Abstract. We describe two obstruction theories for a given topological space \( X \) to be a loop space, the first requiring a given \( H \)-space structure on \( X \), and the second not. Both are defined in terms of higher homotopy operations.

1. Introduction

An \( H \)-space is a topological space \( X \) with a multiplication map \( m : X \times X \to X \) and identity \( * \in X \). The motivating example is a topological group \( G \), which from the point of view of homotopy theory is just a loop space: \( G \simeq \Omega BG = map_*(S^1, BG) \). The question of whether a given \( H \)-space \( X \) is, up to homotopy, a loop space, and thus a topological group (cf. [Mi1, §3]), has been studied from a variety of viewpoints – see [A, Ba, DoL, F, H, Ma2, St1, St2, Ste, Su, Z], and the surveys in [St3], [St4, §1], and [Ke, Part II]. Here we address this question from the aspect of homotopy operations.

As is well known, the homotopy groups of a space \( X \) have Whitehead products and composition operations defined on them; in addition, there are various higher order operations on \( \pi_*X \), such as Toda brackets; and the totality of these actually determine the (weak) homotopy type of \( X \) (cf. [Bl4, §7.17]). Therefore, they should enable us (in theory) to answer any homotopy-theoretic question about \( X \) – in particular, whether it is homotopy equivalent to a loop space. It is the purpose of this note to explain in what sense this can actually be done, using two possible approaches:

First, we explain how an \( H \)-space structure on \( X \) can be used to define the action of the primary homotopy operations on the shifted homotopy groups \( G_* = \pi_{*-1}X \) (which are isomorphic to \( \pi_*Y \) if \( X \simeq \Omega Y \)). This action will behave properly with respect to composition of operations if \( X \) is homotopy-associative, and will lift to a topological action of the monoid of all maps between spheres if and only if \( X \) is a loop space (see Proposition 5.6 below for the precise statement). The obstructions to having such a topological action may be stated in terms of the obstruction theories for realizing \( \Pi \)-algebras and their morphisms described in [Bl4].

A more concrete approach, which does not require a given \( H \)-space structure on \( X \), yields the following:

**Theorem A**: If \( X \) is a CW complex such that all Whitehead products vanish in \( \pi_*X \), then \( X \) is homotopy equivalent to a loop space if and only if all the higher homotopy
operations $\langle \psi \rangle$ defined in §7.10 vanish coherently.

The higher homotopy operations in question depend only on maps between wedges of spheres, and take value in homotopy groups of spheres. They are constructed by means of a certain collection of convex polyhedra, defined in §7.1 below, which may be of independent interest.

1.1. notation and conventions. $T_*$ will denote the category of pointed CW complexes with base-point preserving maps, and by a space we shall always mean an object in $T_*$, which will be denoted by a boldface letter: $X$, $S^n$, and so on. The base-point will be written $* \in X$. The full subcategory of 0-connected spaces will be denoted by $T_0$.

The space of Moore loops on $Y \in T_0$ will be denoted by $\Omega Y$. This is homotopy equivalent to the usual loop space: that is, the space of pointed maps $\text{map}_*(S^1, Y)$ (see [W, III, Corollary 2.19]). The reduced suspension of $X$ is denoted by $\Sigma X$.

$S_*$ will denote the category of pointed simplicial sets and pointed maps; its objects will be denoted by boldface letters $K, L, \ldots$. The subcategory of fibrant simplicial sets (Kan complexes) will be denoted by $S^k_*$, and that of reduced Kan complexes by $S^k$. $[K] \in T_*$ will denote the geometric realization of a simplicial set $K \in S_*$, while $S X \in S^k_*$ will denote the singular simplicial set associated to a space $X \in T_*$. $G$ denotes the category of simplicial groups. (See [Ma1, §§3,14,15,17] for the definitions).

For each of the categories $C = T_0, S^k_*, S^k_0$, or $G$, we will denote by $[X, Y]_C$ (or simply $[X, Y]$, if there is no danger of confusion) the set of pointed homotopy classes of maps $X \to Y$ (cf. [Ma1, §5] and [K1, §3]). The constant pointed map will be written $c_*$, or simply $*$. The homotopy category of $C$, whose objects are those of $C$, and whose morphisms are homotopy classes of maps in $C$, will be denoted by $hoC$. The adjoint functors $S$ and $| \cdot |$ induce equivalences of categories $hoT_* \approx hoS^k_*; \quad$ similarly $hoS^k_0 \approx hoG$ under the adjoint functors $G, W$ (see §5.1 below).

For any $X \in T_*$, we denote by $P^n X$ the $n$-th Postnikov approximation to $X$, so that $\pi_i X \cong \pi_i P^n X$ for $i \leq n$ and $\pi_i P^n X = 0$ for $i > n$.

**Definition 1.2.** A $H$-space structure for a space $X \in T_*$ is a choice of an $H$-multiplication map $m : X \times X \to X$ such that $m \circ i = \nabla$, where $i : X \vee X \hookrightarrow X \times X$ is the inclusion, and $\nabla : X \vee X \to X$ is the fold map (induced by the identity on each wedge summand). If $X$ may be equipped with such an $m$, we say that $\langle X, m \rangle$ (or just $X$) is an $H$-space. Note that if we only have $m \circ i \sim \nabla$, we can find a homotopic map $m' \sim m$ such that $m' \circ i = \nabla$ (since $X$ is assumed to be well-pointed).

An $H$-space $\langle X, m \rangle$ is homotopy-associative if $m \circ (m, id_X) \sim m \circ (id_X, m) : X \times X \times X \to X$. It is an $H$-group if it is homotopy-associative and has a (two-sided) homotopy inverse $i : X \to X$ with $m \circ (i \times id_X) \circ \Delta \sim c_* \sim m \circ (id_X \times i) \circ \Delta$, (where $\Delta : X \to X \times X$ is the diagonal). In fact, any connected homotopy-associative $H$-space is an $H$-group (cf. [W, X, Theorem 2.2]).

If $\langle X, m \rangle$ and $\langle Y, n \rangle$ are two $H$-spaces, a map $f : X \to Y$ is called an $H$-map if $n \circ (f \times f) \sim f \circ m : X \times X \to Y$. The set of pointed homotopy classes of $H$-maps $X \to Y$ will be denoted by $[X, Y]_H$.

One similarly defines $H$-simplicial sets and simplicial $H$-maps in the category $S_*$.  

1.3. organization. In section 2 we review some background material on $\Pi$-algebras; in section 3 we explain how an $H$-space structure on $X$ determines the $\Pi$-algebra
structure of its potential delooping; and in section 4 we show how the obstruction theory of [Bl4] for II-algebras and their morphisms may be used to determine whether \( X \) is a loop space. In section 5 we show, in the context of simplicial groups, that the II-algebra structure on \( \pi_{s-1}X \) can be made “topological” if and only if \( X \) is a loop space (Proposition 5.6).

In section 6 we define a certain cosimplicial simplicial space up-to-homotopy, which can be rectified if and only if \( X \) is a loop space, and prove a technical result (Theorem 6.4), which also yields a simplification of the general theory for realizing II-algebras described in [Bl4]. Finally, in section 7 we construct a certain collection of face-codegeneracy polyhedra, which are used to define the higher homotopy operations referred to in Theorem A (=Theorem 7.12). We also show (Example 7.15) how the theorem may be used in reverse to calculate a certain tertiary operations in \( \pi_*S^7 \).

In an appendix (section 8) we describe a specific construction of a simplicial resolution of the II-algebra \( G_* = \pi_{s-1}X \).

2. II-ALGEBRAS

In this section we recall some known facts on the primary homotopy operations and their relation to the \( H \)-space question. First, some definitions and notation:

**Definition 2.1.** A **II-algebra** is a graded group \( G_* = \{G_k\}_{k=1}^\infty \) (abelian in degrees \( > 1 \)), together with an action on \( G_* \) of the primary homotopy operations (i.e., compositions and Whitehead products, including the “\( \pi_1 \)-action” of \( G_1 \) on the higher \( G_n \)'s, as in [W, X, §7]), satisfying the usual universal identities. See [Bl1, §3] or [Bl2, §2.1] for a more explicit description. A **morphism of II-algebras** is a homomorphism of graded groups \( \phi : G_* \to G'_* \) which commutes with all the operations. II-algebras form a category, which will be denoted II-Alg.

It will sometimes be convenient to denote \( G_k \), the \( k \)-degree of the II-algebra \( G_* \), by \( D_kG_* \).

**Definition 2.2.** We say that a space \( X \) realizes an (abstract) II-algebra \( G_* \) if there is an isomorphism of II-algebras \( G_* \cong \pi_*X \). (There may be non-homotopy equivalent spaces realizing the same II-algebra – cf. [Bl4, §7.18]). Similarly, an abstract morphism of II-algebras \( \phi : \pi_*X \to \pi_*Y \) (between realizable II-algebras) is realizable if there is a map \( f : X \to Y \) such that \( \pi_*f = \phi \).

**Definition 2.3.** The **free** II-algebras are those isomorphic to \( \pi_*W \), for some (possibly infinite) wedge of spheres \( W \): More precisely, let \( L \) be a graded set \( \{L_k\}_{k=1}^\infty \), and let \( W = \bigvee_{k=1}^\infty \bigvee_{x \in L_k} S^k_x \), where each \( S^k_x \) is a \( k \)-sphere. Then we say that \( \pi_*W \) is the free II-algebra generated by \( L \). We shall consider each element \( x \in L_k \) to be an element of \( \pi_*W \), by identifying it with that generator of \( \pi_kW \) which represents the inclusion \( S^k_x \to W \). The free II-algebra \( \pi_*W \) is called finite if \( L \) is finite.

**Fact 2.4.** If we let \( \Pi \) denote the homotopy category of wedges of spheres, and \( \mathcal{F} \subset \Pi-Alg \) the full subcategory of free II-algebras, then the functor \( \pi_* : \Pi \to \mathcal{F} \) is an equivalence of categories. Thus any II-algebra morphism \( \phi : G_* \to G'_* \) is uniquely realizable, if \( G_* \) and \( G'_* \) are free II-algebras – and in fact only \( G_* \) need be free.

**Definition 2.5.** Let \( T : \Pi-Alg \to \Pi-Alg \) be the “free II-algebra” comonad (cf. [M, VI, §1]), defined \( TG_* = \coprod_{k=1}^\infty \coprod_{g \in G_k \setminus \{0\}} \pi_*S^k_g \). The counit \( \varepsilon = \varepsilon_{G_*} : TG_* \to G_* \).
is defined by \( \iota^k_{(g)} \mapsto g \) (where \( \iota^k_{(g)} \) is the canonical generator of \( \pi_* S^k \)), and the
comultiplication \( \vartheta = \vartheta_G : T G_* \to T^2 G_* \) is induced by the natural transformation
\( \bar{\vartheta} : id_F \to T|_F \) defined by \( x_k \mapsto \iota^k_{(x_k)} \).

**Definition 2.6.** An *abelian* \( \Pi \)-algebra is one for which all Whitehead products vanish.

These are indeed the abelian objects of \( \Pi Alg \) — see [Bl2, §2]. If \( X \) is an \( H \)-space, then \( \pi_* X \) is an abelian \( \Pi \)-algebra (cf. [W, X, (7.8)]).

3. Secondary \( \Pi \)-algebra structure

We now describe how an \( H \)-space structure on \( X \) determines the \( \Pi \)-algebra structure
of a (potential) classifying space.

3.1. the James construction. For any \( X \in \mathcal{T}_* \), let \( JX \) be the James reduced
product construction, with \( \lambda : JX \to \Omega \Sigma X \) the homotopy equivalence of [W, VII, (2.6)], and \( j_X : X \to JX \) and \( i_X : X \to \Omega \Sigma X \) the natural inclusions.

If \( \langle X, m \rangle \) is an \( H \)-space, there is a retraction of spaces \( \tilde{m} : JX \to X \) (with
\( \tilde{m} \circ j_X = id_X \)), defined

\[
\tilde{m}(x_1, x_2, \ldots, x_n) = m(\ldots m(m(x_1, x_2), x_3), \ldots, x_n)
\]

(cf. [J1, Theorem 1.8]).

