THE σ-ORIENTATION IS AN H_∞ MAP

MATTHEW ANDO, MICHAEL J. HOPKINS, AND NEIL P. STRICKLAND

Abstract. In [AHS01] the authors introduced the notion of an *elliptic spectrum*, and constructed a natural map from the Thom spectrum $MU\langle 6\rangle$ to any elliptic spectrum. $MU\langle 6\rangle$ is an H_∞ ring spectrum, and in this paper we show that if $\mathbf{E}$ is a $K(2)$-local, H_∞ elliptic spectrum, then the σ-orientation is a map of H_∞ ring spectra.

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1. Introduction

In [Hop95, AHS01], we introduced the notion of an *elliptic spectrum* and showed that any elliptic spectrum $(\mathbf{E}, C, t)$ admits a *canonical* $MU\langle 6\rangle$ orientation

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{E},C,t)} \mathbf{E}$$

called the *σ-orientation* (see also §19). We conjectured that the spectrum TMF of "topological modular forms" of Hopkins and Miller admits an $MO\langle 8\rangle$ orientation, such that for any elliptic spectrum $(\mathbf{E}, C, t)$ the diagram

$$\begin{tikzcd} MO\langle 8\rangle \arrow[r] & TMF \arrow[d] \\ MU\langle 6\rangle \arrow[u] \arrow[r, "\sigma(\mathbf{E},C,t)"] & \mathbf{E} \end{tikzcd}$$

commutes. For more about the conjecture, see [Hop95] and the introduction to [AHS01].

The conjecture seems now to be within reach, although that is the subject of another paper in preparation. The proof depends on the following feature of the σ-orientation, which was not proved in [AHS01]. Let C_0 be a supersingular elliptic curve over a perfect field k of characteristic $p > 0$, and let $\mathbf{E}$ be the even periodic ring spectrum associated to the universal deformation of the formal group of C_0 (so it is a form of E_2). The Serre-Tate theorem endows $\mathbf{E}$ with the structure of an elliptic spectrum (see §17.5), and so a map of ring spectra

$$\sigma : MU\langle 6\rangle \to \mathbf{E}. \tag{1.1}$$

Goerss and Hopkins, building on work of Hopkins and Miller, have shown that $\mathbf{E}$ is an E_∞ ring spectrum [GH02]; it is classical that $MU\langle 6\rangle$ is. We need to know that the map (1.1) is an H_∞ map. We prove that in this paper. (The paper of Goerss and Hopkins has not yet been published. Our result depends only on the existence of the H_∞ structure, and so a cautious statement of the our result is that if $\mathbf{E}$ is an H_∞ ring spectrum, then the map (1.1) is H_∞. See Remark 14.12.)

In Part 1, we study the general problem of showing that an orientation

$$MU\langle 2k\rangle \xrightarrow{g} \mathbf{E}$$

is H_∞, i.e. that for each n the diagram

$$\begin{tikzcd} D_n MU\langle 6\rangle \arrow[r, "D_n g"] \arrow[d] & D_n\mathbf{E} \arrow[d] \\ MU\langle 6\rangle \arrow[r, "g"] & \mathbf{E} \end{tikzcd} \tag{1.2}$$

commutes up to homotopy. Our analysis is based on [And95], which treats the case of $MU\langle 0\rangle$, the Thom spectrum associated to $BU\langle 0\rangle = \mathbb{Z}\times BU$. We review that case in §5, in a form which generalizes easily to $MU\langle 6\rangle$.

Briefly, suppose that $\mathbf{E}$ is a homotopy commutative ring spectrum with the property that $\pi_{\mathrm{odd}}\mathbf{E} = 0$ and $\pi_2\mathbf{E}$ contains a unit, so $\pi_0\mathbf{E}^{\mathbb{C}P^\infty_+}$ is the ring of functions on a formal group $G = G_{\mathbf{E}}$ over $S = \pi_0\mathbf{E}$. If

$$MU\langle 0\rangle \xrightarrow{g} \mathbf{E}$$

is an orientation, then the composition

$$(\mathbb{C}P^\infty)^L \to MU\langle 0\rangle \xrightarrow{g} \mathbf{E}$$

represents a trivialization s_g of the ideal sheaf $\mathcal{I}_G(0)$ of functions on G which vanish at the identity, that is, a coordinate on the formal group G. The association

$$g \mapsto s_g$$

gives a bijection between $MU\langle 0\rangle$-orientations on $\mathbf{E}$ and coordinates on G.

Suppose in addition that $\pi_0\mathbf{E}$ is a complete Noetherian local ring of residue characteristic $p > 0$, and the height of the formal group G is finite. In §3, following [And95], we show that an H_∞ structure on $\mathbf{E}$ adds the following structure to the formal group G. Given a map (of complete local rings) $i : S \to R$, a finite abelian group A, and a level structure (in the sense of [Dri74]; see §10)

$$\ell : A \to i^*G(R), \tag{1.3}$$

there is a map $\psi_\ell : S \to R$, and an isogeny $f_\ell : i^*G \to \psi_\ell^* G$ with kernel A. (The behavior of this structure with respect to variation in A gives *descent data for level structures* as described in Definition 3.1 or Proposition 13.14.)

If s is the coordinate on G associated to an orientation g, then the H_∞ structure gives *two* coordinates on $\psi_\ell^* G$: one ($\psi_\ell^* s$) comes from pulling back along ψ_ℓ; the other ($N_\ell i^* s$) is obtained from the invariant function

$$\prod_{a \in A} T_a^* i^* s \tag{1.4}$$

on i^*G by descent along the isogeny f_ℓ (see §12). In §5.2, we show that these two coordinates arise from the two ways of navigating the diagram

$$\begin{array}{ccccc} (BA^* \times \mathbb{C}P^\infty)^{V_{\mathrm{reg}} \otimes L} & \longrightarrow & D_n MU\langle 0\rangle & \xrightarrow{D_n g} & D_n\mathbf{E} \\ & & \downarrow & & \downarrow \\ & & MU\langle 0\rangle & \xrightarrow{g} & \mathbf{E}, \end{array}$$

where $|A| = n$ and V_{reg} denotes the regular representation of A^* (a key point is that (1.4) is the euler class of the bundle $V_{\mathrm{reg}} \otimes L$ associated to the orientation g). It follows that if g is an H_∞ map, then

$$\psi_\ell^* s = N_\ell i^* s. \tag{1.5}$$

This condition is equivalent to the condition in [And95]; see Remark 5.14.

In §6 we modify the discussion of §5 to handle $MU\langle 6\rangle$-orientations. If

$$MU\langle 6\rangle \xrightarrow{g} \mathbf{E}$$

is an orientation, then the composition

$$((\mathbb{C}P^\infty)^3)^{\prod_i (1-L_i)} \to MU\langle 6\rangle \xrightarrow{g} \mathbf{E}$$

represents a *cubical structure* s_g on the line bundle $\mathcal{I}_G(0)$; in [AHS01], we showed that the assignment $g \mapsto s_g$ is a bijection between the set of $MU\langle 6\rangle$-orientations of $\mathbf{E}$ and the set $C^3(G; \mathcal{I}_G(0))$ of cubical structures.

As before, a cubical structure s on $\mathcal{I}_G(0)$ gives rise to *two* cubical structures $\psi_\ell^* s$ and $\tilde{N}_\ell s$ on $\psi_\ell^* \mathcal{I}_G(0)$. If $s = s_g$ is the cubical structure associated to an $MU\langle 6\rangle$-orientation g, then these two cubical structures correspond to the two ways of navigating the diagram (1.2). If g is an H_∞ orientation, then the cubical structure s must satisfy the equation

$$\psi_\ell^* s = \tilde{N}_\ell i^* s. \tag{1.6}$$

In Proposition 7.1 we show that the necessary conditions (1.5) and (1.6) are sufficient if we suppose in addition that p is not a zero divisor in $\mathbf{E}$. We have given a direct proof, but our argument amounts to showing that that for $k \leq 3$, the character map of [HKR00] for $\mathbf{E}^0 D_p(BU\langle 2k\rangle_+)$ is injective.

Thus we have reduced the problem of checking whether the orientation (1.1) is H_∞ to the problem of checking the equation (1.6). That problem is mostly a matter of recalling the construction of the sigma orientation; we do that in Part 4. Here are the main points.

Definition 1.7. An *elliptic spectrum* consists of

(1) an even, periodic, homotopy commutative ring spectrum $\mathbf{E}$;
(2) an elliptic curve C over $\operatorname{spec} \pi_0 \mathbf{E}$; and
(3) an isomorphism of formal groups

$$t : G_{\mathbf{E}} \cong \widehat{C}$$

over $\operatorname{spec} \pi_0 \mathbf{E}$.

The Theorem of the Cube (or Abel's Theorem, for that matter) shows that an elliptic curve has a *unique* cubical structure $s(C/S)$. If C is the elliptic curve associated to an elliptic spectrum $(\mathbf{E}, C, t)$, then

$$(t^3)^* \widehat{s}(C/S)$$

is a cubical structure on $\mathcal{I}_{G_{\mathbf{E}}}(0)$; the associated $MU\langle 6\rangle$-orientation is the σ-orientation. (The name comes from the fact that if C is a complex elliptic curve, then there is a simple formula for $s(C/S)$ in terms of the Weierstrass σ-function, which shows that the σ-orientation for the Tate curve is the Witten genus. See [AHS01])

Now suppose that $(\mathbf{E}, C, t)$ is an elliptic spectrum, that $\mathbf{E}$ is an H_∞ spectrum, and that $\pi_0 \mathbf{E}$ is a complete Noetherian local ring of residue characteristic $p > 0$. Suppose for each level structure

$$A \xrightarrow{\ell} i^* G(R)$$

we are given an isogeny of elliptic curves

$$h_\ell : i^* C \to \psi_\ell^* C$$

with kernel A, such that

$$t^* \widehat{h}_\ell = f_\ell$$

(Such structure, with compatibility with variation in A, is called an H_∞ *elliptic spectrum* in Definition 19.4). The uniqueness of the cubical structure $s(C/S)$ implies that

$$\psi_\ell^* s(C/S) = N_\ell i^* s(C/S).$$

Thus we have the

Proposition 1.8 (19.5). *If $(\mathbf{E}, C, t)$ is an H_∞ elliptic spectrum, and p is regular in $\pi_0 \mathbf{E}$, then the σ-orientation*

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{E}, C, t)} \mathbf{E}$$

is an H_∞ map. □

The Serre-Tate Theorem implies that the spectrum associated to the universal deformation of C_0 is an H_∞ elliptic spectrum (Corollary 17.15), and so the Proposition implies our result.

