_3\) are primitive. Thus \(\tilde{S}_3: H_7(\Omega \Sigma \mathbb{R}P^4_2) \to H_7(\Omega B)\) is the Lie bracket and \(\beta_1(\alpha_7) = [u_3, x_2^2]\). Now

\[
\phi_1(\beta_{11}) = \text{ad}^2(u_4)(u_3) + l_1[[x_2, u_3], [x_2, u_4]] + l_2[[x_2, v_4], x_2], u_3] + l_3 \text{ad}^2(x_2)(\text{ad}(u_4)(u_3)) + l_4 \text{ad}^4(x_2)(u_3) + l_5 \text{ad}^3(u_3)(x_2),
\]

\[
\phi_1(\beta_8) = S_3^3 \phi_1(\beta_{11}) = [x_2, u_3],
\]

\[
\phi_1(\beta_{10}) = S_3 \phi_1(\beta_{11}) = [x_2, v_4], u_3] + l_3 \text{ad}^3(x_2)(u_3),
\]

\[
\phi_1(\alpha_{10}) = \overline{\text{ad}}^2(u_4)(x_2) + k_1[x_2, u_3]^2 + k_2 \overline{\text{ad}}(u_4)(u_3), u_3] + k_3[[u_3, x_2^2], u_3] + k_4 \text{ad}^2(x_2)([u_4, x_2]),
\]

\[
\phi_1(\alpha_8) = [u_4, x_2^2] + [x_2, u_3^2] + k_2[x_2, u_3^3],
\]

\[
\phi_1(\alpha_9) = [\overline{\text{ad}}(u_4)(u_3), x_2] + k_4 \text{ad}^3(x_2)(u_3).
\]

**Step 2.** Let \(\phi_2: \text{sk}_8(L_3) \wedge S^3 \to \Omega B\) be the Samelson product of

\[
\phi_1|_{\text{sk}_8(L_3)}: \text{sk}_8(L_3) \longrightarrow \Omega B
\]
and \( u_3: \mathbb{S}^3 \to \Omega B \). Then
\[
\phi_2^* (\alpha_7 \otimes \iota_3) = [u_3, x_2^3], \\
\phi_2^* (\alpha_8 \otimes \iota_3) = [u_3, x_2^3] + (k_2 + 1)[x_2, u_3^2], \\
\phi_2^* (\beta_8 \otimes \iota_3) = [x_2, u_3^2].
\]

**Step 3.** We give a decomposition of \( \tilde{H}_*(\sk_9(L_3)) \) in this step. Observe that \( \tilde{H}_*(\sk_9(L_3)) \) has a basis \( \alpha_7, \alpha_8, \beta_8 \) with \( \text{Sq}^1 \alpha_8 = \alpha_7 \). Thus
\[
\sk_9(L_3) \simeq \mathbb{S}^8 \vee P^8(2).
\]
Note that \( \tilde{H}_*(\sk_9(L_3)) \) has a basis \( f \alpha_7, \alpha_8, \alpha_9, \beta_9, \beta_9 \) with \( \text{Sq}^2 \alpha_9 = \alpha_7, \text{Sq}^1 \alpha_8 = \alpha_7 \) and \( \text{Sq}^1 \beta_9 = \beta_8 \). There is a cofibre sequence
\[
\begin{array}{ccc}
\mathbb{S}^8 \vee \mathbb{S}^8 & \xrightarrow{\psi} & \mathbb{S}^8 \vee P^8(2) \\
& \searrow & \downarrow \\
& & \sk_9(L_3)
\end{array}
\]
Let \( \psi_1 \) be the composite \( \mathbb{S}^8 \vee \mathbb{S}^8 \xrightarrow{\psi} \mathbb{S}^8 \vee P^8(2) \xrightarrow{\text{proj.}} P^8(2) \). Recall that
\[
[S^8 \vee \mathbb{S}^8, P^8(2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]
We have \([\psi_1] = \epsilon_1 \tilde{\eta}^{(1)} + \epsilon_2 \tilde{\eta}^{(2)}\) for some \( \epsilon_i = 0, 1 \). Since \( \text{Sq}^2 \alpha_9 \neq 0 \) in \( H_*(\sk_9(L_3)) \), we have \((\epsilon_1, \epsilon_2) \neq (0, 0)\), that is, \([\psi_1] = \tilde{\eta}^{(1)}, \tilde{\eta}^{(2)}\) or \( \tilde{\eta}^{(1)} + \tilde{\eta}^{(2)}\). Let \( i: \mathbb{S}^8 \to \mathbb{S}^8 \vee \mathbb{S}^8 \) be \( \iota_8^{(2)}, \iota_8^{(1)} \) or \( \iota_8^{(1)} + \iota_8^{(2)} \) for the cases where \([\psi_1] = \tilde{\eta}^{(1)}, \tilde{\eta}^{(2)}\) or \( \tilde{\eta}^{(1)} + \tilde{\eta}^{(2)}\), respectively. Then \( \psi_1 \circ i \) is null homotopic and so there is a homotopy commutative diagram of cofibre sequences
\[
\begin{array}{c}
\mathbb{S}^8 \xrightarrow{[k]} \mathbb{S}^8 \xrightarrow{i} \mathbb{S}^8 \vee \mathbb{S}^8 \xrightarrow{\psi} \mathbb{S}^8 \vee P^8(2) \xrightarrow{\text{proj.}} \sk_9(L_3) \\
\downarrow j \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\Sigma^4 \mathbb{R}P^5 \xrightarrow{\theta} P^{10}(2)
\end{array}
\]
Let \( v \) be a basis for \( H_9(P^9(k)) \). Then \( \text{Sq}^2 \alpha_9 (v) = 0 \) and \( j_* (v) = \beta_9 \). Thus \( \text{Sq}^1 \alpha_9 (v) \neq 0 \) and so \( k = 2l \) with \( l \equiv 1 \mod 2 \). Consider the cofibre sequence
\[
P^9(2) \xrightarrow{j} \sk_9(L_3) \xrightarrow{\Sigma^4 \mathbb{R}P^5} P^{10}(2).
\]
There is a homotopy commutative diagram
\[
\begin{array}{c}
\Sigma^4 \mathbb{R}P^5 \xrightarrow{\theta} P^{10}(2) \\
\downarrow \text{pinch} \downarrow \theta' \\
S^9 \xrightarrow{=} S^9.
\end{array}
\]
Note that \( \theta_* : H_\ast(\Sigma\mathbb{R}P^5_3) \to H_\ast(P^{10}(2)) \) is zero. Thus \( \theta' \) is null homotopic and so is \( \theta \). Observe that the 9-skeleton \( \text{sk}_9(L_3) \) is the 14-skeleton of the homotopy fibre of \( \theta \) and \( \text{sk}_9(L_3) \) is a suspension. We have \( \text{sk}_9(L_3) \cong P^{10}(2) \vee \Sigma^4\mathbb{R}P^3_3 \).

**Step 4.** Let \( \phi_3 : \Sigma^4\mathbb{R}P^5_3 \land S^2 \to \Omega\Sigma\mathbb{R}P^4_2 \) be Samelson product of the map \( x_2 : S^2 \to \Omega\Sigma\mathbb{R}P^4_2 \) and the composite

\[
\Sigma^4\mathbb{R}P^5_3 \subset \text{sk}(L_3) \subset L_3 \xrightarrow{\phi_1} \Omega B \to \Omega\Sigma\mathbb{R}P^4_2.
\]

Then \( \bar{\phi}_3 \) lifts to \( \Omega B \). Let \( \phi_3 : \Sigma^4\mathbb{R}P^5_3 \land S^2 \to \Omega B \) be a lifting of \( \bar{\phi}_3 \). Let \( x_7, x_8, x_9 \) be a basis for \( H_\ast(\Sigma^4\mathbb{R}P^5_3) \). Let \( f : \Sigma^4\mathbb{R}P^5_3 \to L_3 \) be the inclusion. Then \( f_\ast(x_7) = \alpha_7, f_\ast(x_8) = \alpha_9 \) and \( f_\ast(x_8) = \alpha_8 + \epsilon \beta_8 \) for some \( \epsilon = 0, 1 \). Then

\[
\phi_{3\ast}(x_7 \otimes x_2) = [\phi_{1\ast}(\alpha_7), x_2] = \text{ad}^3(x_2)(u_3),
\]

\[
\phi_{3\ast}(x_8 \otimes x_2) = [\phi_{1\ast}(\alpha_8), x_2] + \epsilon [\phi_{1\ast}(\beta_8), x_2] = [[v_4, x_2^2], x_2] + (1 + \epsilon + k_2)([x_2, u_3^2], x_2),
\]

\[
\phi_{3\ast}(x_9 \otimes x_2) = [\phi_{1\ast}(\alpha_9), x_2][\text{ad}(u_4)(u_3), x_2] + k_4 \text{ad}^4(x_2)(u_3) - 2
\]

Note that \( \text{ad}^4(x_2)(u_3) \) is spherical.

**Step 5.** Let \( \phi_4 : S^{11} \to \Omega B \) be a map such that \( \phi_{4\ast}(\iota_{11}) = \text{ad}^4(x_2)(u_3) \) and let \( \pi_1 : S^3 \to \Omega B \) be a map such that \( \phi_{1\ast}(\iota_3) = u_3 \). Let \( \phi_0 : \Omega\Sigma Y^7 \to \Omega B \) be the \( H \)-map induced by the inclusion \( Y^7 \to \Omega \). We obtain a map

\[
\phi : S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}P^5_3 \times S^{11} \to \Omega B
\]

by taking the product of the maps \( \phi_j \) for \( -1 \leq j \leq 4 \). Now we show that \( \phi \) induces an isomorphism in homology. Let \( Z \) denote \( S^3 \times \Omega\Sigma Y^7 \times \Omega\Sigma L_3 \times P^{11}(2) \times \Sigma^6\mathbb{R}P^5_3 \times S^{11} \). Let \( V \) be the image of \( \phi_{1\ast} : PH_{15}(Z) \to PH_{15}(\Omega B) \). Then Clearly \( V_r = PH_r(\Omega B) \) for \( r \leq 8 \). Since \( V_9 \) is generated by

\[
[[x_2, v_4], u_3] + l_3 \text{ad}^3(x_2)(u_3), [[\text{ad}(u_4)(u_3), x_2] + k_4 \text{ad}^3(x_2)(u_3), \text{ad}^3(x_2)(u_3),
\]

we have \( V_9 = PH_9(\Omega B) \). Observe that \( V_{10} \) is generated by

\[
[x_2, u_3]^2, [[\text{ad}(u_4)(u_3), u_3] + l_2[[u_3, x^2_2], u_3],
\]

\[
\text{ad}^2(v_4)(x_2) + k_1[x_2, u_3]^2 + k_2[\text{ad}(u_4)(u_3), u_3] + k_3[[u_3, x_2^2], u_3] + k_4 \text{ad}^2(x_2)(u_4),
\]

\[
[u_3, x_2^2], u_3] + (1 + \epsilon + k_2)([x_2, u_3^2], x_2).
\]

