NOTE ON THE MINIMAL SIMPLICIAL SET MODELS FOR $p$-LOCAL $H$-SPACES

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ABSTRACT. We show that any minimal simplicial set model for $\Omega \mathbb{C}P^n$ localized rationally does not have the $A_\infty$-homotopy type of a minimal simplicial group for $n \geq 2$. On the other hand, we show that there is a (non-commutative) minimal simplicial group $G$ such that $BG$ is a minimal simplicial set model for the rational sphere $S^{2n}$ for $n \geq 1$. For $p$-local spaces with $p > 2$, there is a strictly commutative (but non-associative in general) multiplication on the minimal simplicial set model of an $H$-space.

1. Introduction

It is well known that every fibrant simplicial set $X$ is homotopy equivalent to a minimal simplicial set $Y$ and so we can regard the minimal simplicial set $Y$ as a minimal model of the space $X$. If $X$ is a loop space, the product structure on the minimal simplicial set model for $X$ turns to be interesting and has been studied in [1, 6, 8, 9]. In general, there does not exist a strict associative multiplication on minimal simplicial set models. On the other hand the multiplication on the minimal simplicial set model has many nice properties [8]. In this note, we provide more examples of the product structure on a minimal simplicial set model.

In the first case, we look at rational spaces. If $X$ is a path connected rational CW-complex, then $\Omega X$ is homotopy equivalent to a product of the Eilenberg-MacLane spaces $K(\pi_n(X), n)$ for $n \geq 0$, see [2]. In this case we do have a (strictly associative and strictly commutative) multiplication on the minimal simplicial set model for $\Omega X$. However, in general, the $A_\infty$-homotopy type of $\Omega X$ is different from that of a product of $K(\pi_n(X), n)$. Note that if $\Omega X$ has the $A_\infty$-homotopy type of a simplicial group $G$ if and only if $X$ has the homotopy type of $BG$. We give examples to show that in rational case $\Omega X$ may not has the $A_\infty$-homotopy type of a minimal simplicial group.

The examples we look at are $\Omega \mathbb{C}P^n$ considered as $J_n(S^2)$, the $n$-th filtration piece of the James construction of $S^2$.

Theorem 1.1. If $n \geq 2$, then $\Omega \mathbb{C}P^n$, localized rationally, does not have the $A_\infty$-homotopy type of a minimal simplicial group.
Notice that \( \mathbb{C}P^n \simeq J_n(S^2) \) localized rationally, where \( J(X) \) is the James construction of \( X \) and \( \{J_n(X)\} \) is the James filtration (word filtration) of \( J(X) \). Theorem 1.1 follows from the following slightly more general statement.

**Theorem 1.2.** If \( n \geq 1 \) and \( k \geq 2 \), then \( \Omega J_k(S^{2n}) \), localized rationally, does not have the \( A_\infty \)-homotopy type of a minimal simplicial group.

**Theorem 1.3.** Let \( n \geq 1 \) and let \( S^n \) be the rational \( n \)-sphere. Then \( \Omega S^n \) does have the \( A_\infty \)-homotopy type of a minimal simplicial group.

If \( n \) is odd, then \( \Omega S^n \simeq K(\mathbb{Q}, n - 1) \) as \( A_\infty \)-spaces. Thus this theorem is obvious when \( n \) is odd. If \( n \) is even, then the rational \( n \)-sphere \( S^n \) is a two stage Postnikov system with a non-trivial Postnikov invariant. Thus a minimal simplicial group \( G \) with the \( A_\infty \)-homotopy type of \( \Omega S^n \) must be non-commutative. We will construct a specific minimal simplicial group \( G(n) \) which has the \( A_\infty \)-homotopy type of \( \Omega S^n \).

Next we show by examples that there exist strictly commutative (but non-associative) multiplication on certain minimal simplicial set models.

**Theorem 1.4.** Let \( p > 2 \) be an odd prime and let \( X \) be a path-connected \( p \)-local \( H \)-space. Then there exists a strictly commutative (but non-associative in general) multiplication on the minimal simplicial set model of \( X \).