Corollary 1.9 (19.6). *If $(\mathbf{E}, C, t)$ is the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve over a perfect field of characteristic $p > 0$, then the σ-orientation*

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{E}, C, t)} \mathbf{E}$$

is H_∞.

We have analyzed H_∞ ring spectra using the algebraic geometry of group schemes and in particular level structures, and we have analyzed orientations (i.e. Thom isomorphisms) using the algebraic geometry of line bundles. Part 2 describes the relationship to topology. §8 discusses level structures and the cohomology of abelian groups; this is a variation of [HKR00]. §9 expresses some familiar results about the even-periodic cohomology of Thom complexes in the language of line bundles.

The construction of the homomorphism ψ_ℓ and the isogeny f_ℓ and the proof of the sufficiency of the equations (1.5) and (1.6) depend on two technical results (Propositions 10.15 and 10.23) about level structures. We prove those results in Part 3. In order to make the discussion more self-contained, we also recall there some standard results about level structures, primarily from [Dri74, KM85, Str97].

2. Notation

2.1. **Groups.** If X is an object in some category with products, and $J \subseteq I$ is an inclusion of sets, the projection map $X^I \to X^J$ will be denoted π_J. The set J will often be indicated by the sequence of its elements. For example, π_{23} will denote projection to product of the 2^{nd} and 3^{rd} factors. If $\sigma : I \to I$ is an automorphism, the symbol π_σ refers to the induced automorphism of X^I.

If X is a commutative group object, then the symbol μ_J will denote the map $X^J \to X$ obtained by composing π_J with the iterated multiplication. In punctual notation,

$$\mu_{23}(a_1, a_2, a_3, \dots) = a_2 a_3$$

and so forth.

If X is a commutative group object in a category of objects over a base S, then the symbol $0 : S \to X$ will stand for the identity section, and we shall generally abbreviate to π the symbol for the structural map

$$\pi_\emptyset : X \to S.$$

2.2. **Formal schemes and formal groups.** As in [AHS01], we view affine schemes as representable functors from rings to sets, and define a *formal scheme* to be a filtered colimit of affine schemes; the value of the colimit is the colimit of the values

$$(\operatorname*{colim}_\alpha X_\alpha)(R) = \operatorname*{colim}_\alpha X_\alpha(R). \tag{2.1}$$

In this paper we make one important modification to the notation (2.1). Recall from [EGAI, **0**, 7.1.2] that a *preadmissible* ring is a linearly topologized ring which contains an *ideal of definition*: an open ideal I such that, for all open neighborhoods V of zero, $I^n \subset V$ for some $n > 0$. An *admissible* ring is preadmissible ring which is complete and separated. If R is an admissible ring, then the ideals of definition form a fundamental system of neighborhoods of 0. The *formal spectrum* of R is the formal scheme [EGAI, **I**, 10.1.2,10.6]

$$\operatorname{spf} R \overset{\text{def}}{=} \operatorname*{colim}_J \operatorname{spec} R/J,$$

where the colimit is over the ideals of definition. In fact we shall only need the case that R is a local ring; a local ring is admissible if it is complete and separated in its adic topology. If R is an admissible ring and if X is a formal scheme, then we define $X(R)$ to be the set of *natural transformations*

$$\operatorname{spf} R \to X.$$

Thus if R is admissible then $\hat{\mathbb{A}}^1(R)$ is the set of *topologically* nilpotent elements of R, rather than just the set of nilpotent elements. Similarly

$$(\operatorname{spf} R')(R)$$

is the set of continuous ring homomorphisms from R' to R.

A *local scheme (of residue characteristic p)* is a scheme of the form $\operatorname{spec} R$, where R is a local ring (of residue characteristic p). An *local formal scheme (of residue characteristic p)* is a formal scheme of the form $\operatorname{spf} R$, where R is an admissible local ring (of residue characteristic p).

Let S be a formal scheme. A *formal group scheme* over S is a commutative group in the category of formal schemes over S. If A is a finite abelian group, then A_S will denote the constant formal group scheme over S given by A. A *formal group* over S is a formal group scheme which is locally isomorphic to $S \times \hat{\mathbb{A}}^1$ as a pointed formal scheme over S. If R is complete local ring, then a formal group over R means a formal group over $\operatorname{spf} R$. If G is a formal group over R and

$$j : R \to R'$$

is a map of complete local rings, then with the pull-back diagram

$$\begin{array}{ccc} j^*G & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ \operatorname{spf} R' & \xrightarrow{j} & \operatorname{spf} R \end{array}$$

in mind, we write j^*G for the resulting formal group over R'.

We shall be primarily interested the case of a formal group G of finite height over a local formal scheme S whose closed point S_0 is the spectrum of a perfect field of characteristic $p > 0$. Let G_0 be the fiber of G over S_0, i.e. the pull-back in the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{i} & G \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S. \end{array}$$

By construction, $(G/S, i, \mathrm{id}_{S_0})$ is a deformation (14.1) of G_0.

If S is *Noetherian*, then this deformation is classified by a pull-back diagram

$$\begin{array}{ccc} G & \longrightarrow & G' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S', \end{array}$$

where $(G'/S', f_{\text{univ}}, j_{\text{univ}})$ is the universal deformation of G_0 (in the sense of [LT66]; see §14). Various facts about G/S then follow from facts about G'/S' by change of base.

It will be convenient for us to have these facts available also in the case that S is not Noetherian, but that G/S arises from a pull-back diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ S & \xrightarrow{i} & T \end{array} \tag{2.2}$$

where T is Noetherian.

Definition 2.3. A formal group G over a formal scheme S is *Noetherian* if there is a pull-back diagram of the form (2.2), such that T is Noetherian, and the induced isomorphism of formal schemes over S

$$G \to i^*H$$

is a group homomorphism.

The main example is $\mathbf{E} \cong E_n$ is the spectrum associated to the universal deformation of a formal group of height n over a perfect field k of residue characteristic p, X is a space, $\mathbf{F} = \mathbf{E}^{X_+}$, and G is the formal group of $\mathbf{F}$ over $\pi_0\mathbf{F}$.

2.3. **Ideal sheaves associated to divisors.** If G is a formal group or elliptic curve over a (formal) scheme S, and D is a divisor on G, then we shall write $\mathcal{I}(D)$ for the corresponding sheaf of ideals, the inverse of the sheaf which is usually denoted $\mathcal{O}(D)$. For example, if $w \in G(R)$ then $\mathcal{I}(w)$ means the ideal of functions on G which *vanish* at w. More generally, if W is a finite set and

$$\ell : W \to G(R)$$

is a map of sets, then we will write $\mathcal{I}(\ell)$ for the ideal associated to the divisor

$$\{\ell\} = \sum_{w \in W} \{\ell(w)\},$$

so

$$\mathcal{I}(\ell) \cong \bigotimes_{w \in W} \mathcal{I}(\ell(w)).$$

2.4. **Spectra.** The category of spectra over a universe U will be denoted $\mathbb{S}_U$. The category $\mathbb{S}_U$ is enriched over the category Spaces_+ of pointed topological spaces, and our notation will reflect this. Thus, the object $\mathbb{S}_U(E, F)$ will refer to a pointed topological space, and for a pointed space X, E^X is the function object (spectrum). What would in category theory denoted $E \otimes X$ will in this case be denoted $E \wedge X$. There are natural homeomorphisms of pointed spaces

$$\mathrm{Spaces}_+(X, \mathbb{S}_U(E, F)) \cong \mathbb{S}_U(E \wedge X, F) \cong \mathbb{S}_U(E, F^X). \tag{2.4}$$

If V is a vector bundle over a space X, then X^V will refer to the pointed space which is the Thom complex of V. When V is a virtual bundle, then X^V will refer to the Thom *spectrum* of V, arranged so that the "bottom cell" is in the virtual (real) dimension of V. With this convention, the Thom spectrum of an "honest" vector bundle is the suspension spectrum of the Thom complex, so no real problem should come up when regarding an actual vector bundle as a virtual.

We write V_{std} for (the vector bundle over $B\Sigma_n$ associated to) the standard complex representation of Σ_n, and if A is an abelian group, then V_{reg} will denote (the vector bundle over BA associated to) the complex regular representation of A.

2.5. **Even periodic ring spectra.** A (homotopy commutative) ring spectrum $\mathbf{E}$ will be called *even* if $\pi_{\mathrm{odd}}\mathbf{E} = 0$, and *periodic* if $\pi_2\mathbf{E}$ contains a unit. A ring spectrum $\mathbf{E}$ will be called *homogeneous* if it is a homotopy commutative algebra spectrum over an even periodic ring spectrum. We will be particularly interested in homogeneous spectra $\mathbf{E}$ in which the ring $\pi_0\mathbf{E}$ is preadmissible in some natural topology (possibly discrete). If $\mathbf{E}$ is a such a spectrum, then we write $\widehat{\pi_0}\mathbf{E}$ for the separated completion of $\pi_0\mathbf{E}$, and we define

$$S_{\mathbf{E}} \stackrel{\mathrm{def}}{=} \mathrm{spf}(\widehat{\pi_0}\mathbf{E})$$

for the formal scheme defined by $\widehat{\pi}_0\mathbf{E}$.

Let $\mathbf{E}$ be such a spectrum, and let X be a space. If $\{X_\alpha\}$ is the set of compact subsets of X and $\{I_k\}$ is the set of ideals of definition of $\pi_0\mathbf{E}$, then $\pi_0\mathbf{E}^{X_+}$ is preadmissible in the topology defined by the kernels of the maps

$$\pi_0\mathbf{E}^{X_+} \to \left(\pi_0\mathbf{E}^{(X_\alpha)_+}\right)/I_k,$$

and we define $X_{\mathbf{E}}$ to be the formal scheme

$$X_{\mathbf{E}} = \mathrm{spf}\, \widehat{\pi}_0\mathbf{E}^{X_+};$$

this gives a covariant functor from spaces to formal schemes over $S_{\mathbf{E}}$. If $\mathbf{F} = \mathbf{E}^{X_+}$ then

$$S_{\mathbf{F}} = X_{\mathbf{E}},$$

and we shall use these notations interchangeably.

The most important example of these constructions is that $\mathbf{E} \cong E_n$ is the spectrum associated to the universal deformation of a formal group of height n over a perfect field k of characteristic $p > 0$, so

$$\pi_0\mathbf{E} \cong \mathbb{W}k[\![u_1, \ldots, u_{n-1}]\!],$$

and X is a space with the property that $H_*(X, \mathbb{Z})$ is concentrated in even degrees. In that case, the natural map of rings

$$\pi_0\mathbf{E}^{X_+} \to \mathcal{O}(X_{\mathbf{E}}) = \widehat{\pi}_0\mathbf{E}^{X_+}$$

is an isomorphism, but in general all we have is a surjective map.

