Transfers for ramified coverings in homology and cohomology*

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Abstract

Making use of a modified version, due to McCord, of the Dold-Thom construction of ordinary homology, we give a simple topological definition of a transfer for ramified covering maps in homology with arbitrary coefficients. The transfer is induced by a suitable map between topological groups. We also define a cohomology transfer which is dual to the homology transfer. This duality allows us to show that our homology transfer coincides with the one given by L. Smith. With our definition of the homology transfer we can give simpler proofs of the properties of the known transfer and of some new ones.

1 Introduction

In [16] L. Smith introduced a general class of finite ramified covering maps and constructed for them a transfer in ordinary homology. Later on, in [6] A. Dold gave an alternative construction and characterized ramified covering maps as maps between orbit spaces of the action of a finite group and a subgroup, and giving a modified definition of the transfer. Both definitions are algebraic in nature. These transfers have the property that when composed with the homomorphism induced by the projection of the ramified covering

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map, they yield multiplication by the multiplicity of the covering in the homology of the base space.

There are previous definitions of both the homology and the cohomology transfers for maps between orbit spaces of certain actions of a finite group and a subgroup, see Bredon [4] (and also tom Dieck [5]). These definitions depend on the equivariant structure of the spaces involved. This is analogous to the classical definition of the transfer for standard covering maps of Eckmann [8], which is given at the level of chain complexes, thus it is rather algebraic in nature.

In this paper we use a modified version, due to McCord [13], of the Dold-Thom construction of ordinary homology to produce a topological transfer for general ramified covering maps. Namely, we define a transfer that is a map between some topological groups associated to the total and to the base space of the ramified covering map. This codifies in a sense the fact that a transfer can be seen as a multivalued map. We also define a cohomology transfer using models of Eilenberg-Mac Lane spaces that have the structure of topological abelian groups. We apply either transfer to give some results about the homology or cohomology of orbit maps of the action of a group and a subgroup of finite index. In contrast to previous transfers, whose definition, and therefore also their computation, is more complicated than the definition of the homomorphisms induced by the projections of the (ramified) covering maps, our transfer definitions are somehow simpler than the latter, making their computation easy (see (6.1), for instance).

In Section 2, for the benefit of the reader, we recall the construction of McCord's topological groups and the nice main results of McCord's paper. These are essential for our definition of the homology transfer and to prove its properties. They also provide models of the Eilenberg-Mac Lane spaces which are (weak) topological abelian groups. This fact will be used later to construct the cohomology transfer. In Section 3 we recall the definition of a ramified covering, and in Section 4 we define the homology transfer and prove some of its properties. In Section 5 we define the cohomology transfer and prove some of its properties. In Section 6 we use the transfers to prove some results about group actions. For instance, we study the homology and the cohomology of the orbit space of an arbitrary action of a compact Lie group in terms of the orbit space of the action of the connected component of the identity element. In Section 7 we show that the homology and the cohomology transfers are dual with respect to the Kronecker product in homology and cohomology. Finally, in Section 8 we prove that our homology transfer coincides with

Smith's transfer.

2 McCord's topological groups

In this section we recall briefly the spaces $B(G, X)$ introduced by McCord. We find it convenient to use $F(X, G)$ as an alternative notation. Details can be seen in [13] or [1, 6.3.20ff]. In this section we shall work in the category of compactly generated weak Hausdorff spaces (see [13] or [12]) and therein will be all spaces considered.

Definition 2.1. Let G be a topological abelian group and let X be a pointed topological space with base point $* \in X$. We define $F(X, G)$ as the set of all functions $u : X \longrightarrow G$ such that $u(*) = 0$ and $u(x) = 0$ for all but a finite number of elements $x \in X$. If these elements are $x_1, \ldots, x_n$ and the values of u at each of them are $g_1, \ldots, g_n$, it is sometimes convenient to write u as $\sum_{i=1}^{n} g_i x_i = g_1 x_1 + \cdots + g_n x_n$. In particular, for any $x \in X$, $x \neq 0$, one may see gx as the element in $F(X, G)$ whose value at x is g and whose value elsewhere is 0 ($g* = 0$). Of course, these elements gx generate the group $F(X, G)$. Taking $G = R$ to be a commutative ring with 1 and $x \in X$, then $x \in F(X, R)$ can be interpreted as the function whose value at x is 1 and whose value elsewhere is 0. This furnishes $F(X, R)$ with a canonical inclusion $X \hookrightarrow F(X, R)$. In this case, the elements $x \in F(X, R)$ generate $F(X, R)$ freely as an R-module.

Remark 2.2. In this paper we shall mainly restrict ourselves to abelian groups G with the discrete topology, however we work in this section with general topological abelian groups.

The set $F(X, G)$ has a topology that turns it into a pointed space with base point $0 \in F(X, G)$ the constant function with value 0, as we see below. It is also an abelian group with the obvious addition. It is in fact a (weak) topological abelian group. Consider the natural filtration

$$F_0(X, G) \subset F_1(X, G) \subset \cdots \subset F(X, G)\,,$$

where $F_n(X, G)$ consists of those functions u that are nonzero on at most n points in X. If $G = R$, then one can define $F_{\frac{1}{2}}(X, R) = X \subset F_1(X, R)$, as explained above. The topology can then be defined as follows. For each n, take the surjection $k(G \times X)^n \longrightarrow F_n(X, G)$ given by mapping $(g_1, x_1, \ldots, g_n, x_n)$

to $\sum_{i=1}^{n} g_i x_i$. Here $k(G \times X)^n$ is the product of n copies of $G \times X$, furnished with the compactly generated product topology, and $F_n(X, G)$ is given the corresponding quotient topology. Then provide $F(X, G)$ with the weak topology (of the union).

Given a pointed map $\varphi : X \longrightarrow Y$ and a (continuous) homomorphism $\alpha : G \longrightarrow H$, one has a unique pointed map $F(\varphi, \alpha) : F(X, G) \longrightarrow F(Y, H)$ given by

$$F(\varphi, \alpha)\left(\sum_{i=1}^{n} g_i x_i\right) = \sum_{i=1}^{n} \alpha(g_i)\varphi(x_i)\,. \tag{2.3}$$

In other words, $F(\varphi, \alpha)(u)$ is the function whose values at $y \in Y$ are 0 unless $y = \varphi(x)$ and $u(x) \neq 0$; in this case, $F(\varphi, \alpha)(u)(y) = \sum_{\varphi(x)=y} \alpha(u(x))$. This definition turns F into a covariant bifunctor from the category $\mathcal{T}\mathrm{op}_* \times \mathcal{T}\mathrm{opab}$ of pairs consisting of a pointed topological space and a topological abelian group to the category $\mathcal{T}\mathrm{opab}$ of topological abelian groups.

We shall denote $F(\varphi, 1_G)$ simply by φ_* and $F(\mathrm{id}_X, \alpha)$ by α_*. The following will be useful properties of the functor F:

(a) There is a natural isomorphism of topological abelian groups

$$F(X \wedge Y, G) \longrightarrow F(Y, F(X, G))$$

given by mapping $u = \sum g_i(x_i \wedge y_i)$ to $\sum (g_i x_i) y_i$, where $g_i x_i \in F(X, G)$ is as described above (see [13, 6.13]).

(b) There is a natural H-isomorphism (i.e., a homotopy equivalence that is also a homomorphism) of H-groups

$$F(Y, G) \longrightarrow \Omega F(\Sigma Y, G)$$

(if G is well pointed), where Ω means the loop space and Σ the (reduced) suspension given by $\Sigma Y = \mathbb{S}^1 \wedge Y$. This H-isomorphism yields a group isomorphism

$$\sigma : [X, F(Y, G)]_* \longrightarrow [X, \Omega F(\Sigma Y, G)]_* \cong [\Sigma X, F(\Sigma Y, G)]_*\,,$$

where $[-,-]_*$ denotes pointed homotopy classes. We call this the *suspension isomorphism* (see [13, 10.4]).

By (b), $F(\mathbb{S}^q, G) \xrightarrow{\simeq} \Omega F(\mathbb{S}^{q+1}, G)$, and since $F(\mathbb{S}^0, G) = G$, we obtain the following.

Theorem 2.4. ([13, 10.6]) *The space $F(\mathbb{S}^q, G)$ is an Eilenberg–Mac Lane space of type (G, q) that has the structure of an abelian group.* ■

Note 2.5. This is not the first construction of Eilenberg-Mac Lane spaces that yields topological abelian groups (for instance, Milnor [14] shows that his construction of $K(G,q)$ yields always a weak topological abelian group; he shows that if $K(G,q)$ is a countable CW-complex, then it is a topological abelian group). However, McCord's construction is a very convenient one and is easy to give.

Using property (b), we have a long exact sequence for the homotopy groups $\pi_q(F(A,G))$, $\pi_q(F(X,G))$, and $\pi_q(F(X \cup CA, G)$ for a pair (X, A) of the same homotopy type of a CW-pair . This and the previous theorem prove the following.