**Definition 3.3.** Let \( X \) be an \( H \)-space. Given homotopy classes of maps \( \alpha \in [\Sigma A, \Sigma B] \) and \( \beta \in [B, X] \), we define the *derived composition* \( \alpha \ast \beta \in [A, X] \) as follows:

Choose representatives \( f : \Sigma A \to \Sigma B \) and \( g : B \to X \) for \( \alpha, \beta \) respectively, and let \( \lambda^{-1} : \Omega \Sigma X \to JX \) be any homotopy inverse to \( \lambda \). Then \( \alpha \ast \beta \) is represented by
the composite

\[
\begin{array}{c}
A \xrightarrow{i_A} \Omega \Sigma A \xrightarrow{\Omega \alpha} \Omega \Sigma B \xrightarrow{\lambda^{-1}} JB \xrightarrow{J\beta} JX \xrightarrow{\tilde{m}} X.
\end{array}
\]

**Fact 3.4.** Note that if \( \alpha = \Sigma \bar{\alpha} \) for some \( \bar{\alpha} : A \to B \), then \( \alpha \ast \beta = \bar{\alpha} \# \beta \) (this is
well-defined, because \( X \) is an \( H \)-space).

We shall be interested in the case where \( B \) is a wedge of spheres and \( A = S^r \), so \( \ast \) associates a class \( \omega \ast (\beta_1, \ldots, \beta_k) \in \pi_{n_1} X \) to any \( k \)-ary homotopy operation
\( \omega^\# : \pi_{n_1+1}(-) \times \ldots \times \pi_{n_k+1}(-) \to \pi_{n_1+1}(-) \) and collection of elements \( \beta_i \in \pi_{n_i} X \)
\((i = 1, \ldots, k)\).

In particular, if \( \omega : S^{p+q+1} \to S^{p+1} \vee S^{q+1} \) represents the Whitehead product, one
may define a “Samelson product” \( \omega^\ast (-,-) : \pi_p X \times \pi_q X \to \pi_{p+q} X \) for any \( H \)-space \( X \),
even without assuming associativity or the existence of a homotopy inverse (compare
[W, X, §5]).

However, in general this \( \omega^\ast (-,-) \) need not enjoy any of the usual properties of
the Samelson product (bi-additivity, graded-commutativity, Jacobi identity — cf. [W, X, Theorems 5.1 & 5.4]). To ensure that they hold, one needs further assumptions on
\( X \).

First, we note the following homotopy version of [W, VII, Theorem 2.5], which
appears to be folklore:
Lemma 3.5. If $\langle X, n \rangle$ is a homotopy-associative $H$-space, then any map $f : A \to X$ extends to an $H$-map $\hat{f} : JA \to X$, which is unique up to homotopy.

Proof. Given $f : A \to X$, define $\hat{f} : JA \to X$ by

$$\hat{f}(x_1, \ldots, x_r) = m(\ldots m(m(f(x_1), f(x_2)), f(x_3)), \ldots, f(x_r)).$$

This is an $H$-map by [N, Lemma 1.4]. Now let $\hat{g} : JA \to X$ be another $H$-map, with a homotopy $H : f \simeq g$ $\text{Def} \hat{g} \circ j_A$. Since $\hat{g}$ is an $H$-map, there is a homotopy $G : n \circ (\hat{g} \times \hat{g}) \simeq \hat{g} \circ m$ (where $m : JA \times JA \to JA$ is the $H$-multiplication). Moreover, by [N, Lemma 1.3(a)] we may assume $G$ is stationary on $JA \vee JA$.

For each $r \geq 0$, let $J_r A$ denote the $r$-th stage in the construction of $JA$, with $j_s : J_s A \hookrightarrow J_r A$ and $j^s : J_s A \hookrightarrow JA$ the inclusions, starting with $J_0 A = *$ and $J_1 A = A$. We define $T_r A$ to be the pushout in the following diagram:

$$\begin{array}{ccc}
J_{r-1} A & \xrightarrow{i} & J_{r-1} A \times A \\
\downarrow {\bar{f}_r}^{-1} & & \downarrow \bar{q}_r \\
J_r A & \xrightarrow{l} & T_r A
\end{array}$$

for $r \geq 1$ (so $T_1 A = A \vee A$); then $J_{r+1} A$ is the pushout in:

$$\begin{array}{ccc}
T_r A & \xrightarrow{\psi_r = (i_1, j_r^{-1} \times id)} & J_r A \times A \\
\downarrow {\varphi_r} & & \downarrow \bar{q}_{r+1} \\
J_r A & \xrightarrow{j_r} & J_{r+1} A
\end{array}$$

Now let $\hat{f}_r = \hat{f} |_{J_r A}$ and $\hat{g}_r = \hat{g} |_{J_r A}$; we shall extend $H : f \simeq g$ to a homotopy $\hat{H} : \hat{f} \simeq \hat{g}$ by inductively constructing homotopies $\hat{H}_r : \hat{f}_r \simeq \hat{g}_r$ (starting with $\hat{H}_1 = H$) such that $\hat{H}_r | J_{r-1} A = \hat{H}_{r-1}$: let $n_r : X^r \to X$ denote the $n$-fold multiplication $n_r(x_1, \ldots, x_r) = n(\ldots n(x_1, x_2), \ldots, x_r)$ and $q_r : A^r \to J_r A$ the quotient map, so that $n^r \circ f' = \hat{f}_r \circ q_r$.

As a first approximation, define $\hat{H}_{r+1} : \hat{f}_r \times f \simeq \hat{g}_r \times g$ on $J_r A \times A$ in the above pushout to be the sum of homotopies $\hat{H}_{r+1} = n \circ (\hat{H}_r \times H) + G \circ (j^r \times j_A)$.

This does not quite agree with $\hat{H}_r \circ \varphi_r$ on $T_r A$, but since $G$ is stationary on $JA \vee JA$ we have $\hat{H}_{r+1} | J_r A = n \circ (\hat{H}_r \times id) + (\text{stationary}) = \hat{H}_r + (\text{stationary})$ and $\hat{H}_{r+1} | J_{r-1} A \times A = n \circ (\hat{H}_{r-1} \times H) + G \circ (j^{r-1} \times j_A) = \hat{H}_r$.

Since $\hat{H}_1 = H$, we see that $\hat{H}_{2 | T_r A} = (H + (\text{stationary}), H)$, while $H \circ \varphi_1 = (H, H)$. Thus we may assume by induction that there is a homotopy of homotopies $F : \hat{H}_{r+1} | T_r A \simeq \hat{H}_r \circ \varphi_r$. Since $T_r A \hookrightarrow J_r A \times A$ is a cofibration, the inclusion

$$T_r A \times I^2 \cup (J_r A \times A) \times (\{0, 1\} \times I \cup I \times \{0\}) \hookrightarrow (J_r A \times A) \times I^2$$

is a trivial cofibration, and thus we may use the homotopy extension property to obtain a new homotopy $\hat{F}$ on $(J_r A \times A) \times I^2$ which restricts to $\hat{H}_{r+1} : \hat{f}_r \times f \simeq \hat{g}_r \times g$ on
Corollary 3.6. If $X$ is a homotopy-associative $H$-space, then for any $A \in \mathcal{T}$, the inclusion $j_A : A \to JA$ induces a bijection $j_A^* : [JA, X]_H \xrightarrow{\sim} [A, X]_{\mathcal{T}}$.

Proof. Since $X$ is homotopy-associative $H$-space, the retraction $\tilde{m} = \tilde{i}_d_X : JX \to X$ is an $H$-map, by the Lemma, so we may define $\phi : [A, X]_{\mathcal{T}} \to [JA, X]_H$ by $\phi([f]) = [\tilde{m} \circ J(f) \circ j_A] = [f]$. On the other hand, given an $H$-map $g : JA \to X$ we have $\tilde{m} \circ J(g \circ j_A) \circ j_A \simeq g \circ j_A$, which implies that $\tilde{m} \circ J(g \circ j_A) \simeq g$ by Lemma 3.5. Thus also $\phi(j_A([g])) = [g]$. \hfill $\square$

3.7. notation. If $X$ is a homotopy-associative $H$-space, we shall write $\pi_i^H X$ for $[\Omega^i S^i, X]_H = [JS^{i-1}, X]_H \cong \pi_{i-1} X$.

Proposition 3.8. If $X$ is a homotopy-associative $H$-space, then $\alpha \ast (\beta \ast \gamma) = (\alpha \ast \beta) \ast \gamma$ for any $\alpha \in [\Sigma A, \Sigma B]$, $\beta \in [\Sigma B, \Sigma C]$, and $\gamma \in [C, X]$.

Proof. It suffices to consider $\alpha = id_{\Sigma B}$, and so to show that

$$\begin{array}{ccl}
\Omega \Sigma B & \xrightarrow{\Omega \beta} & \Omega \Sigma C \\
\Omega \Sigma (\beta \ast \gamma) & \downarrow \tilde{m} & \Omega \Sigma X \\
\Omega \Sigma X & \xrightarrow{\tilde{m}} & X
\end{array}$$

commutes up to homotopy (where $\tilde{m}$ is the composite $\Omega \Sigma X \xrightarrow{\lambda^{-1}} JX \xrightarrow{\tilde{m}} X$) — or, since $\beta \ast \gamma$ is defined to be the composite $\tilde{m} \circ \Omega \Sigma \gamma \circ \Omega \beta \circ i_B$, that the two composites $\phi = \tilde{m} \circ \Omega \Sigma \gamma \circ \Omega \beta$ and $\psi = \tilde{m} \circ \Omega \Sigma \tilde{m} \circ (\Omega \Sigma)^2 \gamma \circ \Omega \Sigma \beta \circ \Omega \Sigma i_B$ are homotopic.

Now if $X$ is a homotopy-associative $H$-space, then $\tilde{m}$ is an $H$-map by Lemma 3.5, so $\phi, \psi : \Omega \Sigma B \to X$ are $H$-maps. By Corollary 3.6 it suffices to check that $\phi \circ i_B \sim \psi \circ i_B$ — i.e., that $\tilde{m} \circ \Omega \Sigma \gamma \circ \Omega \beta \circ i_B$ is homotopic to the composition of

$$B \xrightarrow{i_B} \Omega \Sigma B \xrightarrow{\Omega \Sigma \tilde{m}} \Omega \Sigma (\Sigma \gamma) \xrightarrow{\Omega \Sigma \tilde{f}} \Omega \Sigma X \xrightarrow{\tilde{m}} X.$$ 

But $\Omega \Sigma \gamma \circ \Omega \beta \circ i_B$ is adjoint to $(\Sigma \gamma) \circ \beta$, while the composition of

$$B \xrightarrow{i_B} \Omega \Sigma B \xrightarrow{\Omega \Sigma \tilde{m}} \Omega \Sigma (\Sigma \gamma) \xrightarrow{\Omega \Sigma \tilde{f}} \Omega \Sigma X$$

is adjoint to $\Sigma (\tilde{m} \circ \Omega \Sigma \gamma \circ \Omega \beta \circ i_B)$ which is equal to $\Sigma (\tilde{m} \circ (\Sigma \gamma) \circ \beta)$, (where $\tilde{f}$ denotes the adjoint of $f$). Since for any $f : Y \to Z$ the adjoint of $\Sigma f$ is $\Omega \Sigma f \circ i_Y$, we see $\tilde{m} \circ \Sigma \tilde{f} \sim f$, which completes the proof. \hfill $\square$

It is readily verified that when $X \simeq \Omega Y$, the secondary composition is the adjoint of the usual composition in $\pi_* Y$; Thus we have:

Corollary 3.9. If $X$ is an $H$-group, then the graded abelian group $G_*$, defined by $G_k = \pi_k^H X \cong \pi_{k-1} X$ (with $\gamma \in G_k$ corresponding to $\gamma \in \pi_{k-1} X$), has a $\Pi$-algebra structure defined by the derived compositions: that is, if $\psi \in \pi_k(S^{1} \vee \ldots \vee S^n)$ and $\gamma_j \in G_{t_j}$ for $1 \leq j \leq n$, then $\psi^\#(\gamma_1, \ldots, \gamma_n) \overset{\text{def}}{=} \omega \ast (\gamma_1, \ldots, \gamma_n) \in G_k$. If $X \simeq \Omega Y$, then $G_*$ is isomorphic as a $\Pi$-algebra to $\pi_* Y$. \hfill $\square$
Definition 3.10. For any $H$-group $\langle X, m \rangle$, the $\Pi$-algebra $G_\ast$ of Corollary 3.9 will be called the delooping of $\pi_\ast X$, and denoted by $\Omega^{-1}\pi_\ast X$ (so in particular $\Omega^{-1}\pi_0Y \cong \pi_0Y$).