If $\mathbf{E}$ is a homogeneous ring spectrum, then it is complex orientable, and

$$G_{\mathbf{E}} = (\mathbb{C}P^\infty)_{\mathbf{E}}$$

is a formal group over $S_{\mathbf{E}}$.

Part 1. H_∞ orientations

3. Algebraic geometry of even H_∞ ring spectra

3.1. Descent data for level structures. Let $\mathbf{E}$ be a homogeneous ring spectrum. In this section we investigate the additional structure which adheres to $G_{\mathbf{E}} = (\mathbb{C}P^\infty)_{\mathbf{E}}/S_{\mathbf{E}}$ when $\mathbf{E}$ is an H_∞ spectrum (see §A). In order to make precise statements, it is convenient to suppose that $\pi_0\mathbf{E}$ is a complete local ring of residue characteristic $p > 0$, and that $G_{\mathbf{E}}$ if a Noetherian formal group (2.3) of finite height. In that case, we shall show that an H_∞ structure on $\mathbf{E}$ determines "descent data for level structures" on $G_{\mathbf{E}}$. In §13 we shall give a definition of this notion in the usual language of descent; the definition we give there is equivalent to the following.

Definition 3.1. Let G be a formal group over a formal scheme S. *Descent data for level structures on* G assign to every map of formal schemes $i : T = \operatorname{spf} R \to S$, finite abelian group A, and level structure (Definition 10.9)

$$\ell : A_T \to i^*G, \tag{3.2}$$

a map of formal schemes $\psi_\ell : T \to S$ and an isogeny $f_\ell : i^*G \to \psi_\ell^* G$ with kernel A, satisfying the following.

(1) If $j : T \to T'$ is a map of formal schemes and

$$j^*\ell : A_{T'} \to j^*i^*G$$

is the resulting level structure, then

$$\psi_{j^*\ell} = j \circ \psi_\ell,$$

and

$$f_{j^*\ell} = j^* f_\ell.$$

(2) If $B \subseteq A$, then with the notation

$$\begin{array}{ccccc} B & \longrightarrow & A & \longrightarrow & A/B \\ {\scriptstyle \ell'}\big\downarrow & & {\scriptstyle \ell}\big\downarrow & & \big\downarrow{\scriptstyle \ell''} \\ i^*G & = & i^*G & \xrightarrow[f_{\ell'}]{} & \psi_{\ell'}^*G, \end{array} \tag{3.3}$$

we have

$$\begin{aligned} \psi_{\ell''} &= \psi_\ell : T \to S \\ f_\ell &= f_{\ell''} \circ f_{\ell'} : i^*G \to \psi_\ell^* G = \psi_{\ell''}^* G. \end{aligned} \tag{3.4}$$

(3) If ℓ is the inclusion of the trivial subgroup, then f_ℓ and ψ_ℓ are the identity maps. Among other things this implies that if ℓ and ℓ' differ by an automorphism of A, then $f_\ell = f_{\ell'}$.

We shall write (ψ, f) for such descent data. Formal groups with descent data for level structures form a category; if G/S and G'/S' are two formal groups with descent data for level structures, then a *map* from G'/S' to G/S is a pull-back

$$\begin{array}{ccc} G' & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ S' & \longrightarrow & S \end{array}$$

in the category of formal schemes, such that the induced isomorphism

$$G' \to S' \times_S G$$

is a group homomorphism, and such that the descent data for G pull back to the descent data for G'.

Let $\mathcal{C}$ be the category whose objects are homogeneous ring spectra $\mathbf{E}$ with the property that $\pi_0\mathbf{E}$ is a complete local ring of positive residue characteristic and $G_{\mathbf{E}}$ is a Noetherian formal group (2.3) of finite height, and whose morphisms are maps $f : \mathbf{E} \to \mathbf{F}$ of ring spectra with the property that $\pi_0 f$ is a map of local rings. Let $H_\infty\mathcal{C}$ be the subcategory of $\mathcal{C}$ consisting of H_∞ ring spectra and H_∞ maps. We shall construct the dotted arrow in the diagram

$$\begin{tikzcd} H_\infty\mathcal{C} \arrow[r, dashed] \arrow[d] & \text{(formal groups with descent data)} \arrow[d] \\ \mathcal{C} \arrow[r] & \text{(formal groups.)} \end{tikzcd} \tag{3.5}$$

The main result is Theorem 3.26.

3.2. **Descent data from H_∞ ring spectra.** The basic operation on the homotopy groups of an H_∞-ring spectrum is the transformation

$$D_n : \pi_0\mathbf{E} \to \pi_0\mathbb{S}_U(D_nS^0, \mathbf{E}) = \pi_0\mathbf{E}^{B\Sigma_n+}.$$

This map is multiplicative in the sense that $D_n(fg) = D_n(f)D_n(g)$, but it is not quite additive. In fact, it follows from Proposition A.4 that

$$D_n(f+g) = \sum_{i+j=n} \mathrm{Tr}_{ij}\, D_i(f)D_j(g). \tag{3.6}$$

If $\mathbf{E}$ is a spectrum such that $\pi_0\mathbf{E}$ is a complete local ring of residue characteristic $p > 0$, and the formal group $G_\mathbf{E}$ is Noetherian (2.3) and of finite height, then there is a slightly more convenient operation to work with. Suppose that A is a finite abelian group, and let A^* be its group of complex characters. With these hypotheses, the natural map (8.2)

$$(BA^*)_\mathbf{E} \to \underline{\mathrm{hom}}(A, G_\mathbf{E}) \tag{3.7}$$

is an isomorphism. Define a functor $D_A : \mathbb{S}_U \to \mathbb{S}_U$ by

$$D_A(X) = \mathcal{L}(U^{A^*}, U) \underset{A^*}{\wedge} X^{(A^*)}, \tag{3.8}$$

where $X^{(A^*)}$ denotes the external smash product

$$\bigwedge_{\alpha\in A^*} X \in \mathrm{Ob}\,\mathbb{S}_{U^{A^*}}$$

(We will also have use for the functor on pointed spaces given by the the analogue of (3.8)).

Definition 3.9. Given a complete local ring R and a level structure (10.9)

$$A_{\mathrm{spf}\,R} \xrightarrow{\ell} i^*G,$$

we define $\psi_\ell^\mathbf{E} : \pi_0\mathbf{E} \to R$ to be the map given by the composition

$$\pi_0\mathbf{E} \xrightarrow{D_A} \pi_0\mathbb{S}_U(D_AS^0, \mathbf{E}) = \pi_0\mathbf{E}^{BA^*_+} \to \mathcal{O}((BA^*)_\mathbf{E}) \cong \mathcal{O}(\underline{\mathrm{hom}}(A, G)) \xrightarrow{\chi_\ell} R$$

where χ_ℓ is the map classifying the homomorphism ℓ as in (8.4), and the isomorphism comes from (3.7)

Lemma 3.10 ([And95])**.** *The map $\psi_\ell^\mathbf{E}$ is a continuous ring homomorphism.*

Proof. $\psi_\ell^\mathbf{E}$ is certainly multiplicative. It's additive because equation (3.6) and the double coset formula imply that $\psi_\ell^\mathbf{E}(x+y) - \psi_\ell^\mathbf{E}(x) - \psi_\ell^\mathbf{E}(y)$ is a sum of elements in the image of the transfer map from proper subgroups of A^*. The result therefore follows from Proposition 8.5.

To see that $\psi_\ell^\mathbf{E}$ is continuous, note that

$$\mathcal{O}((BA^*)_\mathbf{E}) \cong \mathcal{O}(\underline{\mathrm{hom}}(A, G_\mathbf{E})) \tag{3.11}$$

is a local ring by Proposition 10.8. It suffices to show that for y in the maximal ideal of $\pi_0\mathbf{E}$, D_Ay is in the maximal ideal of $\mathcal{O}((BA^*)_\mathbf{E})$. Since, modulo the augmentation ideal of $\pi_0\mathbf{E}^{BA^*_+}$ we have

$$D_Ay = y^{|A|},$$

it follows that $\psi_\ell^\mathbf{E}$ is continuous. $\square$

Remark 3.12. It is not necessary to use a finite group A. The initial ring R over which we may define a ring homomorphism

$$\psi : \pi_0\mathbf{E} \to \pi_0\mathbf{E}^{B(\Sigma_k)_+} \to R$$

as above is the quotient of $\pi_0\mathbf{E}^{B(\Sigma_k)_+}$ by the ideal generated by the images of transfers from proper subgroups of Σ_k. Strickland [Str98] shows that

$$\operatorname{spf}\left(\pi_0 E_n^{B(\Sigma_k)_+}/\text{proper transfers}\right)$$

is the scheme of "subgroups of order k of G_{E_n}". The analysis of this paper can be carried through in that case.

The operation ψ_ℓ is clearly natural in the sense that given a map $f : \mathbf{E} \to \mathbf{F}$ of H_∞ spectra of the indicated kind, with the property that $\pi_0 f$ is continuous, then the level structure ℓ gives a level structure

$$A_{\operatorname{spf} S} \xrightarrow{\ell} j^*G_{\mathbf{F}},$$

where S and j are defined by

$$\pi_0 F \xrightarrow{j} S = R \underset{i,\widehat{\pi}_0\mathbf{E},\widehat{\pi}_0 f}{\widehat{\otimes}} \pi_0\mathbf{F},$$

and the diagram

$$\begin{tikzcd}
S & \pi_0\mathbf{F} \arrow[l, "\psi_\ell^{\mathbf{F}}"'] \\
R \arrow[u] & \pi_0\mathbf{E} \arrow[l, "\psi_\ell^{\mathbf{E}}"'] \arrow[u, "\pi_0 f"']
\end{tikzcd}$$

commutes.