Corollary 2.6. *Let G be a discrete abelian group and let (X, A) be a pair of spaces of the same homotopy type of a* CW*-pair. Then the homotopy groups*

$$H_q(X, A; G) = \pi_q(F(X \cup CA, G))$$

define an ordinary homology theory with coefficients in G. In particular, the groups $\widetilde{H}_q(X; G) = \pi_q(F(X, G))$ provide the reduced homology groups. Moreover, the groups of pointed homotopy classes

$$H^q(X, A; G) = [X \cup CA, F(\mathbb{S}^q, G)]_*$$

define an ordinary cohomology theory with coefficients in G. In particular, the groups $\widetilde{H}^q(X; G) = [X, F(\mathbb{S}^q, G)]_*$ provide the reduced cohomology groups. (If $A \hookrightarrow X$ is a cofibration, one may of course replace $X \cup CA$ by X/A.) ■

Note that the groups of unpointed homotopy classes $H^q(X; G) = [X, F(\mathbb{S}^q, G)]$ provide the unreduced cohomology groups.

Remark 2.7. If we assume that X is paracompact (instead of compactly generated weak Hausdorff of the same homotopy type of a CW-complex), then the groups $[X, F(\mathbb{S}^q, G)]_*$ yield the *Čech cohomology groups* $\check{H}^q(X; G)$ (see [11]).

Lemma 2.8. *The map $\varepsilon : F(X,G) \longrightarrow G$ given by $\sum_{i=1}^m g_i x_i \mapsto \sum_{i=1}^m g_i$ is well defined and continuous. In particular, $\varepsilon : F(\mathbb{S}^0, G) \longrightarrow G$ is a homeomorphism.*

Proof: This follows easily from the fact that the restriction $\varepsilon_n : F_n(X,G) \longrightarrow G$ of ε is continuous, since its composite with the identification $(X \times G)^n \longrightarrow F_n(X,G)$ is obviously continuous. ■

Another useful property of the functor F is that one has a well-defined continuous pairing

$$F(X,G) \times F(Y,H) \longrightarrow F(X \wedge Y, G \otimes H) \tag{2.9}$$

given by

$$\left(\sum_i g_i x_i, \sum_j h_j y_j \right) \longmapsto \sum_{i,j} (g_i \otimes h_j)(x_i \wedge y_j)\,,$$

(see [13, 11.6]). If, in particular, $G = H = R$ is a commutative ring with 1, with $m : R \otimes R \longrightarrow R$ as the ring multiplication, then composing (2.9) with m_*, we obtain another pairing

$$F(X,R) \times F(Y,R) \longrightarrow F(X \wedge Y, R)\,. \tag{2.10}$$

Using (2.10), one obtains products in homology and cohomology. We shall be interested in the following.

Proposition 2.11. *One has cap-products*

$$H^q(X;R) \otimes H_k(X;R) \xrightarrow{\frown} H_{k-q}(X;R)\,,$$

if X is 0-connected and $q \leq k$, and

$$H^q(X;R) \otimes H_k(X;R) \xrightarrow{\frown} H^{q-k}(X;R)\,,$$

if $k \leq q$. In particular, if $k = q$ one has a Kronecker product

$$H^q(X;R) \otimes H_q(X;R) \xrightarrow{\langle -,- \rangle} R\,. \tag{2.12}$$

Proof: Taking smash-products and the pairing (2.9) we have

$$\begin{array}{ccc}
[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_* & \longrightarrow & [X^+ \wedge \mathbb{S}^k, F(\mathbb{S}^q, R) \wedge F(X^+, R)]_* \\
& \overset{\kappa}{\searrow} & \downarrow \\
& & [\Sigma^k X^+, F(\Sigma^q X^+, R)]_* .
\end{array}$$

If $q \leq k$, using σ^{-q} of property (b), we desuspend q times. Composing κ with the homomorphism

$$[\Sigma^{k-q} X^+, F(X^+, R)]_* \longrightarrow [\mathbb{S}^{k-q}, F(X^+, R)]_*$$

induced by the pointed inclusion $\mathbb{S}^0 \longrightarrow X^+$ that sends -1 to some point x_{-1} in the path-connected space X, we obtain the homology $\frown$-product

$$\frown : [X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_* \longrightarrow [\mathbb{S}^{k-q}, F(X^+, R)]_* .$$

On the other hand, if $k \leq q$, using σ^{-k}, we desuspend k times. And then, composing κ with the homomorphism

$$[X^+, F(\Sigma^{q-k} X^+, R)]_* \longrightarrow [X^+, F(\mathbb{S}^{q-k}, R)]_*$$

induced by the obvious map $X^+ \longrightarrow \mathbb{S}^0$, we obtain the cohomology $\frown$-product

$$\frown : [X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_* \longrightarrow [X^+, F(\mathbb{S}^{q-k}, R)]_* .$$

In order to obtain the Kronecker product $\langle -, - \rangle$, we take $q = k$ and consider the composite

$$[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_* \overset{\frown}{\longrightarrow} [X^+, F(\mathbb{S}^0, R)]_* \longrightarrow [\mathbb{S}^0, F(\mathbb{S}^0, R)]_* = R ,$$

where the last arrow is induced by the pointed inclusion $\mathbb{S}^0 \longrightarrow X^+$, and the equality follows from the bijection $\varepsilon : F(\mathbb{S}^0, R) \longrightarrow R$ given in Lemma 2.8.

■

3 Ramified coverings

We recall L. Smith's definition of a ramified covering map (see [16]). We shall need the concept of *nth symmetric product* of Y defined by $\mathrm{SP}^n Y = Y^n / \Sigma_n$, where Σ_n acts on the product of n copies of Y by permuting the coordinates. We denote its elements by $\langle y_1, y_2, \ldots, y_n \rangle$.

Definition 3.1. An *n-fold ramified covering map* is a continuous map $p : E \longrightarrow X$ together with a *multiplicity function* $\mu : E \longrightarrow \mathbb{N}$ such that the following hold:

(i) The fibers $p^{-1}(x)$ are finite (discrete), $x \in X$.

(ii) For each $x \in X$, $\sum_{e \in p^{-1}(x)} \mu(e) = n$.

(iii) The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$ given by

$$\varphi_p(x) = \langle \underbrace{e_1, \ldots, , e_1}_{\mu(e_1)}, \ldots, \underbrace{e_m, \ldots, , e_m}_{\mu(e_m)} \rangle ,$$

where $p^{-1}(x) = \{e_1, \ldots, e_m\}$, is continuous.

Remark 3.2. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ, one can construct an n-fold ramified covering map $p^+ : E^+ \longrightarrow X^+$, where $Y^+ = Y \sqcup \{*\}$ for any space Y and p^+ extends p by defining $p^+(*) = *$ and the multiplicity function μ^+ extends μ by setting $\mu^+(*) = n$. More generally, given a (closed) subspace $A \subset X$, one can construct an n-fold ramified covering map $p' : E' \longrightarrow X/A$, where $E' = E/p^{-1}A$, p' is the map between quotients and the multiplicity function μ' coincides with μ off $p^{-1}A$ and is extended by setting $\mu'(*) = n$, if $*$ is the base point onto which $p^{-1}A$ collapses.

Another useful construction is the following. Let $\overline{E} = E \sqcup X$ and $\overline{p} : \overline{E} \longrightarrow X$ be such that $\overline{p}|_E = p$ and $\overline{p}|_X = \mathrm{id}_X$. Then $\overline{p}$ is an $(n+1)$-fold ramified covering map with the obvious multiplicity function.

On the other hand, given a map $F : Y \longrightarrow X$, one can construct the *induced n-fold ramified covering map* $F^*(p) : F^*(E) \longrightarrow Y$ by taking the pullback $F^*(E) = \{(y, e) \in Y \times E \mid F(y) = p(e)\}$ and $F^*(p) = \mathrm{proj}_Y$. The *induced multiplicity function* $F^*(\mu) : F^*(E) \longrightarrow \mathbb{N}$ is given by $F^*(\mu)(y, e) = \mu(e)$. Call $\widetilde{F} : F^*(E) \longrightarrow E$ the projection proj_E.

Examples 3.3. Typical examples of ramified covering maps are the following:

1. Standard covering maps with finitely many leaves.

2. Orbit maps $E/\Gamma' \longrightarrow E/\Gamma$ for actions of a finite group Γ on a space E and $\Gamma' \subset \Gamma$. They can be considered as $[\Gamma : \Gamma']$-fold ramified covering

maps by taking $\mu(e\Gamma') = [\Gamma_e : \Gamma'_e]$, where Γ_e and Γ'_e denote the isotropy subgroups of $e \in E$ for the action of Γ and the restricted action of Γ', and $[\Gamma : \Gamma']$ and $[\Gamma_e : \Gamma'_e]$ denote the corresponding indexes. In fact, Dold [6] proves that all ramified covering maps are of this form for $\Gamma = \Sigma_n$ and $\Gamma' = \Sigma_{n-1}$.