Thus \( V_{10} = PH_{10}(\Omega B) \). Now \( V_{11} \) is generated by

\[
[[x_2, u_3], [x_2, v_4]],
\]

\[
\text{ad}^2(v_4)(u_3) + l_1[[x_2, u_3], [x_2, v_4] + l_2[[x_2, v_4], x_2], u_3] + l_3 \text{ad}^2(x_2)[\text{ad}(v_4)(u_3)] +
\]

\[
+ l_4 \text{ad}^4(x_2)(u_3) + l_5 \text{ad}^3(u_3)(x_2),
\]

\[
[[x_2, u_3], u_3], \text{ad}^2(x_2)[\text{ad}(v_4)(u_3) + k_4 \text{ad}^4(x_2)(u_3),
\]

\[
[[v_4, x_2^2], u_3] + (k_2 + 1)[[x_2, u_3^2], u_3], \text{ad}^4(x_2)(u_3). \]

Thus \( V_{11} = PH_{11}(\Omega B) \). Note that \( \dim PH_r(Z) = \dim PH_r(\Omega B) \) for \( r \leq 11 \). Thus \( \phi_* : PH_{15}(Z) \to PH_{15}(\Omega B) \) is an isomorphism and so \( \phi_* : H_r(Z) \to H_r(\Omega B) \) is a monomorphism for \( r \leq 11 \). Since \( \dim H_r(Z) = \dim H_r(\Omega B) \) for \( r \leq 11 \), the map \( \phi_* : H_r(Z) \to H_r(\Omega B) \) is an isomorphism for \( r \leq 11 \). We finish the proof now. \( \square \)
Corollary 6.29. There is a decomposition
\[ \pi_r(B) \cong \pi_{r-1}(S^3) \oplus \pi_r(\Sigma Y^7) \oplus \pi_r(\Sigma L_3) \oplus \pi_r(P^{12}(2)) \oplus \pi_r(\Sigma^7 \mathbb{R}P^3) \]
for \( r \leq 11 \).

Observe that \( \Sigma Y^7 \) is 5-connected.

Proposition 6.30. The homotopy groups \( \pi_r(\Sigma Y^7) \) for \( 6 \leq r \leq 11 \) are as follows.
1. \( \pi_6(\Sigma Y^7) = \mathbb{Z}/2 \).
2. \( \pi_7(\Sigma Y^7) = \mathbb{Z}/8 \).
3. \( \pi_8(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).
4. \( \pi_9(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).
5. \( \pi_{10}(\Sigma Y^7) = \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).
6. \( \pi_{11}(\Sigma Y^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \).

Proof. 1. \( \pi_r(\Sigma Y^7) \) for \( r \leq 9 \): Observe that \( \tilde{H}_*(Y^7) \) has a basis \([x_2, u_3], u_3^2, [x_2, v_4] \) and \( \overline{\text{ad}}(v_4)(u_3) \) with
\[ Sq_2[x_2, v_4] = [x_2, u_3], Sq_2 \overline{\text{ad}}(v_4)(u_3) = [x_2, u_3] \quad \text{and} \quad \beta_2 \overline{\text{ad}}(v_4)(u_3) = u_3^2. \]
Thus \( Y^7 \) is the homotopy cofibre of \( (4, \hat{\eta}) : S^6 \longrightarrow S^6 \vee P^7(2) \) and so there is a cofibre sequence
\[ S^7 \xrightarrow{(4, \hat{\eta})} S^7 \vee P^7(2) \longrightarrow \Sigma Y^7 \longrightarrow S^8. \]
Since \( S^7 \vee P^7(2) \) is the 12-skeleton of the homotopy fibre of the pinch map \( \Sigma Y^7 \rightarrow S^8 \), the sequence
\[ \pi_r(S^7) \xrightarrow{(4, \hat{\eta})} \pi_r(S^7 \vee P^7(2)) \longrightarrow \pi_r(\Sigma Y^7) \longrightarrow \pi_{r-1}(S^7) \longrightarrow \cdots \]
is exact for \( r \leq 11 \). Thus \( \pi_6(\Sigma Y^7) \cong \pi_6(P^7(2)) = \mathbb{Z}/2 \). By expecting the exact sequence \( \pi_7(S^7) \xrightarrow{(4, \hat{\eta})} \pi_7(S^7) \oplus \pi_7(P^7(2)) \pi_7(\Sigma Y^7) \longrightarrow 0 \), we have \( \pi_7(\Sigma Y^7) \cong \mathbb{Z}/8 \) and there is an exact sequence
\[ \pi_8(S^7) \xrightarrow{(4, \hat{\eta})} \pi_8(S^7) \oplus \pi_8(P^7(2)) \longrightarrow \pi_8(\Sigma Y^7) \longrightarrow 0. \]
Observe that \( (4, \hat{\eta})_* (\eta_7) = 2\alpha \), where \( \alpha \) is a generator for \( \pi_8(P^7(2)) \cong \mathbb{Z}/4 \). We obtain \( \pi_8(\Sigma Y^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and there is an exact sequence
\[ \pi_9(S^7) \xrightarrow{(4, \hat{\eta})} \pi_9(S^7) \oplus \pi_9(P^7(2)) \longrightarrow \pi_9(\Sigma Y^7) \longrightarrow 0. \]
Since \( (4, \hat{\eta})_* : \pi_9(S^7) \rightarrow \pi_9(S^7) \oplus \pi_9(P^7(2)) \) is zero, we have
\[ \pi_9(\Sigma Y^7) \cong \pi_9(S^7) \oplus \pi_9(P^7(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]

2. \( \pi_{10}(\Sigma Y^7) \): Now there is an exact sequence
\[ \pi_{10}(S^7) \xrightarrow{(4, \hat{\eta})} \pi_{10}(S^7) \oplus \pi_{10}(P^7(2)) \longrightarrow \pi_{10}(\Sigma Y^7) \longrightarrow \pi_9(S^7) \longrightarrow 0, \]
where \( (4, \hat{\eta})_* (\nu_7) = 4\nu_7 + \hat{\eta}_*(\nu_7) = 4\nu_7 + j_*(\eta_6 \circ \nu_7) = 4\nu_7 \) and \( j : S^6 \rightarrow S^7 \) is the inclusion of bottom cell. Thus there is a short exact sequence
\[ 0 \longrightarrow \pi_{10}(S^7)/4 \oplus \pi_{10}(P^7(2)) \longrightarrow \pi_{10}(\Sigma Y^7) \longrightarrow \pi_9(S^7) \longrightarrow 0 \]
Note that \( \pi_{10}(P^7(2)) \cong \pi_{10}(P^7(2)) \cong \mathbb{Z}/2 \). There is a short exact sequence
\[ 0 \longrightarrow \mathbb{Z}/4 \oplus \mathbb{Z}/2 \longrightarrow \pi_{10}(\Sigma Y^7) \longrightarrow \mathbb{Z}/2 \longrightarrow 0. \]
We need to solve the group extension problem. Consider the homotopy commutative diagram of cofibre sequences

\[
\begin{array}{ccccccc}
P^7(2) & \xrightarrow{[2]} & P^7(2) & \xrightarrow{j_1} & P^6(2) \land P^2(2) & \xrightarrow{p_1} & P^8(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{(4, \bar{\eta})} & S^7 \lor P^7(2) & \xrightarrow{j_2} & \Sigma Y^7 & \xrightarrow{} & S^8 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{[2]} & S^7 & \xrightarrow{} & P^8(2) & \xrightarrow{} & S^8 \\
\end{array}
\]

Let \( \alpha \in \pi_{10}(P^6(2) \land P^2(2)) \) such that \( p_2 \circ p_{1*}(\alpha) = \eta_8^2 \). Then \( 2\alpha \) is the composite

\[
S^{10} \xrightarrow{\alpha} P^6(2) \land P^2(2) \xrightarrow{p_1} P^8(2) \xrightarrow{\eta \land \text{id}} P^7(2) \xrightarrow{j_1} P^6(2) \land P^2(2).
\]

Thus \( 2\alpha = j_{1*}(\bar{\eta}^3) \), where \( \bar{\eta}^3 \in \pi_{10}(P^7(2)) \cong \mathbb{Z}/2 \) is the generator. Hence \( j_{2*}(\bar{\eta}^3) \) is also divisible by 2 in \( \pi_{10}(\Sigma Y^7) \) and so \( \pi_{10}(\Sigma Y^7) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).

3. \( \pi_{11}(\Sigma Y^7) \): Consider the exact sequence

\[
\pi_{11}(S^7) = 0 \longrightarrow \pi_{11}(S^7) \oplus \pi_{11}(P^7(2)) \longrightarrow \pi_{11}(\Sigma Y^7) \longrightarrow 2 \cdot \pi_{10}S^7 \longrightarrow 0.
\]

There is a short exact sequence

\[
0 \longrightarrow \pi_{11}(P^7(2)) \longrightarrow \pi_{11}(\Sigma Y^7) \longrightarrow 2 \cdot \pi_{10}(S^7) \longrightarrow 0
\]

or

\[
0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{11}(\Sigma Y^7) \longrightarrow \mathbb{Z}/4 \longrightarrow 0.
\]

Thus \( \pi_{11}(\Sigma Y^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \) or \( \mathbb{Z}/8 \). We show that \( \pi_{11}(\Sigma Y^7) \neq \mathbb{Z}/8 \). From this, we will obtain \( \pi_{11}(\Sigma Y^7) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \).