Remark 3.11. Note that Corollary 3.9 provides us with an algebraic obstruction to delooping a space $X$: if there is no way of putting a $\Pi$-algebra structure on the graded abelian group $G_\ast = \pi_\ast X$ which is consistent with Fact 3.4, then $X$ is not a loop space, or even a homotopy-associative $H$-space. (This is of course assuming that the $\Pi$-algebra $\pi_\ast X$ is abelian -- otherwise $X$ cannot even be an $H$-space.)

Example 3.12. Consider the $\Pi$-algebra $G_*\ast$ defined by $G_2 = \mathbb{Z}\langle x \rangle$, (i.e., $x$ generates the cyclic group $G_2$), $G_3 = \mathbb{Z}/2\langle \eta_2^\# x \rangle$, $G_4 = \mathbb{Z}/2\langle \eta_3^\# \eta_2^\# x \rangle$, and $G_5 = \mathbb{Z}/2\langle \eta_4^\# \eta_3^\# \eta_2^\# x \rangle$, with $G_t = 0$ for $t \neq 2, 3, 4, 5$ and all Whitehead products zero.

There can be no homotopy-associative $H$-space $X$ with $\pi_\ast X \cong G_\ast$, since the $\Pi$-algebra $G'_\ast = \Omega^{-1}G_\ast$ cannot be defined consistently: we would have $G'_2 = \mathbb{Z}\langle \bar{x} \rangle$, $G'_4 = \mathbb{Z}/2\langle \eta_3^\# \bar{x} \rangle$, $G'_5 = \mathbb{Z}/2\langle \eta_4^\# \eta_3^\# \bar{x} \rangle$, and $G'_6 = \mathbb{Z}/2\langle \eta_5^\# \eta_4^\# \eta_3^\# \bar{x} \rangle$ by Fact 3.4; but $\pi_3S^3 = \mathbb{Z}/12\langle \alpha \rangle$ with $6\alpha = \eta_5^\# \eta_4^\# \eta_3^\# \eta_2^\# \bar{x}$, and thus $\alpha^\# \bar{x} \in G'_6$ cannot be defined consistently with the fact that $(6\alpha)^\# \bar{x} \neq 0$.

(We do not claim that $G_\ast$ is realizable; but the obstructions to realizing $G_\ast$ by a space $X \in T_\ast$ require secondary (or higher order) information, while the obstructions to its realization by an $H$-group are primary.)

4. Simplicial $\Pi$-algebras and spaces

We next recall some background on simplicial $\Pi$-algebras and spaces and bisimplicial groups:

Definition 4.1. A simplicial object over any category $\mathcal{C}$ is a sequence of objects $\{X_n\}_{n=0}^\infty$ in $\mathcal{C}$, equipped with face maps $d_i : X_n \to X_{n-1}$ and degeneracies $s_j : X_n \to X_{n+1}$ $(0 \leq i, j \leq n)$, satisfying the usual simplicial identities ([Ma1, §1.1]).

The category of simplicial objects over $\mathcal{C}$ is denoted by $s\mathcal{C}$. An augmented simplicial object $X_\ast \to A$ over $\mathcal{C}$ is a simplicial object $X_\bullet \in s\mathcal{C}$, together with an augmentation $\varepsilon : X_0 \to A$ in $\mathcal{C}$ such that $\varepsilon \circ d_1 = \varepsilon \circ d_0$.

Definition 4.2. A simplicial $\Pi$-algebra $A_\bullet$ is called free if for each $n \geq 0$ there is a graded set $T^n \subseteq A_n$ such that $A_n$ is the free $\Pi$-algebra generated by $T^n$, and each degeneracy map $s_j : A_n \to A_{n+1}$ takes $T^n$ to $T^{n+1}$.

A free simplicial resolution of a $\Pi$-algebra $G_*\ast$ is defined to be an augmented simplicial $\Pi$-algebra $A_\bullet \to G_*\ast$, such that $A_\bullet$ is a free simplicial $\Pi$-algebra, the homotopy groups of the simplicial group $D_kA_\bullet$ vanish in dimensions $n \geq 1$, and the augmentation induces an isomorphism $\pi_0(D_kA_\bullet) \cong G_k$.

Such resolutions always exist, for any $\Pi$-algebra $G_\ast$ -- see [Q1, II, §4], or the explicit construction in [Bl1, §4.3].

4.3. Simplicial spaces. Let $W_\bullet \in sT_\ast$ be a simplicial space: its realization (or homotopy colimit) is a space $\|W_\bullet\| \in T_\ast$ constructed by making identifications in $\prod_{n=0}^\infty W_n \times \Delta[n]$ according to the face and degeneracy maps of $W_\bullet$ (cf. [Se, §1]).

For any simplicial space $W_\bullet$, there is a first quadrant spectral sequence with

$$E_{s,t}^2 = \pi_s(\pi_t W_\bullet) \Rightarrow \pi_{s+t}\|W_\bullet\|$$
Definition 4.5. An augmented simplicial space \( W_\bullet \to X \) is called a resolution of \( X \) by spheres if each \( W_n \) is homotopy equivalent to a wedge of spheres, and \( \pi_* W_\bullet \to \pi_* X \) is a free simplicial resolution of \( \Pi \)-algebras (Def. 4.2).

Using the above spectral sequence, we see that the natural map \( W_0 \to \| W_\bullet \| \) then induces an isomorphism \( \pi_* X \cong \pi_* \| W_\bullet \| \), so \( \| W_\bullet \| \simeq X \).

Definition 4.6. A simplicial object over \( \mathrm{ho} T \), say \( W_\bullet \in \mathrm{ho} T \), is, by definition, a sequence of spaces \( W_0, W_1, \ldots \) with homotopy classes of maps: \( \delta_i \in [W_n, W_{n-1}] \), and so on, for the face maps and degeneracies, satisfying the simplicial identities (in \( \mathrm{ho} T \)). By making choices of actual maps representing each of these homotopy classes we obtain an actual diagram over \( T \), which we shall denote by \( ^h W_\bullet \): of course, the simplicial identities now hold only up to homotopy, in general. Such a \( ^h W_\bullet \) will be called a simplicial space up-to-homotopy. Note we can apply the functor \( T \mapsto \mathrm{Alg} \) to obtain a simplicial \( \Pi \)-algebra \( W_\bullet = \pi_* (^h W_\bullet) \in \mathrm{ho} \Pi \mathrm{-Alg} \).

If we can change the maps and spaces of \( ^h W_\bullet \) up to homotopy in such a way as to obtain a simplicial space “on the nose”, say \( V_\bullet \in \mathrm{ho} T \), we call this a rectification of \( W_\bullet \) or \( ^h W_\bullet \). For our purposes we need not worry over the precise definition of “changing \( ^h W_\bullet \) up to homotopy” (see, e.g., [DwKS, §2.2]); all we require is that \( \pi_* (^h W_\bullet) \) and \( \pi_* V_\bullet \) be isomorphic simplicial \( \Pi \)-algebras.

Similarly in \( \mathrm{ho} S^h \) or \( \mathrm{ho} G \).

4.7. rectifying simplicial resolutions. This suggests one possible approach to determining whether an \( H \)-group \( X \) is equivalent to a loop space:

First, choose some free simplicial \( \Pi \)-algebra \( A_\bullet \) resolving \( G_\bullet \) (for one possible construction, see Appendix 8). By Remark 2.4, the free simplicial \( \Pi \)-algebra \( A_\bullet \) corresponds to a unique simplicial object \( W_\bullet \in \mathrm{ho} T \) over the homotopy category, with each \( W_n \) homotopy equivalent to a wedge of spheres, such that \( \pi_* W_\bullet \cong A_\bullet \); this \( W_\bullet \) may be represented by a simplicial space up-to-homotopy \( ^h W_\bullet \) (see §4.6).

As usual, \( W_\bullet \) may be rectified if and only if \( ^h W_\bullet \) can be made \( \infty \)-homotopy commutative — that is, if and only if one can find a sequence of homotopies for the simplicial identities among the face and degeneracy maps, and then homotopies between these, and so on (cf. [BoaV, Corollary 4.21 & Theorem 4.49]). An obstruction theory for this was described in [Bl4, §5-6]; see Remark 6.5 below. If the obstructions vanish, \( ^h W_\bullet \) may be replaced by a (strict) simplicial space \( V_\bullet \), and by §4.3 we have \( \| V_\bullet \| \cong G_\bullet \), so \( Y \stackrel{\text{Def}}{=} \| V_\bullet \| \) is a candidate for the delooping of the given \( H \)-group \( X \).

Now we apply the obstruction theory of [Bl4, §7] to check whether the \( \Pi \)-algebra isomorphism \( \pi_* \Omega Y \cong \pi_* X \) (cf. Corollary 3.9) may be realized as a map of spaces. If so, \( \Omega Y \simeq X \), so our given \( H \)-group \( X \) is indeed a loop space; if not, we must try other rectifications of \( ^h W_\bullet \).

5. A simplicial group version

For our purposes it will be convenient to work at times in the category \( \mathcal{G} \) of simplicial groups. First, we recall some basic definitions and facts:
5.1. simplicial groups. Let $F : S \to G$ denote the free group functor of [Mi2, §2]; this is the simplicial version of the James construction, and in particular $|FK| \simeq J|K|$.

Let $G : S \to G$ be Kan’s simplicial loop functor (cf. [Ma1, Def. 26.3]), with $\bar{W} : G \to S_0^G$ its adjoint, the Eilenberg-Mac Lane classifying space functor (cf. [Ma1, §21]).

Then $|GK| \simeq \Omega|K|$ and $|K| \simeq |\bar{W}GK|$. Moreover, unlike $T_*$, where we have only a (weak) homotopy equivalence, in $G$ there is a canonical isomorphism $\phi : FK \simeq G\Sigma K$ (cf. [Cu, Prop. 4.15]), and there are natural bijections

$$\text{Hom}_{S_0}(\Sigma L, \bar{W}FK) \simeq \text{Hom}_{\bar{G}}(G\Sigma L, FK) \overset{\phi^*}{\to} \text{Hom}_G(FL, FK) \simeq \text{Hom}_{S_0}(L, FK)$$

for any $L \in S_0$ (induced by the adjunctions), and similarly for homotopy classes of maps.

Thus, we may think of $FS^n$ as the simplicial group analogue of the $r$-sphere; in particular, if $K$ is in $G$, or even if $K$ is just an associative $H$-simplicial set which is a Kan complex, we shall write $\pi^H_i K$ for $[FS^{i-1}, K]_H$ (compare §3.7). Similarly, $F^n$ is the $G$ analogue of the $n$-disc in the sense that any nullhomotopic map $f : FS^{n-1} \to K$ extends to $F^n$.

**Remark** 5.3. The same facts as in §4.3 hold also if we consider bisimplicial groups (which we shall think of as simplicial objects $G \in sG$) instead of simplicial spaces. In this case the realization $\|W_*\|$ should be replaced by the diagonal diag$(G_*)$, and the spectral sequence corresponding to (4.4) is due to Quillen (cf. [Q2]).

The above definitions provide us with a functorial simplicial version of the derived composition of §3.3:

**Definition 5.4.** If $K \in S^k$ is an $H$-simplicial set which is a Kan complex, one again has a retraction of simplicial sets $\tilde{m} : FK \to K$, defined as in (3.2). Given a homomorphism of simplicial groups $f : FA \to FB$ and a map of simplicial sets $g : B \to K$, the composite $\tilde{m} \circ Fg \circ f : FA \to K$ will be denoted by $f \star g$.

