A classification of cohomology transfers for ramified coverings*

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Abstract

We construct a cohomolgy transfer for n-fold ramified covering maps. Then, we define a very general concept of transfer for ramified covering maps and prove a classification theorem for these transfers. This generalizes Roush's classification of transfers for n-fold ordinary covering maps. We characterize those representable cofunctors which admit a family of transfers for ramified covering maps that have two naturality properties, as well as normalization and stability. This is analogous to Roush's characterization theorem for the case of ordinary covering maps. Finally, we classify these families of transfers and construct some examples. In particular, we extend the determinant function in $\mathrm{GL}(k,\mathbb{C})$ to a transfer.

1 Introduction

In [3], we defined a transfer for ramified covering maps in ordinary cohomology. We start this paper by giving a transfer homomorphism $t^p : h(E) = [E,\mathcal{H}] \longrightarrow h(X) = [X,\mathcal{H}]$ for any topological abelian monoid $\mathcal{H}$ and any ramified covering map $p : E \longrightarrow X$. In particular, if $\mathcal{H}$ is an Eilenberg-Mac Lane space (modelled by a topological abelian group), then we have

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the cohomology transfer. This transfer is an example of what we shall call (h,k)-transfers, where h and k are representable functors from the homotopy category of spaces to the category of sets, represented by spaces $\mathcal{H}$ and $\mathcal{K}$ (not necessarily topological abelian groups or H-spaces). We use the properties of the transfer in ordinary cohomology to define the concept of a general (h,k)-transfer for ramified covering maps. We give a classification of these transfers that extends the classification of transfers for ordinary covering maps given by Roush [8]. In particular, the set of (h,k)-transfers has a canonical group structure, when k is group-valued. Our results are applied to the study of transfer families and their classification and to conclude that there are (h,h)-transfers if and only if $\mathcal{H}$ is a weak product of Eilenberg-Mac Lane spaces. This is particularly interesting, since if one finds an (h,h)-transfer family, then it follows that $\mathcal{H}$ has to be a weak product of Eilenberg-Mac Lane spaces.

The structure of the paper is as follows. In Section 2 we recall the definition of a ramified covering map given by Smith [9] and define our (h,h)-transfer for $h(-)=[-,\mathcal{H}]$. In Section 3 we give the definition of a general (h,k)-transfer and study its properties. We prove that there is a one-to-one correspondence between (h,k)-transfers for n-fold ramified covering maps and elements in $k(\mathrm{SP}^n\mathcal{H})$. We prove further that there are nontrivial transfers for n-fold ramified covering maps in singular cohomology (for large n) only when the dimensions of the cohomology groups are the same, and that these transfers are classified by the integers. In Section 4 we compare our transfers with transfers for ordinary covering maps and prove that our classification extends Roush's classification. In Section 5 we consider families of (h,k)-transfers for n-fold ramified covering maps for all n and give their classification. Namely, we prove that there is a one-to-one correspondence between families of (h,k)-transfers for ramified covering maps and elements in $\lim_n k(\mathrm{SP}^n\mathcal{H})$. Analogously to Roush's characterization theorem for the case of ordinary covering maps, we give a characterization of those representable functors which admit a family of transfers. We also show that for singular cohomology, all transfers are determined by the transfers for 2-fold ramified covering maps. We finish the section by giving examples of transfers for functors that are not cohomology theories. In particular, we extend the determinant function $\det : \mathrm{GL}(k,\mathbb{C}) \longrightarrow \mathbb{C}^*$, which yields an element in $H^2(\mathrm{BGL}(k,\mathbb{C}))$, to a transfer for ramified covering maps $\tau : \mathrm{Vect}_k^{\mathbb{C}}(-) \longrightarrow \mathrm{Vect}_1^{\mathbb{C}}(-)$. Finally, in Section 6, we study transfers for $h(-)=k(-)=H^1(-;\mathbb{Z})$ and prove that the transfers for ordinary covering maps are the same as those for ramified covering maps, i.e., that in this case, one can extend in a unique way the transfers for ordinary covering maps to transfers for ramified covering maps. We conclude that for each n, the group of transfers for n-fold ramified covering maps in 1-cohomology is isomorphic to the group of transfers for ordinary covering maps, and both are isomorphic to $\mathbb{Z}$.

2 Transfers for n-fold ramified covering maps

We start by recalling L. Smith's definition of a ramified covering map (see [9]). We shall need the concept of nth symmetric power of Y defined by

$$\mathrm{SP}^n Y = \underbrace{Y \times \cdots \times Y}_{n} / \Sigma_n \,,$$

where Σ_n represents the nth symmetric group acting on the product $Y \times \cdots \times Y$ by permuting coordinates. It is sometimes convenient to view $\mathrm{SP}^n Y$ as $\left\{\sum_{i=1}^k m_i y_i \in F_n(Y,\mathbb{Z}) \mid m_i \geq 0,\, \sum_{i=1}^k m_i \leq n\right\}$, where $F_n(Y,\mathbb{Z})$ denotes the McCord classifying space (see [3] for a thorough discussion on this). Denote the elements of $\mathrm{SP}^n Y$ either by $\sum_{i=1}^k m_i y_i$, or by $\langle y_1', y_2', \ldots, y_n' \rangle$, where there are possible repetitions, for instance, $y_1' = \cdots = y_{m_1}' = y_1$, $y_{m_1+1}' = \cdots = y_{m_1+m_2}' = y_2, \ldots, y_{m_1+\cdots+m_{k-1}+1}' = \cdots = y_{n'}' = y_k$, $n' = \sum_{i=1}^k m_i$, although the order is irrelevant, since these elements are really nonempty sets with at most n members.

Definition 2.1. An *n-fold ramified covering map* is a continuous map $p : E \longrightarrow X$ together with a *multiplicity function* $\mu : E \longrightarrow \mathbb{N}$ such that the following hold:

(i) The fibers $p^{-1}(x)$ are finite (discrete), $x \in X$.

(ii) For each $x \in X$, $\sum_{e \in p^{-1}(x)} \mu(e) = n$.

(iii) The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E$ given by $\varphi_p(x) = \sum_{e \in p^{-1}(x)} \mu(e)e$ is continuous.

Remark 2.2. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ, one can construct an n-fold ramified covering map $p^+ : E^+ \longrightarrow X^+$, where $Y^+ = Y \sqcup \{*\}$ for any space Y and p^+ extends p by defining $p^+(*) = *$ and the multiplicity function μ^+ extends μ by setting

$\mu^+(*) = n$. More generally, given a (closed) subspace $A \subset X$, one can construct an n-fold ramified covering map $p' : E' \longrightarrow X/A$, where $E' = E/p^{-1}A$, p' is the map between quotients and the multiplicity function μ' coincides with μ off $p^{-1}A$ and is extended by setting $\mu'(*) = n$, if $*$ is the base point onto which $p^{-1}A$ collapses.

Another useful construction is the following. Let $\overline{E} = E \sqcup X$ and $\overline{p} : \overline{E} \longrightarrow X$ be such that $\overline{p}|_E = p$ and $\overline{p}|_X = \mathrm{id}_X$. Then $\overline{p}$ is an $(n+1)$-fold ramified covering map with the obvious multiplicity function.

On the other hand, given a map $F : Y \longrightarrow X$, one can construct the *induced n-fold ramified covering map* $F^*(p) : F^*(E) \longrightarrow Y$ by taking the pullback $F^*(E) = \{(y,e) \in Y \times E \mid F(y) = p(e)\}$ and $F^*(p) = \mathrm{proj}_Y$. The *induced multiplicity function* $F^*(\mu) : F^*(E) \longrightarrow \mathbb{Z}$ is given by $F^*(\mu)(y,e) = \mu(e)$. Call $\widetilde{F} : F^*(E) \longrightarrow E$ the projection proj_E.

Examples 2.3. Typical examples of these ramified covering maps are orbit maps $E \longrightarrow E/G$ of actions of a finite group G on a space E. They can be considered as $|G|$-fold ramified covering maps by taking $\mu(e) = |G_e|$, where G_e denotes the isotropy subgroup of $e \in E$ and $|H|$ denotes the order of a group H.

It will be of particular interest to consider the following example. Let B be a space and $\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n B$, where $\overline{n} = \{1, 2, \ldots, n\}$ and $\times_{\Sigma_n}$ represents the twisted product, be given by $\pi_B\langle b_1, b_2, \ldots, b_n; i\rangle = \langle b_1, b_2, \ldots, b_n\rangle$. Then π_B is an n-fold ramified covering map with multiplicity function $\mu_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{Z}$ given by $\mu_B\langle b_1, b_2, \ldots, b_n; i\rangle = \#\{j \mid b_j = b_i\}$ (see [9]).

Definition 2.4. Let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function μ. If $\mathcal{H}$ is a topological abelian group, define

$$t^p : [E, \mathcal{H}] \longrightarrow [X, \mathcal{H}] \quad \text{by} \quad t^p([\widetilde{\alpha}]) = [\alpha]\,,$$

where $\alpha(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}(e)$, $x \in X$.

Let X be a pointed space and G be an abelian (topological) group. Denote by $F(X, G)$ the McCord topological group of functions $u : X \longrightarrow G$ such that $u(*) = 0$ and $u(x) = 0$ for all but finitely many elements in X. This has the structure of a topological group (see [7] or [3]). If $\mathcal{H}$ is an Eilenberg-Mac Lane space of type $K(G, q)$, for instance given by $F(\mathbb{S}^q, G)$, then t^p is the *cohomolgy transfer*

$$\tau^p : \widetilde{H}^q(E; G) \longrightarrow \widetilde{H}^q(X; G)\,.$$

Here $\widetilde{H}^q$ stands for ordinary cohomology when the spaces involved have the homotopy type of CW-complexes, or for Čech cohomology if they are paracompact Hausdorff, provided that either G is countable or the spaces are compactly generated (see [5]).

EXAMPLE 2.5. For the ramified covering map $\pi_B : B^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n B$ of 2.3, the cohomology transfer is as follows. If $\mathcal{H}$ is a topological abelian group, then we have similarly that

$$t^{\pi_B} : [B^n \times_{\Sigma_n} \overline{n}, \mathcal{H}] \longrightarrow [\mathrm{SP}^n B, \mathcal{H}]$$

is given by $t^p([\widetilde{\alpha}]) = [\alpha]$, where

$$\alpha\langle b_1, \ldots, b_n\rangle = \sum_{l=1}^{n} \widetilde{\alpha}\langle b_1, \ldots, b_n; l\rangle \, .$$

REMARK 2.6. Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{Z}$, and a subspace $A \subset X$, we have the *restricted ramified covering map* $p_A : E_A \longrightarrow A$, $E_A = p^{-1}A$, and the *quotient ramified covering map* $p' : E' \longrightarrow X/A$, as given in Remark 2.2. The following diagram clearly commutes.

$$\begin{array}{ccccc} E_A & \hookrightarrow & E & \twoheadrightarrow & E' \\ {\scriptstyle p_A}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ A & \hookrightarrow & X & \twoheadrightarrow & X/A \, . \end{array}$$

Then one has cohomology transfers for each covering map. In particular, if A is a subcomplex of a CW-complex X, one has a *relative cohomology transfer* $t^p : H^n(E, E_A; G) \longrightarrow H^n(X, A; G)$ that fits together into a transformation of the long cohomology exact sequences of the pairs.

The following propositions establish the fundamental properties of the transfer.

Proposition 2.7. *If $p : E \longrightarrow X$ is an n-fold ramified covering map, then the composite*

$$[X, \mathcal{F}] \xrightarrow{p^*} [E, \mathcal{F}] \xrightarrow{t^p} [X, \mathcal{F}]$$

is multiplication by n.

Proof: If $[\alpha] \in [X, \mathcal{F}]$, then $t^p p^*(\alpha) = t^p(\alpha \circ p) : X \longrightarrow \mathcal{H}$, and $t^p(\alpha \circ p)(x) = \sum_{p(e)=x} \mu(e)\alpha p(e) = \left(\sum_{p(e)=x} \mu(e)\right) \alpha(x) = n \cdot \alpha(x)$. Thus $t^p p^*([\alpha]) = n \cdot [\alpha]$. ■

As a consequence, we obtain Theorem 5.4 in [3]. We also obtain the following.

Proposition 2.8. *Let $\mathbb{Z}_n$ act on X^n by cyclic permutation of coordinates, and take the quotient map $p : X^n \longrightarrow X^n/\mathbb{Z}_n$. If the prime q does not divide n, then $p^* : H^l(X^n/\mathbb{Z}_n; \mathbb{Z}_q) \longrightarrow H^l(X^n; \mathbb{Z}_q)$ is a split monomorphism.*

Proof: The map $p : X^n \longrightarrow X^n/\mathbb{Z}_n$ is an n-fold ramified covering map. Take its transfer given by the additive structure of $K(\mathbb{Z}_q, l)$. Then $\tau^p \circ p^*$ is multiplication by n, thus an isomorphism. Hence p^* is a split monomorphism. ■

The invariance under pullbacks is given by the following.