Consider the homotopy commutative diagram

\[
\begin{array}{ccccccc}
P^8(2) & \xrightarrow{\eta \land \text{id}} & P^7(2) & \xrightarrow{} & \mathbb{C}P^2 \land P^5(2) & \xrightarrow{} & P^9(2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{\bar{\eta}} & P^7(2) & \xrightarrow{} & \Sigma R\mathbb{P}^5_3 & \xrightarrow{} & S^8 \\
\downarrow \text{proj.} & & \downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{(4, \bar{\eta})} & S^7 \lor P^7(2) & \xrightarrow{j_2} & \Sigma Y^7 & \xrightarrow{} & S^8 \\
\end{array}
\]

where the rows are cofibre sequences. Observe that \( P^7(2) \) is the 12-skeleton of the homotopy fibre of the pinch map \( \mathbb{C}P^2 \land P^5(2) \longrightarrow P^9(2) \). There is an exact sequence

\[
\pi_{11}(P^8(2)) \xrightarrow{\eta \land \text{id}} \pi_{11}(P^7(2)) \longrightarrow \pi_{11}(\mathbb{C}P^2 \land P^5(2)) \longrightarrow \pi_{11}(P^9(2)).
\]

Let \( \alpha_n \) be a generator for \( \pi_{n+1}(P^n(2)) \cong \mathbb{Z}/4 \) for \( n \geq 3 \). Note that \( \pi_{11}(P^8(2)) = \mathbb{Z}/2 \) and \( p_* (\alpha_8 \circ \eta^2) = \eta^3 = 4\nu \) in \( \pi_{11}(S^8) \), where \( p : P^8(2) \to S^8 \) is the pinch map. Thus \( \pi_{11}(P^8(2)) \) is generated by \( \alpha_8 \circ \eta^2 \). Also note that \( \pi_9(P^7(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) generated
by \( \alpha_7 \circ \eta \) and \( j_*(\nu_6) \), where \( j : S^6 \to P^7(2) \) the inclusion, and \( p_*((\eta \land \text{id}) \circ \alpha_8) = \eta^2 \).

We have

\[
(\eta \land \text{id}) \circ \alpha_8 = \alpha_7 \circ \eta + \epsilon \cdot j_*(\nu_6)
\]

and

\[
(\eta \land \text{id}) \circ \alpha_8 \circ \eta = \alpha_7 \circ \eta^2 + \epsilon j_*(\nu_6 \circ \eta) = \alpha_7 \circ \eta^2
\]

for some \( \epsilon = 0, 1 \). Thus

\[
(\eta \land \text{id}) \circ \alpha_8 \circ \eta^2 = \alpha_7 \circ \eta^3 = \alpha_7 \circ 4 \circ \nu_8 = 0.
\]

Hence \( (\eta \land \text{id})_* : \pi_{11}(P^8(2)) \to \pi_{11}(P^7(2)) \) is zero and so there is an exact sequence

\[
0 \longrightarrow \pi_{11}(P^7(2)) = \mathbb{Z}/2 \longrightarrow \pi_{11}(\text{CP}^2 \land P^5(2)) \longrightarrow \pi_{11}(P^9(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

Now suppose that \( \pi_{11}(\Sigma^7 Y) = \mathbb{Z}/8 \). Then \( \tilde{j}_2(\beta) \) is divisible by 4, where \( \beta \in \pi_{11}(P^7(2)) \) the generator. From the commutative diagram

\[
\begin{array}{ccc}
\pi_{11}(P^7(2)) & \longrightarrow & \pi_{11}(\text{CP}^2 \land P^5(2)) \\
\downarrow & & \downarrow f_* \\
\pi_{11}(P^7(2)) & \overset{j_2_*}{\longrightarrow} & \pi_{11}(\Sigma^7 Y)
\end{array}
\]

the element \( f_* \circ (j_2)_*(\beta) \) is divisible by 4 in \( \pi_{11}(\text{CP}^2 \land P^5(2)) \). By the exact sequence above, \( 4\pi_{11}(\text{CP}^2 \land P^5(2)) = 0 \). This is a contradiction. Thus \( \pi_{11}(\Sigma^7 Y) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \).

This completes the calculation. \( \square \)

Now we compute \( \pi_*(\Sigma L_3) \). Observe that \( \Sigma L_3 \) is 7-connected.

**Proposition 6.31.** The homotopy groups \( \pi_r(\Sigma L_3) \) for \( 8 \leq r \leq 11 \) are as follows.

1. \( \pi_8(\Sigma L_3) = \mathbb{Z}/2 \).
2. \( \pi_9(\Sigma L_3) = \mathbb{Z}/2 \).
3. \( \pi_{10}(\Sigma L_3) = \mathbb{Z}/2 \).
4. \( \pi_{11}(\Sigma L_3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

**Proof.**

1. \( \pi_8(\Sigma L_3) \): By Step 3 of Proposition 6.28, \( \text{sk}_{10}(\Sigma L_3) \simeq P^{10}(2) \lor \Sigma^5 \mathbb{R}P^5_3 \).

Thus \( \pi_8(\Sigma L_3) \cong \pi_8(\Sigma^5 \mathbb{R}P^5_3) \cong \mathbb{Z}/2 \).

2. \( \pi_9(\Sigma L_3) \): Consider the exact sequence

\[
\pi_9(S^9) \overset{\eta}{\longrightarrow} \pi_9(P^9(2)) \longrightarrow \pi_9(\Sigma^5 \mathbb{R}P^5_3) \longrightarrow 0.
\]

We have \( \pi_9(\Sigma^5 \mathbb{R}P^5_3) = 0 \) and so \( \pi_9(\Sigma L_3) = \mathbb{Z}/2 \).

3. A decomposition of \( \text{sk}_{11}(\Sigma L_3) \): There is a cofibre sequence

\[
S^{10} \lor S^{10} \overset{\phi}{\longrightarrow} P^{10}(2) \lor \Sigma^5 \mathbb{R}P^5_3 \longrightarrow \text{sk}_{11}(\Sigma L_3).
\]

Let \( \phi_1 \) be the composite \( S^{10} \lor S^{10} \overset{\phi}{\longrightarrow} P^{10}(2) \lor \Sigma^5 \mathbb{R}P^5_3 \overset{\text{proj}}{\longrightarrow} P^{10}(2) \). Note that \( [S^{10} \lor S^{10}, P^{10}(2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and \( P^{10}(2) \) is not a retract of \( \text{sk}_{11}(\Sigma L_3) \). There exists an inclusion \( i : S^{10} \to S^{10} \lor S^{10} \) such that \( i_* : H_{10}(S^{10}) \to H_{10}(S^{10} \lor S^{10}) \) is a
monomorphism and $\phi_1 \circ i$ is null homotopic. Thus there is a homotopy commutative diagram

where the columns are cofibre sequences and the rows except the terms in the last column are cofibre sequences. We claim that $\psi$ is null.

First we compute $\pi_{10}(A)$. Consider the exact sequence

$$
\pi_{10}(S^0) \xrightarrow{-\eta_*} \pi_{10}(P^9(2)) \xrightarrow{j_*} \pi_{10}(\Sigma^5\mathbb{RP}_3^5) \xrightarrow{p_*} \pi_9(S^0) \xrightarrow{} \mathbb{Z}/2 \xrightarrow{} 0.
$$

Note that $\eta_*(\eta_9) = 2\alpha_9$. We have

$$
\pi_{10}(\Sigma^5\mathbb{RP}_3^5) = \{j_*(\alpha_9)\} \oplus \{\iota_9\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}
$$

with $p_*(\iota_9) = 2\iota_9$. Observe that $g_*: H_*(A) \to H_*(\text{sk}_{11}(\Sigma L_3))$ is a monomorphism. Thus $H_*(A)$ has a basis $x_8, x_9, x_{10}, x_{11}$ with $S_{q_8^1}x_{11} = x_{10}$, $S_{q_9}x_{11} = x_9$, $S_{q_8}x_9 = x_8$ and $S_{q_8}^2x_{10} = x_8$. The composite

$$
S^{10} \xrightarrow{\theta} \Sigma^5\mathbb{RP}_3^5 \xrightarrow{\text{pinch}} S^{10}
$$

is of degree $\pm 2$. Thus

$$
[\theta] = \pm \iota_9 + \epsilon j_*(\alpha_9)
$$

in $\pi_{10}(\Sigma^5\mathbb{RP}_3^5)$ for some $\epsilon = 0, 1$. Consider the exact sequence

$$
\pi_{10}(S^{10}) \xrightarrow{\theta_*} \pi_{10}(\Sigma^5\mathbb{RP}_3^5) \xrightarrow{} \pi_{10}(A) \xrightarrow{} 0
$$

with $\theta_*(\iota_9) = \pm \iota_9 + \epsilon j_*(\alpha_9)$. Thus $\pi_{10}(A) = \mathbb{Z}/2$ generated by $q \circ j_*(\alpha_9)$.

Now suppose that $\psi: S^{10} \to A$ is essential. Then $[\psi] = q \circ j_*(\alpha_9)$ and $\psi$ maps into the subcomplex $P^9(2)$ of $A$ up to homotopy. Thus there is a homotopy commutative
diagram

\[
\begin{array}{cccccc}
S^{10} & \xrightarrow{\alpha_9} & P^9(2) & \xrightarrow{\phi} & C' & \xrightarrow{\theta} S^{11} \\
S^{10} & \xrightarrow{\psi} & A & \xrightarrow{f} & C & \xrightarrow{\eta} S^{11} \\
\end{array}
\]

where \( \phi = q \circ j \) and the rows are cofibre sequences. Let \( u_9 \) and \( v_{10} \) be the basis for \( \tilde{H}_*(P^{10}(2)) \). Note that \( j'_* \) is a monomorphism. Let \( j'_*(v_{10}) = l_1 \alpha_{10} + l_2 \beta_{10} \). Since \( Sq^2 v_{10} = 0 \), we have \( l_1 = 0 \) and so \( j'_*(v_{10}) = \beta_{10} \). It follows that \( j'_*(u_9) = \beta_9 \). Now \( H_*(C) \) has a basis \( p'_*(\alpha_8), p'_*(\alpha_9), p'_*(\alpha_{10}), p'_*(\alpha_{11}) \) and \( p'_*(\beta_{11}) \). Observe that \( H_*(C') \) has a basis \( \{y_8, y_9, y_{11}\} \) \( Sq^2 y_{11} = y_9 \) and \( Sq^1 y_9 = y_8 \). Since \( f_*(y_8) = p'_*(\alpha_8) \) in \( H_*(C) \), we have \( f_*(y_9) = p'_*(\alpha_9) \). Let

\[
f_*(y_{11}) = k_1 p'_*(\alpha_{11}) + k_2 p'_*(\beta_{11}).
\]

Then

\[
0 \neq f_*(y_9) = f_*(Sq^2 y_{11}) = Sq^2 f_*(y_{11}) = k_1 Sq^2 p'_*(\alpha_{11}) + k_2 Sq^2 p'_*(\beta_{11}) = k_1 p'_*(\alpha_9) + k_2 p'_*(\beta_9) = k_1 p'_*(\alpha_9)
\]

Thus \( k_1 = 1 \) and so \( f_*(y_{11}) = p'_*(\alpha_{11}) + k_2 p'_*(\beta_{11}) \). It follows that

\[
0 = f_*(Sq^1 y_{11}) = Sq^1 f_*(y_{11}) = Sq^1 p'_*(\alpha_{11}) + Sq^1 k_2 p'_*(\beta_{11}) = p'_*(\alpha_{10}) \neq 0
\]

This is a contradiction. Thus \( \psi: S^{10} \to A \) is null homotopic and so is \( j' \circ \eta \). Hence \( j': P^{10}(2) \to sk_{11}(\Sigma L_3) \) extends to \( j: \Sigma^6 \mathbb{RP}_3 \to sk_{11}(\Sigma L_3) \). Now the composite

\[
\Sigma^6 \mathbb{RP}_3 \xrightarrow{j} sk_{11}(\Sigma L_3) \to \Sigma^6 \mathbb{RP}_3
\]

is a homotopy equivalence and therefore \( sk_{11}(\Sigma L_3) \simeq \mathbb{CP}^2 \vee \mathbb{CP}^2 \).