The multiplication is not homotopic to the original multiplication on \( X \) in general. But note that any two multiplications on an \( H \)-space are homotopic after looping. This theorem means that the multiplication on the loop space of an \( H \)-space can be deformed to be strictly commutative. This theorem is the homotopy theory analogy of the Jordan algebra in the sense that one can make a new strictly commutative multiplication from an old non-commutative multiplication when the power map \( 2: X \longrightarrow X \) is a homotopy equivalence.

**Corollary 1.5.** Let \( p > 2 \) be an odd prime. Then there exists a strictly commutative (but non-associative if \( n \geq 1 \)) multiplication on the simplicial set model for \( S^{2n+1} \) localized at \( p \) for each \( n \geq 0 \).

These theorems reveal that in the odd prime case the homotopy groups could contain the obstructions only for associativity (as we may assume that the multiplication is already commutative) in the minimal simplicial set models of loop spaces. Since we know that it is impossible to have an associative multiplication in the minimal simplicial set models of loop spaces of a finite complex, it seems that it is reasonable (in the odd prime cases) to consider the "Jordan" product on the minimal simplicial set models.

By [8, Proposition 1.2], there exist unique left inverse and right inverse in any connected minimal simplicial \( H \)-sets. Let \( X \) be a connected commutative (non-associative) minimal simplicial \( H \)-set. Then one can define the associator \( (x, y, z) = \)
((xy)z)(x(yz))^{-1} and so one can define the higher associators. Thus $X$ is filtered by the associator length. By [8, Theorem 1.4], any connected minimal simplicial $H$-set is nilpotent with respect to the associator length. In particular, the nilpotency degree (with respect to the associator length) of the $(n+1)$-st Postnikov section of $X$ is at most one bigger that the nilpotency degree of the $n$-th Postnikov section of $X$ and so if it is strictly bigger (by one), then the elements of dimension $n+1$ which are given by the highest associator in the $(n+1)$-st Postnikov section are (some of) the elements in the $(n+1)$-st homotopy group.

Roughly speaking, some “relations” for associativity are already given in [8, Theorem 1.4]. We should point out that Stasheff’s associahedra [5] for $A_n$-spaces also gives some “relations” for associativity. Further “relations” for associativity need to be studied. The “algebraic” structure on the minimal models of loop spaces remains, for the most part, a mystery.

**Corollary 1.6.** Let $p > 2$ be an odd prime. Then there exists a strictly commutative (but non-associative in general) multiplication on the simplicial set models for loop spaces and Postnikov systems of $S^{2n+1}$ localized at $p$ for each $n \geq 0$.

We have to point out that George Whitehead has given examples of minimal simplicial sets in which the multiplication is commutative but non-associative. But we do not exactly know Whitehead’s examples and do not find Whitehead’s paper. Theorem 1.4 might be more or less known by Whitehead years ago.

The multiplication on loop spaces of 2-local spaces sounds very wild in the sense that any multiplication on minimal simplicial set models of many interesting spaces like the loop spaces of odd spheres is neither commutative nor associative.

The author is indebted to helpful discussions with James Stasheff. Whitehead’s examples were told by him. Theorem 1.4 was inspired by Whitehead’s examples.

2. Proofs

2.1. The construction $F^M(X)$.

**Definition 2.1.** Let $M$ be a simplicial monoid and let $S$ be a pointed simplicial set with a base point $\ast$. The simplicial monoid $F^M(S)$ is defined by

$$F^M(S)_n = \prod_{x \in S_n} (M_n)_x / \approx,$$

where $(M_n)_x$ is a copy of $M_n$, where $\prod^\ast$ is the free product of monoids and where the product equivalent relation $\approx$ is generated by

$$g_{x_0} \approx 1$$
for the elements $g$ in $M_n$ with base point index $x_0$. The face and degeneracy functions are induced by the homomorphism

$$d_j : (M_n)_x \cong M_n \rightarrow M_{n-1} \cong (M_{n-1})_{d_j}$$

and

$$s_j : (M_n)_x \cong M_n \rightarrow M_{n+1} \cong (M_{n+1})_{s_j}$$

for $0 \leq j \leq n$.

