

Modules, comodules and cotensor products over Frobenius algebras

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Abstract

We characterize noncommutative Frobenius algebras A in terms of the existence of a coproduct which is a map of left A^e -modules. We show that the category of right comodules over A , relative to this coproduct, is isomorphic to the category of right modules. This isomorphism enables a reformulation of the cotensor product of Eilenberg and Moore as a functor of modules rather than comodules.

We prove that the cotensor product $M \square N$ of a right A -module M and a left A -module N is isomorphic to the vector space of homomorphisms from a particular right A^e -module D to $M \otimes N$, viewed as a right A^e -module. Some of the properties of D are investigated, and some sample calculations are given. Finally, we show that when A is commutative or semisimple, the cotensor product $M \square N$ and its derived functors are given by the Hochschild cohomology over A of $M \otimes N$.

Keywords: Frobenius algebra, comodule, cotensor product, Hochschild cohomology

1 Introduction

Eilenberg and Moore originally introduced the cotensor product $M \square N$ and its derived functors $\text{Cotor}(M, N)$ on comodules M, N as tools for the calculation of the homology of the fiber space in a fibration [5]. This paper investigates these functors in the context where the coalgebra is a Frobenius algebra (defined in section 2).

The Frobenius case is not far removed from that of Eilenberg and Moore, whose coalgebra is the set of normalized singular chains in some space X ; in the presence of sufficient flatness, all the relevant constructions yield exactly the same data upon passing to homology [*ibid.*]. When the space X under

consideration is compact and oriented, its homology is in fact a Frobenius algebra.

Nevertheless, our approach diverges from that of Eilenberg and Moore in an important way. The results presented here rest on a new characterization of Frobenius algebras as algebras possessing a coassociative comultiplication $\delta: A \rightarrow A \otimes A$, with counit, which is a map of regular bimodules. (This is formulated slightly differently as theorem 2.1 below.) This comultiplication is decidedly different from the one used by Eilenberg and Moore. The relationship between the two coproducts will be discussed elsewhere.

The Frobenius algebra coproduct, and in particular the element $\delta(1_A)$, has already begun to find its place in a variety of contexts. In two dimensional topological quantum field theory, it gives rise to the handle operator [1]. In quantum cohomology it provides a generalization of the classical Euler class [2]. It also plays an important role in the study of quantum Yang-Baxter equations and serves as a separability idempotent [3]. Here, we will consider left and right submodules of $A \otimes A$ generated by $\delta(1_A)$. These will be discussed more later in this section.

The bimodule property of the Frobenius algebra coproduct implies another important property of Frobenius algebras, appearing as theorem 3.3: The category of right modules over a Frobenius algebra A is isomorphic to the category of right comodules over A . This result makes it possible to view Eilenberg and Moore's functors on comodules as functors on modules. Now, using the Snake Lemma, one can show that the cotensor product is left exact in both variables. (This also follows from theorem 4.6, of course.) This suggests that $M \square N$ should be expressible as a module of homomorphisms from some module D to $M \otimes N$. In fact, this is the case, as stated in theorem 4.6. The concern is to develop a satisfactory understanding of the module D .

Specifically, D denotes the right A^e -submodule of $A \otimes A$ generated by $\delta(1_A)$. This is not the same as the left A^e -submodule $\delta(A)$ of $A \otimes A$ generated by $\delta(1_A)$. The latter module is a very natural object to consider, since δ itself is a left A^e -module map, but the importance of D in this context is somewhat surprising. Under certain conditions, delineated in 4.3 and 4.3.1, D and $\delta(A)$ are in fact the same up to a canonical involution. But in other cases, such as those presented below as examples 4.1 and 4.2, quite the contrary is true. For instance, in example 4.2, $\delta(A)$ is four dimensional, whereas D is eight dimensional.

There are two important corollaries to the main results discussed above. One (4.6.1 below) is that the right derived functors of the cotensor product $M \square N$, *i.e.* $\text{Cotor}^*(M, N)$, are in fact the modules $\text{Ext}^*(D, M \otimes N)$. The

other (4.6.2 below) is that when A is commutative, the cotensor product $M \square N$ and its derived functors are given by the Hochschild cohomology over A of $M \otimes N$.