4. \( \pi_{10}(\Sigma L_3) \):

\[
\pi_{10}(sk_{11}(\Sigma L_3)) \simeq \pi_{10}(A) \oplus \pi_{10}(\Sigma^6 \mathbb{RP}_3) = \mathbb{Z}/2 \oplus 0 = \mathbb{Z}/2.
\]

5. \( \pi_{11}(\Sigma L_3) \): By expecting the Steenrod operation on \( \tilde{H}_*(A) \), we have

\[
A \simeq \mathbb{CP}^2 \wedge P^7(2).
\]

From the exact sequence

\[
\pi_{11}(P^{10}(2)) \xrightarrow{\phi} \pi_{11}(P^9(2)) \to \pi_{11}(A) \to \pi_{10}(P^{10}(2)) \to \pi_{10}(P^9(2)),
\]

we have \( \pi_{11}(A) = \mathbb{Z}/2 \) and \( \pi_{11}(P^9(2)) \to \pi_{11}(A) \) is onto. Consider the cofibre sequence

\[
S^{11} \xrightarrow{\theta} A \vee \Sigma^6 \mathbb{RP}_3 \to \Sigma L_3.
\]
Since \( Sq^i \neq 0 \) in \( H_*(\Sigma L_3) \), the composite
\[
S^{11} \xrightarrow{\theta} A \vee \Sigma^6 \mathbb{RP}_3^{\text{proj}} \xrightarrow{\text{proj}} A
\]
is a generator for \( \pi_{11}(A) = \mathbb{Z}/2 \). Note that \( Sq^i \beta_{11} = \beta_{10} \). The composite
\[
S^{11} \xrightarrow{\theta} A \vee \Sigma^6 \mathbb{RP}_3^{\text{proj}} \xrightarrow{\text{proj}} \Sigma^6 \mathbb{RP}_3^{\text{proj}} \xrightarrow{p} S^{11}
\]
is of degree 2. Observe that
\[
\pi_{11}(\Sigma^6 \mathbb{RP}_3^5) = \{j_*(\alpha_{10})\} \oplus \{t_{11}\} = \mathbb{Z}/2 \oplus \mathbb{Z}
\]
with \( p_*(t_{11}) = 2t_{11} \). There is an exact sequence
\[
\mathbb{Z} \xrightarrow{\theta^*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{} \pi_{11}(\Sigma L_3) \xrightarrow{} 0
\]
with the composite
\[
\mathbb{Z} \xrightarrow{\theta^*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\text{proj}} \mathbb{Z}
\]
of degree 1. Thus \( \pi_{11}(\Sigma L_3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). This completes the calculation.

Now we are ready to compute \( \pi_*(\Sigma \mathbb{RP}_2^4) \). Consider the homotopy commutative diagram

\[
\begin{array}{cccccc}
\Omega S^5 & \xrightarrow{\delta} & F & \xrightarrow{i} & \Sigma \mathbb{RP}_2^4 & \xrightarrow{} S^5 \\
E & & & & \Omega S_3 & \\
S^4 (\eta_1 [2]) & \xrightarrow{} S^3 \vee S^4 & \xrightarrow{} \Sigma \mathbb{RP}_2^4 & \xrightarrow{} S^5
\end{array}
\]

where the top row is fibre sequence and the bottom row is cofibre sequence. By Lemmas 6.18 and 6.23 and Proposition 6.28, there is a homotopy commutative diagram

\[
\begin{array}{cccccc}
\Omega^2 S^5 & \xrightarrow{\delta} & \Omega F & \xrightarrow{\phi} & \Omega S^3 \times S^3 \times \Omega S^7 \times \Omega \Sigma L_3 \times P^{11}(2) \times \Sigma^6 \mathbb{RP}_3^5 \times S^{11} \\
\Omega S^4 (\Omega(\eta_1 [2])) & \xrightarrow{} \Omega S^3 \times S^3 \times \Omega S^7 \cong \Omega S^3 \times \Omega S^4,
\end{array}
\]
where \( \psi = (\text{id}_{\Omega S^3}, \text{id}_{S^3}, \Omega j) \), \( \phi \) is a homotopy equivalence through the 11-skeleton and \( \Omega j_*(\iota_6) = u_3^2 \) in \( H_*(\Omega S^7) \).

**Lemma 6.32.** [23, pp.217] The homotopy group \( \pi_{11}(SU(3)) = \mathbb{Z}/4 \).

**Theorem 6.33.** The homotopy groups \( \pi_r(\Sigma \mathbb{R}P^4_2) \) for \( r \leq 11 \) are as follows.

1. \( \pi_3(\Sigma \mathbb{R}P^4_2) = \mathbb{Z} \).
2. \( \pi_4(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/4 \).
3. \( \pi_5(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2 \).
4. \( \pi_6(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).
5. \( \pi_7(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/8 \).
6. \( \pi_8(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4 \).
7. \( \pi_9(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2^{\oplus 4} \).
8. \( \pi_{10}(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \).
9. \( \pi_{11}(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2^{\oplus 4} \oplus \mathbb{Z}/4^{\oplus 2} \).

**Proof.** 1. \( \pi_r(\Sigma \mathbb{R}P^4_2) \) for \( r \leq 7 \): Since \( L_3 \) is 6-connected, the sequence

\[
\ldots \longrightarrow \pi_r(\Sigma \mathbb{R}P^4_2) \longrightarrow \pi_{r-1}(\Sigma \mathbb{R}P^4_2) \longrightarrow \pi_{r-2}(\Sigma \mathbb{R}P^4_2) \longrightarrow \pi_{r-3}(\Sigma \mathbb{R}P^4_2) \longrightarrow \ldots
\]

is exact for \( r \leq 8 \). Hence \( \pi_3(\Sigma \mathbb{R}P^4_2) = \mathbb{Z} \). Note that \( \delta_*(\iota_5) = 2\iota_3 \). We have \( \pi_4(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/4 \). Observe that \( \delta_*(\eta_5) = \delta_*(E\eta_4) = \eta_3^2 \). Thus

\[
\pi_5(\Sigma \mathbb{R}P^4_2) = \mathbb{Z}/2.
\]

Note that \( \delta_*(\eta_3^2) = \delta_*(E^2\eta_3^2) = \eta_3^2 + 0 + 0 = \eta_3^2 \). Thus

\[
\pi_6(\Sigma \mathbb{R}P^4_2) = \pi_6(S^3)/2 \oplus \pi_5(S^3) \oplus \pi_6(S^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

Notice that \( \delta_*(\nu_5) = \delta_*(E\nu_4) = (1_{\Omega S^3}, 1_{S^3}, j)_* \circ \Omega(\eta, [2])_*(\nu_4) \). Now

\[
\Omega(\eta, [2])_*(\nu_4) = \Omega\eta_*(\nu_4) + \Omega[2]_*(\nu_4) = \eta_3 \circ \nu_4 + 4\nu_4 - E\nu'.
\]

Thus

\[
\delta_*(\nu_5) = \eta_3 \circ \nu_4 - \nu' + 4j_*(\iota_7),
\]

where \( j_*(\iota_7) \) is a generator for \( \pi_7(\Sigma S^7) = \mathbb{Z}/8 \). It follows that

\[
\pi_7(\Sigma \mathbb{R}P^4_2) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2,
\]

which are generated by \( \phi_\ast \circ j_*(\iota_7) \) and \( \phi_\ast(\nu') \). Furthermore

\[
\text{Ker}(\delta_\ast: \pi_8(S^5) \to \pi_6(\Omega S^3 \times S^3 \times \Omega S^7)) = 4 \cdot \pi_8(S^5)
\]

because \( 2\delta_*(\nu_5) = 2\nu' \neq 0 \).

2. \( \pi_8(\Sigma \mathbb{R}P^4_2) \): Consider the exact sequence

\[
\ldots \longrightarrow \pi_9(S^5) \longrightarrow \pi_9(S^3) \oplus \pi_7(S^3) \oplus \pi_8(S^7) \oplus \pi_8(\Sigma L_3) \longrightarrow \pi_8(\Sigma \mathbb{R}P^4_2) \longrightarrow \pi_8(S^5) \longrightarrow 0
\]

with \( \delta_*(\nu_5 \circ \eta_8) = \delta_*(\nu_5) \circ \eta_8 = \eta_3 \circ \nu_4 \circ \eta_7 + \nu' \circ \eta_6 = \nu' \circ \eta_6^2 + \nu' \circ \eta_6 \neq 0 \). Recall that \( \pi_8(S^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and \( \pi_8(\Sigma L_3) = \mathbb{Z}/2 \). There is a short exact sequence

\[
0 \longrightarrow (\pi_8(S^3) \oplus \pi_7(S^3))/\{\nu' \circ \eta_6^2 + \nu' \circ \eta_6\} \oplus \pi_8(S^7) \oplus \pi_8(\Sigma L_3) \longrightarrow \pi_8(\Sigma \mathbb{R}P^4_2) \longrightarrow 4 \cdot \pi_8(S^5) \longrightarrow 0
\]

or \( 0 \longrightarrow \mathbb{Z}/2^{\oplus 4} \longrightarrow \pi_8(\Sigma \mathbb{R}P^4_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \).
Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega^2 F & \longrightarrow & \Omega^2 \mathbb{R}P^4 & \overset{p}{\longrightarrow} & \Omega^2 S^5 & \overset{\delta}{\longrightarrow} & \Omega F \\
\Omega^2 S^3 \times \Omega S^3 & \longrightarrow & X & \longrightarrow & S^3 & \overset{(\eta, 2)}{\longrightarrow} & \Omega S^3 \times S^3.
\end{array}
\]