Proposition 2.9. *If $p : E \longrightarrow X$ is an n-fold ramified covering map, $\mathcal{F}$ is a topological abelian group, and $F : X \longrightarrow Y$ is continuous, then the following diagram commutes.*

$$\begin{array}{ccc} [E, \mathcal{F}] & \xrightarrow{t^p} & [X, \mathcal{F}] \\ \downarrow{\scriptstyle \widetilde{F}^*} & & \downarrow{\scriptstyle F^*} \\ [F^*(E), \mathcal{F}] & \xrightarrow[t^{F^*(p)}]{} & [Y, \mathcal{F}]\,. \end{array}$$

Proof: Let $\widetilde{\alpha} : E \longrightarrow \mathcal{F}$ represent an element in $[E, \mathcal{F}]$. Then the map

$$y \longmapsto \sum_{F_*(p)(y,e)=y} F^*(\mu)(y,e)\widetilde{\alpha}(y,e) = \sum_{p(e)=F(y)} \mu(e)\widetilde{\alpha}(y,e)$$

that represents $t^{F^*(p)}\widetilde{F}^*([\widetilde{\alpha}])$, clearly represents also $F^* t^p([\widetilde{\alpha}]) \in [Y, \mathcal{F}]$. ■

As a consequence, we obtain Theorem 5.5 in [3].

One further property of the cohomology transfer that will be useful below is the following.

Proposition 2.10. *Let $f : B \longrightarrow C$ be continuous and consider the commutative diagram*

(2.11)
$$\begin{array}{ccc} B^n \times_{\Sigma_n} \overline{n} & \xrightarrow{f^n \times_{\Sigma_n} 1_{\overline{n}}} & C^n \times_{\Sigma_n} \overline{n} \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \mathrm{SP}^n B & \xrightarrow[\mathrm{SP}^n f]{} & \mathrm{SP}^n C\,. \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} [C^n \times_{\Sigma_n} \overline{n}, \mathfrak{F}] & \xrightarrow{t^{\pi_C}} & [\mathrm{SP}^n C, \mathfrak{F}] \\ {\scriptstyle (f^n \times_{\Sigma_n} 1_{\overline{n}})^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{SP}^n f)^*} \\ [B^n \times_{\Sigma_n} \overline{n}, \mathfrak{F}] & \xrightarrow[t^{\pi_B}]{} & [\mathrm{SP}^n B, \mathfrak{F}]\,. \end{array}$$

The *proof* is fairly routinary and follows easily using the description of the transfer given in Example 2.5. ■

In 2.7 we computed the composite $t^p \circ p^*$. The opposite composite $p^* \circ t^p$ is also interesting. An immediate computation yields the following.

Proposition 2.12. (Cf. [3, 5.6]) *Let $p : E \longrightarrow X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$[E, \mathfrak{F}] \xrightarrow{t^p} [X, \mathfrak{F}] \xrightarrow{p^*} [E, \mathfrak{F}]$$

is given as follows. Take $[\varphi] \in [E, \mathfrak{F}]$, then $p^* t^p[\varphi]$ is represented by the map $\varphi' : E \longrightarrow \mathfrak{F}$ given by

$$\varphi'(e) = \sum_{p(e')=p(e)} \mu(e')e'\,.$$

■

In the case of an action of a finite group G on E and $X = E/G$, we have the following consequence.

Corollary 2.13. (Cf. [3, 5.7]) *If $[\varphi] \in [E, \mathfrak{F}]$, then $p^* t^p[\varphi] = [\varphi'] \in [E, \mathfrak{F}]$, where*

$$\varphi'(e) = \sum_{g \in G} ge\,.$$

Proof: Just observe that the element ge is repeated in the sum $\mu(e) = |G_e|$ times. ∎

Remark 2.14. Considering an action of H on E and a subgroup $K \subset H$, one has different ramified covering maps as depicted in

$$\begin{array}{ccc} & E & \\ {\scriptstyle q_K}\swarrow & & \searrow{\scriptstyle q_H} \\ E/K & \xrightarrow[q_H^K]{} & E/H\,. \end{array}$$

One may easily compute several combinations of the functions induced by these covering maps in homotopy sets and their transfers.

Another interesting property of the transfer is the relationship given by computing the transfer of the composition of two ramified covering maps. Before giving it we need the following.

Definition 2.15. Let $p : Y \longrightarrow X$ be an n-fold ramified covering map, with multiplicity function $\mu : Y \longrightarrow \mathbb{N}$ and let $q : Z \longrightarrow Y$ be an m-fold ramified covering map, with multiplicity function $\nu : Z \longrightarrow \mathbb{N}$. Then the composite $p \circ q : Z \longrightarrow X$ is an mn-fold ramified covering map, with multiplicity function $\xi : Z \longrightarrow \mathbb{N}$ given by $\xi(z) = \nu(z)\mu(q(z))$. In order to verify that this composite is indeed an mn-fold ramified covering map, consider the *wreath product* $\Sigma_n \wr \Sigma_m$, defined as the semidirect product of Σ_n and $(\Sigma_m)^n$, where Σ_n acts on $(\Sigma_m)^n$ by permuting the n factors. We have an action $(Z^m \times \cdots \times Z^m) \times \Sigma_n \wr \Sigma_m \longrightarrow Z^m \times \cdots \times Z^m$ given by $(\zeta_1, \ldots, \zeta_n) \cdot (\sigma, \tau_1, \ldots, \tau_n) = (\zeta_{\sigma(1)} \cdot \tau_1, \ldots, \zeta_{\sigma(n)} \cdot \tau_n)$, where $\zeta_i \in Z^m$. Then we have the following diagram, where all maps are open

$$\begin{array}{ccc} Z^m \times \cdots \times Z^m & \xrightarrow{q\times\cdots\times q} & Z^m/\Sigma_m \times \cdots \times Z^m/\Sigma_m \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ (Z^m)^n/\Sigma_n \wr \Sigma_m & \dashrightarrow & \mathrm{SP}^n(\mathrm{SP}^m Z)\,. \end{array}$$

One may easily show that π is compatible with $\pi' \circ (q \times \cdots \times q)$. Therefore, there is a homeomorphism $X^{mn}/\Sigma_n \wr \Sigma_m \approx \mathrm{SP}^n(\mathrm{SP}^m Z)$ and hence one has a canonical quotient map $\rho : \mathrm{SP}^n(\mathrm{SP}^m Z) \longrightarrow \mathrm{SP}^{mn} Z$. Then one can easily verify that $\varphi_{p \circ q} = \rho \circ \mathrm{SP}^n(\varphi_q) \circ \varphi_p : X \longrightarrow \mathrm{SP}^n(\mathrm{SP}^m Z) \xrightarrow{\rho} \mathrm{SP}^{mn} Z$. Thus $\varphi_{p \circ q}$ is continuous.

The cohomology transfer behaves well with respect to composite ramified covering maps.

Proposition 2.16. *The following holds:*

$$\tau^{p\circ q} = \tau^p \circ \tau^q : H^k(Z;G) \xrightarrow{\tau^q} H^k(Y;G) \xrightarrow{\tau_q} H^k(X;G)\,.$$

Proof: We prove that $\tau^{p\circ q} = \tau^p \circ \tau^q : [Z,\mathcal{F}] \longrightarrow [X,\mathcal{F}]$ for any abelian topological group $\mathcal{F}$. Take $w = [h] \in [Z,\mathcal{F}]$, $v = [g] \in [Y,\mathcal{F}]$, $u = [f] \in [Z,\mathcal{F}]$, then $v = \tau^q(w)$ if $g(y) = \sum_{q(z)=y} \nu(z)h(z)$, and $u = \tau^p(v)$ if $f(x) = \sum_{p(y)=x} \mu(y)g(y)$. Hence,

$$f(x) = \sum_{p(y)=x} \mu(y) \sum_{q(z)=y} \nu(z)h(z) = \sum_{pq(z)=x} \mu(q(z))\nu(z)h(z) = \sum_{pq(z)=x} \xi(z)h(z)\,.$$

Therefore, $\tau^q\tau^p(u) = \tau^{p\circ q}(u)$. ∎

Corollary 2.17. *Given an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function μ and an integer l, there is an ln-fold ramified covering map $p_l : E \longrightarrow X$ such that $p_l = p$ and $\mu_l(e) = l\mu(e)$, $e \in E$. Then $\tau^{p_l} = l\tau^p : H^k(E;G) \longrightarrow H^k(X;G)$.*

Proof: Consider the l-fold ramified covering map $q : E \longrightarrow E$ such that $q = \mathrm{id}_E$ and $\nu(e) = l$ for all $e \in E$. Then $p_l = p \circ q$. Then apply Proposition 2.16. ∎

REMARK 2.18. The ln-fold covering map p_l obtained from l is a sort of spurious ramified covering, since the multiplicity of p is artificially multiplied by l. It is interesting to remark that the previous result shows that the transfer of this new ramified covering p_l remains essentially unchanged.

3 General transfers for n-fold ramified covering maps in cohomology

In this section we consider representable contravariant functors h and k, that is, $h(-) = [-,\mathcal{H}]$ and $k(-) = [-,\mathcal{K}]$, where $\mathcal{H}$ and $\mathcal{K}$ are spaces, in order to study general transfers.

Definition 3.1. An (h,k)*-transfer for n-fold ramified covering maps* associates to every n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{N}$, a function $\tau^p : h(E) \longrightarrow k(X)$, with the following two properties.

1. Given a pullback diagram

(3.2)
$$\begin{array}{ccc} F^*E & \xrightarrow{\widetilde{F}} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[F]{} & X\,, \end{array}$$

of n-fold ramified covering maps, the diagram

$$\begin{array}{ccc} h(E) & \xrightarrow{\tau^p} & k(X) \\ {\scriptstyle h(\widetilde{F})}\downarrow & & \downarrow{\scriptstyle k(F)} \\ h(F^*E) & \xrightarrow[\tau^{p'}]{} & k(Y) \end{array}$$

commutes.

2. Given $f : B \longrightarrow C$, then for the diagram (2.11) the following diagram commutes

$$\begin{array}{ccc} h(C^n \times_{\Sigma_n} \overline{n}) & \xrightarrow{\tau^{\pi_C}} & k(\mathrm{SP}^n C) \\ {\scriptstyle (f^n \times_{\Sigma_n} 1_{\overline{n}})^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{SP}^n f)^*} \\ h(B^n \times_{\Sigma_n} \overline{n}) & \xrightarrow[\tau^{\pi_B}]{} & k(\mathrm{SP}^n B)\,, \end{array}$$

REMARK 3.3. Observe that the transfers just defined need not be homomorphisms (even when $\mathcal{H}$ and $\mathcal{K}$ are H-spaces).

NOTE 3.4. Considering the category $\mathcal{R}\mathrm{amcov}_n$ whose objects are n-fold ramified covering maps and whose morphisms are pullback diagrams, one has two functors, namely $\mathcal{E}, \mathcal{X} : \mathcal{R}\mathrm{amcov}_n \longrightarrow \mathcal{T}\mathrm{op}$ such that given a covering map $p : E \longrightarrow X$, $\mathcal{E}(p) = E$ and $\mathcal{X}(p) = X$. Then a transfer is a natural transformation $h \circ \mathcal{E} \stackrel{\cdot}{\longrightarrow} k \circ \mathcal{X}$ (between functors $\mathcal{R}\mathrm{amcov}_n \longrightarrow \mathcal{S}\mathrm{et}$), that also is a natural transformation $h \circ (-)^n \times_{\Sigma_n} \overline{n} \stackrel{\cdot}{\longrightarrow} k \circ \mathrm{SP}^n$ (between functors $\mathcal{T}\mathrm{op} \longrightarrow \mathcal{S}\mathrm{et}$).

If $h = k = H^q(-;G)$, then by 2.9 and 2.10, we have the following.

Proposition 3.5. *The transfer $\tau^p : h(E) \longrightarrow h(X)$ defined in 2.4 is an (h,h)-transfer.* ■

We have the following classification result.

Theorem 3.6.