The construction $F^M(S)$ is written as $S \otimes M$ or $S \otimes M$ in some references since $F^M(S)$ is the reduced tensor product of $S$ and $M$ in Quillen’s sense [4].

**Theorem 2.2** (Theorem 4.9, [7]). There is a homotopy equivalence

$$BM \wedge S \rightarrow BF^M(S)$$

for any pointed simplicial set $S$ and any simplicial monoid $M$.

**Corollary 2.3.** The simplicial group $F^\mathbb{Q}(S^n)$ is a simplicial group model for the rational space of $\Omega S^{n+1}$.

Notice that there is a canonical injection $i : M \wedge S \rightarrow F^M(S)$ which is given by $i(g, x) = g_x$ in the (reduced) free product $F^M(S)$. The simplicial monoid $F^M(S)$ satisfies the following universal property.

**Proposition 2.4.** Let $S$ be a pointed simplicial set and let $M$ and $N$ be simplicial monoids. Let $\phi : M \wedge S \rightarrow N$ be a simplicial map such that

$$\phi(g_1 g_2, x) = \phi(g_1, x) \cdot \phi(g_2, x)$$

for any $g_1, g_2 \in M$ and $x \in S$. Then there is a unique simplicial homomorphism $\tilde{\phi} : F^M(S) \rightarrow N$ such that $\phi = \tilde{\phi} \circ i : M \wedge S \rightarrow N$.

The assertion follows from the definition.

**2.2. Minimal simplicial $H$-sets.** A minimal simplicial $H$-set $X$ is a minimal simplicial set $X$ together with a multiplication $\mu : X \times X \rightarrow X$ such that $\mu$ has a strict unit element $e$. Let $X$ be a (pointed) fibrant simplicial set. An element $x \in X_n$ is called a Moore cycle if $d_j x = *$ for all $j$ [?, ?, ?]. If $X$ is a minimal simplicial set, then the homotopy relation on Moore cycles is trivial and so we can identify Moore cycles with homotopy classes in $\pi_*(X)$.

**Definition 2.5.** Let $X$ be a simplicial $H$-set. The abelianization $X^{ab}$ of $X$ is defined to be the quotient simplicial set of $X$ modulo the product equivalence relation generated by the the relations

1. $ab = ba$
2. $(ab)c \approx (ab)c$

...
for \(a, b, c \in X\). Thus \(X^{ab}\) is an abelian simplicial monoid.

**Definition 2.6.** Let \(X\) be a simplicial \(H\)-set with right (left) inverse. A subsimplicial \(H\)-set \(X'\) of \(X\) is said to be right (left) group-like if \(rx^{-1} \in X'\) for \(x \in X\), i.e., the right inverse of an element in \(X'\) is in \(X'\).

**Theorem 2.7** (Theorem 6.4, [8]). Let \(X\) be a connected minimal simplicial \(H\)-set and let \(p: X \to X^{ab}\) be the quotient map. Let \(S \subseteq \pi_\ast(X)\) be a (graded) subset of \(\pi_\ast(X)\) such that \(p_\ast(S)\) is a set of generators for \(p_\ast(\pi_\ast(X)) = \text{Im}(p_\ast: \pi_\ast(X) \to \pi_\ast(X^{ab}))\). Then the right (or left) group-like subsimplicial \(H\)-set of \(X\) generated by \(S\) is equal to the total space \(X\).

**Theorem 2.8** (Corollary 2.13, [9]; Theorem 1.4, [8]). Let \(G\) be a connected minimal simplicial group and let \(\{P_nG\}\) be the Moore-Postnikov systems of \(G\). Then the short exact sequence of simplicial groups

\[
0 \to F_nG \to P_nG \to P_{n-1}G \to 0
\]

is a central extension for each \(n\).