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Notation and Conventions

All algebras A considered here are assumed to be finite dimensional as a vector space over their coefficient field K , and to possess a multiplicative identity element 1_A . We let $\mu: A \otimes A \rightarrow A$ denote the multiplication map. The symbols A^n will always denote $A^{\otimes n}$, *i.e.* the tensor product of n copies of A , and never the Cartesian product. For any object X , we will use “ X ” or “ \cdot ” to denote the identity map $X \rightarrow X$, and the symbols $\cdot \otimes \cdot$ will be abbreviated “ $\cdot \cdot$ ”.

2 Noncommutative Frobenius Algebras

An algebra A is defined to be a **Frobenius algebra** if it possesses a left A -module isomorphism $\lambda_L: A \rightarrow A^*$ with its vector space dual. Here, A is viewed as the left regular module over itself, and A^* is a left A -module by the action $a \cdot \zeta(b) := \zeta(ab)$ for any $a, b \in A$ and $\zeta \in A^*$. It is easy to show that the existence of the isomorphism of left modules implies the existence of an isomorphism λ_R of right modules, where the right module structures are defined analogously.

There are many equivalent definitions of Frobenius algebras; see [4] for more information. For our purposes, the new characterization of Frobenius algebras presented below is very useful.

Theorem 2.1 *An algebra is a Frobenius algebra if and only if it has a coassociative comultiplication, with counit, which is a map of left regular A^e -modules.*

Here, A^e denotes the ring $A \otimes A^{\text{op}}$, and A has the left A^e -action defined by $(b \otimes b') \cdot a := bab'$.

In many respects, the proof of this result follows the proof of an analogous result for the commutative case, found in [1]. For the sake of space, we merely indicate how this proof differs from the one given there.

Proof. Assume A denotes a Frobenius algebra with left-module isomorphism $\lambda_L: A \rightarrow A^*$. Let $\mu_T: A \otimes A \rightarrow A$ denote the composition $\mu \circ T$, where $T: A \otimes A \rightarrow A \otimes A$ denotes the canonical involution. Define the comultiplication map $\delta_L: A \rightarrow A \otimes A$ to be the composition $(\lambda_L^{-1} \otimes \lambda_L^{-1}) \circ \mu_T^* \circ \lambda_L$. With

the appropriate adjustments, the discussion in [1] shows that the following diagram commutes:

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{\mu} & A \\
 \downarrow \cdot \otimes \delta_L & & \downarrow \delta_L \\
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \cdot} & A \otimes A
 \end{array}$$

In words, δ_L is a map of left modules.

Using the right-module isomorphism $\lambda_R: A \rightarrow A^*$, it is an analogous exercise to define δ_R and show that this comultiplication map is a map of right modules. Let $\epsilon: A \rightarrow K$ denote $\lambda_R(1_A)$. Note that $\lambda_R(1_A) = \lambda_L(1_A)$, and thus that ϵ serves as a counit for both δ_R and δ_L .

Now consider the following diagram:

$$\begin{array}{ccccc}
 & & A & & \\
 & \nearrow \mu & & \searrow \delta_R & \\
 A^2 & \xrightarrow{\delta_R \otimes \cdot} & A^3 & \xrightarrow{\cdot \otimes \mu} & A^2 \\
 \searrow \delta_R \otimes \delta_L & & \downarrow \cdot \otimes A \otimes \delta_L & & \downarrow \delta_L \\
 & & A^4 & \xrightarrow{\cdot \otimes \mu \otimes A} & A^3 & \xrightarrow{\cdot \otimes \epsilon \otimes \cdot} & A^2
 \end{array}$$

This diagram commutes because of the properties of δ_R , δ_L and ϵ mentioned just above. It follows that $\delta_R \circ \mu$ is the same as the composition of maps from the far left down and along the bottom row to the lower righthand corner. A corresponding diagram shows that $\delta_R \circ \mu$ is also the same as that composition, *i.e.* $\delta_R \circ \mu = \delta_L \circ \mu$. Since A has an identity element, we see that $\delta_R = \delta_L$. Define $\delta := \delta_R = \delta_L$. We have just shown that this map $\delta: A \rightarrow A \otimes A$ is a map of bimodules, *i.e.* is an A^e -module map, and has a counit.