(i) *Each class $w \in k(\mathrm{SP}^n\mathcal{H})$ determines an (h,k)-transfer τ^w for n-fold ramified covering maps, and conversely*

(ii) *each (h,k)-transfer τ for n-fold ramified covering maps determines a class $w_\tau \in k(\mathrm{SP}^n\mathcal{H})$. Moreover,*

(iii) *the class associated to τ^w is w, and conversely*

(iv) *the transfer associated to w_τ is τ.*

Proof:
(i) Take a class $w \in k(\mathrm{SP}^n\mathcal{H})$ and let $p : E \longrightarrow X$ be an n-fold ramified covering map with multiplicity function $\mu : E \longrightarrow \mathbb{Z}$. We define $\tau^{w,p} : h(E) \longrightarrow k(X)$ as follows. Given $[f] \in h(E)$, let $\tau^{w,p}[f]$ be the homotopy class of the composite

$$X \xrightarrow{\varphi_p} \mathrm{SP}^n E \xrightarrow{\mathrm{SP}^n f} \mathrm{SP}^n\mathcal{H} \xrightarrow{w} \mathcal{K}\,.$$

In order to show that τ^w is natural, consider the pullback diagram (3.2). The element $k(F) \circ \tau^{w,p}[f]$ is given by the homotopy class of the composite

$$Y \xrightarrow{F} X \xrightarrow{\varphi_p} \mathrm{SP}^n E \xrightarrow{\mathrm{SP}^n f} \mathrm{SP}^n\mathcal{H} \xrightarrow{w} \mathcal{K}\,.$$

On the other hand, the element $\tau^{w,p'} \circ h(\widetilde{F})[f]$ is given by the homotopy class of the composite

$$\begin{array}{ccccccc} Y & \xrightarrow{\varphi_{p'}} & \mathrm{SP}^n(F^*E) & \xrightarrow{\qquad\mathrm{SP}^n(f\circ\widetilde{F})\qquad} & \mathrm{SP}^n\mathcal{H} & \xrightarrow{w} & \mathcal{K} \\ & & {\scriptstyle \mathrm{SP}^n\widetilde{F}}\searrow & & \nearrow{\scriptstyle \mathrm{SP}^n f} & & \\ & & & \mathrm{SP}^n E\,. & & & \end{array}$$

By the functoriality of the construction SP^n, the triangle commutes; therefore, we only have to show the commutativity of the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{F} & X \\ {\scriptstyle \varphi_{p'}}\downarrow & & \downarrow{\scriptstyle \varphi_p} \\ \mathrm{SP}^n(F^*E) & \xrightarrow[\mathrm{SP}^n\widetilde{F}]{} & \mathrm{SP}^n E\,. \end{array}$$

To that end, take $y \in Y$ and consider the fiber $p^{-1}(F(y)) = \{e_1, \dots, e_r\}$. Then

$$\varphi_p \circ F(y) = \langle e_1, \dots, e_1, \dots, e_r, \dots, e_r \rangle \,,$$

where e_i is repeated $\mu(e_i)$ times. Since ${p'}^{-1}(y) = \{(y, e_1), \dots, (y, e_r)\}$ and the multiplicity function of p' is $\mu \circ \widetilde{F}$, we have that

$$\varphi_{p'}(y) = \langle (y, e_1), \dots, (y, e_1), \dots, (y, e_r), \dots, (y, e_r) \rangle \,,$$

where (y, e_i) appears $\mu(e_i)$ times. Therefore,

$$\mathrm{SP}^n \widetilde{F} \circ \varphi_{p'} = \langle e_1, \dots, e_1, \dots, e_r, \dots, e_r \rangle \,,$$

where again e_i appears $\mu(e_i)$ times, and so the diagram commutes.

(ii) Let τ be an (h,k)-transfer for n-fold ramified covering maps and consider the map $\pi : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n\mathcal{H}$. As remarked in 2.3, π is an n-fold ramified covering map. Therefore, we have $\tau^\pi : h(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \longrightarrow k(\mathrm{SP}^n\mathcal{H})$. Let $\alpha : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathcal{H}$ be given by $\alpha\langle a_1, \dots, a_n, i\rangle = a_i$. Then $[\alpha] \in h(\mathcal{H}^n \times_{\Sigma_n} \overline{n})$. We associate to τ the element $w_\tau = \tau^\pi[\alpha] \in k(\mathrm{SP}^n\mathcal{H})$.

(iii) Let $w \in k(\mathrm{SP}^n\mathcal{H})$, and consider the associated transfer τ^w. The class in $k(\mathrm{SP}^n\mathcal{H})$ determined by τ^w is given by $\tau^{w,\pi}[\alpha]$, where $\alpha : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathcal{H}$ is given by $\alpha\langle a_1, \dots, a_n, i\rangle = a_i$. Therefore, $\tau^{w,\pi}[\alpha]$ is the homotopy class of the composite

$$\mathrm{SP}^n\mathcal{H} \xrightarrow{\varphi_\pi} \mathrm{SP}^n(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \xrightarrow{\mathrm{SP}^n\alpha} \mathrm{SP}^n\mathcal{H} \xrightarrow{w} \mathcal{K} \,.$$

Let $a = \langle a_1, \dots, a_1, \dots, a_r, \dots, a_r\rangle$ be an element in $\mathrm{SP}^n\mathcal{H}$, where a_l appears i_l times. Then

$$\varphi_\pi(a) = \langle \underbrace{\langle a, 1\rangle, \dots, \langle a, 1\rangle}_{i_1}, \underbrace{\langle a, 2\rangle, \dots, \langle a, 2\rangle}_{i_2}, \dots, \underbrace{\langle a, r\rangle, \dots, \langle a, r\rangle}_{i_r} \rangle \,.$$

Therefore,

$$\mathrm{SP}^n\alpha \circ \varphi_\pi(a) = \langle \underbrace{a_1, \dots, a_1}_{i_1}, \underbrace{a_2, \dots, a_2}_{i_2}, \dots, \underbrace{a_r, \dots, a_r}_{i_r} \rangle = a \,,$$

so that $\mathrm{SP}^n\alpha \circ \varphi_\pi = 1$. Hence $\tau^{w,\pi}[\alpha] = w$.

(iv) Finally, given an (h,k)-transfer τ, we have $w_\tau = \tau^\pi[\alpha]$. In order to show that $\tau^{w_\tau} = \tau$, consider an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{Z}$ and some element $[f] \in h(E) = [E, \mathcal{H}]$. We shall prove that $\tau^{w_\tau, p}[f] = \tau^p[f]$. For that, consider the following two diagrams

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{\varphi}_p} & E^n \times_{\Sigma_n} \overline{n} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ X & \xrightarrow[\varphi_p]{} & \mathrm{SP}^n E \end{array} \qquad \begin{array}{ccc} E^n \times_{\Sigma_n} \overline{n} & \xrightarrow{f^n \times_{\Sigma_n} 1_{\overline{n}}} & \mathcal{H}^n \times_{\Sigma_n} \overline{n} \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi = \pi_{\mathcal{H}}} \\ \mathrm{SP}^n E & \xrightarrow[\mathrm{SP}^n f]{} & \mathrm{SP}^n \mathcal{H}\,. \end{array}$$

The one on the left-hand side is a pullback diagram while the one on the right-hand side is like (2.11). Hence, by the two properties of the transfer, we have two commutative diagrams

$$\begin{array}{ccc} h(E^n \times_{\Sigma_n} \overline{n}) & \xrightarrow{\tau^p} & k(\mathrm{SP}^n E) \\ {\scriptstyle (\widetilde{\varphi}_p)^*}\downarrow & & \downarrow{\scriptstyle (\varphi_p)^*} \\ h(E) & \xrightarrow[\tau^p]{} & k(X) \end{array} \qquad \begin{array}{ccc} h(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) & \xrightarrow{\tau^\pi} & k(\mathrm{SP}^n \mathcal{H}) \\ {\scriptstyle (f^n \times_{\Sigma_n} 1_{\overline{n}})^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{SP}^n f)^*} \\ h(E^n \times_{\Sigma_n} \overline{n}) & \xrightarrow[\tau^{\pi_E}]{} & k(\mathrm{SP}^n E)\,, \end{array}$$

and putting the one on the right-hand side on top of the one on the left-hand side, we obtain

$$\begin{array}{ccc} h(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) & \xrightarrow{\tau^\pi} & k(\mathrm{SP}^n \mathcal{H}) \\ {\scriptstyle (\widetilde{\varphi}_p)^* \circ (f^n \times_{\Sigma_n} 1_{\overline{n}})^*}\downarrow & & \downarrow{\scriptstyle (\varphi_p)^* \circ (\mathrm{SP}^n f)^*} \\ h(E) & \xrightarrow[\tau^p]{} & k(X)\,. \end{array}$$

If we now chase our element $[\alpha] \in h(\mathcal{H}^n \times_{\Sigma_n} \overline{n})$ defined in the proof of (ii) along the top and right-side of the diagram, we obtain $[w_\tau \circ \mathrm{SP}^n f \circ \varphi_p] = \tau^{w_\tau, p}[f]$, while if we chase it along the left-hand and bottom side of the diagram we obtain $\tau^p[f]$. Thus $\tau^{w_\tau, p}[f] = \tau^p[f]$, as desired. ∎

As a consequence of Theorem 3.6, we obtain the following.

Corollary 3.7. *There is a one-to-one correspondence between (h,k)-transfers τ and elements w in $k(\mathrm{SP}^n \mathcal{H})$.* ∎

In the following result we compute w for the cohomology transfer τ^p defined in 2.4.

Proposition 3.8. *Let $\mathcal{H} = F(\mathbb{S}^q, G)$. Then the element $w_\tau \in [\mathrm{SP}^n\mathcal{H}, \mathcal{H}]$ that corresponds to the transfer τ^p is given by*

$$w_\tau\langle a_1, \ldots, a_n\rangle = a_1 + \cdots + a_n\,.$$

Proof: Let $\pi : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n\mathcal{H}$ be the n-fold ramified covering map given above. The transfer $\tau^\pi : h(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \longrightarrow h(\mathrm{SP}^n\mathcal{H})$ is such that $\tau^\pi[\alpha](x) = \sum_{\pi(e)=x} \mu(e)\alpha(e)$. Thus, if $x = \langle a_1, \ldots, a_n\rangle$, then $\pi^{-1}(x) = \{\langle a_1, \ldots, a_n; i\rangle \mid i = 1, \ldots, n\}$. Hence

$$w_\tau(x) = \tau^\pi[\alpha](x) = \sum_{i=1}^{n} \alpha\langle a_1, \ldots, a_n; i\rangle = \sum_{i=1}^{n} a_i\,.$$

∎

Definition 3.9. By Theorem 3.6, given representable functors h and k, we can define the set of transfers from $h(E)$ to $k(X)$ for each n-fold ramified covering map $p : E \longrightarrow X$. We denote this set by $T_n^R(h,k)$. If we assume that the functor k takes values in the category $\mathcal{A}\mathrm{b}$ of abelian groups, then we can give $T_n^R(h,k)$ a group structure as follows. Given $\sigma, \tau \in T_n^R(h,k)$ and an n-fold ramified covering map $p : E \longrightarrow X$, we define the transfer $\sigma + \tau$ by $(\sigma+\tau)^p(a) = \sigma^p(a) + \tau^p(a)$, for every $a \in h(E)$.

Corollary 3.10. *Assume that k takes values in $\mathcal{A}\mathrm{b}$. Then the bijection of Corollary 3.7 gives an isomorphism of abelian groups*

$$T_n^R(h,k) \cong k(\mathrm{SP}^n(\mathcal{H}))$$

Proof: By 3.7, there is a bijection $\psi : T_n^R(h,k) \longrightarrow k(\mathrm{SP}^n(\mathcal{H}))$ given by $\psi(\tau) = w_\tau = \tau^\pi[\alpha]$, as in the proof of 3.6(ii). Then

$$\psi(\sigma+\tau) = (\sigma+\tau)^\pi[\alpha] = \sigma^\pi[\alpha] + \tau^\pi[\alpha] = \psi(\sigma) + \psi(\tau)\,.$$

Therefore, ψ is an isomorphism. ∎

The following is a nice consequence of this corollary.

Proposition 3.11. *Let τ be an (h,k)-transfer and assume that there is a commutative diagram*

$$\begin{array}{ccccc} & & \mathbb{Z} & & \\ & \overset{\mu'}{\nearrow} & & \overset{\mu}{\nwarrow} & \\ E' & & \xrightarrow{\quad q \quad} & & E \\ & \underset{p'}{\searrow} & & \underset{p}{\swarrow} & \\ & & X & & \end{array}$$

of n-fold ramified covering maps, such that $q : E' \longrightarrow E$ is surjective. Then the following triangle commutes:

$$\begin{array}{ccc} h(E) & \xrightarrow{\quad q^* \quad} & h(E') \\ & \underset{\tau^p}{\searrow} \quad \underset{\tau^{p'}}{\swarrow} & \\ & k(X) & \end{array} \tag{3.12}$$

Proof: By the classification result 3.7, there is an element $w = w_\tau \in k(\mathrm{SP}^n\mathcal{H})$ such that for any $p : E \longrightarrow X$ and any element $[f] \in h(X) = [X, \mathcal{H}]$, its transfer is given by the composite

$$\tau^p : X \xrightarrow{\varphi_p} \mathrm{SP}^n E \xrightarrow{\mathrm{SP}^n f} \mathrm{SP}^n\mathcal{H} \xrightarrow{w} \mathcal{K}\,.$$

Consider the following diagram:

$$\begin{array}{ccccc} & & SP^nE' & & \\ & \overset{\varphi_{p'}}{\nearrow} & & \overset{\mathrm{SP}^n(f\circ q)}{\searrow} & \\ X & & \Big\downarrow \mathrm{SP}^n q & & \mathrm{SP}^n\mathcal{H} \xrightarrow{\ w\ } \mathcal{K}\,. \\ & \underset{\varphi_p}{\searrow} & & \underset{\mathrm{SP}^n f}{\nearrow} & \\ & & SP^nE & & \end{array}$$

The triangle on the right-hand side commutes clearly. The one on the left-hand side commutes too, since

$$\varphi_p(x) = \sum_{p(e)=x} \mu(e)e\,, \qquad \varphi_{p'}(x) = \sum_{p'(e')=x} \mu'(e')e'\,,$$

and thus

$$\mathrm{SP}^n q\varphi_{p'}(x) = \sum_{p'(e')=x} \mu'(e')q(e') = \sum_{p(q(e'))=x} \mu(q(e'))q(e') = \sum_{p(e)=x} \mu(e)e\,,$$

where the last equality follows since q is surjective. Hence,

$$\tau^p[f] = \tau^{w,p}[f] = \tau^{w,p'}[f \circ q] = \tau^{p'} q^*[f]\,.$$

■

The following theorem tells in some cases about the existence of transfers.