Then the universal cover \(X(1)\) is the homotopy fibre of \(S^3 \overset{(\eta, 2)}{\longrightarrow} \Omega(S^3(3)) \times S^3\) and so there is a commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega^2(S^3(3)) \times \Omega S^3 & \overset{i}{\longrightarrow} & X(1) & \overset{p'}{\longrightarrow} & S^3 & \overset{(\eta, 2)}{\longrightarrow} & \Omega(S^3(3)) \times S^3 \\
\Omega^2(S^3(3)) & \longrightarrow & \Omega^3 S^5 & \overset{\Delta}{\longrightarrow} & S^3 & \overset{\eta}{\longrightarrow} & \Omega(S^3(3)),
\end{array}
\]

where the bottom fibre sequence is the EHP sequence

\[
S^3 \longrightarrow \Omega(S^3(3)) \longrightarrow \Omega S^5.
\]

Consider the commutative diagram of exact sequence

\[
\begin{array}{ccccccc}
\pi_7(S^3) & \overset{0}{\longrightarrow} & \pi_8(S^3) \oplus \pi_7(S^3) & \overset{i_*}{\longrightarrow} & \pi_6(X) & \overset{p'_*}{\longrightarrow} & \pi_6(S^3) & \overset{(\bar{\eta}_*, 2)}{\longrightarrow} & \pi_7(S^3) \oplus \pi_6(S^3) \\
\pi_7(S^3) & \overset{0}{\longrightarrow} & \pi_8(S^3) & \overset{\pi_*}{\longrightarrow} & \pi_8(S^5) & \overset{\Delta}{\longrightarrow} & \pi_6(S^3) & \overset{\pi_*}{\longrightarrow} & \pi_7(S^3).
\end{array}
\]

There is a short exact sequence

\[
0 \longrightarrow \pi_8(S^3) \oplus \pi_7(S^3) \longrightarrow \pi_6(X) \overset{\nu'_*}{\longrightarrow} 2 \cdot \pi_6(S^3) \longrightarrow 0.
\]

Let \(\alpha \in \pi_6(X)\) such that \(p'_*(\alpha) = 2\nu'\). Then \(f_*(\alpha) = \pm 2\nu_5\) and so \(\pi_*(2\alpha) = \nu' \circ \eta_6^2\) or \(2\alpha = i_*(\nu' \circ \bar{\eta}_6^2) + \epsilon i_*(\nu' \circ \eta_6)\) for some \(\epsilon = 0, 1\). Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega^2 S^3 \times \Omega S^3 & \longrightarrow & X & \longrightarrow & S^3 & \longrightarrow & \Omega S^3 \times S^3 \\
\Omega S^3 & \longrightarrow & S^3 \{2\} & \overset{p''}{\longrightarrow} & S^3 & \overset{2}{\longrightarrow} & S^3.
\end{array}
\]
There is a commutative diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \pi_8(S^3) \oplus \pi_7(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{p'_*} & 2\pi_6(S^3) & \rightarrow & 0 \\
& & \pi'_7 & \quad & j_* & \quad & & & \\
0 & \rightarrow & \pi_7(S^3) & \xrightarrow{g_*} & \pi_6(S^3 \{2\}) & \xrightarrow{p''_*} & 2\pi_6 S^3 & \rightarrow & 0
\end{array}
\]

with \( p''_*(g_*(\alpha)) = 2\nu' \). Recall that \( \Omega(S^3 \{2\}) \simeq (\Omega S^3) \{2\} \). We have

\[
2g_*(\alpha) = j_*(2\nu' \circ \eta) = 0
\]
or \( g_*(2\alpha) = 0 \). Thus \( 0 = \pi'_7(\nu' \circ \eta^2 + \nu' \circ \eta_6) = \nu' \circ \eta_6 \), that is, \( \epsilon = 0 \) and so

\[
2\alpha = i_*(\nu' \circ \eta^2).
\]

Now consider the commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
\pi_9(S^3) & \xrightarrow{\delta_*} & \pi_8(F) & \xrightarrow{\phi_*} & \pi_8(\Sigma \mathbb{RP}_2^4) & \xrightarrow{p_*} & 4\pi_8(S^5) & \rightarrow & 0 \\
| & & | & & | & & | & & |
0 & \rightarrow & \pi_8(S^3) \oplus \pi_7(S^3) & \xrightarrow{i_*} & \pi_6(X) & \xrightarrow{p'_*} & 2\pi_6(S^3) & \rightarrow & 0
\end{array}
\]

Since \( p'_* (\alpha) = 2\nu' \), we have \( p_* (e'_* (\alpha)) = 4\nu_5 \). Since \( 2\alpha = i_*(\nu' \circ \eta^2) \), we have

\[
2e'_* (\alpha) = \phi_*(\nu' \circ \eta^2).
\]

Since \( \delta_*(\nu_5 \circ \eta^2) = \delta(\nu_5 \circ \eta_6) \circ \eta_6 = \nu' \circ \eta^2 \circ \eta_6 + \nu' \circ \eta_6 \circ \eta_6 = \nu' \circ \eta_6^2 \), the element \( \nu' \circ \eta_6^2 \) does not lie in the image of \( \delta_* \) and so

\[
2e'_* (\alpha) = \phi_*(\nu' \circ \eta^2) \neq 0.
\]

Thus \( e'_* (\alpha) \) is of order 4 and so \( \pi_8(\Sigma \mathbb{RP}_2^4) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2^{\oplus 3} \).

3. \( \pi_9(\Sigma \mathbb{RP}_2^4) \): Consider the exact sequence

\[
\pi_{10}(S^5) \xrightarrow{\delta_*} \pi_9(S^3) \oplus \pi_9(S^3) \oplus \pi_9(\Sigma Y^7) \oplus \pi_9(\Sigma L_3) \longrightarrow \pi_9(\Sigma \mathbb{RP}_2^4) \longrightarrow 0
\]

with \( \delta_*(\nu_5 \circ \eta^2) = \delta(\nu_5 \circ \eta_6) \circ \eta_6 = \nu' \circ \eta^2 \circ \eta_6 + \nu' \circ \eta_6 \circ \eta_7 = \nu' \circ \eta_6^2 \).

Thus \( \pi_9(\Sigma \mathbb{RP}_2^4) \cong \pi_9(\Sigma Y^7) \oplus \pi_9(\Sigma L_3) \cong \mathbb{Z}/2^{\oplus 4} \).

4. \( \pi_{10}(\Sigma \mathbb{RP}_2^4) \): Consider the exact sequence

\[
\pi_{11}(S^5) \xrightarrow{\delta_*} \pi_{10}(S^3) \oplus \pi_{10}(S^3) \oplus \pi_{10}(\Sigma Y^7) \oplus \pi_{10}(\Sigma L_3) \oplus \pi_{10}(\Sigma Y^7 \mathbb{RP}_3^5) \longrightarrow \pi_{10}(\Sigma \mathbb{RP}_2^4) \longrightarrow 0.
\]

Recall that \( \pi_9(S^3) = \pi_{10}(S^3) = 0 \) and \( 4 \cdot \pi_{10}(\Sigma Y^7) = 0 \). We have

\[
\delta_*(\nu^2_5) = \delta_*(E\nu^2_4) = j_* \circ \Omega(\eta, 2)_*(\nu^2_1) = j_* (4\nu^2_4) = 0
\]

and so \( \pi_{10}(\Sigma \mathbb{RP}_2^4) \cong \pi_{10}(\Sigma Y^7) \oplus \pi_{10}(\Sigma L_3) \oplus \pi_{10}(\Sigma Y^7 \mathbb{RP}_3^5) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

5. \( \pi_{11}(\Sigma \mathbb{RP}_2^4) \): There is an exact sequence

\[
\pi_{12}(S^5) \xrightarrow{\delta_*} \pi_{11}(S^3) \oplus \pi_{10}(S^3) \oplus \pi_{11}(\Sigma Y^7) \oplus \pi_{11}(\Sigma L_3) \oplus \pi_{11}(P^{12}(2)) \oplus \pi_{11}(\Sigma Y^7 \mathbb{RP}_3^5) \longrightarrow \pi_{11}(\Sigma \mathbb{RP}_2^4) \longrightarrow \pi_{11}(S^5) \longrightarrow 0.
\]
Consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{cccc}
\Omega S^5 & \delta & \rightarrow & F \\
\downarrow & & \downarrow & \downarrow \\
\Omega S^5 & \tilde{\eta} & \rightarrow & S^3 \\
\downarrow & & \downarrow & \downarrow \\
S^5 & & \rightarrow & SU(3) \\
\downarrow & & \downarrow & \downarrow \\
S^5 & & & S^5
\end{array}
\]

By [42, pp.64], \( \eta_4 \circ \sigma''' = 0 \) and so \( \tilde{\eta}_*(\sigma''') = p \circ \eta_4 \circ \sigma''' = 0 \), where \( p: \Omega S^4 \rightarrow S^3 \) is a retraction. Thus

\[ \text{Im}(\delta_*: \pi_{11}(\Omega S^5) \rightarrow \pi_{11}(F)) \subseteq \pi_{10}(\Omega B). \]

Let \( X^4 = \text{sk}_7(\Omega^2 S^5) \). There is a commutative diagram

\[
\begin{array}{ccc}
\pi_{10}(\Omega^2 S^5) & \xrightarrow{H} & \pi_{10}(\Omega^2 S^0) \\
\downarrow i_* & & \downarrow i_* \\
\pi_{10}(X^4) & \xrightarrow{p_*} & \pi_{10}(S^7),
\end{array}
\]

where \( p \) is the pinch map. By Step 5 of Theorem 5.8, \( \pi_{10}(X^4) \xrightarrow{p_*} 4 \cdot \pi_{10}(S^7) \) is onto. By [42, Lemma 5.13], \( \pi_{10}(\Omega^2 S^5) \cong 4 \cdot \pi_{10}(\Omega^2 S^0) \) under the Hopf invariant \( H \) and so \( i_*: \pi_{10}(X^4) \longrightarrow \pi_{10}(\Omega^2 S^5) \) is onto. Let \( \delta' \) be the composite

\[ \Omega^2 S^5 \xrightarrow{\delta} \Omega F \xrightarrow{\text{proj}} \Omega B. \]

Then \( \delta' \) maps \( X^4 \) into the 7-skeleton of \( \Omega B \) up to homotopy and so there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega^2 S^5 & \xrightarrow{\delta'} & \Omega B \\
\downarrow & & \downarrow \\
X^4 & \xrightarrow{\delta''} & S^3 \times \Omega \Sigma Y^7.
\end{array}
\]