The remainder of the proof follows as in [1]. ■

Throughout the sequel, δ and ϵ will denote the comultiplication and counit respectively. Let $\delta(A)$ denote the image of δ .

Corollary 2.1.1 *The map δ is an injection of left A^e -modules.*

Proof. By theorem 2.1, δ is a map of left A^e -modules. Since δ has a counit, it is certainly injective. ■

3 Modules and Comodules

We let $1_A: K \rightarrow A$ denote the map sending 1_K to 1_A . Since X and $X \otimes K$ are canonically isomorphic, for any map $f: X \rightarrow X$ we will abuse notation and write $f \otimes 1_A: X \rightarrow X \otimes A$ instead of $f \otimes 1_A: X \otimes K \rightarrow X \otimes A$. When discussing compositions of maps, the term “switch” will always refer to reversing the order of noninteracting maps.

Suppose M is a right A -module with structure map $m: M \otimes A \rightarrow M$. Define the map $\nabla_m: M \rightarrow M \otimes A$ to be the composition $(m \otimes \cdot) \circ (\cdot \otimes \delta) \circ (\cdot \otimes 1_A)$:

$$M \xrightarrow{\cdot \otimes 1_A} M \otimes A \xrightarrow{\cdot \otimes \delta} M \otimes A^2 \xrightarrow{m \otimes \cdot} M \otimes A$$

Lemma 3.1 *The map ∇_m endows M with the structure of a right A -comodule.*

Proof. It is necessary to show that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\nabla_m} & M \otimes A \\ \downarrow \nabla_m & & \downarrow \cdot \otimes \delta \\ M \otimes A & \xrightarrow{\nabla_m} & M \otimes A^2 \end{array} \quad (1)$$

Expanding each of the occurrences of ∇_m in accordance with the definition of that map yields the outer edge of this diagram:

$$\begin{array}{ccccccc} M & \xrightarrow{\cdot \otimes 1_A} & M \otimes A & \xrightarrow{\cdot \otimes \delta} & M \otimes A^2 & \xrightarrow{m \otimes \cdot} & M \otimes A \\ \downarrow \cdot \otimes 1_A & & \downarrow \cdot \otimes \delta & & \downarrow \cdot \otimes \delta & & \downarrow \cdot \otimes \delta \\ M \otimes A & \xrightarrow{\cdot \otimes \delta} & M \otimes A^2 & \xrightarrow{\cdot \otimes \delta \otimes \cdot} & M \otimes A^3 & \xrightarrow{m \otimes \cdot} & M \otimes A^2 \\ \downarrow \cdot \otimes \delta & & \uparrow \cdot \otimes \mu \otimes \cdot & & \uparrow \cdot \otimes \mu \otimes A \otimes \cdot & & \uparrow m \otimes \cdot \\ M \otimes A^2 & \xrightarrow{\cdot \otimes 1_A \otimes \cdot} & M \otimes A^3 & \xrightarrow{\cdot \otimes A \otimes \delta \otimes \cdot} & M \otimes A^4 & \xrightarrow{m \otimes \cdot} & M \otimes A^3 \\ & \searrow m \otimes \cdot & & & & \nearrow \cdot \otimes \delta \otimes \cdot & \\ & & M \otimes A & \xrightarrow{\cdot \otimes 1_A \otimes \cdot} & M \otimes A^2 & & \end{array}$$

From left to right and top down, the squares inside this large diagram commute for the following reasons: Vacuity, coassociativity of δ , switch, property of the multiplicative identity, δ being a module map, m being a module

map. The hexagon on the bottom is commutative because it only involves a switch.