Theorem 3.13. *Let H^* denote singular cohomology with coefficients in $\mathbb{Z}$. Then*

$$T_n^R(H^r, H^s) \cong \begin{cases} 0 & \text{if } n \geq s > r \quad (s > 0) \\ \mathbb{Z} & \text{if } n \geq s = r \\ 0 & \text{if } n \geq s = r+1. \end{cases}$$

Proof: By 3.6 and 3.7, we have an isomorphism

$$T_n^R(H^r, H^s) \cong H^s(\mathrm{SP}^n(K(\mathbb{Z}, r))\,.$$

By [2, 6.3.24], for any $(r-1)$-connected CW-complex X, the inclusion $X \hookrightarrow \mathrm{SP}^\infty X$ is an $(r+1)$-equivalence. Therefore, $\mathrm{SP}^\infty K(\mathbb{Z}, r)$ is $(r-1)$-connected, and so $\pi_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)) \cong \mathbb{Z}$ and $\pi_{r+1}(\mathrm{SP}^\infty K(\mathbb{Z}, r)) = 0$. By the Hurewicz theorem,

$$\widetilde{H}_i(\mathrm{SP}^\infty K(\mathbb{Z}, r)) = 0 \quad \text{for} \quad i < r\,,$$

$$H_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)) \cong \mathbb{Z} \quad \text{and} \quad H_{r+1}(\mathrm{SP}^\infty K(\mathbb{Z}, r)) = 0\,.$$

By the universal coefficients theorem,

$$H^s(\mathrm{SP}^\infty K(\mathbb{Z}, r)) = 0 \quad \text{for} \quad s < r\,,$$

$$H^r(\mathrm{SP}^\infty K(\mathbb{Z}, r)) \cong \mathrm{Hom}(H_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)); \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}\,.$$

Since $H_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)) \cong \mathbb{Z}$, $\mathrm{Ext}(H_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)); \mathbb{Z}) = 0$, and we have that $H^{r+1}(\mathrm{SP}^\infty K(\mathbb{Z}, r)) \cong \mathrm{Hom}(H_r(\mathrm{SP}^\infty K(\mathbb{Z}, r)); \mathbb{Z}) = 0$. By [11], for any CW-complex X, $H^s(\mathrm{SP}^\infty X) \cong H^s(\mathrm{SP}^n X)$ for $n \geq s$, so the result follows. ■

4 Comparison between transfers for ordinary covering maps and for ramified covering maps

In this section we shall compare our classification of transfers for ramified covering maps given in the previous section with the classification of transfers for n-fold ordinary covering maps obtained by Roush [8]. For a description of his result we follow [1].

Definition 4.1. Take again $h(-) = [-, \mathcal{H}]$ and $k(-) = [-, \mathcal{K}]$ as above. *an (h, k)-transfer for n-fold covering maps* associates to every n-fold covering map $p : E \longrightarrow X$ over a paracompact space X a function $t^p : h(E) \longrightarrow k(X)$, which is natural with respect to pullbacks in the same sense of property 1. in the definition of the (h, k)-transfers for n-fold ramified covering maps (3.1).

Denote by $T_n(h, k)$ the set of transfers for n-fold ordinary covering maps, and let $E\Sigma_n \longrightarrow B\Sigma_n$ be the universal principal Σ_n-bundle. Then we have the following.

Theorem 4.2. (Roush) *There is a bijection*

$$[E\Sigma_n \times_{\Sigma_n} \mathcal{H}^n, \mathcal{K}] \longrightarrow T_n(h, k) .$$

■

Since the transfers for n-fold ramified covering maps are also natural with respect to pullbacks, as just mentioned above, we have a restriction function $r : T_n^R(h, k) \longrightarrow T_n(h, k)$. The following theorem relates both classifications, namely Theorems 4.2 and 3.7.

Theorem 4.3. *Let $\rho : E\Sigma_n \times_{\Sigma_n} \mathcal{H}^n \longrightarrow \mathrm{SP}^n\mathcal{H}$ be given by $\rho\langle e; a_1, \ldots, a_n\rangle = \langle a_1, \ldots, a_n\rangle$. Then the following diagram commutes.*

$$\begin{array}{ccc} [\mathrm{SP}^n\mathcal{H}, \mathcal{K}] & \xleftarrow{\ \cong\ }\!\!\!\!\rightarrow & T_n^R(h, k) \\ {\scriptstyle \rho^*}\downarrow & & \downarrow{\scriptstyle r} \\ [E\Sigma_n \times_{\Sigma_n} \mathcal{H}^n, \mathcal{K}] & \longrightarrow & T_n(h, k) \end{array}$$

Proof: Let $w : \mathrm{SP}^n\mathcal{H} \longrightarrow \mathcal{K}$ be a map and τ^w the transfer for n-fold ramified covering maps associated to it according to Corollary 3.7. Consider $\rho^*[w] = [w \circ \rho]$ and let $p : E \longrightarrow X$ be an n-fold covering map and $g : E \longrightarrow \mathcal{H}$.

The value of the transfer t^p associated to the class $\rho^*[w]$, on $[g]$ is defined as follows. Let $q : \overline{E} \longrightarrow X$ be the principal Σ_n-bundle associated to p, i.e., $\overline{E} = \{(e_1, \ldots, e_n) \in E^n \mid e_i \neq e_j \text{ if } i \neq j; \text{ and } p(e_1) = \cdots = p(e_n)\}$, and $q(e_1, \ldots, e_n) = p(e_1)$. There is a free Σ_n-action on $\overline{E}$ defined by permuting coordinates, and a homeomorphism $\gamma : \overline{E}/\Sigma_n \longrightarrow X$ given by $\gamma\langle e_1, \ldots e_n\rangle = p(e_1)$. Therefore there is a pullback square

$$\begin{array}{ccccccc} & & \overline{E} & \xrightarrow{\beta} & E\Sigma_n & & \\ & & \downarrow & & \downarrow & & \\ X & \xleftarrow{\approx} & \overline{E}/\Sigma_n & \longrightarrow & E\Sigma_n/\Sigma_n & = & B\Sigma_n\,. \end{array}$$

Then $t^p[g] \in k(X) = [X, \mathcal{K}]$ s the class of the composite

$$X \approx \overline{E}/\Sigma_n \xrightarrow{\psi} E\Sigma_n \times_{\sigma_n} E^n \xrightarrow{\mathrm{id} \times_{\Sigma_n} g^n} E\Sigma_n \times_{\Sigma_n} \mathcal{H}^n \xrightarrow{w \circ \rho} \mathcal{K}\,,$$

where $\psi\langle e_1, \ldots, e_n\rangle = \langle\beta(e_1, \ldots, e_n); e_1, \ldots, e_n\rangle$. Now we consider tha following diagram:

$$\begin{array}{ccccccc} \overline{E}/\Sigma_n & \xrightarrow{\psi} & E\Sigma_n \times_{\sigma_n} E^n & \xrightarrow{\mathrm{id} \times_{\Sigma_n} g^n} & E\Sigma_n \times_{\Sigma_n} \mathcal{H}^n & \xrightarrow{w \circ \rho} & \mathcal{K}\,, \\ {\scriptstyle\gamma}\downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle\rho} & \nearrow{\scriptstyle w} & \\ X & \xrightarrow[\varphi_p]{} & \mathrm{SP}^n E & \xrightarrow[\mathrm{SP}^n g]{} & \mathrm{SP}^n \mathcal{H} & & \end{array}$$

where $\rho'\langle a; e_1, \ldots, e_n\rangle = \langle e_1, \ldots, e_n\rangle$. Since $p : E \longrightarrow X$ is an n-fold covering map, $\mu(e) = 1$ for all $e \in E$, and $\{e_1, \ldots, e_n\}$ is the fiber over $p(e_1)$. Therefore, $\varphi_p(\gamma\langle e_1, \ldots, e_n\rangle) = \langle e_1, \ldots, e_n\rangle$. Since $\rho'\psi\langle e_1, \ldots, e_n\rangle = \langle e_1, \ldots, e_n\rangle$, the left-hand side square of the diagram commutes, the middle square as well as the triangle are clearly also commutative. But the class of the composite $w \circ \mathrm{SP}^n g \circ \varphi_p$ is $\gamma(\tau^w)^p[g] = \tau^{w,p}[g]$. Hence $\gamma(\tau^w)^p[g] = t^p[g]$ and thus $\gamma(\tau^w) = t$. ■

5 Transfers for ramified covering maps in cohomology

In this section we shall consider families of (h, k)-transfers for n-fold ramified covering maps for all n. Assume that $\mathcal{H}$ and $\mathcal{K}$ are topological abelian groups, in order to have group structures in the homotopy sets $[X, \mathcal{H}]$ and $[X, \mathcal{K}]$ for all X.

Before stating the relevant definition, consider an n-fold ramified covering map $p : E \longrightarrow X$ with multiplicity function $\mu : E \longrightarrow \mathbb{Z}$. Recall the $(n+1)$-fold ramified covering map $\overline{p} : \overline{E} = E \sqcup X \longrightarrow X$ given in Remark 2.2.

Definition 5.1. An (h,k)-*transfer* τ *for ramified covering maps* consists of an (h,k)-transfer τ_n for n-fold ramified covering maps, for each $n = 1, 2, 3, \ldots$, such that for each n-ramified covering map $p : E \longrightarrow X$ the triangles in the following diagram commute:

$$(5.2)\qquad \begin{array}{ccccc} h(X) & \xleftarrow{\;i_2^*\;} & h(E \sqcup X) & \xrightarrow{\;i_1^*\;} & h(E) \\ & \searrow^{\tau_X} & \Big\downarrow {\scriptstyle \tau_{n+1}^{\overline{p}}} & \swarrow_{\tau_n^p} & \\ & & k(X)\,, & & \end{array}$$

where i_1 and i_2 are the canonical inclusions, and $\tau_X = \tau_1^{\mathrm{id}_X}$, which is just given by a natural transformation $h \stackrel{\cdot}{\longrightarrow} k$.

Observe that since i_1^* is an epimorphism, τ_n^p is determined by $\tau_{n+1}^{\overline{p}}$ for any n-fold ramified covering map $p : E \longrightarrow X$. Therefore, we have an inverse system

$$\cdots \longrightarrow T_{n+1}^R(h,k) \longrightarrow T_n^R(h,k) \longrightarrow \cdots \longrightarrow T_1^R(h,k) = \mathrm{Nat}(h,k)\,,$$

where $\mathrm{Nat}(h,k)$ denotes the natural transformations from h to k. Thus, a transfer for ramified covering maps is an element in $\lim_n T_n^R(h,k) = T_\infty^R(h,k)$. On the other hand, we have another inverse system

$$\cdots \longrightarrow k(\mathrm{SP}^{n+1}\mathcal{H}) \xrightarrow{\;i^*\;} k(\mathrm{SP}^n\mathcal{H}) \longrightarrow \cdots \longrightarrow k(\mathcal{H})\,,$$

where $i : \mathrm{SP}^n\mathcal{H} \hookrightarrow \mathrm{SP}^{n+1}\mathcal{H}$ is the canonical inclusion given by $\langle a_1, \ldots, a_n \rangle \mapsto \langle a_1, \ldots, a_n; * \rangle$. By Corollary 3.10, we have the following.

Theorem 5.3. *There is an isomorphism* $T_\infty^R(h,k) \longrightarrow \lim_n k(\mathrm{SP}^n\mathcal{H})$. *More precisely, the diagram*

$$\begin{array}{ccc} T_{n+1}^R(h,k) & \xrightarrow{\;\cong\;} & k(\mathrm{SP}^{n+1}\mathcal{H}) \\ \Big\downarrow & & \Big\downarrow {\scriptstyle i^*} \\ T_n^R(h,k) & \xrightarrow[\cong]{} & k(\mathrm{SP}^n\mathcal{H}) \end{array}$$

commutes for all n.

Proof: Let τ be an (h,k)-transfer for ramified covering maps. We have to show that for each n, $i^*(w^{\tau_{n+1}}) = w^{\tau_n}$. Recall that $w^{\tau_n} = \tau_n^{\pi_n}(\alpha_n)$, where $\pi_n : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n\mathcal{H}$ is the canonical ramified covering map, and $\alpha_n : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathcal{H}$ is given by $\langle a_1, \ldots, a_n; i\rangle \mapsto a_i$.

Let $p : E \longrightarrow \mathrm{SP}^n\mathcal{H}$ be the $(n+1)$-fold ramified covering map obtained by taking the pullback of π_{n+1} over $i : \mathrm{SP}^n\mathcal{H} \hookrightarrow \mathrm{SP}^{n+1}\mathcal{H}$. Thus we have a commutative square

$$\begin{array}{ccc} h(\mathcal{H}^{n+1} \times_{\Sigma_{n+1}} \overline{n+1}) & \xrightarrow{j^*} & h(E) \\ \downarrow{\scriptstyle \tau_{n+1}^{\pi_{n+1}}} & & \downarrow{\scriptstyle \tau_{n+1}^{p}} \\ k(\mathrm{SP}^{n+1}\mathcal{H}) & \xrightarrow[i^*]{} & k(\mathrm{SP}^n\mathcal{H}) \,, \end{array}$$

where j is the induced inclusion. By the universal property of the pullback, there is a (unique) surjective map $q : (\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \sqcup \mathrm{SP}^n\mathcal{H} \longrightarrow E$ such that $p \circ q|_{\mathcal{H}^n \times_{\Sigma_n} \overline{n}} = \pi_n$ (and $p \circ q|_{\mathrm{SP}^n\mathcal{H}} = \mathrm{id}_{\mathrm{SP}^n\mathcal{H}}$).