Note that if $\bar{f} : \Sigma A \to WFB$ and $\bar{f} : A \to FB$ correspond to $f$ under (5.2), the composite $\tilde{m} \circ Fg \circ \bar{f}$ corresponds to $f \star g$, and represents the derived composition $[\bar{f}] \star [g]$ in $[A, K]_{S_0} \simeq \|A\|, \|K\|_{T_*}$.

**Remark** 5.5. The simplicial version of the $*$ operation defined here is obviously functorial in the sense that $(e^*f) \star g = e^*(f \star g)$ for $e : FC \to FA$ in $G$, and $f \star (g^*h) = (f \star g)^*h$ for any $H$-map $h : (K, m) \to (L, n)$ between fibrant $H$-simplicial sets which is strictly multiplicative (i.e., $n \circ (h \times h) = h \circ m : K \times K \to L$).

However, Proposition 3.8 is still valid only in the homotopy category, and this is in fact the obstruction to $K$ being equivalent to a loop space:

**Proposition 5.6.** If $K$ is an $H$-group in $S^k_*$ such that

$$f \star (g \circ h) = (f \circ g) \circ h \quad \forall f : FA \to FB \text{ and } g : FB \to FC \text{ in } G \text{ and } h : C \to K,$$

then $K$ is $H$-homotopy equivalent to a simplicial group (and thus to a loop space); conversely, if $K \in G$ (in particular, if $K = GL$ for some $L \in S_0$), then (*) holds.
Proof. Assume that \( K \) is an \( H \)-group in \( S^k \) satisfying \((*)\). We shall need a simplicial variant of Chris Stover’s construction of resolutions by spheres (Def. 4.5), so as in [Stv, §2], define a comonad \( L : G \to G \) by

\[
(5.7) \quad LG = \prod_{k=0}^{\infty} \prod_{\phi \in \text{Hom}_G(\mathcal{F}^k, G)} \mathcal{F}^k_{\phi} \cup \prod_{k=0}^{\infty} \prod_{\Phi \in \text{Hom}_G(\mathcal{F}^{k+1}, G)} \mathcal{F}^k_{\Phi},
\]

where \( \mathcal{F}^{k+1} \), the \( G \)-disc indexed by \( \Phi : \mathcal{F}^{k+1} \to G \), is attached to \( \mathcal{F}^k_{\Phi} \), the \( G \)-sphere indexed by \( \phi = \Phi|_{\mathcal{F}^{k+1}} \), by identifying \( \mathcal{F}^{k+1} \mathcal{F}^k_{\Phi} \) with \( \mathcal{F}^k_{\Phi} \) (see §5.1 above). The coproduct here is just the (dimensionwise) free product of groups; the counit \( e : LG \to G \) is “evaluation of indices”, and the comultiplication \( \vartheta : LG \to L^2G \) is as in §2.5.

Now let \( W = \bigvee_{f \in \text{Hom}_G(S^k, K)} S^k_f \cup \bigvee_{k=1}^{\infty} \bigvee_{k=1}^{\infty} e^{k+1}_f \) (the analogue for \( S_* \) of \( LG \), with the corresponding identifications), and let \( z : W \to K \) be the counit map. As in (1) of Appendix 8, \( z \) induces an epimorphism \( z_* : \pi_*W \to \pi_*K \) of \( \Pi \)-algebras. (\( K \) is a Kan complex, but \( W \) is not, so we understand \( \pi_*W \) to be the corresponding free \( \Pi \)-algebra \( \cong \pi_1W \) (cf. §2.3).

As in (2) of Appendix 8, we have an epimorphism of \( \Pi \)-algebras \( \tilde{\zeta} : \pi_*\Sigma W \to G_* \), where \( G_* = \Omega^{-1}\pi_*K \) is the delooping of \( \pi_*K \) – or equivalently, \( \tilde{z} : \pi_*H W \to \pi_*K \), induced by \( \tilde{z} = \tilde{m} \circ Fz : FW \to K \) (cf. §5.4).

Let \( M_n = L^nFW \) for \( n = 0, 1, \ldots \), with face and degeneracy maps determined by the comonad structure maps \( \varepsilon, \vartheta \) except for \( d_n : M_0 \to M_{n-1} \), defined \( d_n = L^{n-1}d \), where \( d : LF \to F \), restricted to a summand \( FA_\alpha \) in \( LF \) (\( A = S^k, e^{k+1} \)), is an isomorphism onto \( FA_\beta \to FW \), where \( \beta : A \to K \) is the composite \( (\alpha \ast z) \circ j_A \).

Because \((*)\) holds on the nose, we may verify that \( \tilde{d} \circ T\tilde{d} = \tilde{d} \circ T\varepsilon : M_2 \to M_0 \), so that \( M_* \) is a simplicial object over \( G \) (just as in the proof of Lemma 8.3 in the Appendix). Moreover, the augmented simplicial \( \Pi \)-algebra \( \pi_*\tilde{\zeta} : G_* \to \vartheta \) is acyclic, by a variant of [Stv, Prop. 2.6] and Lemma 8.5 of the Appendix. Thus by the Quillen spectral sequence (see §5.3) we have \( \pi_*H \text{ diag } L_* \cong \pi_*H K \), and thus setting \( L = \text{ diag } WL_* \cong W \text{ diag } L_* \) we obtain a Kan complex \( L \) such that \( K \cong GL \) – i.e., \( |K| \cong \Omega |L| \).

The converse is clear, since if \( K \in G \) then \( j_A : A \to FA \) induces a one-to-one correspondence between maps \( f : A \to K \) in \( S_* \) and homomorphisms \( \varphi : FA \to K \) in \( G \), by the universal property of \( F \). □

Note that in fact we need only verify that 5.6\((*)\) holds for \( A, B, \) and \( C \) in \( S_* \) which are (weakly) homotopy equivalent to wedges of spheres.

6. The simplicial-cosimplicial construction

One disadvantage of directly applying the obstruction theory of [Bl4] for realizing \( G_* = \Omega^{-1}\pi_*X \) as a method for determining if \( X \) is a loop space is that even the algebraic step – namely, determining the \( \Pi \)-algebra \( G_* \) – depends on a choice of \( H \)-space structure for \( X \), and thus cannot be described purely in terms of homotopy
operations in the classical sense. In this section and the next, we shall describe a more explicit version of the obstruction theory, which does not presuppose such a choice, and is more in the spirit of the obstruction theories of [Bl4] and [Bl5].

**Definition 6.1.** A CW-resolution of a Π-algebra \( G \) is a free simplicial resolution \( A_\bullet \to G_\bullet \) as in §4.2, together with a sequence of free Π-algebras \((\bar{A}_n)_{n=0}^{\infty}\), called a CW-basis for \( A_\bullet \), such that

\[
A_n \cong \prod_{0 \leq \lambda \leq n} \prod_{I \in \mathcal{I}_{\lambda,n}} \bar{A}_{n-\lambda}
\]

where \( \mathcal{I}_{\lambda,n} \) is the set of all sequences of \( \lambda \) non-negative integers \( i_1 < i_2 < \ldots < i_\lambda \) \((i_\lambda < n)\) for \( 0 \leq \lambda \leq n \) (compare [Bl1], §5.1).

For each \( I \in \mathcal{I}_{\lambda,n} \), the copy of \( \bar{A}_{n-\lambda} \) indexed by \( I \) is in the image of the \( \lambda \)-fold degeneracy \( s_I = s_{i_1} \circ \ldots \circ s_{i_\lambda} \), in the obvious sense. The face maps \( d_i : A_n \to A_{n-1} \) are determined by the attaching map \( \partial_0^n = d_0|\bar{A}_n : \bar{A}_n \to A_{n-1} \), the simplicial identities, and the requirement that \( d_i|A_n = 0 \) for \( i \geq 1 \).

Aside from allowing one to construct minimal resolutions, which are convenient for explicit computations (compare [Bl5]), such CW-resolutions also have technical advantages for the obstruction theory of [Bl4, §5-6]:

Let \( A_\bullet \to G_\bullet \) be a CW-resolution, and let \( W_\bullet \in s(\text{ho} \mathcal{T}_s) \) be the simplicial object over \( \text{ho} \mathcal{T}_s \) which corresponds to \( A_\bullet \) under Remark 2.4 — that is,

\[
W_n \cong \bigvee_{0 \leq \lambda \leq n} \bigvee_{I \in \mathcal{I}_{\lambda,n}} (W_{n-\lambda}(I))
\]

where \( W_n \) is a wedge of spheres such that \( \pi_\ast W_n \cong \bar{A}_n \) (and thus \( \pi_\ast W_n \cong A_n \) as Π-algebras, and \( \pi_\ast W_\bullet \cong A_\bullet \) as simplicial Π-algebras). Choose some simplicial space up-to-homotopy \( ^hW_\bullet \) corresponding to \( W_\bullet \) (Def. 4.6). We then have the following

**Theorem 6.4.** Let \( ^hW_\bullet \) be as above; then \( ^hW_\bullet \) may be rectified, and thus \( G_\bullet \) realized as \( \pi_\ast Y_\bullet \), if and only if all the higher homotopy operations \( \langle \delta \rangle \subset [\Sigma^{k-1}W_n, W_{n-k-1}] \) of [Bl4, §5.3] vanish coherently.

**Remark 6.5.** In [Bl4, §6] we required the coherent vanishing of an additional collection of (rather inelegant) higher homotopy operations, corresponding to the degeneracies of \( W_\bullet \), in order for \( G_\bullet \) to be realizable. This requirement is eliminated by Theorem 6.4.

**Proof.** Assume that, for \( W_\bullet \) as above, all \( \langle \delta \rangle \) vanish coherently. By definition, this means there is a compatible collection of maps \( \bar{g}^\delta : P(n-k,\delta) \times W_n \to W_{n-k-1} \) for \( \delta \in D(n-k,n) / \sim \), where each such \( \delta \) is a composite of face maps, and \( P(n-k,\delta) \) is a suitable convex polyhedron. (See [Bl4, §5] for the notation and terminology). In particular, \( g^{d_0} : W_n \to W_{n-1} \) is in the homotopy class determined by \( d_0^\lambda : \bar{A}_n \to A_{n-1} \), and \( g^{d_i} \sim * \) on \( W_n \) for \( i \geq 1 \).

We may then define a compatible collection of maps \( g^\psi : P(\psi) \times W_n \to W_m \) for all simplicial morphisms \( \psi : A_n \to A_m \) (and suitable polyhedra \( P(\psi) \) defined in [Bl4, §6.5]), as follows:

By (6.3), it suffices to define \( g^\psi \) on each wedge summand \( (W_{n-\lambda}(I)) \), where \( I = (i_0, \ldots, i_\lambda) \); since \( s_I = s_{i_1} \circ \ldots \circ s_{i_\lambda} : W_{n-\lambda} \to (W_{n-\lambda}(I)) \) is a homeomorphism, it is enough to define \( \hat{g}^\psi = g^\psi \circ (Id_{P(\psi)} \times s_I) : P(\psi) \times W_{n-\lambda} \to W_m \) for all \( \psi, I \). But for
any vertex \(v \in P(\psi)\) we have a corresponding factorization \(\psi = \lambda_0 \circ \ldots \circ \lambda_1\) (compare §7.3 below), with each \(\lambda_i\) either a face or a degeneracy map, and \(\tilde{g}^\psi|_{\{v\} \times \tilde{W}_{n-\lambda}}\) is required to be the corresponding composition of face and degeneracy maps of \(hW_\ast\), precomposed with \(s_I\).

By definition of \(hW_\ast\), such a simplicial morphism may be computed on \(\tilde{W}_{n-\lambda}\) by using the simplicial identities \(d_is_j = s_jd_{i-1}d_i\) (\(i < j\)), \(d_jd_i = d_{j+1}s_j = id\), and \(d_is_j = sjd_{i+1}\) (\(i > j + 1\)) to bring \(\lambda_0 \circ \ldots \circ \lambda_1 \circ s_I\) into “semi-canonical form” \(s_J \circ d_K\), where we assume \(J = (j_1, \ldots, j_r)\) satisfies \(j_1 < j_2 < \ldots < j_r\), but make no assumptions as to \(K = (k_1, \ldots, k_s)\). We then set \(\tilde{g}^\psi|_{\{v\} \times \tilde{W}_{n-\lambda}}\) equal to \(d_K\) (i.e., the composite of the corresponding chosen representatives for the face maps in \(hW_\ast\)), postcomposed with the embedding \(s_J : \tilde{W}_{n-\lambda-s} \hookrightarrow \tilde{W}_m\).