3. *Branched covering maps* on manifolds, namely open maps $p : M^d \longrightarrow N^d$, where M^d and N^d are orientable closed topological manifolds of dimension d, p has finite fibers and its degree is n. Indeed, Berstein and Edmonds [3] prove that p is of the form $E/\Gamma' \longrightarrow E/\Gamma$, with $[\Gamma : \Gamma'] = n$, so that, by 2., p is in fact an n-fold ramified covering map. An interesting special case of this is given by Montesinos [15] and Hilden [10], who show that for any closed orientable 3-manifold M^3, there is a branched covering map $p : M^3 \longrightarrow \mathbb{S}^3$ of degree 3.

4. It will be of particular interest to consider the following example. Let B be a space and $\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n B$, where $\overline{n} = \{1, 2, \ldots, n\}$ and $\times_{\Sigma_n}$ represents the twisted product, be given by $\pi_B\langle b_1, b_2, \ldots, b_n; i\rangle = \langle b_1, b_2, \ldots, b_n\rangle$. Then π_B is an n-fold ramified covering map with multiplicity function $\mu_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ given by $\mu_B\langle b_1, b_2, \ldots, b_n; i\rangle = \#\{j \mid b_j = b_i\}$ (see [16]).

4 The homology transfer

We shall define now the homolgy transfer. Our spaces in this section will be compactly generated weak Hausdorff spaces.

Definition 4.1. Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ. Define the *pretransfer*

$$t_p : F(X, G) \longrightarrow F(E, G) \quad \text{by} \quad t_p(u) = \widetilde{u}\,,$$

where $\widetilde{u}(e) = \mu(e)u(p(e))$. In other words, if $u = \sum_{i=1}^n g_i x_i \in F(X, G)$, then

$$t_p(u) = \sum_{\substack{p(e)=x_i \\ i=1,\ldots,n}} \mu(e) g_i e\,.$$

Remark 4.2. The pretransfer $t_p : F(X, G) \longrightarrow F(E, G)$ is clearly a homomorphism of topological groups and it is thus convenient to see what it does to generators. Namely, if gx is the function in $F(X,G)$ such that it is zero everywhere, with the exception of x, where its value is g, then it is a generator and the pretransfer satisfies

$$t_p(gx)(e) = \mu(e)gx(p(e)) = \begin{cases} \mu(e)g & \text{if } p(e) = x, \text{ i.e., if } e \in p^{-1}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the only points where $t_p(gx)$ is nonzero are the elements of $p^{-1}(x) = \{e_1, e_2, \dots, e_r\}$, that is,

$$t_p(gx)(e_1) = \mu(e_1)g\,,\ t_p(gx)(e_2) = \mu(e_2)g\,, \dots,\ t_p(gx)(e_r) = \mu(e_r)g\,,$$

and thus

$$t_p(gx) = \mu(e_1)ge_1 + \mu(e_2)ge_2 + \cdots + \mu(e_r)ge_r\,.$$

We shall prove below that t_p is continuous. Hence, on homotopy groups, the map t_p induces the *homolgy transfer*

$$\tau_p : \widetilde{H}_q(X;G) \longrightarrow \widetilde{H}_q(E;G)\,.$$

We have the following.

Proposition 4.3. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, where E and X are pointed spaces. Then the pretransfer $t_p : F(X,G) \longrightarrow F(E,G)$ is continuous.*

Proof: Since $F(X,G)$ has the topology of the union of the subspaces

$$\cdots \subset F_r(X,G) \subset F_{r+1}(X,G) \subset \cdots \subset F(X,G)\,,$$

t_p is continuous if and only if the restriction $t_p|_{F_r(X,G)}$ is continuous for each $r \in \mathbb{N}$. Denote by $k(X \times Y)$ the product of X and Y with the compactly generated topology. Then we have a quotient map $q_r : k((G \times X)^r) \twoheadrightarrow F_r(X,G)$ for each r. Define $\delta : G \times X \longrightarrow F_n(X,G)$ by $\delta(g,x) = t_p q_1(g,x) = t_p(gx)$, and $\alpha : G \times X \longrightarrow (G \times X)^n/\Sigma_n$ by

$$\alpha(g,x) = [\underbrace{(g,e_1),\dots,(g,e_1)}_{\mu(e_1)},\dots,\underbrace{(g,e_m),\dots,(g,e_m)}_{\mu(e_m)}]\,,$$

where $p^{-1}(x) = \{e_1,\dots,e_m\}$. For each $g \in G$, let $i_g : X \longrightarrow G \times X$ be given by $i_g(x) = (x,g)$, and let $j_g : E^n/\Sigma_n \longrightarrow (G \times X)^n/\Sigma_n$ be given by

$j_g[e_1,\dots,e_n]=[(g,e_1),\dots,(g,e_n)]$. Then $\alpha\circ i_g=j_g\circ\varphi_p$, where $\varphi_p:X\longrightarrow E^n/\Sigma_n$. Since j_g and φ_p are continuous and G is discrete, α is continuous and $k(\alpha):k(G\times X)\longrightarrow k((G\times X)^n/\Sigma_n)$ is also continuous. Since $G\times X$ is compactly generated, $k(G\times X)=G\times X$. There is a natural homeomorphism $k((G\times X)^n/\Sigma_n)\approx k((G\times X)^n)/\Sigma_n$ (indeed, it is straightforward to show that the orbit space of the action of a finite group on a compactly generated weak Hausdorff space is again a compactly generated weak Hausdorff space). Therefore, the map $k(\alpha):G\times X\longrightarrow k((G\times X)^n)/\Sigma_n$ is continuous.

The quotient map q_n factors through the quotient map $q_n':k(G\times X)^n\twoheadrightarrow k((G\times X)^n)/\Sigma_n$, yielding the following commutative diagram,

$$\begin{array}{ccc} k((G\times X)^n) & \xrightarrow{q_n'} & k((G\times X)^n)/\Sigma_n \\ {\scriptstyle q_n}\downarrow & \swarrow{\scriptstyle \rho_n} & \\ F_n(X,G)\,, & & \end{array}$$

where ρ_n is also a quotient map.

Now, δ makes the following diagram commute,

$$\begin{array}{ccc} & & k((G\times X)^n)/\Sigma_n \\ & \nearrow{\scriptstyle k(\alpha)} & \downarrow{\scriptstyle \rho_n} \\ G\times X & \xrightarrow[\delta]{} & F_n(X,G)\,, \end{array}$$

therefore, δ is continuous.

In order to show that $t_p|_{F_r(X,G)}$ is continuous, consider the diagram

$$\begin{array}{ccc} k((G\times X)^r) & \xrightarrow{k(\delta^r)} & k(F_n(E,G)\times\cdots\times F_n(E,G)) \\ {\scriptstyle q_r}\downarrow & & \downarrow{\scriptstyle \Sigma_{i=1}^r} \\ F_r(X,G) & \xrightarrow[t_p|_{F_r(X,G)}]{} & F(E,G)\,, \end{array}$$

where $\sum_{i=1}^r$ is the operation in $F(E,G)$, which is a topological abelian group in the compactly generated topology, and hence it is continuous. Since also δ is continuous, and q_r is a quotient map, $t_p|_{F_r(X,G)}$ is continuous. ■

Corollary 4.4. *Let $p:E\longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu:E\longrightarrow\mathbb{N}$, where E and X are pointed* CW*-complexes. Then there is a homology transfer $\tau_p:\widetilde{H}_q(X;G)\longrightarrow\widetilde{H}_q(E;G)$.* ■

REMARK 4.5. Besides the transfer τ_p defined above, for every integer k there is another homology transfer ${}_k\tau$ given by $({}_k\tau)_p(\xi) = k \cdot \tau_p(\xi)$, $\xi \in H_q(X;G)$. This transfer, in turn, is determined by the pretransfer $({}_kt)_p : F(X,G) \longrightarrow F(E,G)$ given by $({}_kt)_p(u) = k \cdot t_p(u)$, $u \in F(X,G)$.