Thus, combining (3.12) and the right-hand side of (5.2) for this case, we have a commutative diagram

$$\begin{array}{ccccc} h(E) & \xrightarrow{q^*} & h((\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \sqcup \mathrm{SP}^n\mathcal{H}) & \xrightarrow{i_1^*} & h(\mathcal{H}^n \times_{\Sigma_n} \overline{n}) \\ & {\scriptstyle \tau_{n+1}^{p}} \searrow & \downarrow{\scriptstyle \tau_{n+1}^{\pi_n'}} & \swarrow {\scriptstyle \tau_n^{\pi_n}} & \\ & & k(\mathrm{SP}^n\mathcal{H}) \,. & & \end{array}$$

Therefore, $i^*(w^{\tau_{n+1}}) = \tau_{n+1}^p j^*(\alpha_{n+1}) = \tau_n^{\pi_n} i_1^* q^* j^*(\alpha_{n+1}) = \tau_n^{\pi_n}(\alpha_n) = w^{\tau_n}$, since $i_1^* q^* j^*(\alpha_{n+1}) = (j \circ q \circ i_1)^*(\alpha_{n+1}) = \alpha_n$, as one easily verifies after observing that $j \circ q \circ i_1 : \mathcal{H}^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathcal{H}^{n+1} \times_{\Sigma_{n+1}} \overline{n+1}$ is the canonical inclusion. ∎

EXAMPLE 5.4. The (h,h)-transfers $\tau_n^p = t^p$ given in Definition 2.4 for each n determine an (h,h)-transfer for ramified covering maps, since for any $[\widetilde{\alpha}] \in h(\overline{E}) = [\overline{E}, \mathcal{H}]$, its images on both sides of Diagram (5.2) are given by $[\widetilde{\alpha}_1] = [\widetilde{\alpha}|_E] \in h(E) = [E, \mathcal{H}]$ and $[\widetilde{\alpha}_2] = [\widetilde{\alpha}|_X] \in h(X) = [X, \mathcal{H}]$. Then $\tau_{n+1}^{\overline{p}}[\widetilde{\alpha}] = [\alpha]$, $\tau_n^p[\widetilde{\alpha}_1] = [\alpha_1]$, and $\tau_X[\widetilde{\alpha}_2] = [\alpha_2]$, where $\alpha(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}(e) + \widetilde{\alpha}(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}_1(e) + \widetilde{\alpha}_2(x)$. Thus $\tau_{n+1}^{\overline{p}}[\widetilde{\alpha}] = \tau_n^p[\widetilde{\alpha}_1] + \tau_X[\widetilde{\alpha}_2]$, and hence Diagram (5.2) commutes in this case.

Assume in what follows that τ is an (h,h)-transfer for ramified covering maps given by an element $[w] \in \lim_n[\mathrm{SP}^n\mathcal{H}, \mathcal{H}]$, $h = [-, \mathcal{H}]$. Supposing that

$(\mathcal{H}, 0)$ is a well-pointed space, then the inclusion $i_n : \mathrm{SP}^{n-1}\mathcal{H} \hookrightarrow \mathrm{SP}^n\mathcal{H}$ is a cofibration and we may thus assume that $[w]$ is given by a family of maps $w_n : \mathrm{SP}^n\mathcal{H} \longrightarrow \mathcal{H}$ such that $w_{n-1} = w_n \circ i_n$. If we further assume that $\tau_X = 1_{h(X)}$. Then we have that $w_1 \simeq \mathrm{id}_{\mathcal{H}}$, and we may suppose from the start that $w_1 = \mathrm{id}_{\mathcal{H}}$. We thus have that the maps w_n determine a map $w : \mathrm{SP}^\infty\mathcal{H} \longrightarrow \mathcal{H}$ that has the property that $w|_{\mathrm{SP}^n\mathcal{H}} = w_n$. In particular, it has the property that $w|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$. We have the following.

Lemma 5.5. *Let $\mathcal{H}$ have the homotopy type of a connected* CW*-complex. If there is a map $w : \mathrm{SP}^\infty\mathcal{H} \longrightarrow \mathcal{H}$ such that $w|_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$, then $\mathcal{H}$ has the homotopy type of a weak product $\widetilde{\prod}_{n\geq 0} K(\pi_n(\mathcal{H}), n)$, of Eilenberg-Mac Lane spaces.*

Proof: The homomorphism $w_* : \pi_n(\mathrm{SP}^\infty\mathcal{H}) \longrightarrow \pi_n(\mathcal{H})$ splits $i_* : \pi_n(\mathcal{H}) \longrightarrow \pi_n(\mathrm{SP}^\infty\mathcal{H})$ for all n. By the Dold-Thom theorem (see [2]), $\pi_n(\mathrm{SP}^\infty\mathcal{H}) \cong \widetilde{H}_n(\mathcal{H}; \mathbb{Z})$ and under this isomorphism, i_* corresponds to the Hurewicz homomorphism. Thus, the Hurewicz homomorphism is a split mono and hence by a theorem of Moore (see Theorem 5.1 in [3]) we have the result. ∎

Hence, by the previous lemma and Theorem 5.3, we have the following consequence.

Theorem 5.6. *Let $\mathcal{H}$ have the homotopy type of a connected* CW*-complex. There is an (h, h)-transfer τ for ramified covering maps such that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$ if and only if $\mathcal{H}$ has the homotopy type of a weak product*

$$\widetilde{\prod}_{n\geq 0} K(\pi_n(\mathcal{H}), n)\,,$$

of Eilenberg-Mac Lane spaces. ∎

Let now $\mathcal{H}$ be an abelian H-group with (strict) neutral element $0 \in \mathcal{H}$, and let its multiplication map be $\nu : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$. We can define a map $\nu_n : \mathcal{H}^n \longrightarrow \mathcal{H}$ by $\nu_0 = 0$, the constant map with value 0, $\nu_1 = \mathrm{id}_{\mathcal{H}}$, and inductively $\nu_n(a_1, \ldots, a_n) = \nu_2(\nu_{n-1}(a_1, \ldots, a_{n-1}), a_n)$. Then we define *multiplication by n in $\mathcal{H}$* as the map $\nu_n \circ \Delta_n : \mathcal{H} \longrightarrow \mathcal{H}$, where $\Delta_n : \mathcal{H} \longrightarrow \mathcal{H}^n$ is the diagonal map.

Proposition 5.7. *Let τ be a transfer for ramified covering maps classified by a family $w_n : \mathrm{SP}^n\mathcal{H} \longrightarrow \mathcal{H}$ and let $\mathcal{H}$ be an abelian H-group with multiplication given by ν, $n \in \mathbb{N}$. Assume, moreover, that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$. Then*

$\tau_n^p : h(E) \longrightarrow h(X)$ *is a homomorphism for every n-fold ramified covering map $p : E \longrightarrow X$ if and only if $w_n \circ \pi_n \simeq \nu_n$, where $\pi_n : \mathcal{H}^n \longrightarrow \mathrm{SP}^n\mathcal{H}$ is the identification.*

Proof: Take $[g], [g'] \in h(E) = [E, \mathcal{H}]$, then $[g] + [g'] = [\nu \circ (g \times g') \circ \Delta_E]$, where $\Delta_E : E \longrightarrow E \times E$ is the diagonal map. Consider the diagram

(5.8)

$$\begin{array}{ccccccccc}
 & \mathrm{SP}^n E & \xrightarrow{\mathrm{SP}^n \Delta_E} & \mathrm{SP}^n(E \times E) & \xrightarrow{\mathrm{SP}^n(g \times g')} & \mathrm{SP}^n(\mathcal{H} \times \mathcal{H}) & \xrightarrow{\mathrm{SP}^n \nu} & \mathrm{SP}^n\mathcal{H} & \\
{\scriptstyle \varphi_p}\nearrow & & & \updownarrow \approx & & \updownarrow \approx & & & \searrow {\scriptstyle w_n} \\
X & & & & & & & & \mathcal{H}\,. \\
{\scriptstyle \Delta_X}\searrow & & & & & & & & \nearrow {\scriptstyle \nu} \\
 & X \times X & \xrightarrow[\varphi_p \times \varphi_p]{} & \mathrm{SP}^n E \times \mathrm{SP}^n E & \xrightarrow[\mathrm{SP}^n g \times \mathrm{SP}^n g']{} & \mathrm{SP}^n\mathcal{H} \times \mathrm{SP}^n\mathcal{H} & \xrightarrow[w_n \times w_n]{} & \mathcal{H} \times \mathcal{H} &
\end{array}$$

Since the two left subdiagrams are always strict commutative, the full diagram is (homotopy) commutative if and only if the right subdiagram is (homotopy) commutative. To see the necessity of this commutativity, just consider the trivial n-fold covering map $p = \mathrm{id}_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H}$ and $g = g' = \mathrm{id}$. Hence, τ_n is a homomorphism if and only if the diagram

(5.9)
$$\begin{array}{ccccc}
\mathrm{SP}^n\mathcal{H} \times \mathrm{SP}^n\mathcal{H} & = & \mathrm{SP}^n(\mathcal{H} \times \mathcal{H}) & \xrightarrow{\mathrm{SP}^n \nu} & \mathrm{SP}^n\mathcal{H} \\
\downarrow {\scriptstyle w_n \times w_n} & & & & \downarrow {\scriptstyle w_n} \\
\mathcal{H} \times \mathcal{H} & & \xrightarrow[\nu]{\qquad\qquad} & & \mathcal{H}
\end{array}$$

is homotopy commutative. Since $w_1 \simeq \mathrm{id}_{\mathcal{H}}$, because $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$, in the case $n = 2$, the diagram means that we have two operations on $\mathcal{H}$ with a common zero, namely ν and $w_2 \circ \pi_2$, that are mutually distributive up to homotopy. By following the proof of Lemma [2, 2.10.10] up to homotopy, one can show that both operations are homotopic, that is, $w_2 \circ \pi_2 \simeq \nu_2$ (Observe that this means that ν factors through $\mathrm{SP}^2\mathcal{H}$ up to homotopy).

In the case $n = 1$, take the trivial 1-fold covering map $p = \mathrm{id}_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H}$ again and take $g = \mathrm{id}$, $g' = 0$. Then the commutativity of Diagram (5.8) shows that $w_1 \simeq \mathrm{id}$, since $\nu \circ (\mathrm{id}, 0) = \mathrm{id}$. Further, using the definition of ν_n and the commutativity up to homotopy of (5.9) one may prove inductively that $w_n \circ \pi_n \simeq \nu_n$. ■

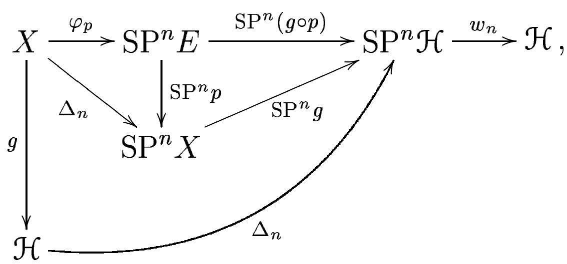
According to Proposition 3.8, we obtain the following.

Corollary 5.10. *If an (h,h)-transfer for ramified covering maps τ such that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$, yields a homomorphism for every n-fold ramified covering and every n, then it is unique (it is namely, the one defined in 2.4), and it has thus the property that for every n-fold ramified covering $p : E \longrightarrow X$, the composite $\tau_n^p \circ p^*$ is multiplication by n.*

Proof: Since $w_n \circ \pi_n \simeq \nu_n$, it follows that w_n is homotopic to the map

$$\langle a_1, \dots, a_n \rangle \longmapsto a_1 + \cdots + a_n \,.$$

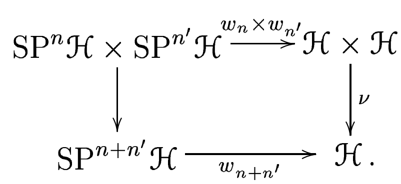
Hence by 3.8, τ is the transfer defined in 2.4 for $\mathcal{H}$. Moreover $\tau_n^p p^*[g] = n[g]$ as follows from the commutativity of the following diagram



where the top row provides $\tau_n^p p^*[g]$, since $w_n \simeq \nu_n$. Note that this follows also explicitly from the very Definition 2.4, as shown in Proposition 2.7. ■

From 5.7, we have the following.