Let \( \delta''' \) be the composite

\[ X^4 \xrightarrow{\delta''} S^3 \times \Omega \Sigma Y^7 \xrightarrow{\text{proj}} \Omega \Sigma Y^7. \]

We show that \( \delta_*: \pi_{10}(X^4) \longrightarrow \pi_{10}(\Omega \Sigma Y^7) \) is zero. Since \( X^4 \) is the homotopy cofibre of \( (\nu,[2]): S^6 \rightarrow S^3 \vee S^6 \) and \( Y^7 \) is 4-connected, there is a homotopy commutative
Recall that \( \pi_{10}(P^7(2)) \cong \pi_{10}^2(P^7(2)) = \mathbb{Z}/2 \) generated by \( \alpha_7 \circ \eta^2 \), where \( \alpha_7 \) is a generator for \( \pi_8(P^7(2)) = \mathbb{Z}/4 \). Since \( \delta_*: H_7(P^7(2)) \to H_7(Y^7) \) is zero, the composite 
\[
P^7(2) \xrightarrow{\delta''} Y^7 \xrightarrow{\text{pinch}} S^7
\]
is null homotopic and so \( \delta'' \) maps into the subcomplex \( P^6(2) \vee S^6 \). Let \( \phi_1 \) be the composite 
\[
P^7(2) \xrightarrow{\delta''} P^6(2) \vee S^6 \xrightarrow{\text{proj.}} P^6(2)
\]
and let \( \phi_2 \) be the composite \( P^7(2) \xrightarrow{\delta''} P^6(2) \vee S^6 \xrightarrow{\text{proj.}} S^6 \). Then \( \phi_2 \) is homotopic to either the constant map or the composite \( P^7(2) \xrightarrow{\eta} S^5 \xrightarrow{j} P^6(2) \), where \( \eta \) is an extension of \( \eta: S^6 \to S^5 \). Then \( \phi_1 \circ \alpha_7 = \eta \circ (k \nu_5) \) for some \( k \) and so \( \phi_1 \circ \alpha_7 \circ \eta^2 = \eta \circ (k \nu_5 \circ \eta^2) \). It follows that \( E(\phi_1 \circ \alpha_7 \circ \eta^2) = 0 \).

**Case 1.** The map \( \phi_1 \) is homotopic to the composite 
\[
P^7(2) \xrightarrow{\text{pinch}} S^7 \xrightarrow{\alpha_6} P^6(2).
\]
Then \( \phi_1 \circ \alpha_7 = \alpha_6 \eta \) and \( \phi_1 \circ \alpha_7 \circ \eta^2 = \alpha_6 \circ \eta^2 = \alpha_6 \circ 4 \nu_7 = 0 \).

**Case 2.** The map \( \phi_1 \) is homotopic to the composite \( P^7(2) \xrightarrow{\eta} S^5 \xrightarrow{j} P^6(2) \), where \( \eta \) is an extension of \( \eta: S^6 \to S^5 \). Then \( \phi_1 \circ \alpha_7 = \eta \circ (k \nu_5) \) for some \( k \) and so \( \phi_1 \circ \alpha_7 \circ \eta^2 = \eta \circ (k \nu_5 \circ \eta^2) \). It follows that \( E(\phi_1 \circ \alpha_7 \circ \eta^2) = 0 \).

**Case 3.** The map \( \phi_1 = \eta \wedge \text{id}: P^7(2) \to P^6(2) \). Then \( \phi_1 \circ \alpha_7 = \alpha_6 \circ \eta + \epsilon \nu_5 \) for some \( \epsilon = 0, 1 \) and \( \phi_1 \circ \alpha_7 \circ \eta^2 = \alpha_6 \eta^2 + \epsilon \nu_5 \circ \eta^2 = \epsilon \nu_5 \circ \eta^2 \). It follows that \( E(\phi_1 \circ \alpha_7 \circ \eta^2) = 0 \).

Thus, in any case, we have 
\[
E \circ \delta'' : \pi_{10}(P^7(2)) \to \pi_{10}(\Omega \Sigma Y^7)
\]
is zero and so is \( \delta_*: \pi_{10}(X^4) \to \pi_{10}(\Omega \Sigma Y^7) \).

Now we show that the boundary \( \delta_*: \pi_{12}(S^5) \to \pi_{11}(F) \) is zero. Since \( \pi_{10}(S^3) \) is zero, \( \delta_*: \pi_{10}(X^4) \to \pi_{10}(S^3 \wedge \Omega \Sigma Y^7) \) is zero. By using the fact that \( \pi_{10}(X^4) \to \pi_{10}(\Omega^2 S^5) \) is onto, we have \( \delta_*: \pi_{10}(\Omega \Sigma Y^7) \to \pi_{10}(\Omega F) \) is zero and so there is a short exact sequence 
\[
0 \to \pi_{11}(S^3) \oplus \pi_{11}(\Sigma Y^7) \oplus \pi_{11}(\Sigma L_3) \oplus \pi_{11}(P^{12}(2)) \to \pi_{11}(\Sigma \mathbb{R}P^4) \xrightarrow{\nu_3} \pi_{11}(S^5) \to 0.
\]

We are going to solve the group extension problem now. Let \( \alpha \in \pi_{11}(\Sigma \mathbb{R}P^4) \) such that \( p_*(\alpha) = \nu_3^2 \) and let \( 2\alpha = \beta_1 + \beta_2 + \beta_3 + \beta_4 \), where \( \beta_1 \in \pi_{11}(S^3) \), \( \beta_2 \in \pi_{11}(\Sigma Y^7) \), \( \beta_3 \in \pi_{11}(\Sigma L_3) \) and \( \beta_4 \in \pi_{11}(P^{12}(2)) \).

**First we show that** \( \beta_4 = 0 \). Consider the Hurewicz homomorphism 
\[
h: \pi_{10}(\Omega \Sigma \mathbb{R}P^4) \to H_{10}(\Omega \Sigma \mathbb{R}P^4).
\]
Then
\[ 0 = h(2\alpha) = h(\beta_1) + h(\beta_2) + h(\beta_3) + h(\beta_4). \]
If \( \beta_4 \neq 0 \), then \( \beta_4 \) is a generator for \( \pi_{11}(P^{12})(2) = \mathbb{Z}/2 \) and \( h(\beta_4) = [[x_2, u_3], u_3] \). Note that \( h(\beta_1), h(\beta_2) \) and \( h(\beta_3) \) lie in \( H_*(S^3), H_*(\Omega\Sigma Y^7) \) and \( H_*(\Omega\Sigma L_3) \), respectively. There is a contradiction. Thus \( \beta_4 = 0 \).

Now we show that \( \beta_3 = 0 \). Let \( \phi \) be the composite
\[ \Omega\Sigma L_3 \longrightarrow \Omega B \longrightarrow \Omega \Sigma \mathbb{RP}_2^4 \xrightarrow{H_3} \Omega \Sigma (\mathbb{RP}_2^4)^{(3)} \longrightarrow \Omega\Sigma L_3. \]
Then \( \phi_*: H_7(\Omega\Sigma L_3) \longrightarrow H_7(\Omega\Sigma L_3) \) is an isomorphism. Hence
\[ \phi_*: PH_r(\Omega\Sigma L_3) \longrightarrow PH_r(\Omega\Sigma L_3) \]
is an isomorphism for \( r \leq 13 \) and so
\[ \phi_*: \pi_r(\Omega\Sigma L_3) \longrightarrow \pi_r(\Omega\Sigma L_3) \]
is an isomorphism for \( r \leq 12 \). Let \( \psi \) be the composite
\[ \Omega\Sigma\mathbb{RP}_2^4 \xrightarrow{H_3} \Omega\Sigma(\mathbb{RP}_2^4)^{(3)} \longrightarrow \Omega\Sigma L_3. \]
From the commutative diagram
\[
\begin{array}{ccc}
\Omega\Sigma\mathbb{RP}_2^4 & \xrightarrow{H_3} & \Omega\Sigma(\mathbb{RP}_2^4)^{(3)} \\
\downarrow & & \downarrow \\
\Omega S^3 & \xrightarrow{H_3} & \Omega\Sigma S^6,
\end{array}
\]
we have \( \psi_*(\beta_1) = 0 \). Recall that \( \pi_{11}(\Sigma Y^7) \cong \pi_{11}(P^7(2)) \oplus 2 \cdot \pi_{10}(S^7) = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \).

By Theorem 5.13, \( j_*(\omega_6) \) is a generator for \( \pi_{11}(P^7(2)) = \mathbb{Z}/2 \). From the homotopy commutative diagram
\[
\begin{array}{ccc}
S^{10} & \xrightarrow{j_*(\omega_6)} & \Omega\Sigma\mathbb{RP}_2^4 \xrightarrow{H_3} \Omega\Sigma(\mathbb{RP}_2^4)^{(3)} \\
\downarrow \omega_6 & & \downarrow \\
\Omega S^6 & \xrightarrow{J([x_2, u_3])} & \Omega\Sigma(S^2 \vee S^3) \xrightarrow{H_3} \Omega\Sigma(S^2 \vee S^3)^{(3)},
\end{array}
\]
we have \( \psi_*(j_*(\omega_6)) = 0 \). By expecting the homotopy commutative diagram
\[
\begin{array}{ccc}
\begin{array}{c}
S^6 \xrightarrow{([4], \tilde{\eta})} S^6 \vee P^6(2) \xrightarrow{\text{pinch}} \delta \xrightarrow{2\nu_7} S^7
\end{array} & \xrightarrow{2\nu_6} & S^9 \xrightarrow{*} S^{10} \xrightarrow{=} S^{10},
\end{array}
\]
the element \( E\delta \) is a generator for the summand \( 2\pi_{10}(S^7) \) in \( \pi_{11}(\Sigma Y^7) \). Since \( \Omega \Sigma \Lambda^3 \) is 6-connected, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega \Sigma \mathbb{R}P_2^4 & \xrightarrow{H_3} & \Omega \Sigma (\mathbb{R}P_2^4)^{(3)} \\
& \downarrow & \downarrow \\
n & \psi' \end{array}
\]

Thus \( \psi_*(E\delta) = \psi'_*(2\nu) = 2\psi'_*(\nu) = 0 \) in \( \pi_{10}(\Omega \Sigma \Lambda^3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). And therefore \( \psi_*(\beta_2) = 0 \). It follows that

\[
0 = \psi_*(2\alpha) = \psi_*(\beta_3) = \phi_*(\beta_3).
\]

Thus \( \beta_3 = 0 \).