It follows that the outer edge forms a commutative square, *i.e.* diagram (1) is commutative. \blacksquare

Suppose now that M is a right A -comodule, with comodule structure map $\nabla: M \rightarrow M \otimes A$. Define the map $m_\nabla: M \otimes A \rightarrow M$ to be the composition $(\cdot \otimes \epsilon) \circ (\cdot \otimes \mu) \circ (\nabla \otimes \cdot)$:

$$M \otimes A \xrightarrow{\nabla \otimes \cdot} M \otimes A^2 \xrightarrow{\cdot \otimes \mu} M \otimes A \xrightarrow{\cdot \otimes \epsilon} M$$

Lemma 3.2 *The map m_∇ endows M with the structure of a right A -module.*

Proof. It is necessary to show that the following diagram commutes:

$$\begin{array}{ccc} M \otimes A^2 & \xrightarrow{m_\nabla \otimes \cdot} & M \otimes A \\ \downarrow \cdot \otimes \mu & & \downarrow m_\nabla \\ M \otimes A & \xrightarrow{m_\nabla} & M \end{array} \quad (2)$$

Expanding each occurrence of m_∇ in accordance with the definition of that map yields the outer edge of the following diagram:

$$\begin{array}{ccccccc} M \otimes A^2 & \xrightarrow{\nabla \otimes \cdot} & M \otimes A^3 & \xrightarrow{\cdot \otimes \mu \otimes \cdot} & M \otimes A^2 & \xrightarrow{\cdot \otimes \epsilon \otimes \cdot} & M \otimes A \\ \downarrow \cdot \otimes \mu & \searrow \nabla \otimes \cdot & \downarrow \nabla \otimes \cdot & \searrow \nabla \otimes \cdot & \downarrow \nabla \otimes \cdot & \searrow \nabla \otimes \cdot & \downarrow \nabla \otimes \cdot \\ & & M \otimes A^3 & \xrightarrow{\cdot \otimes \delta \otimes A \otimes \cdot} & M \otimes A^4 & \xrightarrow{\cdot \otimes A \otimes \mu \otimes \cdot} & M \otimes A^3 & \xrightarrow{\cdot \otimes \epsilon \otimes \cdot} & M \otimes A^2 \\ & & \downarrow \cdot \otimes \mu & \searrow \cdot \otimes \mu \otimes \cdot & \downarrow \cdot \otimes \mu & \searrow \cdot \otimes \delta \otimes \cdot & \downarrow \cdot \otimes \mu & \searrow \cdot \otimes \epsilon \otimes \cdot & \downarrow \cdot \otimes \mu \\ & & & & M \otimes A^2 & \xrightarrow{\cdot \otimes \mu} & M \otimes A & & M \otimes A \\ & & & & \downarrow \cdot \otimes \mu & \searrow \cdot \otimes \cdot & \downarrow \cdot \otimes \epsilon & & \downarrow \cdot \otimes \epsilon \\ M \otimes A & \xrightarrow{\nabla \otimes \cdot} & M \otimes A^2 & \xrightarrow{\cdot \otimes \mu} & M \otimes A & \xrightarrow{\cdot \otimes \epsilon} & M \end{array}$$

The subdiagrams of this diagram are commutative for the following reasons: In the top row of squares, the leftmost square expresses the comodule property of ∇ . The other two squares simply involve switches, as does the large

square on the far left. The square in the center (between the second and third rows of maps) uses the module property of δ . The square to its right uses the counit property of ϵ . The large pentagon on the bottom expresses the associativity of μ . The triangle in the lower right hand corner is vacuous.

It follows that the outer edge forms a commutative square, *i.e.* diagram (2) is commutative. \blacksquare

Lemmas 3.1 and 3.2 show that there are canonical maps between the category of modules over A and the category of comodules over A . In fact, these provide an isomorphism.

Theorem 3.3 *The category of right modules over a Frobenius algebra A is isomorphic to the category of right comodules over A .*

Proof. First we will show that the constructions $m \mapsto \nabla_m$ and $\nabla \mapsto m_\nabla$ are mutual inverses. Then we will show that every module map is a comodule map for the corresponding comodule structures, and vice-versa.