EXAMPLE 4.6. For the ramified covering map $\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n B$ of 3.3, the homology transfer is given as follows. We first compute

$$t_{\pi_B} : F(\mathrm{SP}^n B, G) \longrightarrow F(B^n \times_{\Sigma_n} \overline{n}, G)$$

on the generators. Set

$$b = (\underbrace{b_1, \ldots, b_1}_{i_1}, \underbrace{b_2, \ldots, b_2}_{i_2}, \ldots, \underbrace{b_r, \ldots, b_r}_{i_r}) \in B^n\,,$$

where $i_1 + i_2 + \cdots i_r = n$. Then

$$\pi_B^{-1}\langle b\rangle = \{\langle b_1, i_1\rangle, \langle b_2, i_1 + i_2,\rangle \ldots, \langle b_r, n\rangle\}\,.$$

Therefore,

$$\begin{aligned}
t_{\pi_B}(g\langle x\rangle) &= \mu\langle b, i_1\rangle g\langle b, i_1\rangle + \mu\langle b, i_1+i_2\rangle g\langle b, i_1+i_2\rangle + \cdots + \\
&\quad + \mu\langle b, i_1+i_2+\cdots+i_r\rangle g\langle b, i_1+i_2+\cdots i_r\rangle \\
&= i_1 g\langle b, i_1\rangle + i_2 g\langle b, i_1+i_2\rangle + \cdots + i_r g\langle b, i_1+i_2+\cdots i_r\rangle \\
&= g\,\underbrace{\langle b, i_1\rangle + \langle b, i_1\rangle + \cdots \langle b, i_1\rangle}_{i_1} + \\
&\quad + \underbrace{\langle b, i_1+i_2\rangle + \langle b, i_1+i_2\rangle + \cdots + \langle b, i_1+i_2\rangle}_{i_2} + \cdots + \\
&\quad + \underbrace{\langle b, n\rangle + \langle b, n\rangle + \cdots + \langle b, n\rangle}_{i_r} \\
&= g\langle b, 1\rangle + \cdots + \langle b, i_1\rangle + \langle b, i_1+1\rangle + \ldots \langle b, i_1+i_2\rangle + \cdots + \\
&\quad + \langle b, i_1+i_2+\cdots+i_{r-1}+\cdots+1\rangle + \langle b, n\rangle \\
&= g\langle b, 1\rangle + g\langle b, 2\rangle + \cdots + g\langle b, n\rangle\,,
\end{aligned}$$

hence

$$t_{\pi_B}(g\langle b_1, \ldots, b_n\rangle) = g\langle b_1, \ldots, b_n; 1\rangle) + \cdots + g\langle b_1, \ldots, b_n; n\rangle)\,. \tag{4.7}$$

Thus, in general, if $\beta = \sum_{i=1}^{k} g_i\langle b_1^i, \dots, b_n^i\rangle$, then

$$t_{\pi_B}(\beta) = \sum_{(i,l)=(1,1)}^{(k,n)} g_i\langle b_1^i, \dots, b_n^i; l\rangle\,,$$

since by varying l from 1 to n, the fiber elements over $\langle b_1^i, \dots, b_n^i\rangle$, namely $\langle b_1^i, \dots, b_n^i; l\rangle$, are repeated $\mu_B\langle b_1^i, \dots, b_n^i; l\rangle$ times.

Remark 4.8. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, and a (closed) subspace $A \subset X$, we have the *restricted ramified covering map* $p_A : E_A \longrightarrow A$, $E_A = p^A$, and the *quotient ramified covering map* $\overline{p} : \overline{E} \longrightarrow X/A$, as described in Remark 3.2. The following diagram obviously commutes:

$$\begin{array}{ccccc}
E_A & \hookrightarrow & E & \twoheadrightarrow & \overline{E} \\
{\scriptstyle p_A}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\
A & \hookrightarrow & X & \twoheadrightarrow & X/A\,.
\end{array}$$

Thus the diagram above yields

$$\begin{array}{ccccc}
F(A,G) & \longrightarrow & F(X,G) & \longrightarrow & F(X/A,G) \\
{\scriptstyle t_A}\downarrow & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \overline{t}} \\
F(E_A,G) & \longrightarrow & F(E,G) & \longrightarrow & F(\overline{E},G)\,,
\end{array}$$

where the horizontal arrows are obvious and t_A, t, and $\overline{t}$ are the corresponding pretransfers. Therefore, using $\overline{t}$, we have a *relative homology transfer* $\tau_p : H_n(X, A; G) \longrightarrow H_n(E, E_A; G)$, and by the commutativity of the diagram, also this transfer maps the long exact sequences of (X, A) into the long exact sequence of (E, E_A), provided that the inclusion $A \hookrightarrow X$ is a closed cofibration (in general it is also true by constructing an adequate ramified covering over $X \cup CA$).

The following theorems establish the fundamental properties of the transfer.

Theorem 4.9. *The composite*

$$p_* \circ \tau_p : \widetilde{H}_n(X; G) \longrightarrow \widetilde{H}_n(X; G)$$

is multiplication by n.

The *proof* follows immediately from the following proposition.

Proposition 4.10. *If $p : E \longrightarrow X$ is an n-fold ramified covering map, then the composite*

$$F(X,G) \xrightarrow{t_p} F(E,G) \xrightarrow{p_*} F(X,G)$$

is multiplication by n.

Proof: If $u = \sum_{i=1}^n g_i x_i \in F(X,G)$, then $p_* t_p(u) = p_* t_p\left(\sum_{i=1}^n g_i x_i\right) = \sum_{p(e)=x_i,\, i=1,\ldots,n} \mu(e) g_i x_i = \sum_{i=1}^n g_i x_i \sum_{p(e)=x_i} \mu(e) = n \sum_{i=1}^n g_i x_i = n \cdot u$. ∎

The invariance under pullbacks is given by the following.

Theorem 4.11. *Assume that $F : X \longrightarrow Y$ is continuous and that $g : G \longrightarrow H$ is a homomorphism of discrete abelian groups. Then the following diagram commutes:*

$$\begin{array}{ccc} H_q(Y;G) & \xrightarrow{\tau_{F^*(p)}} & H_q(F^*(E);G) \\ {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle \widetilde{F}_*} \\ H_q(X;G) & \xrightarrow[\tau_p]{} & H_q(E;G)\,, \end{array}$$

where $F^*(E) \longrightarrow Y$ is the n-fold ramified covering map induced by $p : E \longrightarrow X$ over F.

As for the previous theorem, the *proof* follows immediately from the next proposition.

Proposition 4.12. *If $p : E \longrightarrow X$ is an n-fold ramified covering map and $F : X \longrightarrow Y$ is continuous, then the following diagram commutes.*

$$\begin{array}{ccc} F(Y,G) & \xrightarrow{t_{F^*(p)}} & F(F^*(E),G) \\ {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle \widetilde{F}_*} \\ F(X,G) & \xrightarrow[t_p]{} & F(E,G)\,. \end{array}$$

Proof: Let $v = \sum_{i=1}^n g_i y_i \in F(Y,G)$. Then $t_{F^*(p)}(v) \in F(F^*(E),G)$ is such that

$$\begin{aligned}
\widetilde{F}_*\left(t_{F^*(p)}(v)\right) &= \widetilde{F}_*\left(\sum_{\substack{F_*(p)(y,e)=y_i\\ i=1,\dots,n}} F_*(\mu)(y,e)g_i(y,e)\right)\\
&= \sum_{\substack{F_*(p)(y,e)=y_i\\ i=1,\dots,n}} \mu(e)g_i\widetilde{F}(y,e)\\
&= \sum_{\substack{p(e)=F(y_i)\\ i=1,\dots n}} \mu(e)g_i e\\
&= t_p\left(F_*(v)\right)\,.
\end{aligned}$$

■

One further property of the homology transfer that is useful is the following.

Proposition 4.13. *Let $f : B \longrightarrow C$ be continuous and consider the commutative diagram*

(4.14)
$$\begin{array}{ccc}
B^n \times_{\Sigma_n} \overline{n} & \xrightarrow{f^n \times_{\Sigma_n} 1_{\overline{n}}} & C^n \times_{\Sigma_n} \overline{n}\\
{\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_C}\\
\mathrm{SP}^n B & \xrightarrow[\mathrm{SP}^n f]{} & \mathrm{SP}^n C\,.
\end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc}
F(B^n \times_{\Sigma_n} \overline{n}, G) & \xrightarrow{(f^n \times_{\Sigma_n} 1_{\overline{n}})_*} & F(C^n \times_{\Sigma_n} \overline{n}, G)\\
{\scriptstyle t_{\pi_B}}\uparrow & & \uparrow{\scriptstyle t_{\pi_C}}\\
F(\mathrm{SP}^n B, G) & \xrightarrow[(\mathrm{SP}^n f)_*]{} & F(\mathrm{SP}^n C, G)\,.
\end{array}$$

The *proof* is fairly routinary and follows easily using the description of the transfers given in Example 4.6. ■

In 4.10 we computed the composite $p_* \circ t_p$. The opposite composite $t_p \circ p_*$ is also interesting. An immediate computation yields the following.

Proposition 4.15. *Let $p : E \longrightarrow X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$F(E,G) \xrightarrow{p_*} F(X,G) \xrightarrow{t_p} F(E,G)$$

is given by

$$t_p p_*(v)(e) = \sum_{p(e')=p(e)} \mu(e')v(e')\,,$$

for any $v \in F(E,G)$. ∎

In the case of an action of a finite group Γ on E and $X = E/\Gamma$, we have the following consequence.