Now it is clear that the only homotopies needed for the 1-dimensional polyhedra \(P(\psi)\) are those involving the various composite face maps \(d_K\) in the semi-canonical form of \(\psi \circ s_I\), since all other simplicial identities for \(hW_\ast\) hold precisely; by induction on the dimension we see this is true for all \(P(\psi)\), so that the compatible collection \(\{g^\psi\}\) is simply the given compatible collection \(\{\tilde{g}^\psi\}\), post-composed with the appropriate embeddings \(s_J\), and in fact we may collapse the polyhedra \(P(\psi)\) to the face-map polyhedra, or permutohedra, \(P_{n-k}(\delta)\) (cf. [Bl4, §4] and §7.1 below). \(\square\)

**Definition 6.6.** A \(\Delta\)-cosimplicial object \(E^\Delta_\ast\) over a category \(C\) is a sequence of objects \(E^0, E^1, \ldots\), together with coface maps \(d^i : E^n \to E^{n+1}\) for \(1 \leq i \leq n\) satisfying \(d^i \circ d^j = d^j \circ d^{i-1}\) for \(i < j\). Given a cosimplicial object \(E^\ast\) (cf. [BoK, X, 2.1]), we let \(E^\Delta_\ast\) denote the underlying \(\Delta\)-cosimplicial object (obtained by forgetting the codegeneracies).

**6.7. the cosimplicial James construction.** Given a space \(X \in T\), we define a \(\Delta\)-cosimplicial space \(U^\Delta_\ast = U(X)^\Delta_\ast\) by setting \(U^n = X^{n+1}\) (the Cartesian product), and \(d^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, \ast, x_i, \ldots, x_n)\). Note that \(JX = \text{colim} U(X)^\Delta_\ast\) and \(\text{Fact 6.8.}\) If \(\langle X, m \rangle\) is a (strictly) associative \(H\)-space, we can extend \(U^\Delta_\ast\) to a full cosimplicial space \(U^\ast\) by setting \(s^i(x_0, \ldots, x_n) = (x_0, \ldots, m(x_j, x_{j-1}), \ldots, x_n)\).

**Definition 6.9.** Let \(A^\ast\) be a \(CW\)-resolution of the \(\Pi\)-algebra \(\pi_*X = \pi_*U^0\). We construct a \(\Delta\)-cosimplicial augmented simplicial \(\Pi\)-algebra \((E^\Delta_\ast \to \pi_*U^\Delta_\ast)\), such that each \(E^n_\ast\) is a \(CW\)-resolution of \(\pi_*U^n = \pi_*(X^{n+1})\), with \(CW\)-basis \(\{E^n_r\}_{r=0}^\infty\). We start by setting \(E^n_0 = C^n_0 = A^\ast\) for all \(r \geq 0\), and then define \(E^n_r\) by a double induction (on \(r \geq 0\) and then on \(n \geq 0\)) as

\[(6.10)\]

\[E^n_r = \prod_{0 \leq \lambda \leq n} \prod_{I \in \mathcal{I}_{\lambda,n}} [\tilde{C}^{n-\lambda}_r]_I,\]

where \(\mathcal{I}_{\lambda,n}\) is as in (6.2) and \(C^m_0 = 0 = C^0_r\) for all \(m, r \geq 0\).

The coface maps \(d^i : E^{n-1}_r \to E^n_r\) are determined by the cosimplicial identities and the requirement that \(d^i|_{[\tilde{C}^{n-\lambda}_r]_{(i_1, \ldots, i_n)}}\) be an isomorphism onto \([\tilde{C}^{n-\lambda}_r]_{(i_1, \ldots, i_n)}\) if \(i > i_n\).

The only summand in (6.10) which is not defined is thus \([\tilde{C}^r_0]_0\), which we denote simply by \(\tilde{C}^r_0\). We require that it be an \(n\)-th cross-term in the sense that \(d^0_0|_{\tilde{C}^r_0}\) does not factor through the image of any coface map \(d^i : E^{n-1}_r \to E^n_{r-1}\). Other than
that, \( C^n_r \) may be any free \( \Pi \)-algebra which ensures that (6.10) defines a \( CW \)-basis for a \( CW \)-resolution \( E^n_r \to \pi_* U^n \). We shall call the double sequence \((C^n_r)_{n=1}^\infty, r=1\) a cross-term basis for \((E_r^*)\Delta^r\).

Note that \( A_* \) is a retract of \( E^2_* \) in two different ways (under the two coface maps \( d^0, d^1 \)), corresponding to the fact that \( X \) is a retract of \( X \times X \) in two different ways; the presence of the cross-terms \( C^r_2 \) indicates that \( A_* \times A_* \) is a resolution of \( \pi_* X^2 \), but not a free one, while \( A_* \Pi A_* \) is a free simplicial \( \Pi \)-algebra, but not a resolution.

Similarly, \( X \times X \) embeds in \( X^3 \) in three different ways, and so on.

**Example 6.11.** For any \( A_* \to \pi_* X \) we may set \( C^2_1 = \prod S^p_{(x,y)} A_0^{[x]} \prod S^q_{(x,y)} A_1^{[y]} \), with \( d^0 |_{S^p+q-1}^S (x,y) = [t_x, t_y] \) (in the notation of §2.5). The higher cross-terms \( C^r_1 = 0 \for n \geq 3 \), since any \( k \)-th order cross-term element \( z \in \prod_{j=0}^n A_0^{(j)} \) \((k \geq 3)\) is a sum of elements of the form \( z = \sigma^* [\ldots [r^{i_1}_{(x_1)}, r^{i_2}_{(x_2)}, \ldots, r^{i_k}_{(x_k)}], \ldots, r^{i_3}_{(x_3)}], \ldots, r^{i_2}_{(x_2)}], \ldots, r^{i_1}_{(x_1)}] \). Then

\[
z = d_0(\sigma^* [\ldots [r^{i_1+r_2-1}_{(x_1, x_2)}, s_0 r^{i_3}_{(x_3)}], \ldots, s_0 r^{i_2}_{(x_2)}], \ldots, s_0 r^{i_1}_{(x_1)}]).
\]

**Definition 6.12.** Let \( h(W_n)^* \to U^*_\Delta \) be the \( \Delta \)-cosimplicial augmented simplicial space up-to-homotopy which corresponds to \((E_r^*) \to \pi_* U^*_\Delta \) under Fact 2.4. Each \( W^n_r \) is homotopy equivalent to a wedge of spheres, and has a wedge summand \( W^n_r \to W^n_r \) corresponding to the \( CW \)-basis free \( \Pi \)-algebra summand \( E^n_r \to E^n_r \). We let \( C^n_r \) denote the wedge summand of \( W^n_r \) corresponding to \( C^n_r \).

We do not enter here into the question of whether every free simplicial \( \Pi \)-algebra resolution of a *realizable* \( \Pi \)-algebra \( \pi_* Y \) may be realized by a resolution of \( Y \) by spheres as in Definition 4.5 (but see [Bl5, 4.1(a)]). However, every space \( Y \) has a functorial resolution by spheres \( V_*(Y) \to Y \) by [Stv, Prop. 2.6], and one may in fact construct smaller (non-functorial) resolutions, as in [Bl5, 3.12, 4.19]. Thus we may make the following

**Assumption 6.13.** \((E_r^*) \to \pi_* U^*_\Delta \) maps monomorphically into \( \pi_* V_*(U^*_\Delta) \), and \( h(W_n)^* \to U^*_\Delta \) can be rectified so as to yield a strict \( \Delta \)-cosimplicial augmented simplicial space \((W_n)^* \to U^*_\Delta \) realizing \((E_r^*) \to \pi_* U^*_\Delta \).

**Definition 6.14.** Now assume that \( \pi_* X \) is an abelian \( \Pi \)-algebra (Def. 2.6) -- this is the necessary \( \Pi \)-algebra condition in order for \( X \) to be an \( H \)-space -- and let \( \mu : \pi_* X \times \pi_* X \to \pi_* X \) be the morphism of \( \Pi \)-algebras defined levelwise by the group operation (see [Bl5, §2]). This \( \mu \) is of course associative, in the sense that \( \mu \circ (\mu, id) = \mu \circ (id, \mu) : \pi_* (X^3) \to \pi_* X \), so it allows one to extend the \( \Delta \)-cosimplicial \( \Pi \)-algebra \( F^*_\Delta \to \pi_* (U^*_\Delta) \) to a full cosimplicial \( \Pi \)-algebra \( F^* \), defined as in §6.8.

Since \( E^n_r \to E^n_r = \pi_* U^n \) is a free resolution of \( \Pi \)-algebras, the codegeneracy maps \( s^i : F^n_r \to F^{n-1} \) induce maps of simplicial \( \Pi \)-algebras \( s^i_* : E^n_r \to E^{n-1}_r \), unique up to simplicial homotopy, by the universal property of resolutions (cf. [Q1, I, p. 1.14 & II, §2, Prop. 5]). Note, however, that the individual maps \( s^i_* : E^n_r \to E^{n-1}_r \) are not unique, in general; in fact, different choices may correspond to different \( H \)-multiplications on \( X \).

These maps \( s^i \) make \((E_r^*) \to F^*_\Delta \) into a full cosimplicial augmented simplicial \( \Pi \)-algebra \( E^*_r \to F^* \), and thus \( hW^*_r \to U^*_\Delta \) into a cosimplicial augmented simplicial
Proposition 6.15. The cosimplicial simplicial space up-to-homotopy \( hW^* \) of §6.14 may be rectified if and only if \( X \) is homotopy equivalent to a loop space.

Proof. If \( X \) is a loop space, it has a strictly associative \( H \)-multiplication \( m : X \times X \to X \) which induces \( \mu \) on \( \pi_*(-) \) (cf. [Gr, Prop. 9.9]), so \( U^*_\Delta \) extends to a cosimplicial space \( U^* \) by Fact 6.8. Applying the functorial construction of [Stv, §2] to \( U^* \) yields a (strict) cosimplicial augmented simplicial space \( (V_\ast)_\Delta \to U^* \), and since we assumed \( \pi_*W^*_\Delta \) embeds in \( \pi_*V^*_n \) for each \( n \), \( hW^*_\Delta \) may also be rectified.

Conversely, if \( W^*_\Delta \) is a (strict) cosimplicial simplicial space realizing \( E^*_\Delta \), then we may apply the realization functor for simplicial spaces in each cosimplicial dimension \( n \geq 0 \) to obtain \( \|W^*_n\| \simeq U^n = X^{n+1} \) (by (4.4)). The realization of the codegeneracy map \( \|s^0\| : \|W^1_\ast\| \to \|W^0_\ast\| \) induces \( \mu : \pi_k(X^2) \to \pi_kX \), so it corresponds to an \( H \)-space multiplication \( m : X^2 \to X \) (see [Bl5, Prop. 2.7]).

The fact that \( \|W^*_\ast\| \) is a (strict) cosimplicial space means that all composite codegeneracy maps \( \|s^0 \circ s^{j_1} \circ \ldots \circ s^{j_{n-1}}\| : \|W^*_\ast\| \to \|W^0_\ast\| \) are equal, and thus all possible composite multiplications \( X^{n+1} \to X \) (i.e., all possible bracketings in (3.2)) are homotopic, with homotopies between the homotopies, and so on — in other words, the \( H \)-space \( (X, m) \) is an \( A_\infty \) space (see [St3, Def. 11.2]) — so that \( X \) is homotopy equivalent to loop space by [St3, Theorem 11.4]. Note that we only required that the codegeneracies of \( hW^*_\ast \) be rectified; after the fact this ensures that the full cosimplicial simplicial space is rectifiable. \( \square \)

In summary, the question of whether \( X \) is a loop space reduces to the question of whether a certain diagram in the homotopy category, corresponding to a diagram of free \( A_\infty \)-algebras, may be rectified — or equivalently, may be made \( \infty \)-homotopy commutative.