Suppose $m: M \otimes A \rightarrow M$ is a right module structure map. Consider the following diagram:

$$\begin{array}{ccccccc}
M \otimes A & \xrightarrow{\cdot \otimes 1_A \otimes \cdot} & M \otimes A^2 & \xrightarrow{\cdot \otimes \delta \otimes \cdot} & M \otimes A^3 & \xrightarrow{m \otimes \cdot} & M \otimes A^2 \\
\downarrow \cdot \otimes \delta & & \downarrow \cdot \otimes \delta & & \downarrow \dots \otimes \delta & & \downarrow \cdot \otimes \delta \searrow \cdot \otimes \mu \\
M \otimes A^2 & \xrightarrow{\cdot \otimes 1_A \otimes \cdot} & M \otimes A^3 & \xrightarrow{\cdot \otimes \delta \otimes A \otimes \cdot} & M \otimes A^4 & \xrightarrow{m \otimes \cdot} & M \otimes A^3 \\
\downarrow \dots & \swarrow \cdot \otimes \mu \otimes \cdot & & \downarrow \cdot \otimes A \otimes \mu \otimes \cdot & & \downarrow \cdot \otimes \mu \otimes \cdot & \downarrow \cdot \otimes \epsilon \otimes \cdot \\
M \otimes A^2 & \xrightarrow{\cdot \otimes \delta \otimes \cdot} & M \otimes A^3 & \xrightarrow{m \otimes \cdot} & M \otimes A^2 & & M \otimes A \\
& & & & \searrow \cdot \otimes \epsilon \otimes \epsilon & & \downarrow \cdot \otimes \epsilon \\
& & & & & & M \otimes A \xrightarrow{m} M
\end{array}$$

The composition of maps across the top and down the right is nothing other than the definition of the map $m_{\nabla_m}: M \otimes A \rightarrow M$. Since the composition of maps down the left and across the bottom is m itself (by the counit property), the identity $m_{\nabla_m} \equiv m$ will follow if the diagram is commutative. This is in fact the case, because the subdiagrams are commutative for the following reasons: With the exception of those that will now be mentioned explicitly, the subdiagrams are commutative simply because they involve switches. The triangle on the lower left uses the multiplicative unit property.

The square to its right expresses the module property of δ . The square on the far upper right is commutative because it is essentially the outer edge of the following diagram:

$$\begin{array}{ccccc}
 & & A^2 & & \\
 & \cdot \otimes \delta & \downarrow \mu & \mu & \\
 A^3 & & A & & A \\
 & \mu \otimes \cdot & \downarrow \delta & \epsilon \otimes \cdot & \\
 & & A^2 & &
 \end{array}$$

This latter diagram is commutative because the square on the left expresses the module property of δ , and the square on the right express the counit property of ϵ .

It follows that $m_{\nabla m} \equiv m$. Suppose, on the other hand, that $\nabla: M \rightarrow M \otimes A$ is a comodule structure. We now show that $\nabla_{m_{\nabla}} \equiv \nabla$. Consider the following diagram:

$$\begin{array}{ccccccccc}
 M & \xrightarrow{\cdot \otimes 1_A} & M \otimes A & \xrightarrow{\cdot \otimes \delta} & M \otimes A^2 & \xrightarrow{\nabla \otimes \cdot} & M \otimes A^3 & \xrightarrow{\cdot \otimes \mu \otimes A} & M \otimes A^2 \\
 & \searrow \nabla & & \searrow \nabla & & \nearrow \cdot \otimes A \otimes \delta & & \nearrow \cdot \otimes \delta & \downarrow \cdot \otimes \epsilon \otimes \cdot \\
 & & M \otimes A & \xrightarrow{\cdot \otimes 1_A} & M \otimes A^2 & \xrightarrow{\cdot \otimes \mu} & M \otimes A & \xrightarrow{\cdot} & M \otimes A
 \end{array}$$

From left to right, the subdiagrams are commutative for the following reasons: Switch, switch, the module property of δ , the counit property of ϵ . Because the composition of maps across the top and down the right of this diagram is simply the definition of $\nabla_{m_{\nabla}}$, and the composition of maps down the left and across the bottom is just ∇ (by the unit property of 1_A), we see that $\nabla_{m_{\nabla}} \equiv \nabla$.