Corollary 4.16. *For $v \in F(E,G)$ one has $t_p p_*(v)(e) = \sum_{\gamma\in\Gamma} v(\gamma e)$. Therefore, the composite*

$$F(E/\Gamma,G) \xrightarrow{p_*} F(E,G) \xrightarrow{t_p} F(E/\Gamma,G)$$

is given by $t_p p_*(v) = \sum_{\gamma\in\Gamma} \gamma_*(v)$.

Proof. Just observe that the element γe is repeated in the sum $\mu(e) = |\Gamma_e|$ times. □

The two previous results yield the following in homology.

Theorem 4.17. *Let $p : E \longrightarrow X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$H_q(E;G) \xrightarrow{p_*} H_q(X;G) \xrightarrow{\tau_p} H_q(E;G)$$

is given by $\tau_p p_*(y) = y'$, where $y' = [v'] \in \pi_q(F(E,G))$, and

$$v'(s)(e) = \sum_{p(e')=p(e)} \mu(e')v(s)(e')$$

where $y = [v] \in \pi_q(F(E,G))$ and $s \in \mathbb{S}^q$. ∎

Corollary 4.18. *For an action of a finite group Γ on E and $X = E/\Gamma$ one has that the composite*

$$H_q(E;G) \xrightarrow{p_*} H_q(E/\Gamma;G) \xrightarrow{\tau_p} H_q(E;G)$$

is given by $\tau_p p_*(y) = \sum_{\gamma\in\Gamma} \gamma_*(y)$.

Remark 4.19. Considering an action of H on E and a subgroup $K \subset H$, one has different ramified covering maps as depicted in

$$\begin{array}{ccc} & E & \\ {}^{q'_\Gamma}\swarrow & & \searrow^{q_\Gamma} \\ E/\Gamma' & \xrightarrow[q_\Gamma^{\Gamma'}]{} & E/\Gamma\,. \end{array}$$

One may easily compute several combinations of the maps induced by these covering maps and their transfers.

Another interesting property of the transfer is the relationship given by computing the transfer of the composition of two ramified covering maps. Before giving it we need the following.

Definition 4.20. Let $p : Y \longrightarrow X$ be an n-fold ramified covering map, with multiplicity function $\mu : Y \longrightarrow \mathbb{N}$ and let $q : Z \longrightarrow Y$ be an m-fold ramified covering map, with multiplicity function $\nu : Z \longrightarrow \mathbb{N}$. Then the composite $p \circ q : Z \longrightarrow X$ is an mn-fold ramified covering map, with multiplicity function $\xi : Z \longrightarrow \mathbb{N}$ given by $\xi(z) = \nu(z)\mu(q(z))$. In order to verify that this composite is indeed an mn-fold ramified covering map, consider the *wreath product* $\Sigma_n \wr \Sigma_m$, defined as the semidirect product of Σ_n and $(\Sigma_m)^n$, where Σ_n acts on $(\Sigma_m)^n$ by permuting the n factors. We have an action $(Z^m \times \cdots \times Z^m) \times \Sigma_n \wr \Sigma_m \longrightarrow Z^m \times \cdots \times Z^m$ given by $(\zeta_1, \ldots, \zeta_n) \cdot (\sigma, \tau_1, \ldots, \tau_n) = (\zeta_{\sigma(1)} \cdot \tau_1, \ldots, \zeta_{\sigma(n)} \cdot \tau_n)$, where $\zeta_i \in Z^m$. Then we have the following diagram, where all maps are open

$$\begin{array}{ccc} Z^m \times \cdots \times Z^m & \xrightarrow{q \times \cdots \times q} & Z^m/\Sigma_m \times \cdots \times Z^m/\Sigma_m \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ (Z^m)^n/\Sigma_n \wr \Sigma_m & \dashrightarrow & \mathrm{SP}^n(\mathrm{SP}^m Z)\,. \end{array}$$

One may easily show that π is compatible with $\pi' \circ (q \times \cdots \times q)$. Therefore, there is a homeomorphism $X^{mn}/\Sigma_n \wr \Sigma_m \approx \mathrm{SP}^n(\mathrm{SP}^m Z)$ and hence one has a canonical quotient map $\rho : \mathrm{SP}^n(\mathrm{SP}^m Z) \longrightarrow \mathrm{SP}^{mn} Z$. Then one can easily verify that $\varphi_{p \circ q} = \rho \circ \mathrm{SP}^n(\varphi_q) \circ \varphi_p : X \longrightarrow \mathrm{SP}^n(\mathrm{SP}^m Z) \xrightarrow{\rho} \mathrm{SP}^{mn} Z$. Thus $\varphi_{p \circ q}$ is continuous.

The homology transfer behaves well with respect to composite ramified covering maps.

Theorem 4.21. *The following hold:*

$$t_{p\circ q} = t_q \circ t_p : F(X;G) \xrightarrow{t_p} F(Y;G) \xrightarrow{t_q} F(Z;G)\,;$$

$$\tau_{p\circ q} = \tau_q \circ \tau_p : H_k(X;G) \xrightarrow{\tau_p} H_k(Y;G) \xrightarrow{\tau_q} H_k(Z;G)\,.$$

Proof: As before, the second formula follows from the first. So, if $u \in F(X;G)$, $v \in F(Y;G)$, $w \in F(Z;G)$, then $v = t_p(u)$ if $v(y) = \mu(y)u(p(y))$, and $w = t_q(v)$ if $w(z) = \nu(z)v(q(z))$. Hence $(t_q t_p(u))(z) = t_q(\nu(z)v(q(z)) = \nu(z)\mu(q(z))u(pq(z)) = \xi(z)u((p\circ q)(z)) = t_{p\circ q}(u)(z)$. ■

Corollary 4.22. *Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ and an integer l, there is an ln-fold ramified covering map $p_l : E \longrightarrow X$ such that $p_l = p$ and $\mu_l(e) = l\mu(e)$, $e \in E$. Then $t_{p_l} = lt_p : F(X;G) \longrightarrow F(E;G)$ and $\tau_{p_l} = l\tau_p : H_k(X;G) \longrightarrow H_k(E;G)$.*

Proof: Consider the l-fold ramified covering map $q : E \longrightarrow E$ such that $q = \mathrm{id}_E$ and $\nu(e) = l$ for all $e \in E$. Hence $p_l = p \circ q$. Then apply Theorem 4.21. ■

REMARK 4.23. The ln-fold covering map p_l obtained from p is a sort of spurious ramified covering map, since the multiplicity of p is artificially multiplied by l. It is interesting to remark that the previous result shows that the transfer of this new ramified covering map p_l is just the corresponding multiple of the transfer of the original ramified covering map p. Thus on this sort of artificial ramified covering maps, the transfer remains essentially unchanged.

5 THE COHOMOLOGY TRANSFER

In this section we define the chomology transfer and prove some of its properties.

Definition 5.1. Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ, where E and X are compactly generated weak Hausdorff spaces of the same homotopy type of CW-complexes. Define its *cohomology transfer*

$$\tau^p : H^q(E;G) = [E, F(\mathbb{S}^q, G)] \longrightarrow [X, F(\mathbb{S}^q, G)] = H^q(X;G)$$

by $\tau^p([\widetilde{\alpha}]) = [\alpha]$, where $\alpha(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}(e)$, $x \in X$. To see that the map α is continuous and that its homotopy class depends only on the homotopy class of $\widetilde{\alpha}$, observe that α is given by the composite

$$\alpha : X \xrightarrow{\varphi_p} \mathrm{SP}^n E \xrightarrow{\mathrm{SP}^n \widetilde{\alpha}} \mathrm{SP}^n F(\mathbb{S}^q, G) \longrightarrow F(\mathbb{S}^q, G)\,,$$

where the last map is given by the group structure on $F(\mathbb{S}^q, G)$. Using the fact that X has the homotopy type of a CW-complex, similar arguments to those used in the proof of 4.3 show that α is continuous.

We might write τ^p_n instead of τ^p when we wish to remark the multiplicity n of the ramified covering map p.

REMARK 5.2. We might assume that E and X are paracompact spaces instead of compactly generated weak Hausdorff spaces of the same homotopy type of a CW-complex. In this case, the same definition yields a transfer that is a homomorphism between Čech cohomology groups

$$\tau^p : \check{H}^q(E; G) \longrightarrow \check{H}^q(X; G)\,,$$

(see Remark 2.7), provided that G is an at most countable coefficient group.

NOTE 5.3. In order to define the cohomology transfer, the only property of the Eilenberg-Mac Lane spaces given by $F(\mathbb{S}^q, G)$ required, is the fact that they are (weak) topological abelian groups.

Similarly to the homology transfer, the cohomology transfer has the following fundamental properties.

Theorem 5.4. *The composite*

$$\tau^p_n \circ p^* : H^k(X; G) \longrightarrow H^k(X; G)$$

is multiplication by n.