In the language of algebraic geometry, let

$$T = \operatorname{spf} R \xrightarrow{i} S_{\mathbf{E}}.$$

The map $\psi_\ell^{\mathbf{E}}$ is a map of formal schemes

$$\psi_\ell^{\mathbf{E}} : T \to S_{\mathbf{E}},$$

and the naturality is expressed in terms of the commutative diagram

$$\begin{tikzcd}
T \underset{i,S_{\mathbf{E}},S_f}{\times} S_{\mathbf{F}} \arrow[r, "\psi_\ell^{\mathbf{F}}"] \arrow[d] & S_{\mathbf{F}} \arrow[d, "S_f"] \\
T \arrow[r, "\psi_\ell^{\mathbf{E}}"] & S_{\mathbf{E}}.
\end{tikzcd}$$

Making use of the isomorphism

$$T \times_{i,S_{\mathbf{E}},S_f} S_{\mathbf{F}} \cong i^*S_{\mathbf{F}},$$

we find that the map $\psi_\ell^{\mathbf{F}}$ can be factored through a *relative* map

$$\psi_\ell^{\mathbf{F}/\mathbf{E}} : i^*S_{\mathbf{F}} \to \psi_\ell^{\mathbf{E}*} S_{\mathbf{F}} \tag{3.13}$$

as in

$$\begin{tikzcd}
i^*S_{\mathbf{F}} \arrow[dr, "\psi_\ell^{\mathbf{F}/\mathbf{E}}"] \arrow[drr, bend left, "\psi_\ell^{\mathbf{F}}"] \arrow[ddr, bend right] & & \\
& \left(\psi_\ell^{\mathbf{E}}\right)^* S_{\mathbf{F}} \arrow[r, "S_f^*\psi_\ell^{\mathbf{E}}"'] \arrow[d] & S_{\mathbf{F}} \arrow[d, "S_f"] \\
& T \arrow[r, "\psi_\ell^{\mathbf{E}}"'] & S_{\mathbf{E}}.
\end{tikzcd} \tag{3.14}$$

For example, let $G = G_{\mathbf{E}}$, and take $\mathbf{F} = \mathbf{E}^{\mathbb{C}P^\infty_+}$ so that $G = S_{\mathbf{F}}$. There results a map

$$\psi_\ell^{G/\mathbf{E}} : i^*G \to \left(\psi_\ell^{\mathbf{E}}\right)^* G. \tag{3.15}$$

This map turns out to be a homomorphism of groups, as one can see by considering the (H_∞) map $\mathbf{E}^{\mathbb{C}P^\infty_+} \to \mathbf{E}^{(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+}$ coming from $\mu : \mathbb{C}P^{\infty 2} \to \mathbb{C}P^\infty$. We shall eventually show (Proposition 3.21) that $\psi_\ell^{G/\mathbf{E}}$ is an isogeny, with kernel $\ell : A \to i^*G$. In order to give the proof, it is essential to understand the effect of the operation $\psi_\ell^{\mathbf{E}}$ on the cohomology of Thom complexes.

Suppose that V is a virtual bundle over a space X, and write

$$\mathbf{F} = \mathbf{E}^{X_+}.$$

If we start with an element $m \in \pi_0 \mathbb{S}_U\left((X)^V, \mathbf{E}\right)$ and follow the construction of the map $\psi_\ell^{\mathbf{F}}$, we wind up with an element $\psi_\ell^V(m)$ in

$$R \mathop{\widehat{\otimes}}_{\chi_\ell, \widehat{\pi}_0 \mathbf{E}^{BA^*_+}} \widehat{\pi}_0 \mathbb{S}_U\left((BA^* \times X)^{V_{\mathrm{reg}} \otimes V}, \mathbf{E}\right), \tag{3.16}$$

where V_{reg} denotes the regular representation of A^*. As before, this map is additive, and in fact $\psi_\ell^{\mathbf{F}}$-linear:

$$\psi_\ell^V(xm) = \psi_\ell^{\mathbf{F}}(x)\psi_\ell^V(m). \tag{3.17}$$

Let $T = \operatorname{spf} R$; then we have a commutative diagram

$$\begin{tikzcd}
i^*S_{\mathbf{F}} \arrow[r, "\chi_\ell"] \arrow[d] & \underline{\hom}(A, G_{\mathbf{F}}) \arrow[r] \arrow[d] & S_{\mathbf{F}} \arrow[d] \\
T \arrow[r, "\chi_\ell"] \arrow[rr, bend right, "i"'] & \underline{\hom}(A, G_{\mathbf{E}}) \arrow[r] & S_{\mathbf{E}}
\end{tikzcd}$$

in which all the squares are pull-backs. In the language of §9, the element m is a section of the line bundle $\mathbb{L}(V)$ over $S_{\mathbf{F}}$. Elements of (3.16) are sections of

$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V)$$

over $i^*S_{\mathbf{F}}$. Taking into account the linearity (3.17) we find that the map ψ_ℓ^V can be interpreted as a map

$$\psi_\ell^V : \left(\psi_\ell^{\mathbf{F}}\right)^* \mathbb{L}(V) \to \chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V) \tag{3.18}$$

of line bundles over $i^*S_{\mathbf{F}}$.

Lemma 3.19. *The map ψ_ℓ^V has the following properties*

(1) *If m is a trivialization of $\mathbb{L}(V)$, then ψ_ℓ^V is a trivialization of $\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V)$.*
(2) *With the obvious identifications*
$$\psi_\ell^{V_1 \oplus V_2} = \psi_\ell^{V_1} \otimes \psi_\ell^{V_2}.$$
(3) *If $f : Y \to X$ is a map, then*
$$\psi_\ell^{f^*V} = f^* \psi_\ell^V.$$

The most important example is the case $X = \mathbb{C}P^\infty$ and $V = L$, so $S_{\mathbf{F}} = G = G_{\mathbf{E}}$ and $\mathbb{L}(L) = \mathcal{I}_G(0)$ (9.4). Then (9.10) gives an isomorphism

$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes L) \cong \mathcal{I}_{i^*G}(\ell),$$

and so we may think of ψ_ℓ^L as a map

$$\left(\psi_\ell^{\mathbf{F}}\right)^* \mathcal{I}_G(0) \to \mathcal{I}_{i^*G}(\ell)$$

of line bundles over i^*G, or on sections a $\psi_\ell^{\mathbf{F}}$-linear map

$$\Gamma(\mathcal{I}_G(0)) \to \Gamma(\mathcal{I}_{i^*G}(\ell)). \tag{3.20}$$

Proposition 3.21 ([And95]). (1) *The map*

$$\psi_\ell^{G/\mathbf{E}} : i^* G \to \left(\psi_\ell^{\mathbf{E}}\right)^* G$$

of (3.15) *is an isogeny with kernel* $\ell : A \to i^* G$.

(2) *If the isogeny* $\psi_\ell^{G/\mathbf{E}}$ *is used to identify*

$$\left(\psi_\ell^{G/\mathbf{E}}\right)^* I_{(\psi_\ell^{\mathbf{E}})^* G}(0) \cong I_{i^* G}(\ell)$$

as in (12.6), *then the map* ψ_ℓ^L (3.20) *sends a coordinate* x *on* G *to the trivialization* $\left(\psi_\ell^{G/\mathbf{E}}\right)^* \left(\psi_\ell^{\mathbf{E}}\right)^* x$ *of* $I_{i^*G}(\ell)$.

Proof. First, observe that $\psi_\ell^{G/\mathbf{E}}$ is an isogeny of degree $|A|$. This follows from the Weierstrass Preparation Theorem [Lan78, pp. 129–131], because, after reducing modulo the maximal ideal in R, the ring homomorphism

$$(\psi_\ell^{G/\mathbf{E}})^* : R \mathop{\widehat{\otimes}}_{\psi_\ell^{\mathbf{E}}} \widehat{\pi}_0 E^{\mathbb{C}P^\infty +} \to R \mathop{\widehat{\otimes}}_{i} \widehat{\pi}_0 E^{\mathbb{C}P^\infty +}$$

sends a coordinate x to $x^{|A|}$. To see this, note that the composition

$$\pi_0 \mathbf{F} \xrightarrow{D_A} \pi_0 \mathbf{F}^{BA^*_+} \xrightarrow{\pi_0 \mathbf{F}^{S^0 \to BA^*_+}} \pi_0 \mathbf{F}$$

is the map $x \mapsto x^{|A|}$.

To prove the first part of the Proposition, it remains to show that the A is contained in the kernel, i.e. that $\psi_\ell^{G/\mathbf{E}} \ell = 0$, i.e. that if x is a coordinate on $G_{\mathbf{E}}$, then $(\psi_\ell^{G/\mathbf{E}})^* \left(\psi_\ell^{\mathbf{E}}\right)^* x$ vanishes on $\ell(A)$. A coordinate on G is a generator the ideal $\mathcal{I}(0)$, which the zero section identifies with $\mathbb{L}(L)$. The commutativity of the diagram

$$\begin{tikzcd}
\pi_0 \mathbf{E}^{\mathbb{C}P^\infty_+} \arrow[d, "D_A"'] \arrow[dd, bend right=60, "\psi_\ell^{G/\mathbf{E}}"'] & \pi_0 \mathbf{E}^{\mathbb{C}P^{\infty\, L}} \arrow[l, "\pi_0 \mathbf{E}^{\zeta}"'] \arrow[d, "D_A"] \arrow[dd, bend left=60, "\psi_\ell^L"] \\
\pi_0 \mathbf{E}^{BA^* \times \mathbb{C}P^\infty_+} \arrow[d] & \pi_0 \mathbf{E}^{(BA^* \times \mathbb{C}P^\infty)^{V_{\mathrm{reg}} \otimes L}} \arrow[l, "\pi_0 \mathbf{E}^{\zeta}"] \arrow[d] \\
R \widehat{\otimes} \widehat{\pi}_0 \mathbf{E}^{BA^* \times \mathbb{C}P^\infty_+} & R \widehat{\otimes} \widehat{\pi}_0 \mathbf{E}^{(BA^* \times \mathbb{C}P^\infty)^{V_{\mathrm{reg}} \otimes L}} \arrow[l]
\end{tikzcd} \tag{3.22}$$

(in which the tensor products are taken over the ring $\widehat{\pi}_0 \mathbf{E}^{BA^*_+}$) shows that $\psi_\ell^{G/\mathbf{E}}$ takes a function which vanishes at 0 to a function which vanishes on $\ell(A)$. The claim about $\psi_\ell^L x$ also follows from inspection of the diagram (3.22). ☐

In fact Proposition 3.21 gives a simple description of the map ψ_ℓ^V for a general virtual bundle V, and in particular, shows that it is determined by maps which have already been constructed. We shall express the answer in the language of §9, where line bundles of the form $\mathbb{L}(V)$ are computed in terms of divisors. As in §9, it is illuminating to work at the outset with $V \otimes L$ over $X \times \mathbb{C}P^\infty$ and then pull back along the identity section of $G_{\mathbf{F}}$.