### 2.3. Proof of Theorem 1.2.

**Proof of Theorem 1.2.** Suppose that there were a minimal simplicial group \(G\) such that \(G\) had the rational \(A_\infty\)-homotopy type of \(\Omega J_k(S^{2n})\) with \(k \geq 2\).

Let \(p: G \to G^{ab}\) be the abelianization of \(G\). Notice that \(\pi_i(G) = \mathbb{Q}\) for \(i = 2n-1, 2kn-2\) and 0 for others. Notice that \(p_\ast: \pi_\ast(G) \to \pi_\ast(G^{ab})\) is an epimorphism and \(BG \simeq J_k(S^n)\) has a nontrivial Postnikov invariant. Thus \(\pi_{2kn-2}(G^{ab}) = 0\) and \(G^{ab}\) is isomorphic to the minimal simplicial group \(K(\mathbb{Q}, 2n-1)\).

Notice that \(G_j = \{e\}\) for \(j < 2n-1\) and \(G_{2n-1} = \pi_{2n-1}(G) = \mathbb{Q}\). Let \(x \in G_{2n-1}\) be a nonzero element. By Proposition 2.4, there is a unique simplicial homomorphism

\[
\phi: F^\mathbb{Q}(S^{2n-1}) \to G
\]

such that \(\phi(\sigma_{2n-1}) = x\), where \(\sigma_{2n-1}\) is the (only) non-degenerate simplex of dimension \(2n - 1\) in the standard simplicial \(n\)-sphere \(S^n\). By Theorem 2.7 since that \(G^{ab} = K(\mathbb{Q}, 2n - 1)\), \(G\) is generated by \(G_{2n-1}\) as a right (or left) group-like simplicial \(H\)-set. Thus the map \(\phi: F^\mathbb{Q}(S^n) \to G\) is an epimorphism. By Theorem 2.8, there is a central extension

\[
K(\mathbb{Q}, 2kn - 2) \to G \to K(\mathbb{Q}, 2n - 1).
\]

Let \(\{\Gamma^nG\}\) be the lower central series of \(G\) starting with \(\Gamma^1 = G\). Then \(\Gamma^3G = \{e\}\) is a trivial simplicial group and the simplicial homomorphism \(\phi: F^\mathbb{Q}(S^{2n-1}) \to G\)
factors through the quotient simplicial group $F^Q(S^{2n-1})/\Gamma^3(F^Q(S^{2n-1}))$. Thus we have a commutative diagram of central extensions

$$
\begin{array}{ccc}
K(Q, 2kn - 2) & \hookrightarrow & G \\
\phi & & \phi \\
\Gamma^2/\Gamma^3 & \hookrightarrow & F^Q(S^{2n-1})/\Gamma^3 \\
\end{array}
\rightarrow 
K(Q, 2n - 1)
$$

where $\Gamma^k = \Gamma^k(F^Q(S^{2n-1}))$ and $\phi$ is induced by $\phi$. It follows that the simplicial homomorphism $\phi: \Gamma^2/\Gamma^3 \rightarrow K(Q, 2kn - 2)$ is an epimorphism. Notice that the group $(\Gamma^2/\Gamma^3)_{2n+k-1}$ is generated as a $Q$-module by the elements $[s_I\sigma_{2n-1}, s_J\sigma_{2n-1}]$, where $s_I = s_{i_1} s_{i_2} \cdots s_{i_k} \sigma_{2n-1}$ is a degenerate element for $I = (i_1, i_2, \cdots, i_k)$ and where $I$ and $J$ run over all sequences $I = (i_1, i_2, \cdots, i_k)$ with $0 \leq i_1 < i_2 < \cdots < i_k \leq 2n + k - 2$ and $J = (j_1, j_2, \cdots, j_k)$ with $0 \leq j_1 < j_2 < \cdots < j_k \leq 2n + k - 2$. Observe that $[s_I\sigma_{2n-1}, s_I\sigma_{2n-1}]$ is a degenerate element if $I \cap J \neq \emptyset$. Thus $\Gamma^2/\Gamma^3$ is generated as a simplicial group by $(\Gamma^2/\Gamma^3)_j$ for $j \leq 2n - 2$. It follows that $K(Q, 2kn - 2) = \text{Im}(\phi: \gamma^2/\Gamma^3 \rightarrow K(Q, 2kn - 2))$ is generated as a simplicial group by $K(Q, 2kn - 2)_j$ for $j \leq 2n - 2$. Notice that $K(Q, 2kn - 2)_j = \{e\}$ is a trivial group for $j \leq 2n - 2$. It follows that $K(Q, 2kn - 2)$ is a trivial simplicial group. One gets a contradiction here and the assertion follows.