Lemma 5.11. *Let $\mathcal{H}$ be a topological abelian group with multiplication given by $\nu : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$. Let moreover τ be an (h,h)-transfer for ramified covering maps such that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$ and that yields a homomorphism for every n-fold ramified covering and every n. If $w_n : \mathrm{SP}^n \mathcal{H} \longrightarrow \mathcal{H}$ are the classifying maps for τ, then for all n, n', the following diagram commutes up to homotopy:*



Proof: Since $w_n \circ \pi_n \simeq \nu_n$, and $\nu \circ (\nu_n \times \nu_{n'}) = \nu_{n+n'}$, as follows from the associativity of ν, the desired commutativity follows. ■

We obtain immediately the following.

Proposition 5.12. *Let $p : E \longrightarrow X$, resp. $p' : E' \longrightarrow X$, be an n-fold, resp. n'-fold, ramified covering map and take the $(n+n')$-fold ramified covering map $(p, p') : E \sqcup E' \longrightarrow X$ defined py p and p'. If τ is an (h, h)-transfer for ramified covering maps such that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$ and that yields a homomorphism for every n-fold ramified covering and every n, then*

$$\tau_{n+n'}^{(p,p')}[g] = \tau_n^p[f] + \tau_{n'}^{p'}[f'] ,$$

where $[g]$ corresponds to $([f], [f'])$ under the obvious isomorphism

$$[E \sqcup E', \mathcal{H}] \cong [E, \mathcal{H}] \oplus [E', \mathcal{H}] .$$

Proof: The following diagram commutes:

$$\begin{array}{ccccc}
 & \mathrm{SP}^n E \times \mathrm{SP}^{n'} E' & \xrightarrow{\mathrm{SP}^n f \times \mathrm{SP}^{n'} f'} & \mathrm{SP}^{n+n'}\mathcal{H} & \\
{}^{(\varphi_p,\varphi'_p)}\nearrow & \downarrow & & \downarrow & \searrow^{w_n+w_{n'}} \\
X & & & & \mathcal{H}\,, \\
{}_{\varphi_{(p,p')}}\searrow & \downarrow & & \downarrow & \nearrow_{w_{n+n'}} \\
 & \mathrm{SP}^{n+n'}(E\sqcup E') & \xrightarrow[\mathrm{SP}^{n+n'} g]{} & \mathrm{SP}^{n+n'}\mathcal{H} &
\end{array}$$

where $w_n + w_{n'} = \nu \circ (w_n \times w_{n'})$. Hence, the right-hand-side subdiagram commutes by Lemma 5.11. The other two subdiagrams commute obviously, where the vertical arrows are clear.

Following the diagram along the bottom yields $\tau_{n+n'}^{(p,p')}[g]$, while doing it along the top yields $\tau_n^p[f] + \tau_{n'}^{p'}[f']$. ■

In the same situation as above and identifying g with (f, f'), we obtain the following.

Corollary 5.13. *A family $\{\tau_n\}_{n=1,2,\ldots}$ such that each τ_n is an (h, h)-transfer for n-fold ramified covering maps and such that $\tau_1^{\mathrm{id}_X} = 1_{h(X)}$, determines an (h, h)-transfer for ramified covering maps if and only if*

$$\tau_{n+n'}^{(p,p')}[(f, f')] = \tau_n^p[f] + \tau_{n'}^{p'}[f'] .$$

■

In what follows, we shall show that the transfers for ramified covering maps in singular cohomology, i.e., the elements of $\lim_n T_n^R(h,k)$, where $h(X) = H^q(X;G)$ and $k(X) = H^q(X;G')$, are determined by the transfers for 2-fold ramified covering maps.

Theorem 5.14. *The restriction function*

$$r : T_{n+1}^R(H^q(-;G), H^q(-;G')) \longrightarrow T_n^R(H^q(-;G), H^q(-;G'))$$

is an isomorphism for $n \geq 2$ ($q > 0$).

Proof: Since $h = H^q(-;G)$, $\mathcal{H} = K(G;q)$, which is a CW-complex. By [12], $\mathrm{SP}^n\mathcal{H}$ is also a CW-complex and a subcomplex of $\mathrm{SP}^{n+1}\mathcal{H}$. Therefore, we have a (H-coexact) cofibration sequence

$$\mathrm{SP}^n\mathcal{H} \overset{i}{\hookrightarrow} \mathrm{SP}^{n+1}\mathcal{H} \overset{j}{\twoheadrightarrow} \mathrm{SP}^{n+1}\mathcal{H}/\mathrm{SP}^n\mathcal{H}\,.$$

Let us denote by $\widetilde{\mathrm{SP}}^{n+1}\mathcal{H}$ the *reduced symmetric product* of $\mathcal{H}$, i.e., the quotient of the action of Σ_{n+1} on the smash product $\mathcal{H}\wedge\cdots\wedge\mathcal{H}$ ($n+1$ factors). Clearly, $\mathrm{SP}^{n+1}\mathcal{H}/\mathrm{SP}^n\mathcal{H} \approx \widetilde{\mathrm{SP}}^{n+1}\mathcal{H}$. Using the exact cohomology sequence of the cofibration sequence above for the theory $k(X) = H^q(X;G')$ gives us an exact sequence

$$\widetilde{H}^q(\widetilde{\mathrm{SP}}^{n+1}\mathcal{H};G') \xrightarrow{j^*} H^q(\mathrm{SP}^{n+1}\mathcal{H};G') \xrightarrow{i^*} H^q(\mathrm{SP}^n\mathcal{H};G') \xrightarrow{\delta} \widetilde{H}^{q+1}(\widetilde{\mathrm{SP}}^{n+1}\mathcal{H};G')\,,$$

and Theorem 5.3, gives us the commutative diagram

$$\begin{array}{ccc}
H^q(\mathrm{SP}^{n+1}\mathcal{H};G') & \xrightarrow{\quad i^* \quad} & H^q(\mathrm{SP}^n\mathcal{H};G') \\
\cong \updownarrow & & \updownarrow \cong \\
T_{n+1}^R(H^q(-;G), H^q(-;G')) & \xrightarrow[r]{} & T_n^R(H^q(-;G), H^q(-;G'))\,.
\end{array}$$

Since $\mathcal{H} = K(G;q)$ is $(q-1)$-connected, by [12], $\widetilde{\mathrm{SP}}^{n+1}\mathcal{H}$ is $(2n-2+q)$-connected. Therefore, by the Hurewicz isomorphism theorem and the universal coefficient theorem, we have that $\widetilde{H}^q(\widetilde{\mathrm{SP}}^{n+1}\mathcal{H};G')$ and $\widetilde{H}^{q+1}(\widetilde{\mathrm{SP}}^{n+1}\mathcal{H};G')$ are both zero, provided that $n \geq 2$. Hence, r is an isomorphism for $n \geq 2$. ∎

REMARK 5.15. From Proposition 2.8 it follows that in the case that $\mathcal{H} = K(\mathbb{Z}_p, q) = \mathcal{K}$, if p is an odd prime, then w_2 is characterized uniquely up to homotopy by the equation $w_2 \circ \pi_2 \simeq \nu_2$ proved in Proposition 5.7.

In the case of theories other than the ordinary ones (given by Eilenberg-Mac Lane spaces), there are nontrivial transfers. The following is an interesting case.

EXAMPLE 5.16. We analyze transfers for vector bundles. Let $\mathcal{H}$ be $\mathrm{BU}(1)$ and $\mathcal{K}$ be $\mathrm{BU}(n)$. We denote by $\mathrm{Vect}^k_C(X)$ the set of isomorphism classes of numerable complex k-dimensional vector bundles over X. By 3.7, given an n-fold ramified covering map $p : E \longrightarrow X$, there is a bijection between transfers $t_p : \mathrm{Vect}^1_{\mathbb{C}}(E) \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X)$ and elements in $[\mathrm{SP}^n\mathrm{BU}(1), \mathrm{BU}(n)] \cong \mathrm{Vect}^n_{\mathbb{C}}(\mathrm{SP}^n\mathrm{BU}(1))$.

Notice that $\mathrm{BU}(1)$ is a topological abelian group of which one can give a model by defining $\mathrm{BU}(1) = F(\mathbb{S}^1, \mathrm{U}(1))$ (see [7]). This group structure is given in terms of line bundles by the tensor product. Since $\mathrm{BU}(1)$ is an Eilenberg-Mac Lane space of type $K(\mathbb{Z}, 2)$, this group structure also corresponds to the group structure in $H^2(X; \mathbb{Z})$, where one maps each line bundle to its first Chern class.

Consider the map $\nu : \mathrm{BU}(1)^n \longrightarrow \mathrm{BU}(1)$ defined by the product above, and which corresponds to the bundle $\gamma^1 \boxtimes \cdots \boxtimes \gamma^1$, where γ^1 is the universal line bundle. Since $\mathrm{BU}(1)$ is abelian, this map defines a map $\overline{\nu} : \mathrm{SP}^n\mathrm{BU}(1) \longrightarrow \mathrm{BU}(1)$. Now let $\rho : \mathrm{BU}(1) \longrightarrow \mathrm{BU}(n)$ be the classifying map of the Whitney sum $\gamma^1 \oplus \cdots \oplus \gamma^1$ of n copies of γ^1. Then the homotopy class of $\rho \circ \overline{\nu}$ defines a transfer as we mentioned above. To see that this transfer is not trivial, consider the diagonal map $d : \mathrm{BU}(1) \longrightarrow \mathrm{BU}(1)^n$. Denote by λ the tensor product $\gamma^1 \otimes \cdots \otimes \gamma^1$ of n copies of γ^1. Then the map $\rho \circ \overline{\nu} \circ p \circ d = \rho \circ \nu \circ d$ classifies the bundle $\lambda \oplus \cdots \oplus \lambda$ (n copies). By the comments above, $\gamma^1 \mapsto c_1(\gamma^1)$ yields an isomorphism, and $H^2(\mathrm{BU}(1); \mathbb{Z}) \cong \mathbb{Z}$, therefore $c_1(\lambda) = nc_1(\gamma^1) \neq 0$. Hence $c_1(\lambda \oplus \cdots \oplus \lambda) = c_1(\lambda) + \cdots + c_1(\lambda) = n^2 c_1(\gamma^1) \neq 0$, and so the transfer defined is not trivial.

EXAMPLE 5.17. Now, we analyze transfers for principal bundles. Let $\pi : P \longrightarrow E$ be a principal Γ-bundle, where Γ is a topological abelian group, and let $\{U_\alpha \mid \alpha \in \mathcal{I}\}$ be a trivializing open cover of E for π. Let

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma \mid \alpha, \beta \in \mathcal{I}\}$$

be a set of transition functions for the bundle, so that for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$ they satisfy

$$g_{\alpha\,\beta}(x) g_{\beta\,\gamma}(x) = g_{\alpha\,\gamma}(x)\,.$$

The nth product bundle $\pi^n : P^n \longrightarrow E^n$ with fiber Γ^n is a principal bundle with trivializing open cover $\{U_{\alpha_1} \times \cdots \times U_{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathcal{I}^n\}$ and transition functions

$$g_{(\alpha_1,\ldots,\alpha_n)\,(\beta_1,\ldots,\beta_n)} : (U_{\alpha_1} \times \cdots \times U_{\alpha_n}) \cap (U_{\beta_1} \times \cdots \times U_{\beta_n}) =$$

$$(U_{\alpha_1} \cap U_{\beta_1}) \times \cdots \times (U_{\alpha_n} \cap U_{\beta_n}) \xrightarrow{g_{\alpha_1\,\beta_1} \times \cdots \times g_{\alpha_n\,\beta_n}} \Gamma^n\,.$$

Let $q : E^n \longrightarrow \mathrm{SP}^n E$ be the quotient map and define $U_{(\alpha_1,\ldots,\alpha_n)} = q(U_{\alpha_1} \times \cdots \times U_{\alpha_n})$, which is an open set since q is an open map. Given two multi-indexes $(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n) \in \mathcal{I}^n$, define

$$g'_{(\alpha_1,\ldots,\alpha_n)\,(\beta_1,\ldots,\beta_n)} : U_{(\alpha_1,\ldots,\alpha_n)} \cap U_{(\beta_1,\ldots,\beta_n)} \longrightarrow \Gamma$$

by

$$\begin{array}{ccc}
(U_{\alpha_1} \times \cdots \times U_{\alpha_n}) \cap (U_{\beta_1} \times \cdots \times U_{\beta_n}) & \xrightarrow{g_{(\alpha_1,\ldots,\alpha_n)\,(\beta_1,\ldots,\beta_n)}} & \Gamma^n \\
{\scriptstyle q|}\downarrow & & \downarrow \\
U_{(\alpha_1,\ldots,\alpha_n)} \cap U_{(\beta_1,\ldots,\beta_n)} & \xrightarrow{\widetilde{g}} & \mathrm{SP}^n\Gamma \\
& \searrow{\scriptstyle g'_{(\alpha_1,\ldots,\alpha_n)\,(\beta_1,\ldots,\beta_n)}} & \downarrow \\
& & \Gamma\,,
\end{array}$$

where the restriction of q on the left-hand side is a surjective open map, thus an identification. Hence $\widetilde{g}$ is well defined. The map $\mathrm{SP}^n\Gamma \longrightarrow \Gamma$ given by multiplying the n elements of Γ is well defined too, since Γ is abelian, and its composite with $\widetilde{g}$ yields the desired map. In order to see that these maps $g'_{(\alpha_1,\ldots,\alpha_n)\,(\beta_1,\ldots,\beta_n)}$ are transition functions, we check the relevant relation. Take $x \in U_{(\alpha_1,\ldots,\alpha_n)} \cap U_{(\beta_1,\ldots,\beta_n)} \cap U_{(\gamma_1,\ldots,\gamma_n)}$. Thus $x = q(x_1, \ldots, x_n)$, where $x_i \in U_{\alpha_i} \cap U_{\beta_i} \cap U_{\gamma_i}$, $i = 1, \ldots, n$. These elements are well defined, since we are taking ordered multi-indexes (permutations in the multi-indexes yield different transition functions, even with the same open sets in $\mathrm{SP}^n E$). We have

$$
\begin{aligned}
g'_{(\alpha_1,\dots,\alpha_n)\,(\beta_1,\dots,\beta_n)}(x)g'_{(\beta_1,\dots,\beta_n)\,(\gamma_1,\dots,\gamma_n)}(x) &= \\
&= g_{\alpha_1\,\beta_1}(x_1)g_{\beta_1\,\gamma_1}(x_1)\cdots g_{\alpha_n\,\beta_n}(x_n)g_{\beta_n\,\gamma_n}(x_n) \\
&= g_{\alpha_1\,\gamma_1}(x_1)\cdots g_{\alpha_n\,\gamma_n}(x_n) \\
&= g'_{(\alpha_1,\dots,\alpha_n)\,(\gamma_1,\dots,\gamma_n)}(x)\,.
\end{aligned}
$$

Thus these maps are, indeed, transition functions for a new principal Γ-bundle $\pi' : E' \longrightarrow \mathrm{SP}^n E$.