With this in mind, let $F = E^{X_+}$, and let

$$\mathbf{G} = \mathbf{F}^{\mathbb{C}P^\infty_+} = \mathbf{E}^{(\mathbb{C}P^\infty \times X)_+},$$

so $G = G_{\mathbf{F}} = S_{\mathbf{G}}$. Let $D = D_V$ be the divisor on G corresponding to V. We use the isomorphism (Proposition 9.12)

$$t_{V \otimes L} : \mathbb{L}(V \otimes L) \to \mathcal{I}(D^{-1})$$

and the relative map (3.13) to replace the domain of

$$\psi_\ell^{V \otimes L} : \left(\psi_\ell^{\mathbf{G}}\right)^* \mathbb{L}(V \otimes L) \to \chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V \otimes L).$$

with
$$\left(\psi_\ell^{\mathbf{G}}\right)^* \mathcal{I}(D^{-1}) \cong \left(\psi_\ell^{G/\mathbf{F}}\right)^* \left(\psi_\ell^{\mathbf{F}}\right)^* I(D^{-1}).$$
Using the analogous isomorphism (9.9)
$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V \otimes L) \cong \bigotimes_{a\in A} T_a^* i^* \mathcal{I}(D^{-1})$$
to interpret the range, we may think of $\psi_\ell^{V\otimes L}$ as a map
$$\psi_\ell^{V\otimes L} : \left(\psi_\ell^{G/\mathbf{F}}\right)^* \left(\psi_\ell^{\mathbf{F}}\right)^* \mathcal{I}(D^{-1}) \to \mathcal{I}_{i^*G}\left(\sum_a T_a^* D^{-1}\right), \tag{3.23}$$
or, equivalently, as a $\psi_\ell^{\mathbf{G}}$-linear map
$$\mathcal{I}_G(D^{-1}) \to \mathcal{I}_{i^*G}\left(\sum_a T_a^* D^{-1}\right). \tag{3.24}$$

Proposition 3.25. *In the guise of* (3.24) *the map* $\psi_\ell^{V\otimes L}$ *is given by*
$$f \mapsto \left(\psi_\ell^{G/\mathbf{F}}\right)^* \left(\psi_\ell^{\mathbf{F}}\right)^* f.$$

Proof. There are actually two assertions. One is that
$$\langle f\rangle \geq D^{-1} \implies \langle (\left(\psi_\ell^{\mathbf{F}}\right)^* f) \circ \psi_\ell^{G/\mathbf{F}}\rangle \geq \sum T_a^* D^{-1}.$$
The other is that this gives the map $\psi_\ell^{V\otimes L}$. The verification of both assertions follows the lines of the proof of Proposition 3.21. Indeed, everything involved takes Whitney sums in V to tensor products, and commutes with base change in V. It suffices then to verify the case when X is a single point, and V has dimension 1. In this case, the isomorphism t_L is given by the inclusion of the zero section $\mathbb{C}P^\infty \to \mathbb{C}P^{\infty L}$, and the result follows from naturality of he maps ψ_ℓ, as in the diagram (3.22). □

The results of this section assemble to give the following.

Theorem 3.26. *Let* $\mathbf{E}$ *be a homogeneous* H_∞ *ring spectrum. Suppose that* $\pi_0\mathbf{E}$ *is a local ring of residue characteristic* $p > 0$, *and the formal group* $G = G_{\mathbf{E}}$ *is Noetherian and of finite height. The rule which associates to a level structure*
$$\ell : A_{\operatorname{spf} R} \to i^*G \tag{3.27}$$
the map of formal schemes $\psi_\ell^{\mathbf{E}} : \operatorname{spf} R \to S_{\mathbf{E}}$ *and the isogeny*
$$\psi_\ell^{G/\mathbf{E}} : i^*G \to \psi_\ell^* G$$
is descent data for level structures on the formal group $G/S_{\mathbf{E}}$, *and gives the dotted arrow in the diagram* (3.5).

Proof. Lemma 3.10 and Proposition 3.21 show that $\psi_\ell^{\mathbf{E}}$ is a ring homomorphism and $\psi_\ell^{G/\mathbf{E}}$ is an isogeny with kernel $\ell(A)$. The commutativity of the diagrams (A.1) imply the compatibility of the isogenies $\psi_\ell^{G/\mathbf{E}}$ with variation in A, as described in Definition 3.1. □

4. H_∞ structures on Thom spectra of infinite loop spaces

Suppose that $B \to \mathbb{Z}\times BO$ is a homotopy multiplicative map, and let M be the associated Thom spectrum. The spectrum M has a natural multiplication. If $W : X \to B$ is a vector bundle over X with a B-structure, then the Thom complex X^W comes equipped with a canonical M-Thom class
$$\Phi_M(W) : X^W \to M$$

Lemma 4.1. *The Thom class* $\Phi_M(W)$ *has the following properties*

i) *It is multiplicative:* $\Phi_M(W \oplus W') = \Phi_M(W)\Phi_M(W')$

ii) *It is preserved under base change: given $f : X \to Y$,*

$$\Phi_M(f^*W) = f^*\Phi_M(W).$$

□

An infinite loop map

$$B \to \mathbb{Z} \times BO$$

gives, for every vector bundle $W : X \to B$ with a B-structure, a B-structure to the vector bundle $D_n W$ over $D_n X$, and so also to its restriction $V_{\mathrm{reg}} \otimes W$ to $B\Sigma_n \times X$. The Thom spectrum spectrum M is then an E_∞ ring spectrum, whose underlying H_∞-structure is such that if $u_W : X^W \to M$ is the M–Thom class of the B-bundle W, then the composition

$$(B\Sigma_n \times X)^{V_{\mathrm{reg}} \otimes W} \to D_n M \to M$$

is the M-Thom class of $V_{\mathrm{reg}} \otimes W$.

5. A necessary condition for an $MU\langle 0\rangle$-orientation to be H_∞

Let $MU\langle 0\rangle$ be the Thom spectrum of the tautological bundle over $\mathbb{Z} \times BU$, and let $\mathbf{E}$ be a homogeneous ring spectrum. In §5.1 we recall that to give a map of (homotopy commutative) ring spectra

$$g : MU\langle 0\rangle \to \mathbf{E}. \tag{5.1}$$

is to give a coordinate s on $G = G_{\mathbf{E}}$.

In §5.2 we give a necessary condition for the map g to be a map of H_∞ spectra, in the case that $\mathbf{E}$ is an H_∞ ring spectrum, that $\pi_0\mathbf{E}$ is a complete local ring of characteristic $p > 0$, and that the formal group G is Noetherian (2.3) and of finite height. The result may described as follows.

Let $s = s_g$ be the coordinate on G associated to the orientation (5.1). In §3 we showed that the hypotheses on $\mathbf{E}$ give descent data for level structures on G. Given a level structure (10.9)

$$A_T \xrightarrow{\ell} i^*G, \tag{5.2}$$

we get *two* coordinates on the formal group $\left(\psi_\ell^{\mathbf{E}}\right)^* G$: one is just $\left(\psi_\ell^{\mathbf{E}}\right)^* s$, the other is the norm $N_\ell i^* s$ of the coordinate i^*s with respect to the isogeny

$$i^*G \xrightarrow{\psi_\ell^{G/\mathbf{E}}} \psi_\ell^* G$$

as in Proposition 12.4. We show that these two coordinates correspond to the two ways of going around the diagram

$$\begin{array}{ccc} D_A MU\langle 0\rangle & \longrightarrow & D_A\mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 0\rangle & \longrightarrow & \mathbf{E}; \end{array}$$

the main result is Proposition 5.11.

5.1. The spectrum $MU\langle 0\rangle$.

A $BU\langle 0\rangle = \mathbb{Z} \times BU$ bundle over a space X is just a virtual complex vector bundle W, with rank given by the locally constant function

$$X \xrightarrow{W} \mathbb{Z} \times BU \to \mathbb{Z}.$$

The tautological line bundle L over $\mathbb{C}P^\infty$ gives rise to a natural map

$$\Phi_{MU\langle 0\rangle}(L) : \mathbb{C}P^{\infty\,L} \to MU\langle 0\rangle.$$

If $\mathbf{E}$ is an even periodic ring spectrum with formal group $G = G_{\mathbf{E}}$ and

$$g : MU\langle 0\rangle \to \mathbf{E}$$

is a homotopy multiplicative map, then by Proposition 9.14 the composition

$$(\mathbb{C}P^\infty)^L \xrightarrow{\Phi_{MU\langle 0\rangle}(L)} MU\langle 0\rangle \xrightarrow{g} \mathbf{E}$$

is a trivialization s_g of the ideal sheaf $\mathbb{L}(L) \cong \mathcal{I}(0)$ over G, that is, a coordinate on G. The standard result about $MU\langle 0\rangle$-orientations is

Lemma 5.3. *The assignment $g \mapsto s_g$ is a bijection between the set of maps of homotopy commutative ring spectra $MU\langle 0\rangle \to \mathbf{E}$ and coordinates on $G_{\mathbf{E}}$.*

Proof. For $MU = MU\langle 2\rangle$ instead of $MU\langle 0\rangle$ the standard reference is [Ada74]. The minor modifications for $MU\langle 0\rangle$ may be found in [AHS01]. □

It is customary to express Lemma 5.3 in terms of formal group laws. A formal group law is the same thing as a formal group together with a coordinate: the equivalence sends a formal group G over R with multiplication

$$G \times G \xrightarrow{m} G,$$

and coordinate $s \in \mathcal{O}(G)$ to the power series

$$m^* s \in \mathcal{O}(G \times G) \cong R[\![s,t]\!].$$

We shall write (G, s) for this group law.

For example, the tautological map

$$(\mathbb{C}P^\infty)^L \to MU\langle 0\rangle$$

gives a coordinate $s_{MU\langle 0\rangle}$ on $G_{MU\langle 0\rangle}$. (Quillen's Theorem [Qui69] is that $(G_{MU\langle 0\rangle}, s_{MU\langle 0\rangle})$ is the *universal* formal group law.)

The commutative diagram

$$\begin{tikzcd}
\operatorname{spf} \widehat{\pi}_0 \mathbf{E}^{\mathbb{C}P^\infty_+} \arrow[r, "\operatorname{spf} \widehat{\pi}_0 g^{\mathbb{C}P^\infty_+}"] \arrow[d] & \operatorname{spf} \widehat{\pi}_0 MU\langle 0\rangle^{\mathbb{C}P^\infty_+} \arrow[d] \\
\operatorname{spf} \widehat{\pi_0 \mathbf{E}} \arrow[r, "\operatorname{spf} \widehat{\pi}_0 g"] & \operatorname{spec} \pi_0 MU\langle 0\rangle
\end{tikzcd}$$

gives a *relative* map

$$\bar{g} : G_{\mathbf{E}} = \operatorname{spf} \widehat{\pi_0 \mathbf{E}^{\mathbb{C}P^\infty_+}} \to (\operatorname{spf} \widehat{\pi_0 g})^* G_{MU\langle 0\rangle}.$$

Naturality together with the analogous diagram for

$$(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)_+ \to \mathbb{C}P^\infty_+$$

shows that $\bar{g}$ is a homomorphism of formal groups over $S_{\mathbf{E}}$. The construction of s_g shows that $\bar{g}$ is an isomorphism of formal groups, and

$$\bar{g}^* s_{MU\langle 0\rangle} = s_g.$$

In particular, we have the following.