Suppose that M and N are right A modules with module structure maps m and n respectively. In order to verify that a map $f: M \rightarrow N$ of right modules is also a map of right comodules (for the corresponding comodule structures), consider the following diagram:

$$\begin{array}{ccccccc}
 M & \xrightarrow{\cdot \otimes 1_A} & M \otimes A & \xrightarrow{\cdot \otimes \delta} & M \otimes A^2 & \xrightarrow{m \otimes \cdot} & M \otimes A \\
 \downarrow f & & \downarrow f \otimes \cdot & & \downarrow f \otimes \cdot & & \downarrow f \otimes \cdot \\
 N & \xrightarrow{\cdot \otimes 1_B} & N \otimes A & \xrightarrow{\cdot \otimes \delta} & N \otimes A^2 & \xrightarrow{n \otimes \cdot} & N \otimes A
 \end{array}$$

Two of the subdiagrams simply involve switches. The third is commutative because f is a map of modules. Thus, the outer edges form a commutative

diagram as well. But this diagram asserts that f is a map of comodules, where the comodule structure maps are ∇_m and ∇_n .

If $f: M \rightarrow N$ is assumed to be a map of right comodules, where the comodule structure maps are ∇ and ∇' , then by reasoning analogous to that of the previous paragraph, the following diagram shows that f is a map of right modules:

$$\begin{array}{ccccccc}
M \otimes A & \xrightarrow{\nabla \otimes \cdot} & M \otimes A^2 & \xrightarrow{\cdot \otimes \mu} & M \otimes A & \xrightarrow{\cdot \otimes \epsilon} & M \\
\downarrow f \otimes \cdot & & \downarrow f \otimes \cdot & & \downarrow f \otimes \cdot & & \downarrow f \\
N \otimes A & \xrightarrow{\nabla' \otimes \cdot} & N \otimes A^2 & \xrightarrow{\cdot \otimes \mu} & N \otimes A & \xrightarrow{\cdot \otimes \epsilon} & N
\end{array}$$

This completes the proof. ■

With appropriate changes, all the results and proofs in this section apply to left modules and left comodules as well.

4 Cotensor Product

Suppose that M is a right A -module with module structure map m , and that N is a left A -module with module structure map n . By theorem 3.3, M is a right comodule with structure map ∇_m and N is a left comodule with structure map ∇_n . Let ϕ denote the map

$$\phi := \nabla_m \otimes N - M \otimes \nabla_n: M \otimes N \longrightarrow M \otimes A \otimes N.$$

The **cotensor product** [5] $M \square N$ of M and N is defined to be the kernel of ϕ .

Viewing A as both the right and left regular modules over itself (*i.e.* the module structure maps are both μ), we can form $A \square A$. Note that ∇_μ is just the map δ , by the module property of δ .

Proposition 4.1 *The cotensor product $A \square A$ is exactly $\delta(A)$.*

Proof. By the definition of ϕ , it suffices to show that the two maps $(\nabla_\mu \otimes A) \circ \delta$ and $(A \otimes \nabla_\mu) \circ \delta$ are the same. But these two maps are just $(\delta \otimes A) \circ \delta$ and $(A \otimes \delta) \circ \delta$, respectively. These are the same, by the coassociativity of δ . ■

Definition 4.2 Let D denote the right A^e -submodule of $A \otimes A$ generated by $\delta(1_A)$. Note that D and $\delta(A)$ (see corollary 2.1.1 above) are different objects.