Proof: If $[\alpha] \in [X, F(\mathbb{S}^k, G)]$, then $\tau^p p^*(\alpha) = \tau^p(\alpha \circ p) : X \longrightarrow F(\mathbb{S}^k, G)$, and $\tau^p(\alpha \circ p)(x) = \sum_{p(e)=x} \mu(e)\alpha p(e) = \left(\sum_{p(e)=x} \mu(e)\right) \alpha(x) = n \cdot \alpha(x)$. Thus $\tau^p p^*([\alpha]) = n \cdot [\alpha]$. ∎

Theorem 5.5. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map and assume that $F : Y \longrightarrow X$ is continuous. Then the following diagram commutes:*

$$\begin{array}{ccc} H^q(E;G) & \xrightarrow{\tau^p} & H^q(X;G) \\ {\scriptstyle \widetilde{F}^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ H^q(F^*(E);G) & \xrightarrow[\tau^{F^*(p)}]{} & H^q(Y;G)\,, \end{array}$$

where $F^*(p) : F^*(E) \longrightarrow Y$ is the n-fold ramified covering map induced by $p : E \longrightarrow X$ over F.

Proof: Let $\widetilde{\alpha} : E \longrightarrow F(\mathbb{S}^q, G)$ represent an element in $H^q(E;G)$. Then the map

$$y \longmapsto \sum_{F_*(p)(y,e)=y} F^*(\mu)(y,e)\widetilde{\alpha}(y,e) = \sum_{p(e)=F(y)} \mu(e)\widetilde{\alpha}(y,e)$$

that represents $\tau^{F^*(p)}\widetilde{F}^*(\widetilde{\alpha})$, clearly represents also $F^*\tau^p([\widetilde{\alpha}]) \in H^q(Y;G)$. ∎

In 5.4 we computed the composite $\tau^p \circ p^*$. The opposite composite $p^* \circ \tau^p$ is also interesting. As it was the case for the homology transfer, an immediate computation yields the following results for the cohomology transfer.

Proposition 5.6. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$H^q(E;G) \xrightarrow{\tau^p} H^q(X;G) \xrightarrow{p^*} H^q(E;G)$$

is given as follows. Take $[\varphi] \in H^q(E;G) = [E, F(\mathbb{S}^q, G)]$, then $p^*\tau^p[\varphi]$ is represented by the map $\varphi' : E \longrightarrow F(\mathbb{S}^q, G)$ given by

$$\varphi'(e) = \sum_{p(e')=p(e)} \mu(e')e'\,.$$

∎

In the case of an action of a finite group Γ on E and $X = E/\Gamma$, we have the following consequence.

Corollary 5.7. *If $\xi \in H^q(E;G)$, then*

$$p^*\tau^p(\xi) = \sum_{\gamma\in\Gamma} \gamma^*(\xi) \in H^q(E;G),.$$

Proof: Just observe that in the sum the element $\gamma^*(\xi)$ is repeated $\mu(e) = |\Gamma_e|$ times. ■

Generalizations and further properties of the cohomology transfer are studied in [2].

6 Some applications of the transfers

First we start considering a standard n-fold covering map $p : E \longrightarrow X$. In this case, the pretransfer (and thus also the transfer in homology) has a particularly nice definition. Since the multiplicity function $\mu : E \longrightarrow \mathbb{N}$ is constant $\mu(e) = 1$, the transfer $t_p : F(X, G) \longrightarrow F(E, G)$ is given by

$$t_p(u)(e) = u(p(e))\,. \tag{6.1}$$

This fact has a nice consequence.

Theorem 6.2. *Let Γ be a finite group acting freely on a Hausdorff space E. Then the orbit map $p : E \longrightarrow E/\Gamma$ is a standard covering map, and its pretransfer induces an isomorphism*

$$t_p : F(E/\Gamma, G) \xrightarrow{\cong} F(E, G)^\Gamma\,,$$

where the second term represents the fixed points under the induced Γ-action on $F(E, G)$. Consequently, the pretransfer yields an isomorphism

$$H_q(E/\Gamma; G) \xrightarrow{\cong} \pi_q(F(E, G)^\Gamma)\,,$$

for all q.

Proof. We assume that the projection $p : E \longrightarrow E/\Gamma$ maps the base point to the base point. The pretransfer t_p is a monomorphism. Namely, if $t_p(u) = 0$, then, by (6.1), $u(p(e)) = t_p(u)(e) = 0$ for all $e \in E$. Since p is surjective, $u = 0$.

On the other hand, obviously $t_p(u) \in F(E, G)^\Gamma$ for all $u \in F(E/\Gamma, G)$. To see that it is an epimorphism, take any $v \in F(E, G)^\Gamma$. Then $v(e) = v(e\gamma)$ for all $\gamma \in \Gamma$, and thus v determines a well-defined element $u \in F(E/\Gamma, G)$ by $u(e\Gamma) = v(e)$. Then clearly $t_p(u) = v$. □

In what follows, we use the fundamental properties 4.9 and 4.18, and 5.4 and 5.7 of both the homology and the cohomology transfers to prove some results about the homology and cohomology of orbit maps between orbit spaces of the action of a topological group Γ and a subgroup Γ' of finite index on a compactly generated weak Hausdorff space of the same homotopy type of a CW-complex (and a corresponding result in Čech cohomology for a paracompact space).

Before starting we need to recall Dold's definition of an n-fold ramified covering map [6]. It is a finite-to-one map $p : E \longrightarrow X$ together with a continuous map $\psi_p : X \longrightarrow \mathrm{SP}^n E$ such that

(i) for every $e \in E$, e appears in the n-tuple $\psi_p(p(e)) = \langle e_1, \ldots, e_n \rangle$, and

(ii) $\mathrm{SP}^n(p)\psi_p(x) = \langle x, \ldots, x \rangle \in \mathrm{SP}^n X$.

This definition is equivalent to Smith's (see 3.1), by setting $\varphi_p = \psi_p$ and defining $\mu(e)$ as the number of times that e is repeated in $\psi_p(p(e))$.

We have the following interesting result.

Proposition 6.3. *Let Γ be a topological group acting on a space Y on the right and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Then the orbit map $p : Y/\Gamma' \longrightarrow Y/\Gamma$ is an n-fold ramified covering map.*

Proof: There is a commutative diagram

$$\begin{array}{ccc} Y \times \Gamma & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y \times (\Gamma/\Gamma') & \xrightarrow[\nu]{} & Y/\Gamma', \end{array}$$

where the top map is the action and the vertical maps are the quotient maps. Take the adjoint map of ν, $\eta : Y \longrightarrow \mathrm{Map}(\Gamma/\Gamma', Y/\Gamma')$. The function space $\mathrm{Map}(\Gamma/\Gamma', Y/\Gamma')$ has a right Γ-action given as follows. For $f : \Gamma/\Gamma' \longrightarrow Y/\Gamma$, take $(f \cdot \gamma)[\gamma_1] = f(\gamma[\gamma_1]) = f[\gamma\gamma_1]$. The map η is then Γ-equivariant and thus induces a map

$$\overline{\eta} : Y/\Gamma \longrightarrow \mathrm{Map}(\Gamma/\Gamma', Y/\Gamma')/\Gamma\,.$$

On the other hand, if we identify Γ/Γ' with the set $\underline{n} = \{1, \ldots, n\}$, then we have a homeomorphism

$$\mathrm{Map}(\Gamma/\Gamma', Y/\Gamma')/\Gamma \approx \mathrm{Map}(\underline{n}, Y/\Gamma')/\Sigma_n = \mathrm{SP}^n(Y/\Gamma')\,.$$

Let $\psi_p : Y/\Gamma \longrightarrow \mathrm{SP}^n(Y/\Gamma')$ be $\overline{\eta}$ followed by the previous homeomorphism. Then ψ_p satisfies conditions (i) and (ii) and thus p is an n-fold ramified covering map. ∎

We apply the results 4.10 and 4.16 that we have for the pretransfer to the n-fold ramified covering described above to obtain the following.

Proposition 6.4. *Let Y be a space with an action of a topological group Γ and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : F(Y/\Gamma', R) \longrightarrow F(Y/\Gamma, R)$ is a split (continuous) epimorphism. Moreover, if Γ is finite and its order m is invertible in R, then the kernel of p_* is the complement in $F(Y/\Gamma', R)$ of the invariant subgroup $F(Y/\Gamma', R)^\Gamma$ under the induced action of Γ. Thus in this case*

$$F(Y/\Gamma, R) \cong F(Y/\Gamma', R)^\Gamma\,;$$

in particular, if Γ is finite and Γ' is trivial, then $m = n$ and

$$F(Y/\Gamma, R) \cong F(Y, R)^\Gamma\,.$$

Proof: By 4.10 applied to the n-fold ramified covering $p : Y/\Gamma' \longrightarrow Y/\Gamma$, $p_* \circ t_p : F(Y/\Gamma', R) \longrightarrow F(Y/\Gamma', R)$ is multiplication by n, hence it is an isomorphism, and consequently p_* is a split epimorphism. Moreover, if Γ is finite of order m, by 4.16, we have that $t_p \circ p_* : F(Y/\Gamma', R)^\Gamma \longrightarrow F(Y/\Gamma', R)^\Gamma$ is multiplication by m. So, if m is invertible in R, then $p_* : F(Y/\Gamma', R)^\Gamma \longrightarrow F(Y/\Gamma, R)$ is an isomorphism. ∎

As an immediate consequence of the result above, or applying 4.9 and 4.18, we obtain the following two well-known results (cf. [16, 2.5], [4], [5]).