Lemma 5.4. *If*

$$g : MU\langle 0\rangle \to \mathbf{E}$$

is a homotopy multiplicative map, then

$$\pi_0 g : \pi_0 MU\langle 0\rangle \to \pi_0 \mathbf{E}$$

classifies the group law $(G_{\mathbf{E}}, s_g)$. □

To understand the Thom class associated to a general virtual complex vector bundle W over a pointed space X, it is convenient as in §9 to work first with the bundle $W \otimes L$ over $X \times \mathbb{C}P^\infty$, and then pull back along the identity section. So let $\mathbf{F} = \mathbf{E}^{X_+}$, and let $f : \mathbf{E} \to \mathbf{F}$ be the map associated to the map $X \to *$.

The map

$$(X \times \mathbb{C}P^\infty)^{W \otimes L} \to MU\langle 0\rangle \xrightarrow{g} \mathbf{E}$$

represents a trivialization s_W of the line bundle $\mathbb{L}(W \otimes L)$ over $G_{\mathbf{F}} = (\operatorname{spec} \pi_0 f)^* G_{\mathbf{E}}$.

Lemma 5.5. *Suppose that W is a line bundle, and let $b \in G(\pi_0 \mathbf{F})$ be the corresponding point. Under the isomorphism* (9.7)

$$\mathbb{L}(W \otimes L) \cong T_b^* \mathcal{I}(0),$$

s_W is the section

$$s_W = T_b^* (\pi_0 f)^* s_g$$

Proof. The map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ which classifies the tensor product of line bundles is responsible for the group structure of G. □

Now take $X = BA^*$ so that $S_\mathbf{F} = (BA^*)_\mathbf{E}$, and take $W = V_{\text{reg}}$. Let

$$\epsilon : S_\mathbf{F} \to S_\mathbf{E}$$

be the structural map. We have a homomorphism

$$A \to G_\mathbf{F}$$

as in (8.1) and an isomorphism of line bundles over $S_\mathbf{F}$

$$\mathbb{L}(V_{\text{reg}} \otimes L) \cong \bigotimes_{a \in A} T_a^* \epsilon^* \mathcal{I}(0), \tag{5.6}$$

as in (9.10). Lemma 5.5 implies the following.

Proposition 5.7. *Under the isomorphism* (5.6), *we have*

$$s_{V_{\text{reg}}} = \prod_{a \in A} T_a^* \epsilon^* s_g.$$

□

5.2. **Comparing the H_∞ structures.** We continue to suppose that

$$g : MU\langle 0 \rangle \to \mathbf{E}$$

is a map of homotopy commutative ring spectra. Now suppose in addition that $\pi_0 \mathbf{E}$ is a complete local ring of characteristic $p > 0$, and that the formal group $G = G_\mathbf{E}$ is Noetherian (2.3) and of finite height. Let A be a finite abelian group. Proposition 8.3 implies that, with our hypotheses on $\mathbf{E}$, the natural map (8.2)

$$(BA^*)_\mathbf{E} \to \underline{\text{hom}}(A, G)$$

is an isomorphism. If

$$A_T \xrightarrow{\ell} i^* G$$

is a level structure (10.9) with cokernel

$$i^* G \xrightarrow{q} H,$$

then the homomorphism ℓ is classified by a map χ_ℓ making the diagram

$$\begin{tikzcd} T \arrow[r, "\chi_\ell"] \arrow[dr, "i"'] & \underline{\text{hom}}(A, G) \arrow[d, "\epsilon"] \\ & S_\mathbf{E} \end{tikzcd}$$

commute. The isogeny q gives a line bundle $N_q i^* \mathcal{I}_G(0)$ over H, defined so that

$$q^* N_q i^* \mathcal{I}_G(0) \cong \bigotimes_{a \in A} T_a^* i^* \mathcal{I}_G(0),$$

and Proposition 12.4 gives a canonical isomorphism

$$N_q i^* \mathcal{I}_G(0) \cong \mathcal{I}_H(0).$$

After changing base along $\chi_\ell : T \times G \to \underline{\text{hom}}(A, G) \times G$, the isomorphism (5.6) becomes

$$\chi_\ell^* \mathbb{L}(V_{\text{reg}} \otimes L) \cong \bigotimes_{a \in A} T_a^* i^* \mathcal{I}(0), \tag{5.8}$$

and Proposition 5.7 has the following

Corollary 5.9. *With respect to the isomorphism* (5.8), *we have*

$$\chi_\ell^* s_{V_{\mathrm{reg}}} = q^* N_q i^* s_g.$$

□

Now suppose in addition that $\mathbf{E}$ is an H_∞ ring spectrum. The two ways of going around the diagram

$$\begin{array}{ccccc} (BA^* \times \mathbb{C}P^\infty)^{V_{\mathrm{reg}}\otimes L} & \longrightarrow & D_A MU\langle 0\rangle & \xrightarrow{D_A g} & D_A \mathbf{E} \\ & & \downarrow & & \downarrow \\ & & MU\langle 0\rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

give two different trivializations s_{cl} and s_{cc} of $\mathbb{L}(V_{\mathrm{reg}} \otimes L)$ over

$$(BA^* \times \mathbb{C}P^\infty)_{\mathbf{E}} = \underline{\mathrm{hom}}(A, G) \times G.$$

If

$$A_T \xrightarrow{\ell} i^* G$$

is a level structure (10.9), then changing base along $\chi_\ell : T \times G \to \underline{\mathrm{hom}}(A, G) \times G$ and using the isogeny (3.21)

$$\psi_\ell^{G/\mathbf{E}} : i^* G \to \left(\psi_\ell^{\mathbf{E}}\right)^* G$$

gives isomorphisms

$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes L) = \left(\psi_\ell^{G/\mathbf{E}}\right)^* \mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0) \cong I_{i^* G}(\ell) \tag{5.10}$$

as in (9.11). Corollary 5.9 shows that

$$\chi_\ell^* s_{\mathrm{cc}} = \chi_\ell^* s_{V_{\mathrm{reg}}} = \left(\psi_\ell^{G/\mathbf{E}}\right)^* N_{\psi_\ell^{G/\mathbf{E}}} s_g.$$

By definition,

$$\chi_\ell^* s_{\mathrm{cl}} = \psi_\ell^L(s_g),$$

and so with respect to the isomorphism (5.10), Proposition 3.21 gives

$$\chi_\ell^* s_{\mathrm{cl}} = \left(\psi_\ell^{G/\mathbf{E}}\right)^* \left(\psi_\ell^{\mathbf{E}}\right)^* s_g.$$

Thus we have the following

Proposition 5.11. *Let $g : MU\langle 0\rangle \to \mathbf{E}$ be a homotopy multiplicative map, and let $s = s_g$ be corresponding trivialization of $\mathcal{I}_G(0)$. If the map g is H_∞, then for any level structure*

$$A \xrightarrow{\ell} i^* G,$$

the section s satisfies the identity

$$N_{\psi_\ell^{G/\mathbf{E}}} i^* s = \left(\psi_\ell^{\mathbf{E}}\right)^* s, \tag{5.12}$$

in which the isogeny $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$N_{\psi_\ell^{G/\mathbf{E}}} i^* \mathcal{I}_G(0) \cong \mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0).$$

□

Remark 5.13. The Proposition can be stated in terms of formal group laws along the lines of Lemma 5.4. Given an orientation g and a level structure ℓ as in the Proposition, we get two ring homomorphisms

$$\alpha, \beta : \pi_0 MU\langle 0\rangle \to \mathbf{E},$$

namely

$$\alpha : \pi_0 MU\langle 0\rangle \xrightarrow{P_A} \pi_0 MU\langle 0\rangle^{BA_+^*} \xrightarrow{\pi_0 g_+^{BA^*}} \pi_0 \mathbf{E}^{BA_+^*} \xrightarrow{\chi_\ell} R$$

and

$$\beta : \pi_0 MU\langle 0\rangle \xrightarrow{\pi_0 g} \pi_0 \mathbf{E} \xrightarrow{\psi_\ell^{\mathbf{E}}} R.$$

The Proposition implies that α classifies the group law $\left(\left(\psi_\ell^{\mathbf{E}}\right)^* G, N_\ell i^* s_g\right)$, while β classifies the group law $\left(\left(\psi_\ell^{\mathbf{E}}\right)^* G, \left(\psi_\ell^{\mathbf{E}}\right)^* s_g\right)$.

Remark 5.14. The necessary condition of Proposition 5.11 was introduced in [And95], in the case that $\mathbf{E}$ is the spectrum associated to the universal deformation of the Honda formal group of height n. In that case, if one has a level structure

$$(\mathbb{Z}/p)^n \xrightarrow{\ell} i^* G_{\mathbf{E}},$$

one finds that

$$\begin{aligned} \psi_\ell^{\mathbf{E}} &= i \\ \psi_\ell^{G/\mathbf{E}} &= p : G_{\mathbf{E}} \to G_{\mathbf{E}}, \end{aligned}$$

so that equation (5.12) becomes

$$N_p i^* s = i^* s,$$

or after pulling back along p,

$$\prod_{a \in (\mathbb{Z}/p)^n} T_a^* s = p^* s.$$

6. A necessary condition for an $MU\langle 2k\rangle$-orientation to be H_∞

In this section we describe the modifications to Proposition 5.11 needed in the case that $k \geq 1$ and

$$g : MU\langle 2k\rangle \to \mathbf{E}$$

is a homotopy multiplicative map from the Thom spectrum of $BU\langle 2k\rangle$ to $\mathbf{E}$.

A $BU\langle 2k\rangle$-bundle over a space X can be interpreted as an equivalence class of elements $V \in K(X, A)$, with the pair (X, A) having connectivity $(2k-1)$. This makes it clear that if V is a $BU\langle 2\rangle k$-bundle over X and W is any (virtual) vector bundle, then $W \otimes V$ has a canonical $BU\langle 2k\rangle$ structure. For example, it follows that the bundle

$$V = (1 - L_1) \otimes \cdots \otimes (1 - L_k)$$

over $(\mathbb{C}P^\infty)^k$ has a $BU\langle 2k\rangle$-structure (this is also explained in our paper [AHS01]).

Suppose that $\mathbf{E}$ is an even periodic ring spectrum and let $G = G_{\mathbf{E}}$.