7. Polyhedra and higher homotopy operations

As in [Bl4, §4], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, and these may be described combinatorially in terms of certain polyhedra, as follows:

Definition 7.1. The \( N \)-permutohedron \( P^N \) is defined to be the convex hull in \( \mathbb{R}^N \) of the points \( p_\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N)) \), where \( \sigma \) ranges over all permutations \( \sigma \in \Sigma_N \) (cf. [Sc]). It is \( (N-1) \)-dimensional.

For any two integers \( 0 \leq n < N \), the corresponding \( (N,n) \)-face-codegeneracy polyhedron \( P^N_n \) is a quotient of the \( N \)-permutohedron \( P^N \) obtained by identifying two vertices \( p_\sigma \) and \( p_{\sigma'} \) to a single vertex \( \bar{p}_\sigma = \bar{p}_{\sigma'} \) of \( P^N_n \) whenever \( \sigma = (i, i+1)\sigma' \), where \( (i, i+1) \) is an adjacent transposition and \( \sigma(i), \sigma(i+1) > n \).

Since each facet \( A \) of \( P^N \) is uniquely determined by its vertices (see below), the facets in the quotient \( P^N_n \) are obtained by collapsing those of \( P^N \) accordingly.

Note that \( P^N_{N-1} \) is the \( N \)-permutohedron \( P^N \), and in fact the quotient map \( q : P^N \to P^N_n \) is homotopic to a homeomorphism (though not a combinatorial isomorphism, of course) for \( n \geq 1 \). On the other hand, \( P^N_0 \) is a single point. For non-trivial examples of face-codegeneracy polyhedra, see Figures 1 & 2 below.
Fact 7.2. From the description of the facets of the permutahedron given in [GG], we see that \( P_n^N \) has an edge connecting a vertex \( p_\sigma \) to any vertex of the form \( p_{(i,i+1)\sigma} \) (unless \( \sigma(i), \sigma(i+1) > n \), in which case the edge is degenerate).

More generally, let \( \tilde{p}_\sigma \) be any vertex of \( P_n^N \). The facets of \( P_n^N \) containing \( \tilde{p}_\sigma \) are determined as follows:

Let \( P = \{1, 2, \ldots, \ell_i \mid \ell_{i-1} + 1, \ldots, \ell_i \mid \ldots \} \) be a partition of \( 1, \ldots, N \) into \( r \) consecutive blocks, subject to the condition that for each \( 1 \leq j < r \) at least one of \( \sigma(\ell_i), \sigma(\ell_{i+1}) \leq n \). Denote by \( n_i \) the number of \( j \)'s in the \( i \)-th bloc (i.e., \( \ell_{i-1} + 1 \leq j \leq \ell_i \)) such that \( \sigma(j) \leq n \). Then \( P_n^N \) will have a subpolyhedron \( Q(P) \) (containing \( p_\sigma \)) which is isomorphic to the product

\[
P_{n_1}^{\ell_1} \times P_{n_2}^{\ell_2-\ell_1} \times \cdots \times P_{n_{r-1}}^{\ell_r-\ell_{r-1}} \times P_{n_r}^{N-\ell_r-1}.
\]

This follows from the description of the facets of the \( N \)-permutahedron in [Bl4, §4.3].

We denote by \((P_n^N)^{(k)}\) the union of all facets of \( P_n^N \) of dimension \( \leq k \). In particular, for \( n \geq 1 \) we have \( \partial P_n^N = (P_n^N)^{(N-2)} = S^{N-2} \), since the homeomorphism \( \tilde{q} : P_n^N \to P_n^N \) preserves \( \partial P_n^N \).

7.3. factorizations. Given a cosimplicial simplicial object \( E^* \) as in §6.14, any composite face-codegeneracy map \( \psi : E^{n+k}_m \to E^k_{m+\ell} \) may be factored uniquely \( \psi = \phi \circ \theta \)

where \( \theta : E^{n+k}_{m+\ell} \to E^k_{m+\ell} \) may be written \( \theta = s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n} \) for \( 0 \leq j_1 < j_2 < \cdots < j_n < n + k \) and \( \phi : E^k_{m+\ell} \to E^k_{m+\ell} \) may be written \( \phi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n} \) for \( 0 \leq i_1 < i_2 < \cdots < i_n \leq m + \ell \).

Let \( D(\psi) \) denote the set of all possible factorizations of \( \psi \) as a composite of face and codegeneracy maps: \( \psi = \lambda_{n+m} \circ \cdots \circ \lambda_1 \). We define recursively a bijective correspondence between \( D(\psi) \) and the vertices of an \((n+m)\)-permutahedron \( P_n^{n+m} \), as follows (compare [Bl4, Lemma 4.7]):

The canonical factorization \( \psi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n} \circ s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n} \) corresponds to the vertex \( p_{id} \). Next, assume that the factorization \( \psi = \lambda_{n+m} \circ \cdots \circ \lambda_1 \) corresponds to \( p_\sigma \). Then the factorization corresponding to \( p_{\sigma'} \) for \( \sigma = (i, i+1)\sigma' \), is obtained from \( \psi = \lambda_1 \circ \cdots \circ \lambda_{n+m} \) by switching \( \lambda_i \) and \( \lambda_{i+1} \), using the identity \( s^j \circ s^i = s^{j-1} \circ s^j \) for \( i > j \) if \( \lambda_i \) and \( \lambda_{i+1} \) are both codegeneracies, and the identity \( d_i \circ d_j = d_{j-1} \circ d_i \) for \( i < j \) if they are both face maps.

Passing to the quotient face-codegeneracy polyhedron, we see that the vertices of \( P_n^{n+m} \) are now identified with factorizations of \( \psi \) of the form

\[
E^{n+k}_{m+\ell} \xrightarrow{s^{i_1}_{\ell_1}} E^{n+k-1}_{m+\ell} \xrightarrow{s^{i_2}_{\ell_2}} \cdots \xrightarrow{s^{i_n}_{\ell_n}} E^k_{m+\ell} \xrightarrow{\theta_1} E^k_{m+\ell} \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_n} E^k_{m+\ell} \xrightarrow{\theta_0} E^k_m,
\]

where \( \theta_i \) is a composite of face maps (i.e., we do not distinguish the different ways of decomposing \( \theta_i \) as \( d_{k_1} \circ \cdots d_{k_k} \)). The collection of such factorizations of \( \psi \) will be denoted by \( D(\psi)/\sim \), where \( \sim \) is the obvious equivalence relation on \( D(\psi) \). We shall denote the face-codegeneracy polyhedron \( P_n^{n+m} \) with its vertices so labelled by \( P_n^{n+m}(\psi) \). An example for \( \psi = d_0 d_1 s^0 s^1 \) appears in Figure 1.

7.5. notation. For \( \psi : E^{n+k}_{m+\ell} \to E^k_{m+\ell} \) as above, we denote by \( C(\psi) \) the collection of all composite face-codegeneracy maps \( \rho : E^{m(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \to E^k_{\ell(\rho)} \) such that \( \rho \) is of the form \( \rho = \xi_{s} \circ \cdots \circ \xi_{s} (1 \leq s \leq t \leq \nu) \) for some decomposition \( \psi = \xi_{t} \circ \cdots \circ \xi_{1} = \theta_{0} \circ s^{0}_{\theta_{0}} \circ \cdots \circ s^{n}_{\theta_{1}} \circ \cdots \circ \theta_{t} \circ s^{j}_{\theta_{t}} \circ \cdots \circ s^{j}_{\nu} \) of (7.4). That is, we allow only those
subsequences $\lambda_b, \ldots, \lambda_a$ of a factorization $\psi = \lambda_{n+m} \circ \cdots \circ \lambda_1$ in $D(\psi)$ which are compatible with the equivalence relation $\sim$ in the sense that $\lambda_{b+1}$ and $\lambda_b$ are not both face maps, and similarly for $\lambda_{a-1}$ and $\lambda_a$. Such a $\rho$ will be called allowable.

7.6. higher homotopy operations. Given a cosimplicial simplicial space up-to-$\lambda_1$ $X$, a subsequence of maps $\lambda_1 \circ \cdots \circ \lambda_b$ from $X$ to $W_{m(\rho)}(\rho)$ will be called allowable.

**Definition 7.7.** Given a composite face-codegeneracy map $\psi : W_{n+k} \to W_k$ as above, a compatible collection for $C(\psi)$ and $W_{m(\rho)}(\rho)$ is a set $\{g^\rho\}_{\rho \in C(\psi)}$ of maps $g^\rho : P_m(\rho) \to W_{m(\rho)+k(\rho)}$ for each $\rho \in C(\psi)$, satisfying the following condition:

Assume that for such a $\rho \in C(\psi)$ we have some decomposition

$$\rho = \xi_\nu \circ \cdots \circ \xi_1 = \theta_0 \circ \delta_{00} \circ \cdots \circ \delta_{01} \circ \theta_1 \circ \delta_{11} \circ \cdots \circ \delta_{n1} \circ \cdots \circ \delta_{nt}$$

in $D(\rho) / \sim$, as in (7.4), and let $P = \{1, \ldots, \ell_1 \mid \ldots \mid \ell_{i-1} + 1, \ldots, \ell_i \mid \ldots \mid \ell_{r-1} + 1, \ldots, \nu\}$ be a partition of $(1, \ldots, \nu)$ as in §7.2, yielding a sequence of composite face-codegeneracy maps $\rho_i \in C(\rho) \subseteq C(\psi)$ for $i = 1, \ldots, r$.

Let $Q(P) \cong P_{n1}^{\ell_1}(\rho_1) \times \cdots \times P_{ni}^{\ell_i-\ell_{i-1}}(\rho_i) \times \cdots \times P_{nr}^{\nu-\ell_{r-1}}(\rho_r)$ be the corresponding sub-polyhedron of $P_{m(\rho)+m(\rho)}(\rho)$. Then we require that $g^\rho|_{Q(P)} : W_{m(\rho)+k(\rho)} \to W_{m(\rho)+k(\rho)}$ be the composite of the corresponding maps $g^\rho_i$ in the sense that

$$g^\rho(x_1, \ldots, x_r, w) = g^{\rho_1}(x_1, g^{\rho_2}(x_2, \ldots, g^{\rho_r}(x_r, w) \ldots))$$

for $x_i \in P^{\ell_i-\ell_{i-1}}(\rho_i)$ and $w \in W_{m(\rho)+k(\rho)}$.

We further require that if $\rho = \lambda_1$ is of length 1, then $g^\rho$ must be in the prescribed homotopy class of the face or codegeneracy map $\lambda_1$. Thus in particular, for each

**Figure 1.** The face-codegeneracy polyhedron $P^1_2(d_0d_1s^0s^1)$
vertex \( \bar{p}_g \) of \( \mathbf{P}^{n+m}(\psi) \), indexed by a factorization \( \psi = \xi_\nu \circ \ldots \circ \xi_1 \) in \( \mathcal{D}(\psi)/\sim \), the map \( g^\rho|_{\{\bar{p}_g\} \times \mathbf{W}^{n+k}_{m+i+k}} \) represents the class [\( \xi_\nu \circ \ldots \circ \xi_1 \)].

**Fact 7.9.** Any compatible collection of maps \( \{g^\rho\}_{\rho \in \mathcal{C}(\psi)} \) for \( \mathcal{C}(\psi) \) induces a map \( f = f^\psi : \partial \mathbf{P}^{n+m}_n \times \mathbf{W}^{n+k}_{m+i+k} \to \mathbf{W}^k_\ell \) (since all the facets of \( \partial \mathbf{P}^{n+m}_n \) are products of face-codegeneracy polyhedra of the form \( \mathbf{P}^{(\rho)+m(\rho)}(\rho) \) for \( \rho \in \mathcal{C}(\psi) \), and condition (7.8) guarantees that the maps \( g^\rho \) agree on intersections).