Proposition 4.3 *If $\delta(1_A)$ is symmetric, i.e. $T \circ \delta(1_A) = \delta(1_A)$, then D and $T \circ \delta(A)$ are isomorphic as right A^e -modules.*

Proof. For any $a, b, x, y \in A$, we have

$$[T(x \otimes y)] \cdot (a \otimes b) = ya \otimes bx = T[(b \otimes a) \cdot (x \otimes y)] .$$

In our case, the hypothesis on $\delta(1_A)$ and the module property of δ therefore imply that

$$\delta(1_A) \cdot (a \otimes b) = T[(b \otimes a) \cdot \delta(1_A)] = T \circ \delta(ba) . \quad (3)$$

Because A contains 1_A , this equality shows that D and $T \circ \delta(A)$ are identical sets. Define the right A^e -action on $T \circ \delta(A)$ to be the action inherited from the right A^e -action on $A \otimes A$. Equation (3) guarantees that this action is well defined and that the correspondence between D and $T \circ \delta(A)$ is actually an isomorphism of modules. ■

Corollary 4.3.1

1. *If A is commutative then D and $\delta(A)$ are isomorphic as right and left A^e -modules.*
2. *If A is semisimple then D and $T \circ \delta(A)$ are isomorphic as right A^e -modules.*

Proof. If A is commutative then δ is a cocommutative map, by definition. (See the proof of theorem 2.1.) Thus the hypothesis of proposition 4.3 is automatically satisfied, and also $T \circ \delta = \delta$. Of course, in the case of a commutative algebra there is no distinction between left and right regular actions.

By Wedderburn's first structure theorem, to prove the result in the case when A is semisimple it suffices to assume that A is a matrix ring. In that case, A has a Frobenius algebra structure given by the map $\lambda_L(a) := \text{Tr} \circ \mu(a \otimes -)$. Let e_{ij} denote the elementary matrix with the entry 1 in the i, j position. Of course, the set of elementary matrices in A forms a basis. Let e_{ij}^* denote the element of A^* such that $e_{ij}^*(e_{kl}) = \delta_{ik} \delta_{jl}$. It is easy to see that $\lambda_L^{-1}(e_{ij}^*) = e_{ji}$. Applying proposition 5 from [1], we have

$$\delta(1_A) = \sum_{i,j} e_{ij} \otimes \lambda_L^{-1}(e_{ij}^*) = \sum_{i,j} e_{ij} \otimes e_{ji} .$$

Clearly, the hypothesis of proposition 4.3 is satisfied. ■

When A is neither commutative nor semisimple, proposition 4.3 does not necessarily apply. In fact, D and $\delta(A)$ may differ quite strongly.

Example 4.4 Let A denote the exterior algebra on two generators, x and y . Then

$$\delta(1_A) = 1_A \otimes xy + xy \otimes 1_A - x \otimes y + y \otimes x,$$

and $\delta(A)$ has the basis

$$\{\delta(1_A), x \otimes xy + xy \otimes x, y \otimes xy + xy \otimes y, xy \otimes xy\},$$

whereas D has the basis

$$\{\delta(1_A), x \otimes xy - xy \otimes x, y \otimes xy - xy \otimes y, xy \otimes xy\}.$$

Example 4.5 Let A denote the algebra with generators x and y satisfying the following relations: $x^2 = 0$, $y^2 = 0$, $yx = xy + x$. A has the basis $\{1_A, x, y, xy\}$, and is a Frobenius algebra with map $\lambda_L: A \rightarrow A^*$ given by

$$\begin{array}{ll} 1_A \mapsto (xy)^* & xy \mapsto 1_A^* \\ x \mapsto y^* & y \mapsto x^* + (xy)^* \end{array}$$

We have

$$\delta(1_A) = 1_A \otimes xy + xy \otimes 1_A + x \otimes (y - 1) + y \otimes x,$$

and $\delta(A)$ is four-dimensional with the basis

$$\left\{ \begin{array}{l} \delta(1_A), \\ x \otimes xy + xy \otimes x, \\ y \otimes xy + xy \otimes y + x \otimes (y - 1), \\ xy \otimes xy \end{array} \right\},$$

whereas, after some manipulation, D can be seen to have the basis

$$\left\{ \begin{array}{l} x \otimes x, x \otimes xy, xy \otimes x, xy \otimes xy, \\ y \otimes x - x \otimes y + xy \otimes 1_A, \\ 1_A \otimes xy + 2x \otimes y - x \otimes 1_A, \\ 2y \otimes xy - xy \otimes 1_A, \\ y \otimes xy - xy \otimes y \end{array} \right\}.$$