Lemma 6.1. (1) *Proposition 9.12 gives an isomorphism*

$$t_V : \mathbb{L}(V) \cong \Theta^k(\mathcal{I}_G(0)).$$

(2) *For the bundle $L \otimes V$ over ${\mathbb{C}P^\infty}^{k+1}$, Proposition 9.12 gives an isomorphism*

$$t_{L \otimes V} : \mathbb{L}(L \otimes V) \cong \frac{\mu_{12}^* \mathbb{L}(V)}{\hat{\pi}_2 \mathbb{L}(V)}$$

of line bundles over $G_{\mathbf{E}}^{k+1}$.

(3) *In the notation of Lemma 4.1, the $MU\langle 2k\rangle$-Thom class of the bundle $L \otimes V$ over $(\mathbb{C}P^\infty)^{k+1}$ is given by*

$$\Phi_{MU\langle 2k\rangle}(L \otimes V) = \frac{\mu_{12}^* \Phi_{MU\langle 2k\rangle}(V)}{\hat{\pi}_2 \Phi_{MU\langle 2k\rangle}(V)}$$

Proof. The first part follows from the discussion of the line bundles $\mathbb{L}(V)$ in §9. For the second two parts, simply write

$$L \otimes V = (1 - LL_1) \otimes (1 - L_2) \otimes \cdots \otimes (1 - L_k) - (1 - L) \otimes (1 - L_2) \otimes \cdots \otimes (1 - L_k),$$

and use Lemma 4.1. $\square$

The Lemma implies that if $g : MU\langle 2k\rangle \to \mathbf{E}$ is a homotopy multiplicative map, then the composition

$$(\mathbb{C}P^{\infty\, k})^V \to MU\langle 2k\rangle \xrightarrow{g} \mathbf{E}$$

represents a trivialization $s = s_g$ of $\Theta^k(\mathcal{I}_G(0))$ (In fact it is easily seen to be a Θ^k-structure on $\mathcal{I}_G(0)$ in the sense of (15.5)). If $\mathbf{E}$ is an H_∞ ring spectrum, and A is a finite abelian group, then the two ways of going around the diagram

$$\begin{array}{ccccc} (BA^* \times (\mathbb{C}P^\infty)^k)^{V_{\mathrm{reg}}\otimes V} & \longrightarrow & D_A MU\langle 2k\rangle & \xrightarrow{D_A g} & D_A\mathbf{E} \\ & & \downarrow & & \downarrow \\ & & MU\langle 2k\rangle & \xrightarrow{g} & \mathbf{E} \end{array} \tag{6.2}$$

give two different trivializations s_{cl} and s_{cc} of

$$\mathbb{L}(V_{\mathrm{reg}} \otimes V)$$

over

$$\operatorname{spf} \widehat{\pi}_0 \mathbf{E}^{(BA^* \times (\mathbb{C}P^\infty)^k)_+} = \underline{\hom}(A, G) \times G^k.$$

The second part of Lemma 6.1 implies that

$$\mathbb{L}(V_{\mathrm{reg}} \otimes V) \cong \bigotimes_{a\in A} \tilde{T}_a \mathbb{L}(V), \tag{6.3}$$

where $\tilde{T}_a$ is translation operation introduced in (16.1).

If

$$A_T \xrightarrow{\ell} i^* G$$

is a level structure on G (10.9), then after changing base along the map

$$T \times G^k \xrightarrow{\chi_\ell \times G^k} \underline{\hom}(A, G) \times G^k,$$

we have

$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes V) \cong \left(\psi_\ell^{G/\mathbf{E}}\right)^* \tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* \Theta^k(\mathcal{I}_G(0)) \cong \left(\psi_\ell^{G/\mathbf{E}}\right)^* \Theta^k(\mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0)); \tag{6.4}$$

the first isomorphism follows from (6.3) and the definition (16.7) of $\tilde{N}$, while the second isomorphism is from Proposition 16.9. The third part of Lemma 6.1 implies that with respect to this isomorphism we have

$$\chi_\ell^* s_{\mathrm{cc}} = \left(\psi_\ell^{G/\mathbf{E}}\right)^* \tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* s_g.$$

By definition, we have

$$\chi_\ell^* s_{\mathrm{cl}} = \psi_\ell^V(s_g),$$

and with respect to the isomorphism (6.4), Proposition 3.25 gives the equation

$$\chi_\ell^* s_{\mathrm{cl}} = \left(\psi_\ell^{G/\mathbf{E}}\right)^* \left(\psi_\ell^{\mathbf{E}}\right)^* s_g.$$

The analogue of Proposition 5.11 is

Proposition 6.5. *Let $g : MU\langle 2k\rangle \to \mathbf{E}$ be a homotopy multiplicative map, and s corresponding section of $\Theta^k(\mathcal{I}_G(0))$. If the map f is H_∞, then for each level structure*

$$A \xrightarrow{\ell} i^* G$$

the section s satisfies the identity

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} i^* s = \left(\psi_\ell^{\mathbf{E}}\right)^* s,$$

in which the map $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} \Theta^k(\mathcal{I}_{i^*G}(0)) \cong \Theta^k(\mathcal{I}_{(\psi_\ell^{\mathbf{E}})^* G}(0)).$$

□

7. The necessary condition is sufficient for $k \le 3$

Suppose that $\mathbf{E}$ is an even periodic H_∞ spectrum. Suppose that $\pi_0\mathbf{E}$ is an p-regular admissible local ring of residue characteristic p, and the formal group $G = G_{\mathbf{E}}$ is Noetherian (2.3) and of finite height. Suppose that $k \le 3$, and let $g : MU\langle 2k\rangle \to \mathbf{E}$ be a homotopy multiplicative map. Let $s = s_g$ be the corresponding section of $\Theta^k(I_G(0))$.

Proposition 7.1. *The map g is H_∞ if and only if for each level structure*

$$A \xrightarrow{\ell} i^*G,$$

the section s satisfies the identity

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} s = \left(\psi_\ell^{\mathbf{E}}\right)^* i^* s,$$

in which, as in Proposition 6.5, the isogeny $\psi_\ell^{G/\mathbf{E}}$ has been used make the identification

$$\tilde{N}_{\psi_\ell^{G/\mathbf{E}}} \Theta^k(\mathcal{I}_{i^*G}(0)) \cong \Theta^k(\mathcal{I}_{(\psi_\ell^{\mathbf{E}})^*G}(0)).$$

Proof. We must show that, for all n, the diagram

$$\begin{array}{ccc} D_n MU\langle 2k\rangle & \xrightarrow{D_n g} & D_n\mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

commutes. The hypotheses on $\pi_0\mathbf{E}$ and the algebra of the D_n's together with the Sylow structure of the symmetric groups reduce us immediately to checking that the diagram

$$\begin{array}{ccc} D_A MU\langle 2k\rangle & \xrightarrow{D_A g} & D_A\mathbf{E} \\ \downarrow & & \downarrow \\ MU\langle 2k\rangle & \xrightarrow{g} & \mathbf{E} \end{array}$$

commutes when A is a Sylow subgroup of Σ_p [McC86, §7].

Let g_{cl} and g_{cc} be the two ways of navigating this diagram. Each is a generator of $\pi_0\mathbb{S}_U(D_A MU\langle 2k\rangle, \mathbf{E})$, so by the Thom isomorphism their ratio is a generator of

$$\pi_0\mathbb{S}_U(D_A BU\langle 2k\rangle_+, \mathbf{E}).$$

For $k \le 3$, the natural map

$$\pi_0\mathbb{S}_U(D_A BU\langle 2k\rangle_+, \mathbf{E}) \xrightarrow{\Delta^*} \pi_0\mathbb{S}_U((BA^* \times BU\langle 2k\rangle)_+, \mathbf{E})$$

is injective (see e.g. [McC86, 7.3]). Let $\mathbf{F} = \mathbf{E}^{BU\langle 2k\rangle_+}$. Our hypotheses on $\mathbf{E}$ and the fact that for $k \le 3$, $H_*(BU\langle 2k\rangle, \mathbb{Z})$ is concentrated in even degrees (for $BU\langle 6\rangle$ see [Sin68] or [AHS01]), imply that the natural maps induce isomorphisms

$$\begin{aligned} \pi_0\mathbf{E} &\cong \mathcal{O}(S_{\mathbf{E}}) \\ \pi_0\mathbf{F} &\cong \mathcal{O}(S_{\mathbf{F}}) \\ \pi_0\mathbb{S}_U((BA^* \times BU\langle 2k\rangle)_+, \mathbf{E}) &\cong \mathcal{O}(\underline{\mathrm{hom}}(A, G_{\mathbf{F}})). \end{aligned}$$

By Proposition 10.23, it suffices to show that $g_{\mathrm{cl}}/g_{\mathrm{cc}} = 1$ after changing base along the two maps

$$\begin{aligned} \underline{\mathrm{level}}(A, G_{\mathbf{F}}) &\to \underline{\mathrm{hom}}(A, G_{\mathbf{F}}) \\ S_{\mathbf{F}} &\to \underline{\mathrm{hom}}(A, G_{\mathbf{F}}) \end{aligned}$$

classifying respectively the level structure and the zero homomorphism (this is essentially a result of [HKR00]).

After changing base to $\underline{\mathrm{level}}(A, G)$, $g_{\mathrm{cl}}/g_{\mathrm{cc}}$ becomes

$$\left(\left(\psi_\ell^{\mathbf{F}}\right)^* s\right) / \left(\tilde{N}_{\psi_\ell^{G/\mathbf{F}}} i^* s\right),$$

as in Proposition 6.5. The base change $S_{\mathbf{F}} \to \underline{\mathrm{hom}}(A, G)$ corresponds to the augmentation

$$\pi_0 \mathbf{F}^{BA^*_+} \to \pi_0 \mathbf{F},$$

under which each g restricts to the Thom class

$$BU\langle 2k\rangle^{V^p} \xrightarrow{\Phi_{MU\langle 2k\rangle}(V^p)} MU\langle 2k\rangle \xrightarrow{g} \mathbf{E}$$

of V^p, where V is the standard bundle over $BU\langle 2k\rangle$. ◻

Part 2. Even periodic cohomology of abelian groups and Thom complexes

8. Even cohomology of abelian groups

Suppose that $\mathbf{E}$ is a homogeneous ring spectrum, with formal group $G = G_{\mathbf{E}}$, and let A be a finite abelian group. Let A^* be the character group $A^* = \mathrm{hom}(A, \mathbb{C}^\times)$. An element a of A may be viewed as a character of A^*, giving a line bundle V_a over BA^* and so a map $(BA^*)_{\mathbf{E}} \to (\mathbb{C}P^\infty)_{\mathbf{E}} = G$, i.e. a $(BA^*)_{\mathbf{E}}$-valued "point" of G. As a varies we get a map of sets

$$A \xrightarrow{\chi} G((BA^*)_{\mathbf{E}}). \tag{8.1}$$

Since

$$V_{a+b} = V_a \otimes V_b,$$

and since the group structure of G comes from the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

which classifies the tensor product of line bundles, the map χ is a group homomorphism, and so it is classified by a map of of formal schemes

$$(BA^*)_{\mathbf{E}} \xrightarrow{\tilde{\chi}} \underline{\mathrm{hom}}(A, G). \tag{8.2}$$

This map is often an isomorphism. For example, we have the

Proposition 8.3. *If $\pi_0\mathbf{E}$ is a complete local ring of residue characteristic $p > 0$, and if the height of the formal group $G_{\mathbf{E}}$ is finite, then the map $\tilde{\chi}$ is an isomorphism of formal schemes over $S_{\mathbf{E}}$.*

Proof. This formulation of the $\mathbf{E}$-cohomology of abelian groups appeared in [HKR00]. ◻

Suppose that $\tilde{\chi}$ is an isomorphism, and suppose that we have a level structure

$$A_T \xrightarrow{\ell} i^*G$$

over a formal scheme T. The homomorphism ℓ is classified by a map χ_ℓ making the diagram

$$\begin{tikzcd} T \arrow[r, "\chi_\ell"] \arrow[dr, "i"'] & (BA^*)_{\mathbf{E}} \arrow[d] \\ & S_{\mathbf{E}} \end{tikzcd} \tag{8.4}$$

commute.