\begin{proof}

2.4. Proof of Theorem 1.3.

Proof of Theorem 1.3. The proof is given by constructing a minimal simplicial group which has a rational $A_\infty$-homotopy type of $\Omega S^{2n}$.

Consider the simplicial group $F^Q(S^{2n-1})$. Let $\Gamma^k = \Gamma^k(F^Q(S^{2n-1}))$. Then we have a central extension

$$
\begin{array}{ccc}
\Gamma^2/\Gamma^3 & \hookrightarrow & F^Q(S^{2n-1})/\Gamma^3 \\
\end{array}
\rightarrow 
\Gamma^3 \\
\rightarrow 
K(Q, 2n - 1)
$$

Notice that $\Gamma^2/\Gamma^3$ is a simplicial $Q$-module and has the homotopy type of $K(Q, 2n - 2)$. Thus there exists a sub simplicial $Q$-module $V \subseteq \Gamma^2/\Gamma^3$ such that

1) $V$ is contractible;

2) the quotient simplicial $Q$-module $(\Gamma^2/\Gamma^3)/V$ is the minimal simplicial group $K(Q, 2n - 2)$.

Notice that $V$ is contained in the center of $F^Q(S^{2n-1})/\Gamma^3$. Thus $V$ is a normal subsimplicial group of $F^Q(S^{2n-1})/\Gamma^3$. Let

$$
G = (F^Q(S^{2n-1})/\Gamma^3)/V
$$


be the quotient simplicial group of $F^Q(S^{2n-1})/\Gamma^3$ modulo the normal simplicial subgroup $V$. Then the simplicial epimorphism $F^Q(S^{2n-1}) \to G$ is a homotopy equivalences, but the inverse is only an $A_\infty$-homotopy equivalence. The assertion follows.

\[
\begin{align*}
\end{align*}
\]

2.5. Proof of Theorem 1.4. In the following proof, all spaces are localized at $p$ with $p > 2$.

Proof of Theorem 1.4. We may assume that $X$ is a minimal simplicial set. Let $Sp^n(X)$ be the $n$-fold symmetric product of $X$. Notice that there exist a strictly commutative multiplication (with a strict unit) on $X$ if and only if there exist a simplicial map $r : Sp^2(X) \to X$ such that the composite $X \to i \to Sp^2(X) \to X$, where $i$ is the canonical inclusion, is the identity map of $X$ if and only if $X$ is a retract of $Sp^2(X)$. Observe that by giving a retraction $r : Sp^2(X) \to X$ the multiplication given by the composite $X \times X \to Sp^2(X) \to X$ is a strictly commutative multiplication. Thus it suffices to show that $X$ is a retract of $Sp^2(X)$.