Theorem 6.5. *Let Y be a space with an action of a topological group Γ and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : H_q(Y/\Gamma'; R) \longrightarrow H_q(Y/\Gamma; R)$ is a split epimorphism. Moreover, if Γ is finite and its order m is invertible in R, then the kernel of p_* is the complement of $H_q(Y/\Gamma'; R)^\Gamma$ in $H_q(Y/\Gamma'; R)$. Thus in this case*

$$H_q(Y/\Gamma; R) \cong H_q(Y/\Gamma'; R)^\Gamma\,;$$

and in particular,

$$H_q(Y/\Gamma; R) \cong H_q(Y; R)^\Gamma\,.$$

∎

Similarly, 5.4 and 5.7 one has for cohomology the following.

Theorem 6.6. *Let Y be a space with an action of a topological group Γ and let $\Gamma' \subset \Gamma$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p^* : H^q(Y/\Gamma; R) \longrightarrow H^q(Y/\Gamma'; R)$ is a split monomorphism. Moreover, if Γ is finite and its order m is invertible in R, then the image of p^* is $H^q(X; R)^\Gamma$. Thus in this case*

$$H^q(Y/\Gamma; R) \cong H^q(Y/\Gamma'; R)^\Gamma ;$$

and in particular,

$$H^q(Y/\Gamma; R) \cong H^q(Y; R)^\Gamma .$$

■

Remark 6.7. One may take a paracompact space Y with an action of a topological group Γ and obtain for Čech cohomology an analogous result, namely $p^* : \check{H}^q(Y/\Gamma; R) \longrightarrow \check{H}^q(Y/\Gamma'; R)$ is a split monomorphism, and

$$\check{H}^q(Y/\Gamma; R) \cong \check{H}^q(Y/\Gamma'; R)^\Gamma .$$

A nice application of the previous ideas is the following generalization of a well-known result of Grothendieck [9] (in the case $Y = E\Gamma$).

Theorem 6.8. *Let Γ be a compact Lie group and let Γ_1 be the component of $1 \in \Gamma$. Let R be a ring where $n = [\Gamma, \Gamma_1]$ is an invertible element. For an action of Γ on a topological space Y, one has*

$$H_q(Y/\Gamma; R) \cong H_q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

$$H^q(Y/\Gamma; R) \cong H^q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

$$\check{H}^q(Y/\Gamma; R) \cong \check{H}^q(Y/\Gamma_1; R)^{\Gamma/\Gamma_1},$$

the last two according to what kind of a space Y is. ■

7 Duality between the homology and cohomology transfers

In this section we compare the homology transfer with the cohomology transfer.

Given an n-fold ramified cover $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, we can extend it to the n-fold ramified covering map $p^+ : E^+ \longrightarrow X^+$ as explained in Remark 3.2. Consider the cohomology transfer

$$\tau^p : H^q(E;G) = \widetilde{H}^q(E^+;G) \longrightarrow \widetilde{H}^q(X^+;G) = H^q(X;G)\,,$$

and consider also the homology transfer

$$\tau_p : H_q(X;G) = \widetilde{H}_q(X^+;G) \longrightarrow \widetilde{H}_q(E^+;G) = H_q(E;G)$$

as given in Definition 4.1.

Theorem 7.1. *Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$ and E path connected, and let $\tau_p : H_q(X;R) \longrightarrow H_q(E;R)$ and $\tau^p : H^q(E;R) \longrightarrow H^q(X;R)$ be its homology and cohomolgy transfers. If $\xi \in H_q(X;G)$ and $\widetilde{\xi} \in H^q(E;G)$, then*

$$\langle \tau_p(\xi), \widetilde{\xi} \rangle_E = \langle \xi, \tau^p(\widetilde{\xi}) \rangle_X \in R\,,$$

for the Kronecker products for E and X, respectively, and R a commutative ring with 1 *(see* (2.12)*).*

Proof: We have to prove the commutativity of the following diagram:

$$\begin{array}{ccccc}
[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_* & \xrightarrow{\frown} & [X^+, F(\mathbb{S}^0, R)]_* & & \\
\uparrow{\scriptstyle \tau^p \times 1} & & & \searrow & \\
[E^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_* & & & & R \\
\downarrow{\scriptstyle 1 \times \tau_p} & & & \nearrow & \\
[E^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(E^+, R)]_* & \xrightarrow[\frown]{} & [E^+, F(\mathbb{S}^0, R)]_* & &
\end{array}$$

By the naturality of the construction of the pretransfers and the definition of the $\frown$-product (see Proposition 2.11), it is fairly easy to check that this commutativity follows from the commutativity of the following:

$$\begin{array}{ccccc}
[X^+, F(X^+, R)]_* & \longrightarrow & [\mathbb{S}^0, F(X^+, R)]_* & & \\
\uparrow{\scriptstyle \tau^p} & & & \searrow & \\
[E^+, F(X^+, R)]_* & & & & R \\
\downarrow{\scriptstyle \tau_p} & & & \nearrow & \\
[E^+, F(E^+, R)]_* & \longrightarrow & [\mathbb{S}^0, F(E^+, R)]_* & &
\end{array}$$

Let $\delta : E^+ \longrightarrow F(X^+, R)$ be given by $\delta(e) = \sum_{i=1}^{m(e)} r_i(e) x_i(e)$, $e \in E$. Chasing this element δ along the top of the diagram, one easily verifies that it maps to the element

$$d = \sum_{p(e)=x_{-1}} \mu(e) \sum_{i=1}^{m(e)} r_i(e)\,,$$

while chasing it along the bottom of the diagram, it maps to the element

$$d' = \sum_{i=1}^{m(e_{-1})} r_i(e_{-1}) \sum_{p(e_i)=x_i(e_{-1})} \mu(e_i) = n \sum_{i=1}^{m(e_{-1})} r_i(e_{-1})\,.$$

Call $\rho(e) = \sum_{i=1}^{m(e)} r_i(e)$. Since $\rho = \varepsilon \circ \delta$, by 2.8 this defines a continuous map $\rho : E \longrightarrow R$, but since E is path connected and R is discrete, ρ is constant with value $r_\delta \in R$. Hence

$$d = \sum_{p(e)=x_{-1}} \mu(e)\rho(e) = n \cdot r_\delta \quad \text{and} \quad d' = n\rho(e_{-1}) = n \cdot r_\delta\,.$$

Thus $d = d'$ and the diagram commutes. ■

For simplicity, in what follows we omit the coefficient ring R in homology and cohomology. For the Kronecker product $\langle -, - \rangle_Y : H^q(Y) \otimes H_q(Y) \longrightarrow R$ there are induced homomorphisms $\Phi_Y : H^q(Y) \longrightarrow \mathrm{Hom}(H_q(Y), R)$ and $\Psi_Y : H_q(Y) \longrightarrow \mathrm{Hom}(H^q(Y), R)$ for every space Y, given by $\Phi(y)(\eta) = \langle y, \eta \rangle_Y$ and $\Psi(\eta)(y) = \langle y, \eta \rangle_Y$, $y \in H^q(Y)$, $\eta \in H_q(Y)$.

Corollary 7.2. *The following diagrams commute*

$$\begin{array}{ccc} H^q(E) & \xrightarrow{\Phi_E} & \mathrm{Hom}(H_q(E), R) \\ \downarrow{\scriptstyle \tau^p} & & \downarrow{\scriptstyle \mathrm{Hom}(\tau_p, 1)} \\ H^q(X) & \xrightarrow[\Phi_X]{} & \mathrm{Hom}(H_q(X), R)\,, \end{array} \qquad \begin{array}{ccc} H_q(X) & \xrightarrow{\Psi_X} & \mathrm{Hom}(H^q(X), R) \\ \downarrow{\scriptstyle \tau_p} & & \downarrow{\scriptstyle \mathrm{Hom}(\tau^p, 1)} \\ H_q(E) & \xrightarrow[\Psi_E]{} & \mathrm{Hom}(H^q(E), R)\,, \end{array}$$

the one on the right-hand side only if $\tau^p : H^q(E) \longrightarrow H^q(X)$ is a homomorphism (which is rather seldom the case). ■

Remark 7.3. Under suitable conditions Φ or Ψ are isomorphisms, in whose case one of the transfers determines the other.