Assume now that $p : E \longrightarrow X$ is an n-fold ramified covering map and take $\varphi_p : X \longrightarrow \mathrm{SP}^n E$. If $\pi : P \longrightarrow E$ is principal Γ-bundle as above, then define the bundle $\tau^p(\pi) : \tau^p(P) \longrightarrow X$ to be the pullback of $\pi' : E' \longrightarrow \mathrm{SP}^n E$ over φ_p.

If the given principal Γ-bundle $\pi : P \longrightarrow E$ is numerable and is classified by a map $f_\pi : E \longrightarrow B\Gamma$, then $\tau^p(\pi) : \tau^p(P) \longrightarrow X$ is classified by the composite $\nu \circ \mathrm{SP}^n f_\pi \circ \varphi_p : X \longrightarrow B\Gamma$, where $\nu : \mathrm{SP}^n B\Gamma \longrightarrow B\Gamma$ is given by the multiplication in $B\Gamma$. Indeed, if Γ is an abelian group, one can give a model of $B\Gamma$ that is also an abelian (topological) group. For instance, we may take $B\Gamma = F(\mathbb{S}^1, \Gamma)$ as the McCord topological group (see [7, 9.17]).

If we denote by $\mathrm{Prin}_\Gamma(Y)$ the set of (equivalence classes of) numerable principal Γ-bundles over Y, then the transfer of the n-fold ramified covering map $p : E \longrightarrow X$ is a function

$$
\tau^p : \mathrm{Prin}_\Gamma(E) \longrightarrow \mathrm{Prin}_\Gamma(X) \quad \text{or, equivalently,} \quad \tau^p : [E, B\Gamma] \longrightarrow [X, B\Gamma]\,.
$$

According to the classification result 3.7, this transfer corresponds precisely to the element $[\nu] \in [\mathrm{SP}^n B\Gamma, B\Gamma]$.

A modified version of the previous example is the following.

EXAMPLE 5.18. Let $\pi : \xi \longrightarrow E$ be a (numerable) k-dimensional $\mathbb{F}$-vector bundle ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), and let

$$
\{g_{\alpha\,\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k, \mathbb{F}) \mid \alpha, \beta \in \mathfrak{I}\}
$$

be a set of transition functions for this bundle.

Consider the *determinant bundle* $\det \pi : \det \xi \longrightarrow E$, whose transition functions are given by

$$
\{\det \circ g_{\alpha\,\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{F}^* = \mathrm{GL}(1, \mathbb{F}) \mid \alpha, \beta \in \mathfrak{I}\}\,,
$$

where $\det : \mathrm{GL}(k, \mathbb{F}) \longrightarrow \mathbb{F}^*$ is the determinant function. Applying the construction given in Example 5.17 for $\Gamma = \mathbb{F}^*$, we obtain a line bundle $\pi' : \xi' \longrightarrow \mathrm{SP}^n E$ for any n.

As in 5.17, if $p : E \longrightarrow X$ is an n-fold ramified covering map with $\varphi_p : X \longrightarrow \mathrm{SP}^n E$, then the pullback of π' over φ_p defines a bundle $\tau_n^p(\pi) : \tau_n^p(\xi) \longrightarrow X$. Thus we have a transfer

$$\tau_n^p : \mathrm{Vect}_k^{\mathbb{F}}(E) = [E, \mathrm{BGL}(k, \mathbb{F})] \longrightarrow [X, \mathrm{B}\mathbb{F}^*] = \mathrm{Vect}_1^{\mathbb{F}}(\mathrm{X}) \,.$$

To compute the group of transfers of the previous example in the complex case, we need the following.

Lemma 5.19. *Let* $\det_k : \mathrm{U}(k) \longrightarrow \mathrm{U}(1)$ *be the determinant function. Then* $\mathrm{Bdet}_k : \mathrm{BU}(k) \longrightarrow \mathrm{BU}(1)$ *is such that* $[\mathrm{Bdet}] = c_1(\gamma^k)$*, the first Chern class of the universal bundle, thus it is a generator of* $H^2(\mathrm{BU}(k)) \cong \mathbb{Z}$*.*

Proof: Take $\mathrm{BU}(k) = F(\mathbb{S}^1, \mathrm{U}(k))$. Then $\mathrm{Bdet}_k : F(\mathbb{S}^1, \mathrm{U}(k)) \longrightarrow F(\mathbb{S}^1, \mathrm{U}(1))$ is given by $\mathrm{Bdet}_k(u) = \det_k \circ u$. Since $\mathrm{BU}(1) = F(\mathbb{S}^1, \mathrm{U}(1))$ is a $K(\mathbb{Z}, 2)$, we take it to represent the second cohomology groups. If $k = 1$, then $\det_1 : \mathrm{U}(1) \longrightarrow \mathrm{U}(1)$ is the identity, hence

$$[\mathrm{Bdet}_1] = [\mathrm{id}] \in H^2(\mathrm{BU}(1)) = [\mathrm{BU}(1), \mathrm{BU}(1)] \,,$$

which is the generator, since we are dealing with Eilenberg-Mac Lane spaces, and we have that

$$[\mathrm{BU}(1), \mathrm{BU}(1)] \xrightarrow{\cong} \mathrm{Hom}(\pi_2(\mathrm{BU}(1)), \pi_2(\mathrm{BU}(1))) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \,.$$

By induction, we assume that the homotopy class of $\mathrm{Bdet}_k : \mathrm{BU}(k) \longrightarrow \mathrm{BU}(1)$ is $c_1(\gamma^k)$, and consider $H^2(\mathrm{BU}(k+1)) \xrightarrow{Bi^*} H^2(\mathrm{BU}(k))$, where i is the inclusion map. By naturality and stability of the Chern classes, $Bi^* c_1(\gamma^{k+1}) = c_1(\gamma^k)$. Since these classes are the generators, Bi^* is an isomorphism. Recalling that $i : \mathrm{U}(k) \longrightarrow \mathrm{U}(k+1)$ maps a matrix A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, one has that $Bi^*[\mathrm{Bdet}_{k+1}] = [\mathrm{Bdet}_k]$, since

$$\mathrm{Bdet}_{k+1} \circ Bi : x \longmapsto \det_{k+1} iu(x) = \det_{k+1} \begin{pmatrix} u(x) & 0 \\ 0 & 1 \end{pmatrix} = \det_k u(x) \,.$$

Therefore, $[\mathrm{Bdet}_{k+1}] = c_1(\gamma^{k+1})$ and thus it is a generator. ∎

We have the following result.

Theorem 5.20. $T^R(\mathrm{Vect}_k^{\mathbb{C}}(-),\mathrm{Vect}_1^{\mathbb{C}}(-))\cong\mathbb{Z}$ *and the family of transfers* τ_n^p *constructed in Example 5.19 constitute a generator.*

Proof: First we analyze the situation for the transfers for n-fold ramified covering maps. By Corollary 3.10,

$$T_n^R(\mathrm{Vect}_k^{\mathbb{C}}(-),\mathrm{Vect}_1^{\mathbb{C}}(-))\cong H^2(\mathrm{SP}^n\mathrm{BGL}(k,\mathbb{C});\mathbb{Z})\,.$$

Without losing generality, we may write $\mathrm{U}(k)$ instead of $\mathrm{GL}(k,\mathbb{C})$. Making use of the fibration $\mathrm{U}(k-1)\hookrightarrow\mathrm{U}(k)\longrightarrow\mathbb{S}^{2k-1}$, one easily shows that $\pi_1(\mathrm{U}(k))\cong\mathbb{Z}$ for $k\geq 1$. With this and the same arguments used in the proof of 3.13, one has

$$H^2(\mathrm{SP}^n\mathrm{BGL}(k,\mathbb{C});\mathbb{Z})\cong\mathbb{Z}\,.$$

Now, consider the inverse system

$$\cdots\longrightarrow H^2(\mathrm{SP}^3\mathrm{BU}(k))\longrightarrow H^2(\mathrm{SP}^2\mathrm{BU}(k))\longrightarrow H^2(\mathrm{BU}(k))\,. \tag{5.21}$$

To show that all arrows are isomorphisms, take the cofiber sequence $\mathrm{SP}^nX\hookrightarrow\mathrm{SP}^{n+1}X\twoheadrightarrow\overline{\mathrm{SP}}^{n+1}X=\mathrm{SP}^{n+1}X/\mathrm{SP}^nX$. By [12] we have that if X is $(l-1)$-connected, then $\overline{\mathrm{SP}}^{n+1}X$ is $(2n+l-2)$-connected. Therefore, since $\mathrm{BU}(k)$ is 1-connected, in the exact sequence

$$\begin{array}{ccccccc}\widetilde{H}^2(\overline{\mathrm{SP}}^{n+1}\mathrm{BU}(k)) & \to & \widetilde{H}^2(\mathrm{SP}^{n+1}\mathrm{BU}(k)) & \overset{\cong}{\to} & \widetilde{H}^2(\mathrm{SP}^n\mathrm{BU}(k)) & \to & \widetilde{H}^3(\overline{\mathrm{SP}}^{n+1}\mathrm{BU}(k))\\ \| & & & & & & \|\\ 0 & & & & & & 0\end{array}$$

the middle arrow is an isomorphism if $n\geq 2$. In order to see that the last arrow on the right in the inverse system (5.21) is also an isomorphism, we do the following. Take $\mathrm{BU}(k)=F(\mathbb{S}^1,\mathrm{U}(k))$, and take as base point $*$ the function given by $*(s)=\mathbf{1}$, the identity matrix, for all $s\in\mathbb{S}^1$. In particular, $\mathrm{BU}(1)$ is a topological abelian group. On the other hand the inclusions $\mathrm{SP}^n\mathrm{BU}(k)\hookrightarrow\mathrm{SP}^{n+1}\mathrm{BU}(k)$ are given by $\langle u_1\ldots,u_n\rangle\mapsto\langle u_1\ldots,u_n,*\rangle$. By Lemma 5.19, the generator of $H^2(\mathrm{BU}(k))=[\mathrm{BU}(k),\mathrm{BU}(1)]$ as an infinite cyclic group is given by $[\mathrm{Bdet}]$; on the other hand, the homotopy classes of the maps β_n given by the diagrams

$$\begin{array}{ccccc}\mathrm{BU}(k)^n & \xrightarrow{(\mathrm{Bdet})^n} & \mathrm{BU}(1)^n & \longrightarrow & \mathrm{BU}(1)\\ \downarrow & & & \nearrow_{\beta_n} & \\ \mathrm{SP}^n\mathrm{BU}(k) & & & & \end{array}$$

seen as elements in $\widetilde{H}^2(\mathrm{SP}^n\mathrm{BU}(k))$, where the top arrow on the right-hand side is given by the abelian multiplication in $\mathrm{BU}(1)$, obviously map to each other in the inverse system (5.21). In particular, $[\beta_2] \mapsto [\beta_1] = [\mathrm{Bdet}]$. So, the last arrow on the right of the inverse system is surjective, and thus it is an isomorphism. Hence, all arrows are isomorphisms and the elements $[\beta_n]$ are generators of the infinite cyclic groups, and since each τ_n^p corresponds to $[\beta_n]$, the family $\tau^p = \{\tau_n^p\}$ is a transfer for ramified covering maps. Consequently, $T^R(\mathrm{Vect}_k^{\mathbb{C}}(-), \mathrm{Vect}_1^{\mathbb{C}}(-)) \cong \lim_n H^2(\mathrm{SP}^n\mathrm{BU}(k)) \cong \mathbb{Z}$, and τ^p is the generator. ■

6 Transfers in 1-dimensional integral cohomology

We consider $(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z}))$-transfers for n-fold ramified covering maps as well as for n-fold covering maps. We denote by $\Sigma_n \wr \mathbb{Z}$ the wreath product of Σ_n and $\mathbb{Z}$, i.e., the semidirect product of Σ_n with $\mathbb{Z}^n$, where Σ_n acts on $\mathbb{Z}^n$ by permuting the summands. Therefore, the product in $\Sigma_n \wr \mathbb{Z}$ is given by $(\sigma, a_1, \ldots, a_n) \cdot (\tau, b_1, \ldots, b_n) = (\sigma\tau, a_{\tau(1)} + b_1, \ldots, a_{\tau(n)} + b_n)$.