*Continuation of the proof.* Now consider the homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
\Omega F^5\{2\} & \xrightarrow{\Omega f} & \Omega \Sigma \mathbb{R}P_2^4 \\
& \downarrow \Omega F & \downarrow \Omega f \\
\Omega F & \xrightarrow{\Omega \Sigma \mathbb{R}P_2^4} & \Omega S^5 \\
\end{array}
\]

Since \( \Omega f\{1\} : \Omega S^3 \to \Omega P^5(2) \) is null homotopic, \( \Omega f_*(\beta_1) = 0 \). By the relative Samelson product, the composite

\[
\begin{array}{ccc}
\Omega S^6 & \xrightarrow{J([x_2, u_3])} & \Omega \Sigma \mathbb{R}P_2^4 \\
& \downarrow \Omega \Sigma \mathbb{R}P_2^4 & \downarrow \Omega P^5(2) \\
\end{array}
\]

is null homotopic and so \( \Omega f_*(j_*(\omega_6)) = 0 \). Observe that there is a commutative diagram

\[
\begin{array}{ccc}
\Omega B & \xrightarrow{\Omega f} & \Omega F^5\{2\} \\
& \downarrow g & \downarrow g \\
\Omega F & \xrightarrow{\Omega F^5\{2\}} & \Omega X^5 \\
\end{array}
\]

where \( X^5 \) is the homotopy fibre of \( S^7 \xrightarrow{2u_4} S^4 \). Note that \( \pi_{10}(X^5) \cong 2 \cdot \pi_{10}(S^7) \). The element \( g \circ f \circ \delta : S^{10} \to X^5 \) is a generator for \( \pi_{10}(X^5) \). Let \( \beta_2 = k_1 j_*(\omega_6) + k_2 E(\delta) \). Then

\[
f_*(2\alpha) = f_*(\beta_1 + \beta_2) = f_*(\beta_1 + k_1 j_*(\omega_6) + k_2 E(\delta)) = k_2 f_*(E(\delta)).
\]
Consider the commutative diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \pi_{11}(X^5) & \rightarrow & \pi_{11}(P^5(2)) & \rightarrow & \pi_{11}(S^5) & \rightarrow & 0 \\
\downarrow & & \downarrow f_* & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \pi_{11}(F) & \rightarrow & \pi_{11}(\Sigma\mathbb{RP}^4_2) & \rightarrow & \pi_{11}(S^5) & \rightarrow & 0
\end{array}
\]

By Step 5 of Theorem 5.10, the short exact sequence

\[
0 \rightarrow \mathbb{Z}/4 \rightarrow \pi_{11}(P^5(2)) \rightarrow \pi_{11}(S^5) \rightarrow 0
\]
splits. Since \(f_*(E\delta)\) is a generator for \(\mathbb{Z}/4\)-summand in \(\pi_{11}(P^5(2))\), we have \(k_2 \equiv 0(2)\). Let \(\tilde{\alpha} = \alpha + k_2/2E(\delta)\). Then \(p_*(\tilde{\alpha}) = \nu_5^2\) and

\[
2\tilde{\alpha} = 2\alpha + k_2E(\delta) = \beta_1 + k_1\nu_3(\omega_6).
\]

We Show that \(k_1 = 0\). Consider the commutative diagram

\[
\begin{array}{ccccccccc}
\Omega\Sigma S^6 & \overset{J([x_2, u_3])}{\rightarrow} & \Omega\Sigma(S^2 \vee S^3) & \overset{\Omega\Sigma j}{\rightarrow} & \Omega\Sigma\mathbb{RP}^4_2 \\
\downarrow H_2 & & \downarrow & & \downarrow H_2 \\
\Omega\Sigma(S^2 \vee S^3)^{(2)} & \overset{\Omega\Sigma j^{(2)}}{\rightarrow} & \Omega\Sigma(\mathbb{RP}^4_2)^{(2)}.
\end{array}
\]

By [47, Corollary 1.2], the composite \(H_2 \circ J([x_2, u_3])\) is a loop map and so

\[
H_2 \circ \Omega\Sigma j \circ J([x_2, u_3]) = \Omega\lambda,
\]

where \(\lambda: S^6 \to (\mathbb{RP}^4_2)^{(2)}\) is a map with the property that \(\lambda_*([s]) = s(x_2 \otimes u_3 + u_3 \otimes x_2)\) in \(H_*(\Sigma(\mathbb{RP}^4_2)^{(2)})\) and \(s\) is the suspension. Observe that \(s(x_2 \otimes x_2), s(x_2 \otimes u_3)\) and \(s(u_3 \otimes u_3)\) are spherical classes in \(H_*(\Sigma(\mathbb{RP}^4_2)^{(2)})\). There is a map

\[
f: S^6 \vee S^6 \vee S^7 \longrightarrow (\mathbb{RP}^4_2)^{(2)}
\]
such that the elements \(s(x_2 \otimes x_2), s(x_2 \otimes u_3)\) and \(s(u_3 \otimes u_3)\) lie in the image of \(f_*\). Let \(X_1 = C_f\) be the homotopy cofibre of \(f\). Then \(sk_7(X_1) = P^7(2) \vee S^7\) and \(s(x_2 \otimes v_4)\) is spherical in \(H_*(X_1)\). Let \(X_2 = X_1/S^7\) which cancels the element \(s(x_2 \otimes v_4)\). Then \(H_*(X_2)\) has a basis

\[
\{s[x_2, u_3], s[x_2, v_4] + s[u_3, v_4], s[u_3, u_4], s(u_4 \otimes u_4), s(u_3 \otimes v_4)\}.
\]

There is a cofibre sequence

\[
S^7 \vee S^7 \longrightarrow P^7(2) \longrightarrow sk_8(X_2).
\]

Since \([S^7 \vee S^7, P^7(2)] \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2, sk_8(X_2) \cong \Sigma^3\mathbb{RP}^8_3 \vee S^8\). Thus \(s(u_3 \otimes v_4)\) is spherical in \(H_*(X_2)\). By expecting the Steenrod operations, we have \(X_2/S^8 \simeq \mathbb{C}P^2 \wedge P^5(2)\). Thus there is a pinch map \(g: (\mathbb{RP}^4_2)^{(2)} \rightarrow \mathbb{C}P^2 \wedge P^5(2)\) such that \(g_*(s[x_2, u_3]) \neq 0\). Consider the exact sequence

\[
0 \longrightarrow \pi_{11}(P^7(2)) \overset{i_*}{\longrightarrow} \pi_{11}(\mathbb{C}P^2 \wedge P^5(2)) \overset{p_*}{\longrightarrow} \pi_{11}(P^9(2)) \overset{\partial}{\longrightarrow} \pi_{10}P^7(2).
\]

Recall that \(\pi_{11}(P^9(2)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\) generated by \(\alpha_9 \cdot \eta_{10}\) and \(j_*(\nu_8)\). Observe that

\[
i \partial \circ j_*(\nu_8) = j_*(\eta_{10} \circ \nu_8) = 0.
\]
\[ \partial(\alpha_0) = \alpha_\tau \circ \eta + \epsilon j_*(\nu_6). \]

\[ \partial(\alpha_0 \circ \eta_{10}) = \delta_*(\alpha_0) \circ \eta_{10} = \alpha_\tau \circ \eta^2 + \epsilon j_*(\nu_6 \circ \eta) = \alpha_\tau \circ \eta^2. \]

The kernel \( \text{Ker}(\partial: \pi_{11}(P^9(2)) \to \pi_{10}(P^7(2))) \) is generated by \( j_*(\nu_8) \). Let

\[ \beta \in \pi_{11}(\mathbb{C}P^2 \wedge P^5(2)) \]

so that \( p_*(\beta) = j_*(\nu_8) \). Then \( 2\beta \) is the composite

\[ S^{11} \longrightarrow \mathbb{C}P^2 \wedge P^5(2) \overset{\text{pinch}}{\longrightarrow} \Sigma^5\mathbb{C}P^2 \overset{\eta \wedge \text{id}}{\longrightarrow} \Sigma^4\mathbb{C}P^2 \longrightarrow \mathbb{C}P^2 \wedge P^5(2). \]

Note that \( \eta \wedge \text{id} \mid_{S^7} \) is null homotopic and \( \pi_9(S^6) \longrightarrow \pi_9(\Sigma^4\mathbb{C}P^2) \) is onto. Thus \( \eta \wedge \text{id}: \Sigma^5\mathbb{C}P^2 \to \Sigma^4\mathbb{C}P^2 \) is homotopic to the composite

\[ \Sigma^5\mathbb{C}P^2 \overset{\text{pinch}}{\longrightarrow} S^9 \overset{k_{10}}{\longrightarrow} S^6 \longrightarrow \Sigma^4\mathbb{C}P^2. \]

Consider the commutative diagram of pinch maps

\[
\begin{array}{ccc}
\Sigma^5\mathbb{C}P^2 & \xrightarrow{p'} & S^9 \\
\downarrow q & & \downarrow q' \\
\mathbb{C}P^2 \wedge P^5(2) & \xrightarrow{p} & P^9(2)
\end{array}
\]

Since \( p' \circ q_*(\beta) = q'_* \circ p_*(\beta) = q'_* \circ j_*(\nu_8) = 0 \), we have \( 2\beta = 0 \) and so

\[ \pi_{11}(\mathbb{C}P^2 \wedge P^5(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \]

generated by \( j_*(\omega_6) \) and \( j_*(\nu_8) \). Now let \( \mu \) be the composite

\[ \Omega\Sigma\mathbb{R}P^4_2 \overset{H_2}{\longrightarrow} \Omega\Sigma(\mathbb{R}P^4_2)^{(2)} \overset{\Omega g}{\longrightarrow} \Omega(\mathbb{C}P^2 \wedge P^5(2)). \]

Then \( \mu_*(j_*(\omega_6)) = j_*(\omega_6) \) the generator for \( \pi_{11}(P^7(2)) \). From the commutative diagram

\[
\begin{array}{ccc}
\Omega\Sigma\mathbb{R}P^4_2 & \xrightarrow{H_2} & \Omega\Sigma(\mathbb{R}P^4_2)^{(2)} \\
\downarrow \Omega S^3 & & \downarrow \Omega S^5, \\
\Omega S^3 & \xrightarrow{H_2} & \Omega S^5,
\end{array}
\]

we have \( \mu_*(\beta_1) = 0 \) and so

\[ 0 = \mu_*(2\tilde{\alpha}) = \mu_*(\beta_1) + k_1\mu_*(j_*(\omega_6)) = k_1j_*(\omega_6). \]