In particular, D is eight dimensional. These basis elements, in the order shown, are the elements obtained from right action on $\delta(1_A)$ by the following elements of A^e , respectively:

$$\begin{aligned} & \frac{1}{2}(x \otimes 1_A - 1_A \otimes x), \\ & \frac{1}{2}(x \otimes 1_A + 1_A \otimes x) - xy \otimes y - 2y \otimes xy, \\ & xy \otimes y + 2y \otimes xy, \\ & y \otimes xy, \quad 1_A \otimes y - y \otimes 1_A, \\ & 1_A \otimes 1_A - 1_A \otimes y + y \otimes 1_A, \\ & y \otimes 1_A + y \otimes y, \quad y \otimes y \end{aligned}$$

This example shows very clearly that D and $\delta(A)$ are fundamentally different.

Given a right module M and a left module N as above, endow $M \otimes N$ with the right A^e -module structure. Let $\text{Hom}_{A^e}(D, M \otimes N)$ denote the vector space of right A^e -module maps.

Theorem 4.6 *There is a vector space isomorphism*

$$M \square N \cong \text{Hom}_{A^e}(D, M \otimes N).$$

Proof. Note first that an element $f \in \text{Hom}_{A^e}(D, M \otimes N)$ is determined by its value on $\delta(1_A)$, the generator of D .

The following diagram is commutative, since f is a map of modules:

$$\begin{array}{ccccc} K & \xrightarrow{\delta(1_A) \otimes \delta(1_A)} & A^4 & \xrightarrow{(\mu \otimes \cdot) \circ T_{23} - (\cdot \otimes \mu) \circ T_{234}} & A^3 \\ & & \downarrow f \otimes \cdot & & \downarrow f \otimes \cdot \\ & & M \otimes N \otimes A^2 & \xrightarrow{(m \otimes \cdot) \circ T_{23} - (\cdot \otimes n) \circ T_{234}} & M \otimes N \otimes A \end{array}$$

By proposition 4.1, the composition of maps across the top of the diagram is 0. Since the composition of maps from the upper left, down and across the bottom is $T_{23} \circ \phi \circ f[\delta(1_A)]$, it follows that $f[\delta(1_A)] \in M \square N$. Thus, there is a well defined injective map $\sigma: \text{Hom}_{A^e}(D, M \otimes N) \rightarrow M \square N$ sending $f \mapsto f[\delta(1_A)]$. Since each element $e \in M \otimes N$ defines a unique A^e -module map $\tau(e): \delta(1_A) \mapsto e$, restriction of τ to $M \square N$ provides an inverse to σ . ■

Allowing for abuse of notation, define the cotensor product functor \square_A by $\square: M \otimes N \mapsto M \square N$, and let $\text{Cotor}_A^i(M, N)$ denote its right derived functors. Let $H^i(A, -)$ denote the Hochschild cohomology functors.

Corollary 4.6.1 *Over a Frobenius algebra A , the Cotor functor is given by*

$$\text{Cotor}_A^*(M, N) \cong \text{Ext}_{A^e}^*(D, M \otimes N).$$

Proof. In light of theorem 4.6, this is purely a matter of definitions. ■

Corollary 4.6.2 *If A is a commutative or semisimple Frobenius algebra, then cotensor product and its derived functors are Hochschild cohomology, i.e.*

$$\text{Cotor}_A^*(M, N) \cong H^*(A, M \otimes N).$$

Proof. By corollary 4.3.1 we have $\delta(1_A) = T \circ \delta(1_A)$ and $D \cong T \circ \delta(A)$. Since, by corollary 2.1.1, δ is an injective map of left A^e -modules (determined by its value on $\delta(1_A)$), so is $T \circ \delta$. Thus D and A are isomorphic as A^e -modules. It follows from theorem 4.6 that $M \square N \cong \text{Hom}_{A^e}(A, M \otimes N)$. But this is exactly $H^0(A, M \otimes N)$ [6, pg. 301]. Since $H^*(A, -) \cong \text{Ext}_{A^e}^*(A, -)$ [*ibid.* pg. 303], this corollary follows from 4.6.1. ■

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