Lemma 6.1. *The following hold.*

(a) $B(\Sigma_n \wr \mathbb{Z}) = E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n$.

(b) *Let* $f : \Sigma_n \wr \mathbb{Z} \longrightarrow \mathbb{Z}$ *be the homomorphism defined by* $f(\sigma, a_1, \ldots, a_n) = a_1 + \cdots + a_n$. *Then* $Bf : B(\Sigma_n \wr \mathbb{Z}) \longrightarrow B\mathbb{Z}$ *is given by*

$$\varphi : E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n \longrightarrow \mathbb{R}/\mathbb{Z}\,, \quad \textit{where} \quad \varphi\langle y, \bar{t}_1, \ldots, \bar{t}_n\rangle = \bar{t}_1 + \cdots + \bar{t}_n\,.$$

Proof: (a) Consider the space $E\Sigma_n \times \mathbb{R}^n$. The space is contractible and has a free action of $\Sigma_n \wr \mathbb{Z}$ given by

$$(y, t_1, \ldots, t_n) \cdot (\sigma, a_1, \ldots, a_n) = (y \cdot \sigma, t_{\sigma(1)} + a_1, \ldots, t_{\sigma(n)} + a_n)\,.$$

Therefore, $B(\Sigma_n \wr \mathbb{Z}) = (E\Sigma_n \times \mathbb{R}^n)/(\Sigma_n \wr \mathbb{Z})$. Now consider the following diagram

$$\begin{array}{ccc} E\Sigma_n \times \mathbb{R}^n & \xrightarrow{\mathrm{id} \times q^n} & E\Sigma_n \times (\mathbb{R}/\mathbb{Z})^n \\ \downarrow & & \downarrow \\ (E\Sigma_n \times \mathbb{R}^n)/(\Sigma_n \wr \mathbb{Z}) & \dashrightarrow & E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n\,. \end{array}$$

Since the quotient map $q : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is a covering map, so is also $\mathrm{id} \times q^n$; in particular, it is a quotient map as are also the two vertical maps. Thus they clearly define a homeomorphism $(E\Sigma_n \times \mathbb{R}^n)/(\Sigma_n \int \mathbb{Z}) \longrightarrow E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n$ given by $\langle y, t_1, \ldots, t_n \rangle \mapsto \langle y, \bar{t}_1, \ldots, \bar{t}_n \rangle$.

(b) Consider the action $\Sigma_n \int \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ given by

$$(\sigma, a_1, \ldots, a_n) \cdot k = f(\sigma, a_1, \ldots, a_n) + k = a_1 + \cdots a_n + k\,.$$

By part (a), we have a principal $\Sigma_n \int \mathbb{Z}$-bundle $p : E\Sigma_n \times \mathbb{R}^n \longrightarrow E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n$, where $p(y, t_1, \ldots, t_n) = \langle y, \bar{t}_1, \ldots, \bar{t}_n \rangle$. Then Bf classifies the associated principal $\mathbb{Z}$-bundle $\bar{p} : (E\Sigma_n \times \mathbb{R}^n) \times_{\Sigma_n \int \mathbb{Z}} \mathbb{Z} \longrightarrow E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n$.

Now consider the following diagram

$$\begin{array}{ccc} (E\Sigma_n \times \mathbb{R}^n) \times_{\Sigma_n \int \mathbb{Z}} \mathbb{Z} & \xrightarrow{\psi} & \mathbb{R} \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle q} \\ E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n & \xrightarrow[\varphi]{} & \mathbb{R}/\mathbb{Z}\,, \end{array}$$

where $\psi\langle (y, t_1, \ldots, t_n), k \rangle = t_1 + \cdots + t_n + k$. Clearly this is a morphism of principal $\mathbb{Z}$-bundles; therefore, $Bf \simeq \varphi$. ■

Lemma 6.2. $H^1(E\Sigma_n \times_{\Sigma_n} K(\mathbb{Z}, 1)^n; \mathbb{Z}) \cong \mathbb{Z}$.

Proof: Since $K(\mathbb{Z}, 1) = B\mathbb{Z}$ and by 6.1, $E\Sigma_n \times_{\Sigma_n} B\mathbb{Z}^n = B(\Sigma_n \int \mathbb{Z})$, we have that $H^1(E\Sigma_n \times_{\Sigma_n} K(\mathbb{Z}, 1)^n; \mathbb{Z}) \cong H^1(B(\Sigma_n \int \mathbb{Z}); \mathbb{Z}) \cong \mathrm{Hom}(\Sigma_n \int \mathbb{Z}, \mathbb{Z})$. Let $\iota : \mathbb{Z}^n \hookrightarrow \Sigma_n \int \mathbb{Z}$ be the inclusion given by $\iota(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$, and consider $\iota^* : \mathrm{Hom}(\Sigma_n \int \mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$. Let $F : \Sigma_n \int \mathbb{Z} \longrightarrow \mathbb{Z}$ be a homomorphism and assume that $\iota^*(F) = 0$. Then $F(1, a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Since any element $(\sigma, a_1, \ldots, a_n) \in \Sigma_n \int \mathbb{Z}$ can be written as $(\sigma, a_1, \ldots, a_n) = (\sigma, 0, \ldots, 0) \cdot (1, a_1, \ldots, a_n)$, we have that $F(\sigma, a_1, \ldots, a_n) = F(\sigma, 0, \ldots, 0) + F(1, a_1, \ldots, a_n) = F(\sigma, 0, \ldots, 0)$. But σ is an element of Σ_n, that is a finite subgroup of $\Sigma_n \int \mathbb{Z}$ and the codomain is free, hence $F(\sigma, 0, \ldots, 0) = 0$. Therefore, $F = 0$, so that ι^* is a monomorphism.

Let e_i be the element in $\mathbb{Z}^n$ whose coordinates are all zero, except the ith one that is equal to 1. Let $\tau \in \Sigma_n$ be the permutation given by $\tau(i) = j$ and $\tau(k) = k$ for $k \neq i, j$. Then $(1, e_i) \cdot (\tau, 0) = (\tau, e_j) = (\tau, 0) \cdot (1, e_j)$. Hence $F((1, e_i) \cdot (\tau, 0)) = F((\tau, 0) \cdot (1, e_j))$. But $F((1, e_i) \cdot (\tau, 0)) = F(1, e_i)$ and $F((\tau, 0) \cdot (1, e_j)) = F(1, e_j)$. Therefore, $F(1, e_i) = F(1, e_j)$. Since we have an isomorphism $\psi : \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^n$, given by $\psi(f) = (f(e_1), \ldots, f(e_n))$, then $\mathrm{im}(\psi \circ \iota^*)$ is the diagonal subgroup in $\mathbb{Z}^n$, that is isomorphic to $\mathbb{Z}$. Moreover, the canonical element Ω in $\mathrm{Hom}(\Sigma_n \wr \mathbb{Z}, \mathbb{Z})$ given by $\Omega(\sigma, a_1, \ldots, a_n) = a_1 + \cdots + a_n$ is a generator because $\psi\iota^*(\Omega) = (\iota^*(\Omega)(e_1), \ldots, \iota^*(\Omega)(e_n)) = (1, \ldots, 1)$. ■

As a consequence of Lemma 6.2, we have that the $(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z}))$-transfers for n-fold ramified covering maps are exactly the same as the $(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z}))$-transfers for n-fold ordinary covering maps. Moreover, there is exactly one transfer for each integer. In other words, we have the following.

Theorem 6.3. *The restriction*

$$r : T_n^R(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z})) \longrightarrow T_n(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z}))$$

is an isomorphism, and both groups are isomorphic to $\mathbb{Z}$.

Proof: By 4.3, we have a commutative diagram

$$\begin{array}{ccc} T_n^R(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z})) & \xrightarrow{\cong} & H^1(\mathrm{SP}^n(\mathbb{R}/\mathbb{Z});\mathbb{Z}) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \rho^*} \\ T_n(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z})) & \xrightarrow[\cong]{} & H^1(E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n;\mathbb{Z})\,. \end{array}$$

By [2, 5.2.23], the canonical inclusion $j : \mathbb{R}/\mathbb{Z} \hookrightarrow \mathrm{SP}^n(\mathbb{R}/\mathbb{Z})$ is a homotopy equivalence. Let $w : \mathrm{SP}^n(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}/Z$ be the map defined by $w\langle \bar{t}_1, \ldots, \bar{t}_n \rangle = \bar{t}_1 + \cdots + \bar{t}_n$. Then $j^*[w] = [w \circ j] = [\mathrm{id}]$, which is the generator of $H^1(\mathbb{R}/\mathbb{Z});\mathbb{Z})$. Therefore, $H^1(\mathrm{SP}^n(\mathbb{R}/\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}$, with generator given by $[w]$.

By Lemma 6.1(a),

$$H^1(E\Sigma_n \times_{\Sigma_n} (\mathbb{R}/\mathbb{Z})^n;\mathbb{Z}) \cong H^1(B(\Sigma_n \wr \mathbb{Z});\mathbb{Z}) = [B(\Sigma_n \wr \mathbb{Z}), B\mathbb{Z}]\,.$$

By [2, 6.4.6],

$$[B(\Sigma_n \wr \mathbb{Z}), B\mathbb{Z}] \cong \mathrm{Hom}(\pi_1(B(\Sigma_n \wr \mathbb{Z})), \pi_1(B\mathbb{Z})) = \mathrm{Hom}(\Sigma_n \wr \mathbb{Z}, \mathbb{Z})\,,$$

and by Lemma 6.2, $\mathrm{Hom}(\Sigma_n \wr \mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, with a generator $f : \Sigma_n \wr \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(\sigma, a_1, \ldots, a_n) = a_1 + \cdots + a_n$. Therefore, by the naturality of the homotopy equivalence $\Omega BG \simeq G$ for any discrete group G, the generator of $H^1(B(\Sigma_n \wr \mathbb{Z});\mathbb{Z})$ is given by Bf. By Lemma 6.1(b), $Bf \simeq \varphi$. Since $\rho^*[w] = [w \circ \rho]$, and $w\rho\langle y, \bar{t}_1, \ldots, \bar{t}_n \rangle = w\langle \bar{t}_1, \ldots, \bar{t}_n \rangle = \bar{t}_1 + \cdots + \bar{t}_n$, $w \circ \rho = \varphi$. Therefore, ρ^* is an isomorphism and therewith r is also an isomorphism. ■

As an immediate consequence we have the following.

Corollary 6.4. *There is an isomorphism* $T_n^R(H^1(-;\mathbb{Z}), H^1(-;\mathbb{Z})) \longrightarrow \mathbb{Z}$*. The canonical transfer* τ*, as given in 2.4, corresponds to* $1 \in \mathbb{Z}$*. For any other integer* k *the corresponding transfer is* ${}_k\tau$ *given by* $({}_k\tau)_n^p(\eta) = k(\tau_n^p(\eta))$ *for any* n*-fold ramified covering map* $p : E \longrightarrow X$ *and any element* $\eta \in H^1(E;\mathbb{Z})$*.*

■

References

[1] J. F. ADAMS, *Infinite Loop Spaces*, Annals of Math. Studies No. 90, Princeton Univ. Press, Princeton, NJ 1978

[2] M. AGUILAR, S. GITLER, C. PRIETO, *Algebraic Topology from a Homotopical Viewpoint*, Universitexts, Springer-Verlag, New York Berlin Heidelberg 2002

[3] M. AGUILAR, C. PRIETO, Transfers for ramified coverings in homology and cohomology, 2003 (to appear).

[4] A. DOLD, Ramified coverings, orbit projections and symmetric powers, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 65–72

[5] P. J. HUBER, Homotopical cohomology and Čech cohomology, *Math. Annalen* **144** (1961), 73–76.

[6] J. P. MAY, *A Concise Course in Algebraic Topology,* Chicago Lecture Notes in Mathematics, The University of Chicago Press, Chicago and London 1999.

[7] M.C. MCCORD, Classifying spaces and infinite symmetric products, *Trans. Amer. Math. Soc.* **146** (1969), 273–298

[8] F. W. ROUSH, *Transfer in Generalized Cohomology Theories*, Pure and Applied Mathematics, Akadémiai Kiadó, Budapest 1999

[9] L. SMITH, Transfer and ramified coverings, *Math. Proc. Camb. Phil. Soc.* **93** (1983), 485–493

[10] E. H. SPANIER, Function spaces and duality, *Ann. of Math.* **70** (1959), 338–378

[11] N. STEENROD, Cohomology operations, and obstructions to extending continuous functions, *Advances in Math.* **8** (1972), 371–416

[12] P. J. Welcher, Symmetric products and the stable Hurewicz homomorphism, *Illinois J. Math.* **24** (1980), 527–544

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