Thus \( k_1 = 0 \) and so \( 2\tilde{\alpha} = \beta_1 \in \pi_{11}(S^3) = \mathbb{Z}/2. \)
Continuation for solving the group extension problem. Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{11}(B) & \xrightarrow{\phi_1} & \pi_{11}(F) \\
\downarrow & & \downarrow \\
\pi_{11}(B) & \xrightarrow{\phi_2} & \pi_{11}(\Sigma \mathbb{RP}^4) \\
\downarrow & & \downarrow \\
\pi_{11}(B) & \xrightarrow{q_*} & \pi_{11}(SU(3)) \\
\downarrow & & \downarrow \\
\pi_{11}(S^5) & \xrightarrow{=} & \pi_{11}(S^5),
\end{array}
\]

where \(\phi_1\) and \(\phi_2\) are monomorphisms. Let \(G\) be the subgroup of \(\pi_{11}(\Sigma \mathbb{RP}^4)\) generated by \(\tilde{\alpha}\) and \(\epsilon_3\). Then \(G\) has four elements and \(q_*|G: G \to \pi_{11}(SU(3))\) is onto. Thus \(G \cong \pi_{11}(SU(3))\) and so

\[
\pi_{11}(\Sigma \mathbb{RP}^4) \cong \pi_{11}(SU(3)) \oplus \pi_{11}(B) = \pi_{11}(SU(3)) \oplus \mathbb{Z}/2\oplus \mathbb{Z}/4.
\]

We finish the proof now. \(\square\)

6.3.2. The Homotopy Groups \(\pi_*(\mathbb{CP}^2 \wedge P^6(2))\). Now we compute the homotopy groups of other factors in the decomposition of \(\Omega^3_0(P^3(2))\). Observe that \(\mathbb{CP}^2 \wedge P^6(2)\) is 6-connected.

**Lemma 6.34.** The homotopy groups of \(\pi_r(\mathbb{CP}^2 \wedge P^6(2))\) for \(7 \leq r \leq 11\) are as follows.

1) \(\pi_7(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2\).
2) \(\pi_8(\mathbb{CP}^2 \wedge P^6(2)) = 0\).
3) \(\pi_9(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2\).
4) \(\pi_{10}(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2\).
5) \(\pi_{11}(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2\).

**Proof.** Consider the cofibre sequence

\[
P^9(2) \xrightarrow{\eta \wedge \text{id}} P^8(2) \longrightarrow \mathbb{CP}^2 \wedge P^6(2) \longrightarrow P^{10}(2) \xrightarrow{\eta \wedge \text{id}} P^9(2).
\]

The sequence

\[
\pi_r(P^9(2)) \xrightarrow{\eta \wedge \text{id}} \pi_r(P^8(2)) \longrightarrow \pi_r(\mathbb{CP}^2 \wedge P^6(2)) \longrightarrow \pi_{r-1}(P^9(2)) \longrightarrow \cdots
\]

is exact for \(r \leq 11\). Thus

\[
\pi_7(\mathbb{CP}^2 \wedge P^6(2)) \cong \pi_7(P^8(2)) = \mathbb{Z}/2 \quad \text{and} \quad \pi_8(\mathbb{CP}^2 \wedge P^6(2)) = 0.
\]

Consider the short exact sequence

\[
0 \longrightarrow \pi_9(P^9(2)) \xrightarrow{\eta \wedge \text{id}} \pi_9(P^8(2)) \longrightarrow \pi_9(\mathbb{CP}^2 \wedge P^6(2)) \longrightarrow 0.
\]

We have \(\pi_9(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2\) and there is an exact sequence

\[
\pi_{10}(P^9(2)) \xrightarrow{\eta \wedge \text{id}} \pi_{10}(P^8(2)) \longrightarrow \pi_{10}(\mathbb{CP}^2 \wedge P^6(2)) \longrightarrow 0.
\]
Thus $\pi_{10}(\mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2$ and there is an exact sequence

$$\pi_{11}(P^9(2)) \xrightarrow{\eta \wedge \text{id}} \pi_{11}(P^8(2)) \rightarrow \pi_{11}(\mathbb{CP}^2 \wedge P^6(2)) \rightarrow 2 \cdot \pi_{10}(P^9(2)) \rightarrow 0.$$ 

Notice that $(\eta \wedge 1) \circ \alpha_0 = \alpha_8 \circ \eta + \epsilon j_\ast(\nu_7)$. Then

$$(\eta \wedge 1) \circ \alpha_0 \circ \eta = \alpha_8 \circ \eta^2 + \epsilon j_\ast(\nu_7 \circ \eta) = \alpha_8 \circ \eta^2.$$ 

Thus $\eta \wedge \text{id}_*: \pi_{11}(P^9(2)) \rightarrow \pi_{11}(P^8(2)) = \mathbb{Z}/2$ if onto and so

$$\pi_{11}(\mathbb{CP}^2 \wedge P^6(2)) \cong 2\pi_{10}(P^9(2)) = \mathbb{Z}/2.$$ 

This completes the calculation.

\[\square\]

**Lemma 6.35.** The homotopy groups $\pi_r((\mathbb{CP}^2)^2 \wedge P^6(2))$ for $9 \leq r \leq 11$ are as follows.

1) $\pi_9(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2$.
2) $\pi_{10}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) = 0$.
3) $\pi_{11}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

**Proof.** Clearly $\pi_9(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) \cong \pi_9(\mathbb{CP}^2 \wedge P^8(2)) = \mathbb{Z}/2$. There is an exact sequence

$$\pi_{10}(\mathbb{CP}^2 \wedge P^9(2)) \rightarrow \pi_{10}(\mathbb{CP}^2 \wedge P^8(2)) \rightarrow \pi_{10}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) \rightarrow 0.$$ 

Thus $\pi_{10}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) = 0$ and there is an exact sequence

$$\pi_{11}(\mathbb{CP}^2 \wedge P^9(2)) \rightarrow \pi_{11}(\mathbb{CP}^2 \wedge P^8(2)) \rightarrow \pi_{11}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2))$$

$$\rightarrow \pi_{10}(\mathbb{CP}^2 \wedge P^9(2)) \rightarrow 0.$$ 

Recall that $\pi_{11}(\mathbb{CP}^2 \wedge P^9(2)) = 0$. There is a short exact sequence

$$0 \rightarrow \pi_{11}(\mathbb{CP}^2 \wedge P^8(2)) \rightarrow \pi_{11}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) \rightarrow \pi_{10}(\mathbb{CP}^2 \wedge P^9(2)) \rightarrow 0.$$ 

From the commutative diagram

\[
\begin{array}{ccc}
\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2) & \xrightarrow{[2]} & \mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2) \\
\downarrow & & \downarrow \\
\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge S^6 & \xrightarrow{\text{id} \wedge \eta} & \mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge S^5 \\
\downarrow & & \downarrow \\
\mathbb{CP}^2 \wedge S^{10} & \xrightarrow{\text{id} \wedge k \nu_7} & \mathbb{CP}^2 \wedge S^7,
\end{array}
\]

we have $2 \cdot \pi_{11}(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^6(2)) = 0$ and hence the result.

\[\square\]

Now we list a table for $\pi_r(P^3(2))$ with $r \leq 11$. In the following table, an integer $n$ indicates a cyclic group $\mathbb{Z}/n$, the symbol “+” the direct sum of groups and $n^k$ indicates the direct sum of $k$-copies of cyclic group $\mathbb{Z}/n$. 

\[
\begin{array}{|c|c|}
\hline
r & \pi_r(P^3(2)) \\
\hline
9 & \mathbb{Z}/2 \\
10 & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
11 & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\hline
\end{array}
\]
Theorem 6.36. $\pi_2(P^3(2)) = \mathbb{Z}/2$, $\pi_3(P^3(2)) = \mathbb{Z}/4$, $\pi_4(P^3(2)) = \mathbb{Z}/4$ and $\pi_r(P^3(2))$ for $5 \leq r \leq 11$ is as follows

<table>
<thead>
<tr>
<th>$\pi_r(P^3(2))$</th>
<th>$G_1 \oplus$</th>
<th>$G_2 \oplus$</th>
<th>$G_3 \oplus$</th>
<th>$G_4 \oplus$</th>
<th>$G_5 \oplus$</th>
<th>$G_6 \oplus$</th>
<th>others</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 5$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r = 6$</td>
<td>2</td>
<td>$2^3$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r = 7$</td>
<td>4</td>
<td>$2 + 8$</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r = 8$</td>
<td>2</td>
<td>$2^3 + 4$</td>
<td>$2^2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r = 9$</td>
<td>2</td>
<td>$2^4$</td>
<td>$2^2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$r = 10$</td>
<td>0</td>
<td>$2^2 + 4^2$</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$r = 11$</td>
<td>0</td>
<td>$2^4 + 4^2$</td>
<td>$2^2$</td>
<td>2</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^3$</td>
</tr>
</tbody>
</table>

where $G_1 = \pi_{r-1}(S^3)$, $G_2 = \pi_r(\Sigma\mathbb{RP}^4)$, $G_3 = \pi_r(P^6(2))$, $G_4 = \pi_r(\mathbb{CP}^2 \wedge P^6(2))$, $G_5 = \pi_r(P^9(2))$, and $G_6 = \pi_r(\mathbb{CP}^2 \wedge \mathbb{CP}^2 \wedge P^k(2))$.

**Appendix A. The Table of the Homotopy Groups of $\Sigma^n\mathbb{RP}^2$**

The homotopy groups of the mod 2 Moore spaces, up to $8 +$ dimension, are listed in the following table. In the following table, an integer $k$ indicates a cyclic group $\mathbb{Z}/k\mathbb{Z}$, the symbol " + " indicates the direct sum of groups, $k^l$ indicates the direct sum of $l$-copies of the cyclic group $\mathbb{Z}/k\mathbb{Z}$. 
<table>
<thead>
<tr>
<th>$r \backslash n$</th>
<th>$n=3$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>$\geq 12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r=-1$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$2^3$</td>
<td>2</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$2^5$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$2^7$</td>
<td>$2^3$</td>
<td>2</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$2^{10}$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$2^5$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^4$</td>
<td>$2^4$</td>
<td>$2^4$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$2^{14}$</td>
<td>$2^5$</td>
<td>$2^2$</td>
<td>$2^4$</td>
<td>$2^3$</td>
<td>$2^3$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td>$2^2$</td>
<td></td>
</tr>
</tbody>
</table>

**References**


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