**Definition 7.10.** Given \( ^h \mathbf{W}^\bullet_{\ast} \) as in \( \S 6.14 \), for each \( k \geq 2 \) and each composite face-codegeneracy map \( \psi : \mathbf{W}^{n+k}_{m+i+k} \to \mathbf{W}^k_\ell \), the \( k \)-th order homotopy operation associated to \( ^h \mathbf{W}^\bullet_{\ast} \) and \( \psi \) is a subset \( \langle \langle \psi \rangle \rangle \) of the track group \( [\Sigma^{n-m-2}\mathbf{W}^{n+k}_{m+i+k}, \mathbf{W}^k_\ell] \), defined as follows:

Let \( S \subseteq [\partial \mathbf{P}^{n+m}_n \times \mathbf{W}^{n+k}_{m+i+k}, \mathbf{W}^k_\ell] \) be the set of homotopy classes of maps \( f = f^\psi : \partial \mathbf{P}^{n+m}_n \times \mathbf{W}^{n+k}_{m+i+k} \to \mathbf{W}^k_\ell \) which are induced as above by some compatible collection \( \{g^\rho\}_{\rho \in \mathcal{C}(\psi)} \) for \( \mathcal{C}(\psi) \).

Now choose a splitting

\[
(7.11) \quad \partial \mathbf{P}^{n+m}_n(\psi) \times \mathbf{W}^{n+k}_{m+i+k} \cong \Sigma^{n-m-2} \times \mathbf{W}^{n+k}_{m+i+k} \cong (\Sigma^{n-m-2} \wedge \mathbf{W}^k_\ell) \vee \mathbf{W}^k_\ell
\]

and let \( \langle \langle \psi \rangle \rangle \subseteq [\Sigma^{n-m-2}\mathbf{W}^{n+k}_{m+i+k}, \mathbf{W}^k_\ell] \) be the image of the subset \( S \) under the resulting projection.

It is clearly a necessary condition in order for the subset \( \langle \langle \psi \rangle \rangle \) to be non-empty that all the lower order operations \( \langle \langle \rho \rangle \rangle \) vanish (i.e., contain the null class) for all \( \rho \in \mathcal{C}(\psi) \setminus \{\psi\} \) – because otherwise the various maps \( g^\rho : \mathbf{P}^{(\rho)+m(\rho)}(\rho) \times \mathbf{W}^{(\rho)+m(\rho)}(\rho) \to \mathbf{W}^k_\ell(\rho) \) cannot even extend over the interior of \( \mathbf{P}^{(\rho)+m(\rho)}(\rho) \). A sufficient condition is that the operations \( \langle \langle \rho \rangle \rangle \) vanish coherently, in the sense that the choices of compatible collections for the various \( \rho \) be consistent on common subpolyhedra (see [Bl4, \S 5.7] for the precise definition, and [Bl4, \S 5.9] for the obstructions to coherence).

On the other hand, if \( ^h \mathbf{W}^\bullet_{\ast} \) is the cosimplicial simplicial space up-to-homotopy of \( ^h \mathbf{E}^\bullet_{\ast} \) corresponding to the cosimplicial simplicial II-algebra \( (E^\bullet_{\ast})^\Delta \) with the CW-basis \( \{E^r_r\}_r, n=0 \}, \) then the vanishing of the homotopy operation \( \langle \langle \psi|_{\mathbf{C}^p} \rangle \rangle - \psi \) restricted to the \( (n, r) \)-cross-term – implies the vanishing of \( \langle \langle \psi \rangle \rangle \), for any \( \psi : \mathbf{W}^{n+k}_{m+i+k} \to \mathbf{W}^k_\ell \) (assuming lower order vanishing). This is because outside of the wedge summand \( \mathbf{C}^p_n \), the map \( \psi \) is determined by the maps \( \rho \in \mathcal{C}(\psi) \) and the coface and degeneracy maps of \( ^h \mathbf{W}^\bullet_{\ast} \), which we may assume to \( \infty \)-homotopy commute by induction and 6.13 respectively.

We may thus sum up the results of this section, combined with Proposition 6.15, in:

**Theorem 7.12.** Let \( \pi_* \mathbf{X} \) be an abelian II-algebra (for some space \( \mathbf{X} \in \mathcal{T}_* \)). Then \( \mathbf{X} \) is homotopy equivalent to a loop space if and only if all the higher homotopy operations \( \langle \langle \psi|_{\mathbf{C}^p} \rangle \rangle \) defined above vanish coherently.

**Remark 7.13.** As observed in \( \S 6.7 \), for any \( \mathbf{X} \in \mathcal{T}_* \) the space \( J \mathbf{X} \) is the colimit of the \( \Delta \)-cosimplicial space \( U(\mathbf{X})^\Delta_n \), and in fact the \( n \)-th stage of the James construction, \( J_n \mathbf{X} \), is the (homotopy) colimit of the \( (n-1) \)-coskeleton of \( U_n^\Delta \mathbf{X} \). Thus if we think of the sequence of higher homotopy operations “in the simplicial direction” as obstructions to the validity of 5.6(*) (up to \( \infty \)-homotopy commutativity), then the \( n \)-th cosimplicial
In particular, if we fix $k = \ell = 0$, $n = 1$ and proceed by induction on $m$, we are computing the obstructions for the existence of an $H$-multiplication on $X$, as in [Bl5]. (Thus if $X$ is endowed with an $H$-space structure to begin with, they must all vanish.)

Observe that the face-codegeneracy polyhedron $P^n_1$ is an $(n - 1)$-cube, as in Figure 2, rather than the $(n - 1)$-simplex we had in [Bl5, §4] – so the homotopy operations we obtain here are more complicated. This is because they take value in the homotopy groups of spheres, rather than those of the space $X$.

**Figure 2.** The face-codegeneracy polyhedron $P^n_1(d_0d_1d_2s^0)$

As a corollary to Theorem 7.12 we may deduce the following result of Hilton (cf. [H, Theorem C]):

**Corollary 7.14.** If $\langle X, m \rangle$ is a $(p - 1)$-connected $H$-space with $\pi_iX = 0$ for $i \geq 3p$, then $X$ is a loop space, up to homotopy.

**Proof.** Choose a CW-resolution of $\pi_*X$ which is $(p - 1)$-connected in each simplicial dimension, and let $E^\bullet_r$ be as in §6.9. By definition of the cross-term $\Pi$-algebras $C^n_r$ in §6.9, they must involve Whitehead products of elements from all lower order cross-terms; but since $X$ is an $H$-space by assumption, all obstructions of the form $\langle \psi|_{C^n_r} \rangle$ vanish (see §7.13). Thus, the lowest dimensional obstruction possible is a third-order operation $\langle \psi|_{C^n_r} \rangle$ ($r \geq 2$), which involves a triple Whitehead product and thus takes value in $\pi_1^{\ast}W_3^\ast$ for $i \geq 3p$. If we apply the $(3p - 1)$-Postnikov approximation functor to $hW^\ast$ in each dimension, to obtain $hZ^\ast$, all obstructions to rectification vanish, and from the spectral sequence of (4.4) we see that obvious map $X = ||W^\ast|| \to ||Z^\ast||$ induces an isomorphism in $\pi_i$ for $i < 3p$. Since $||Z^\ast||$ is a loop space by Theorem 7.12, so is its $(3p - 1)$-Postnikov approximation, namely $X$.  

**Example 7.15.** The 7-sphere is an $H$-space (under the Cayley multiplication, for example), but none of the 120 possible $H$-multiplications on $S^7$ are homotopy-associative; the first obstruction to homotopy-associativity is a certain “separation element” in $\pi_{21}S^7$ (cf. [J2, Theorem 1.4 and Corollary 2.5]).
Since \( \pi_*S^7 \) is a free \( \Pi \)-algebra, it has a very simple \( CW \)-resolution \( A_* \rightarrow \pi_*S^7 \), with \( A_0 \cong \pi_*S^7 \) (generated by \( i^7 \)), and \( A_r = 0 \) for \( r \geq 1 \). A cross-term basis (§6.9) for the cosimplicial simplicial \( \Pi \)-algebra \( E_\bullet \) of §6.14 is then given in dimensions < 24 by:

- \( C^1 \cong \pi_*S^{13} \), with \( \bar{d}_0i^{13} = [d^0i^7, d^1i^7] \);
- \( C^2 \cong \pi_*S^{19} \), with \( \bar{d}_0i^{19} = [d^0i^{13}, s_0d^2d^1i^7] - [d^1i^{13}, s_0d^2d^0i^7] + [d^2i^{13}, s_0d^1d^0i^7] \);
- \( C^r \) is at least 24-connected for all other \( n, r \).

We set \( s^i_j|_{C^r} = 0 \) for all \( n \leq 2 \); this determines \( E_\bullet \) in degrees \( \leq 21 \) and cosimplicial dimensions \( \leq 2 \).

By Remark 7.13, the two secondary operations \( \langle d_0s^0|C^1 \rangle \) and \( \langle d_1s^0|C^1 \rangle \) must vanish; on the other hand, by Corollary 7.14 all obstructions to \( S^7 \) being a loop space are in degrees \( \geq 21 \), so the only relevant cross-term is \( C^2 \), with three possible third-order operations \( \langle \psi|C^2 \rangle \), for \( \psi = d_0d_1s^0s^1 \), \( d_0d_2s^0s^1 \), or \( d_1d_2s^0s^1 \). The corresponding face-codegeneracy polyhedra \( P^4_2(\psi) \) is as in Figure 1.

It is straightforward to verify that the operations \( \langle \psi|C^2 \rangle \) are trivial for \( \psi = d_0d_1s^0s^1 \) or \( d_1d_2s^0s^1 \) (in fact, many of the maps \( g^\rho \), for \( \rho \in C(\psi) \), may be chosen to be null). On may also show that there is a compatible collection \( \{ g^\rho \}_{\rho \in C(\varphi)} \) for \( \varphi = d_0d_1s^0s^1 \), in the sense of §7.7, so that the corresponding subset \( \langle \varphi|C^2 \rangle \subseteq \pi_{21}S^7 \) is non-empty; in fact, it contains the only possible obstruction to the 21-Postnikov approximation for \( S^7 \) to be a loop space.

The existence of the tertiary operation \( \langle \varphi|C^2 \rangle \) corresponds to the fact that the element \( [[i^7, i^7], i^7] - [[i^7, i^7], i^7] + [[i^7, i^7], i^7] \in \pi_{21}S^7 \) is trivial for three different reasons: because of the Jacobi identity, because all Whitehead products vanish in \( \pi_*S^7 \), and because of the linearity of the Whitehead product — i.e., \( [0, \alpha] = 0 \).

On the other hand, we know that there is a 3-primary obstruction to the homotopy-associativity of any \( H \)-multiplication on \( S^7 \), namely the element \( \sigma^1_{14}7_7 \in \pi_{21}S^7 \) (see [J2, Theorem 2.6]). We deduce that \( 0 \notin \langle \varphi|C^2 \rangle \), and in fact (modulo 3) this tertiary operation consists exactly of the elements \( \pm \sigma^1_{14}7_7 \).

For a detailed calculation of such higher order operations using simplicial resolutions of \( \Pi \)-algebras, see [Bl5, §4.13].

Remark 7.16. Our approach to the question of whether \( X \) is a loop space is clearly based on, and closely related to, the classical approaches of Sugawara and Stasheff (cf. [St1, St2, Stu]). One might wonder why Stasheff’s associahedra \( K_i \) (cf. [St1, §2.6]) do not show up among the face-codegeneracy polyhedra we describe above. Apparently this is because we do not work directly with the space \( X \), but rather with its simplicial resolution, which may be thought of as a “decomposition” of \( X \) into wedges of spheres.

8. APPENDIX: CONSTRUCTING SIMPLICIAL RESOLUTIONS

The obstruction theory of section 4 required a \( \Pi \)-algebra resolution \( A_* \) of \( \Omega^{-1}\pi_*X \) as the initial algebraic ingredient. There are of course many possible constructions of such resolutions; for practical purposes a minimal \( CW \)-resolution (as in §6.1) is the most convenient. We here describe an approach specifically geared towards \( \Omega^{-1}\pi_*X \),
in line with section 3, because it may help to explain the simplicial group analogue used in the proof of Proposition 5.6.