Let $X$ be a path-connected space. Let $J(X)$ be the James construction of $X$ and $J_n(X)$ be the sub spaces of $J(X)$ with word length less than or equal to $n$. Then there is a homotopy commutative diagram of cofibre sequences

\[
\begin{align*}
X & \to J_2(X) \to X \wedge X \\
X & \to Sp^2(X) \to X \wedge_{\Sigma_2} X \\
* & \to C \rightleftharpoons C
\end{align*}
\]

where $C$ is the homotopy cofibre of $J_2(X) \to Sp^2(X)$. Let $\mathbb{Z}_{(p)}$ be the $p$-localization of integers $\mathbb{Z}$ and $S$ be a pointed simplicial set. We write $\hat{\mathbb{Z}}_{(p)}(S)$ for the reduced free simplicial $\mathbb{Z}_{(p)}$-module generated by $S$, that is, $\hat{\mathbb{Z}}_{(p)}(S) = \mathbb{Z}_{(p)}(S)/\mathbb{Z}_{(p)}(*)$, where $*$ is the base point. Consider the transfer simplicial map $tr : \hat{\mathbb{Z}}_{(p)}(X \wedge_{\Sigma_2} X) = \hat{\mathbb{Z}}_{(p)}(X) \otimes_{\Sigma_2} \hat{\mathbb{Z}}_{(p)}(X)$ defined by

\[
tr(x \wedge y) = \frac{1}{2}(x \wedge y + y \wedge x)
\]

for $x, y \in X$. It follows that $\hat{\mathbb{Z}}_{(p)}(\Sigma X \wedge X) \simeq \hat{\mathbb{Z}}_{(p)}(C) \oplus \hat{\mathbb{Z}}_{(p)}(\Sigma(X \wedge_{\Sigma_2} X))$. 

\[
\begin{align*}
\end{align*}
\]
Let $T: J_2(X) \to J_2(X)$ be the map defined by
\[ T(xy) = yx \]
for $x, y \in X$. Notice that the set of homotopy classes $[\Sigma J_2(X), Y]$ is a group for any $Y$ under the multiplication induced by the comultiplication on $\Sigma J_2(X)$. Let $\beta_2: \Sigma J_2(X) \to \Sigma J_2(X)$ be the map defined by $\beta_2 = 1/2(\Sigma(T) - \text{id}_{\Sigma J_2(X)})$. Let $L_2 = \text{hocolim}_\beta \Sigma J_2(X)$. By using the decomposition $\mathbb{Z}(p)(\Sigma X \wedge X) \cong \mathbb{Z}(p)(C) \oplus \mathbb{Z}(p)(\Sigma(X \wedge \Sigma_2 X))$, it is a routine work to check that the composite
\[ C \to \Sigma J_2(X) \to L_2 = \text{hocolim}_\beta \Sigma J_2(X) \]
is a $\mathbb{Z}(p)$-homology equivalence and so is a homotopy equivalence. From the homotopy commutative diagram
\[
\begin{array}{ccc}
\ast & \to & C \\
\downarrow & & \downarrow \\
\Sigma X & \to & \Sigma J_2(X) & \to & \Sigma X \wedge X \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma X & \to & \Sigma Sp^2(X) & \to & \Sigma Sp^2(X),
\end{array}
\]
we have that $\Sigma Sp^2(X)$ is a homotopy retract of $\Sigma J_2(X)$ over $\Sigma X$. It follows that $\Sigma X$ is a homotopy retract of $\Sigma Sp^2(X)$. Let $r: \Sigma Sp^2(X) \to \Sigma X$ be a retraction. Let $\phi: J(X) \to X$ be the map given by
\[ \phi(x_1 x_2 \cdots x_n) = (\cdots (x_1 \cdot x_2) \cdots) \cdot x_{n-1} \cdot x_n \]
for $x_j \in X$, that is, $\phi(x_1 x_2 \cdots x_n)$ is the multiplication of $x_1, x_2, \ldots, x_n$ in $X$ from left to right. Then the composite
\[ X \hookrightarrow Sp^2(X) \hookrightarrow \Omega \Sigma Sp^2(X) \xrightarrow{\Omega(r)} \Omega \Sigma X \cong J(X) \xrightarrow{\phi} X \]
is a homotopy equivalence. Thus $X$ is a homotopy retract of $Sp^2(X)$. Notice that the injection $X \hookrightarrow Sp^2(X)$ is a cofibration. Thus $X$ is retract of $Sp^2(X)$. The assertion follows.
References


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