8 Comparison with Smith's transfer

In this section we show that the transfer defined in [16] coincides with ours if we take $\mathbb{Z}$-coefficients. To that end, we first recall his definition of the transfer. It makes use of a result of Moore, that we state below. Recall that the *weak product* $\widetilde{\prod}_{n=1}^{\infty} X_n$ of a family of pointed spaces is the colimit over n of the directed system of spaces

$$X_1 \hookrightarrow X_1 \times X_2 \hookrightarrow X_1 \times X_2 \times X_3 \hookrightarrow \cdots ,$$

where the inclusions are given by letting the last coordinate be the base point. Moore's result, as it appears in [18], is as follows.

Theorem 8.1. (Moore) *A connected space X is weakly homotopy equivalent to the weak product $\widetilde{\prod}_{n\geq 1} K(\pi_n(X), n)$ of Eilenberg-Mac Lane spaces if and only if the Hurewicz homomorphism $h_n : \pi_n(X) \longrightarrow \widetilde{H}_n(X;\mathbb{Z})$ is a split monomorphism for all $n \geq 1$.* ■

Suppose that $\rho_n : \widetilde{H}_n(X) = \widetilde{H}_n(X;\mathbb{Z}) \longrightarrow \pi_n(X)$ is a left inverse of h_n. The Kronecker product defined in Section 2 determines an epimorphism

$$\widetilde{H}^n(X;\pi_n(X)) \longrightarrow \mathrm{Hom}(\widetilde{H}_n(X), \pi_n(X)) .$$

Let $[\xi_n] \in \widetilde{H}^n(X;\pi_n(X)) = [X, K(\pi_n(X), n)]_*$ be some preimage of ρ_n. Then the family of pointed maps (ξ_n) defines the weak homotopy equivalence of the previous theorem.

Corollary 8.2. *If X is a connected topological abelian monoid of the same homotopy type of a* CW*-complex, then there is a homotopy equivalence $X \longrightarrow \widetilde{\prod}_{n\geq 1} K(\pi_n(X), n)$.*

Proof: Since X is a topological abelian monoid, there is a retraction $r : \mathrm{SP}^{\infty}X \longrightarrow X$ given by the retractions

$$r_n : \mathrm{SP}^n X \longrightarrow X , \qquad r_n\langle x_1, x_2, \ldots, x_n\rangle = x_1 + x_2 + \cdots + x_n .$$

Recall, on the other hand, that by the Dold-Thom theorem one has an isomorphism $\pi_n(\mathrm{SP}^{\infty}X) \cong \widetilde{H}_n(X)$, so that the inclusion $i : X \hookrightarrow \mathrm{SP}^{\infty}X$ defines the Hurewicz homomorphism (see [1]). Since $r \circ i = \mathrm{id}_X$, the homomorphism $\rho_n = r_* : \widetilde{H}_n(X) = \pi_n(\mathrm{SP}^{\infty}X) \longrightarrow \pi_n(X)$ provides a left inverse of the Hurewicz homomorphism h_n. Hence, by Moore's theorem, we obtain the result. ■

REMARK 8.3. Note that in the proof above, it is enough to assume that X is a *weak* topological abelian monoid, i.e., that the product in X is continuous on compact sets.

For any space E, the space $\mathrm{SP}^\infty E$ is a weak topological abelian monoid. Thus we have the following.

Corollary 8.4. *For a connected space E of the same homotopy type of a CW-complex, there is a natural homotopy equivalence $w_E : \mathrm{SP}^\infty E \longrightarrow K(\widetilde{H}_*(E)) = \prod_{n=1}^{\infty} K(\widetilde{H}_n(E), n)$.* ■

The definition of Smith's transfer is as follows. Given an n-fold ramified cover $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, consider the following composite:

$$\widehat{p} : X \xrightarrow{\varphi_p} \mathrm{SP}^n E \longrightarrow \mathrm{SP}^\infty E \xrightarrow{\simeq} K(\widetilde{H}_*(E))\,.$$

This map defines a family of elements $[\widehat{p}] \in \widetilde{H}^*(X; \widetilde{H}_*(E))$. On the other hand, the Kronecker product determines a homomorphism

$$\psi : \widetilde{H}^*(X; \widetilde{H}_*(E)) \longrightarrow \mathrm{Hom}(\widetilde{H}_*(X), \widetilde{H}_*(E))\,.$$

Smith's transfer is the image $p_\sharp : \widetilde{H}_*(X) \longrightarrow \widetilde{H}_*(E)$ of $[\widehat{p}]$ under the homomorphism ψ.

Theorem 8.5. *Let $p : E \longrightarrow X$ be an n-fold ramified cover with multiplicity function $\mu : E \longrightarrow \mathbb{N}$. Then $p_\sharp = \tau_p : \widetilde{H}_*(X; \mathbb{Z}) \longrightarrow \widetilde{H}_*(E; \mathbb{Z})$, where τ_p is the transfer in reduced homology.*

Proof: Consider the following commutative diagram.

$$\begin{array}{ccccccc}
[E, \mathrm{SP}^n E]_* & \longrightarrow & [E, \mathrm{SP}^\infty E]_* & \xrightarrow{\cong} & \widetilde{H}^*(E, \widetilde{H}_*(E)) & \longrightarrow & \mathrm{Hom}(\widetilde{H}_*(E), \widetilde{H}_*(E)) \\
\downarrow{\scriptstyle \tau^p} & & \downarrow{\scriptstyle \tau^p} & & \downarrow{\scriptstyle \tau^p} & & \downarrow{\scriptstyle \mathrm{Hom}(\tau_p, 1)} \\
[X, \mathrm{SP}^n E]_* & \longrightarrow & [X, \mathrm{SP}^\infty E]_* & \xrightarrow{\cong} & \widetilde{H}^*(X, \widetilde{H}_*(E)) & \longrightarrow & \mathrm{Hom}(\widetilde{H}_*(X), \widetilde{H}_*(E))
\end{array}$$

The two squares on the left-hand side, where τ^p represents the cohomology transfer, commute obviously. The one on the right-hand side commutes by Corollary 7.2. Take $[i] \in [E, \mathrm{SP}^n E]_*$, where $i : E \hookrightarrow \mathrm{SP}^n E$ is the canonical inclusion. Chasing $[i]$ down and then right on the bottom of the diagram, we obtain $p_\sharp$, while chasing it to the right on the top of the diagram and then down, we obtain τ_p. This is true, because the image of $[i]$ along the top row of the diagram is the identity homomorphism $1 \in \mathrm{Hom}(\widetilde{H}_*(E), \widetilde{H}_*(E))$. This follows from the naturality of the Kronecker product, since by Corollary 8.2, we have an explicit description of the weak homotopy equivalence that defines the isomorphism in the middle arrow. ■

References

[1] M. Aguilar, S. Gitler, C. Prieto, *Algebraic Topology from a Homotopical Viewpoint*, Universitexts, Springer-Verlag, New York Berlin Heidelberg 2002.

[2] M. Aguilar, C. Prieto, A classification of cohomology transfers for ramified coverings, 2003 (to appear).

[3] I. Berstein, A. L. Edmonds, The degree and branch set of a branched covering, *Inventiones Math.* **45** (1978), 213–220.

[4] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York and London 1972.

[5] T. tom Dieck, *Transformation Groups*, de Gruyter, Berlin, New York 1987

[6] A. Dold, Ramified coverings, orbit projections and symmetric powers, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 65–72.

[7] A. Dold, R. Thom, Quasifaserungen und unendliche symmetrische Produkte, *Annals of Math.* **67** (1958), 239–281.

[8] B. Eckmann, On complexes with operators, *Proc. Nat. Acad. Sci.* **39** (1953), 35–42

[9] A. Grothendieck, 'Sur quelques points d'algebre homologique' *Tôhoku Math. J.* 9 (1957), 119–221

[10] H. M. Hilden, Every closed orientable 3-manifold is a 3-fold branched covering space of $\mathbb{S}^3$, *Bull. Amer. Math. Soc.* **80** (1974), 1243–1244.

[11] P. J. Huber, Homotopical cohomology and Čech cohomology, *Math. Annalen* **144** (1961), 73–76.

[12] J. P. May, *A Concise Course in Algebraic Topology,* Chicago Lecture Notes in Mathematics, The University of Chicago Press, Chicago and London 1999.

[13] M. C. McCord, Classifying spaces and infinite symmetric products, *Trans. Amer. Math. Soc.* **146** (1969), 273–298.

[14] J. W. Milnor, The geometric realization of a semi-simplicial complex, *Annals of Math.* **65** (1957), 357–362

[15] J. M. Montesinos, Three-manifolds as 3-fold branched covers of $\mathbb{S}^3$, *Quart. J. Math. Oxford Ser.* (2), **27** (1976), 85–94

[16] L. Smith, Transfer and ramified coverings, *Math. Proc. Camb. Phil. Soc.* **93** (1983), 485–493.

[17] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math., Springer-Verlag, New York Heidelberg Berlin 1978.

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