Given an $H$-group $X$, we may proceed as follows to construct a free simplicial $\Pi$-algebra resolving $G_* = \Omega^{-1} \pi_* X$:

1. Choose a space $W$ which is homotopy equivalent to a wedge of spheres, and a map $z : W \to X$ which induces a surjection $z_* : \pi_* W \to \pi_* X$. We may assume that $z_*(t^k(\alpha)) = \alpha \in \pi_k X$, where $t^k(\alpha) \in \pi_k S^k(\alpha)$ is the canonical generator of the free $\Pi$-algebra $\pi_* S^k(\alpha)$, and $\pi_* W \cong \coprod_{k, \alpha} \pi_* S^k(\alpha)$ is a coproduct in $\Pi$-$\text{Alg}$ of such free $\Pi$-algebras for various $k$ and $\alpha$.

For example, we could let $W = \bigvee_{k=1}^{\infty} \bigvee_{\alpha \in \pi_k X \setminus \{0\}} \pi_* S^k(\alpha)$, so that $\pi_* W \cong T(\pi_* X)$; then by Fact 2.4 there is a map $z : W \to X$, unique up to homotopy, which realizes the counit $\varepsilon_{\pi_* X} : T(\pi_* X) \to \pi_* X$.

2. Since $X$ is an $H$-group, we may let $G_* = \Omega^{-1} \pi_* X$ denote the delooping provided by Corollary 3.9, and define a morphism of $\Pi$-algebras $\tilde{\zeta} : \pi_* \Sigma W \to G_*$ by setting $\tilde{\zeta}(t^{k+1}(\alpha)) = \tilde{\alpha} \in G_{k+1}$, where $t^{k+1}(\alpha)$ generates the summand $\pi_* \Sigma S^k(\alpha)$ in $\pi_* \Sigma W$.

More explicitly, any $\beta \in \pi_k \Sigma W$ is given as $\beta = \psi^\#(\tilde{t}(\alpha_1), \ldots, \tilde{t}(\alpha_n))$ where $\psi$ is some $n$-ary homotopy operation and $\tilde{t}(\alpha_j) = t^{m_j+1}(\alpha_j)$ is as above; then

$$ (8.1) \quad \tilde{\zeta}(\beta) = \psi^\#(\tilde{\zeta}(\tilde{t}(\alpha_1)), \ldots, \tilde{\zeta}(\tilde{t}(\alpha_n))) = \psi^\#(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = \psi^* (\alpha_1, \ldots, \alpha_n). $$

In fact, $\tilde{\zeta}$ may be identified with $\tilde{z}_* : \pi_*^H(JW) \to \pi_*^H X \cong G_*$ (using (3.7.3)), where $\tilde{z} : JW \to X$ is the $H$-map $\tilde{m} \circ J(z)$ of (3.1). Thus $\tilde{\zeta}(\beta) = \beta^* \tilde{\zeta}$, where $\tilde{\zeta}$ denotes the homotopy class of $z : W \to X$.

Note that $\tilde{\zeta} : \pi_* \Sigma W \to G_* :$ is also surjective, since for every $\gamma \in \pi_n X$ there is an $\omega \in \pi_n S^k$ such that $\gamma = \omega^*(z_*(t(\alpha)))$ for some $S^k(\alpha) \to W$ (since $\gamma$ is unary since $X$ is an $H$-space), and then

$$ \tilde{\gamma} = \omega^*(z_*(t(\alpha))) = (\Sigma \omega)^*(z_*(t(\alpha))) = (\Sigma \omega)^* \tilde{z}_* t(\alpha) = (\Sigma \omega)^* \tilde{\alpha} = \tilde{\zeta}((\Sigma \omega)^* t(\alpha)) $$

by Fact 3.4 and Corollary 3.9.

3. Set $A_0 = \pi_* \Sigma W \in \Pi$-$\text{Alg}$, and for each $n \geq 1$ let $A_n = T^n A_0$. Define face maps $d_i : A_n \to A_{n-1}$ by $d_i = T^i(\varepsilon_{T^n-1})$, and degeneracies $s_j : A_n \to A_{n+1}$ by $s_j = T^j(\partial_{T^n-1})$ for $0 \leq i, j \leq n - 1$ (compare [Go, App., §3]), with $s_n^0 = T^n \partial$ for $\partial : A_0 \to TA_0$ as in Definition 2.5.

4. We define a morphism of $\Pi$-algebras $\tilde{d} : A_1 \to A_0$ by letting $\tilde{d}(t^k(\beta)) = t^k(\beta \ast \zeta)$ for each $\beta \in D_k A_0$ — that is, if $\beta = \psi^*(\tilde{t}(\alpha_1), \ldots, \tilde{t}(\alpha_n))$ as above, then by (8.1) we have

$$ (8.2) \quad \tilde{d}(t^k(\beta)) = t^k(\beta \ast \zeta) = t^k(\psi(\alpha_1, \ldots, \alpha_n)). $$

We then set $d_n^0 : A_n \to A_{n-1}$ to be $T^{n-1} \tilde{d}$ for all $n \geq 1$.

**Lemma 8.3.** $A_* = ((A_n)_{n=0}^{\infty}, (d_n^0), (s_j^0)) \xto{\tilde{\zeta}} G_*$ is an augmented simplicial $\Pi$-algebra.
Proof. All the simplicial identities, except for those involving \(d^n_i\), follow as usual from the identities for the comonad \(\langle T, \varepsilon, \theta \rangle\) (cf. [Go, App., §3] or [BoK, I, §4.1]).

Now let \(t^k_\gamma\) be a generator for a coproduct summand \(\pi_*S^k_\gamma\) in \(A_2 = T A_1\), where \(\gamma \in D_k A_1\). Then \((T \varepsilon A_0)^k_\gamma = t^k_\gamma\), and thus \((d \circ T \varepsilon A_0)^k_\gamma = t^k_\gamma\) \(\in D_k A_0\), where \((\varepsilon A_0 \circ \gamma) \star \zeta\) is in \(D_k G_*\).

Write \(\gamma = \omega^\#(i_{\beta_1}, \ldots, i_{\beta_m})\) for some \(m\)-ary homotopy operation \(\omega^\#\) and \(i_{\beta_i} = \bar{t}_{\beta_i}\) a generator of a summand \(\pi_* S_{\beta_i}\) in \(A_1 = T A_0\), where \(\beta_i \in D_k A_0\) \((i = 1, \ldots, m)\). Then \(\varepsilon A_0 \circ \gamma = \omega^\#(\beta_1, \ldots, \beta_m)\).

Therefore, if we write each \(\beta_i\) \((i = 1, \ldots, m)\) as \(\beta_i = \psi_i^\#(i_{\alpha_1}, \ldots, i_{\alpha_n})\) (for \(\bar{t}_{\alpha_j}\) as in (2)), then \(\varepsilon A_0 \circ \gamma = (\omega^\#(\psi_1, \ldots, \psi_n))^k_\gamma(i_{\alpha_1}, \ldots, i_{\alpha_n})\), so

\[
(\varepsilon A_0 \circ \gamma) \star \zeta = (\omega^\#(\psi_1, \ldots, \psi_n))^k_\gamma(i_{\alpha_1} \lor \cdots \lor i_{\alpha_n})) \star \zeta
= (\omega^\#(\psi_1, \ldots, \psi_n)) \star (\alpha_1, \ldots, \alpha_n) = \omega \star ((\psi_1, \ldots, \psi_n)) \star (\alpha_1, \ldots, \alpha_n)
\]

by (8.2), and thus

\[
(\bar{d} \circ T \varepsilon A_0)^k_\gamma = (\omega^\#(\psi_1, \ldots, \psi_n) \star (\alpha_1, \ldots, \alpha_n)) \in D_k A_0.
\]

On the other hand \((T \bar{d})^k_\gamma = t^k_{\omega^\#(\bar{t}_{\beta_1}, \ldots, \bar{t}_{\beta_m})}\) by definition of \(T\), and

\[
\bar{d} \gamma = \bar{d}(\omega^\#(\bar{t}_{\beta_1}, \ldots, \bar{t}_{\beta_m})) = \omega^\#(\bar{d}(\bar{t}_{\beta_1}), \ldots, \bar{d}(\bar{t}_{\beta_m})) = \omega^\#(\bar{t}_{\beta_1 \star \zeta}, \ldots, \bar{t}_{\beta_m \star \zeta})
\]
since \(\bar{d}\) is a morphism of \(\Pi\)-algebras.

Thus \((d \circ T \bar{d})^k_\gamma = (T \bar{d})^k_\gamma = t^k_{\omega^\#(\bar{t}_{\beta_1 \star \zeta}, \ldots, \bar{t}_{\beta_m \star \zeta})}\) again by (8.2).

Since \(\beta_i = \psi_i^\#(i_{\alpha_1}, \ldots, i_{\alpha_n})\) for \(i = 1, \ldots, m\), we have \(\beta_i \star \zeta = \psi_i \star (\alpha_1, \ldots, \alpha_n)\), so

\[
(\omega^\#(\bar{t}_{\beta_1 \star \zeta}, \ldots, \bar{t}_{\beta_m \star \zeta})) \star \zeta = \omega^\#(\psi_1 \star (\alpha_1, \ldots, \alpha_n), \ldots, \psi_m \star (\alpha_1, \ldots, \alpha_n))
= \omega \star ((\psi_1 \lor \cdots \lor \psi_m)) \star (\alpha_1, \ldots, \alpha_n)
\]

by Proposition 3.8, so by (8.4) we see that \(\bar{d} \circ T \bar{d} = \bar{d} \circ T \varepsilon : A_2 \rightarrow A_0\).

Since \(\tilde{\zeta} \circ \varepsilon A_0 = \tilde{\zeta} \circ \bar{d}\) and \(\bar{d} \circ \bar{d} = i d_{A_0}\) (by Fact 3.4), the remaining simplicial identities follow from the fact that \(\varepsilon\) and \(\theta\) are natural transformations. \(\blacksquare\)

**Lemma 8.5.** The augmented simplicial \(\Pi\)-algebra \(A_\bullet \xrightarrow{\zeta} G_*\) is acyclic.

**Proof.** For any \(n \geq 0\) we may represent any \(x \in \pi_n D_k A_\bullet\) by a normalized cycle \(\beta \in D_k A_n\) with \(d_i \beta = 0\) for \(0 \leq i \leq n\) (cf. [Ma1, §17]), and consider \(t^k_\beta \in D_k A_{n+1}\):

\[
d d t^k_\beta = \beta, \text{ while } d_i t^k_\beta = T^i \varepsilon_{A_{n-1}} t^k_\beta = T^n \varepsilon_{A_{n-1}} t^k_\beta = t^k_\beta = t^k_{d_i \beta} = 0 \text{ (see Definition 2.5) for } 1 \leq i \leq n. \text{ If } n \geq 1 \text{ then also } d_{n+1} t^k_\beta = T^n \varepsilon_{A_{n-1}} t^k_\beta = t^k_{d_{n+1} \beta} = t^k_\beta = 0, \text{ so } \beta \text{ is a normalized boundary and thus } x = 0 \text{ in } \pi_n D_k A_\bullet.
\]

For \(n = 0\) by (2) we know \(\tilde{\zeta} : A_0 \rightarrow G_*\) is surjective. Given \(\beta \in D_k \text{Ker}(\tilde{\zeta}) \subseteq D_k A_0\), we may assume \(\beta = \omega^\#(\bar{t}_{\alpha_1}, \ldots, \bar{t}_{\alpha_n})\), and thus

\[
0 = \tilde{\zeta}(\beta) = \tilde{\zeta}(\omega^\#(\bar{t}_{\alpha_1}, \ldots, \bar{t}_{\alpha_n})) = \omega^\#(\tilde{\zeta}(\bar{t}_{\alpha_1}), \ldots, \tilde{\zeta}(\bar{t}_{\alpha_n})) = \omega^\#(\alpha_1, \ldots, \alpha_n) = \beta \star \zeta
\]

by (8.2).
Thus again \( d_0^k(\beta) = \beta \), while \( d_1^k = d^k = e(\beta) = d_0^k = 0 \), so \( [\beta] = 0 \) in \( \pi_0 D_k A \), which shows that \( \tilde{\zeta} \) indeed induces an isomorphism \( \pi_0 A \cong G \). □

REFERENCES


University of Haifa, 31905 Haifa, Israel

E-mail address: blanc@mathcs2.haifa.ac.il