Proposition 8.5. *If G is Noetherian (2.3), and if $A'^* \subset A^*$ is a proper subgroup, then the composite map of $\pi_0\mathbf{E}$-modules*

$$\pi_0 \mathbf{E}^{BA'^*_+} \xrightarrow{\textit{transfer}} \pi_0 \mathbf{E}^{BA^*_+} \xrightarrow{\chi_\ell} \mathcal{O}(T)$$

is zero.

Proof. It suffices to consider the case that

$$\ell : A \to i^*G$$

is the tautological level structure over $\underline{\mathrm{level}}(A, G)$.

If A is not a p-group then $\underline{\mathrm{level}}(A, G)$ is empty and the result is trivial.

Suppose that $A' = 0$ and $A = \mathbb{Z}/p$. Let $t \in \pi_0\mathbf{E}^{\mathbb{C}P^\infty_+}$ be a coordinate, and let F be the resulting group law. Then

$$\pi_0\mathbf{E}^{BA^*_+} \cong \pi_0\mathbf{E}[\![t]\!]/[p]_F(t)$$

and $\tau : \pi_0\mathbf{E}^{BA'^*_+} = \pi_0\mathbf{E} \to \pi_0\mathbf{E}^{BA^*}$ is given by

$$\tau(1) = \langle p \rangle(t).$$

(see e.g. [Qui71]), and so the result follows from the isomorphism (10.22)

$$\mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p, G_{\mathbf{E}})) \cong \pi_0\mathbf{E}[\![t]\!]/\langle p \rangle(t).$$

For the general case, we may suppose that $A'^* \subsetneq A^*$ is maximal, and so we have a pull-back diagram

$$\begin{tikzcd} BA'^* \arrow[r, "i'"] \arrow[d, "j'"'] & B0 \arrow[d, "j"] \\ BA^* \arrow[r, "i"] & BC^* \end{tikzcd}$$

where $C \subsetneq A$ is cyclic of order p. The commutativity of the diagram

$$\begin{tikzcd} \pi_0\mathbf{E}^{BA'^*_+} \arrow[d, "\tau"'] & \pi_0\mathbf{E}^{B0_+} \arrow[l, "i'^*"'] \arrow[d, "\tau"] \arrow[dd, bend left, "0"] \\ \pi_0\mathbf{E}^{BA^*_+} \arrow[d, "\pi"'] & \pi_0\mathbf{E}^{BC^*_+} \arrow[l, "i^*"'] \arrow[d] \\ \mathcal{O}(\underline{\mathrm{level}}(A, G)) & \mathcal{O}(\underline{\mathrm{level}}(C, G)) \arrow[l] \end{tikzcd}$$

implies that

$$\pi(\tau(1)) = 0.$$

The result follows, since $\pi_0\mathbf{E}^{BA'^*_+}$ is a cyclic $\pi_0\mathbf{E}^{BA^*_+}$-module via j'^*, and τ is a map of $\pi_0\mathbf{E}^{BA^*_+}$-modules. □

9. Cohomology of Thom spectra

Suppose that X is a space, and that V is a complex vector bundle over X. The $\pi_0\mathbf{E}^{X_+}$-module $\pi_0\mathbb{S}_U(X^V, \mathbf{E})$ is free of rank one (since $\mathbf{E}$ is complex orientable) and so can be interpreted as the module of sections of a line bundle $\mathbb{L}(V)$ over $X_{\mathbf{E}}$. The fact that the Thom complex of an external Whitney sum is the smash product of the Thom complexes gives rise to a canonical isomorphism

$$\mathbb{L}(V \oplus W) \cong \mathbb{L}(V) \otimes \mathbb{L}(W) \tag{9.1}$$

This property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles; we define

$$\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}. \tag{9.2}$$

If $f : X \to Y$ is a map, and V is a virtual bundle over Y, then there is an isomorphism

$$\pi_0\mathbb{S}_U(X^{f^*V}, \mathbf{E}) \cong \left(\pi_0\mathbf{E}^{X_+}\right) \underset{\pi_0 E^{Y_+}}{\otimes} \pi_0\mathbb{S}_U(Y^V, \mathbf{E})$$

In terms of algebraic geometry, this means that there is a natural isomorphism

$$\mathbb{L}(f^*V) \cong (f_{\mathbf{E}})^* \, \mathbb{L}(V). \tag{9.3}$$

Here is a series of examples which lead to a fairly complete understanding of the functor $\mathbb{L}(V)$.

(1) If L denotes the canonical line bundle over $\mathbb{C}P^\infty$, then the zero section identifies $\pi_0\mathbb{S}_U(\mathbb{C}P^{\infty\, L}, \mathbf{E})$ with the augmentation ideal in $\pi_0\mathbf{E}^{\mathbb{C}P^\infty_+}$, and so we have an isomorphism

$$\mathbb{L}(L) \cong \mathcal{I}(0). \tag{9.4}$$

(2) Suppose that V is a line bundle over X, classified by a map $b : X \to \mathbb{C}P^\infty$. In terms of algebraic geometry, the map b defines an X_E-value point $b = b_{\mathbf{E}}$ of G. It follows from (9.3) that

$$\mathbb{L}(V) \cong b^*\mathcal{I}(0) \cong 0^*\mathcal{I}(-b). \tag{9.5}$$

(3) Taking X to be a point and V to be the trivial complex line bundle in (2), we have

$$\mathbb{L}(V) \cong 0^*\mathcal{I}(0). \tag{9.6}$$

Now $\mathbb{L}(V)$ is the sheaf associated to $\pi_2\mathbf{E}$, while $0^*\mathcal{I}(0)$ is the sheaf of cotangent vectors at the origin of G, isomorphic to the sheaf ω_G of invariant differentials on G. If $f : \mathbf{E} \to \mathbf{F}$ is an $\mathbf{E}$-algebra (e.g. $\mathbf{F} = \mathbf{E}^{X_+}$), this gives an interpretation of the homotopy group $\pi_{2k}\mathbf{F}$ as the sections of $f^*\omega_G$.

(4) If V is the trivial bundle of dimension k, then by (9.6) and (9.1), $\mathbb{L}(V)$ is just ω^k.

(5) If $V = (1 - L)$ is the reduced canonical line bundle over $\mathbb{C}P^\infty$, then using (9.2), (9.4), and (9.6) we have

$$\mathbb{L}(V) \cong \pi^*0^*\mathcal{I}(0) \otimes \mathcal{I}(0)^{-1} = \Theta^1(\mathcal{I}(0)),$$

where $\pi : G_{\mathbf{E}} \to S_{\mathbf{E}}$ is the structural map and Θ^1 is defined in (15.2).

(6) With the notation of example (2) consider bundle $V \otimes L$ over $X \times \mathbb{C}P^\infty$. Then the line bundle $\mathbb{L}(V \otimes L)$ is pulled back from $\mathcal{I}_G(0)$ along the map

$$X_{\mathbf{E}} \times G \xrightarrow{b\times 1} G \times G \xrightarrow{\mu} G.$$

It follows that

$$\mathbb{L}(V \otimes L) \cong T_b^*\mathcal{I}(0) = \mathcal{I}_{X_{\mathbf{E}}\times G}(-b). \tag{9.7}$$

(7) More generally, suppose that $V = \sum n_i L_i$ is a virtual sum of line bundles over X. The line bundles L_i define points b_i of G over $X_{\mathbf{E}}$, and the bundle V determines the divisor $D = \sum n_i\{b_i\}$. It follows using (9.1) that

$$\mathbb{L}(V \otimes L) = \mathcal{I}_{X_{\mathbf{E}}\times G}(D^{-1}), \tag{9.8}$$

where $D^{-1} = \sum n_i\{b_i^{-1}\}$.

(8) In fact, by the splitting principle, the line bundle $\mathbb{L}(V \otimes L)$ can be computed in this manner even when V is not a virtual sum of line bundles. Indeed, by the splitting principle, there is a map $f : F \to X$ with the properties that $f_{\mathbf{E}}$ is finite and faithfully flat, and f^*V is a virtual sum of line bundles. The line bundle $\mathbb{L}(f^*(V) \otimes L)$ can then be computed as $\mathcal{O}(D^{-1})$ as above. But then it is easy to check that the divisor D descends to $X_{\mathbf{E}} \times G$, even though none of its points do.

(9) Let A be a finite abelian group. An element $a \in A$ can be regarded as a character of A^*. Let V_a be the associated line bundle over BA^*. Recall (8.1) that this construction defines a group homomorphism

$$\chi : A \to G(BA^*_{\mathbf{E}}).$$

The line bundle $\mathbb{L}(V_a \otimes V \otimes L)$ over $BA^*_{\mathbf{E}} \times X_{\mathbf{E}} \times G$ is

$$\mathbb{L}(V_a \otimes V \otimes L) \cong T_a^*\mathcal{I}(D^{-1});$$

taking V to be the trivial line bundle over a point gives

$$\mathbb{L}(V_a \otimes L) \cong T_a^*\mathcal{I}(0) = I(a^{-1})$$

(10) Now let

$$V_{\text{reg}} = \bigoplus_{a\in A} V_a$$

be the regular representation of A^*. Over the scheme $(BA^*)_{\mathbf{E}} \times G$, the line bundle associated to the Thom complex of $V_{\text{reg}} \otimes V \otimes L$ is

$$\mathbb{L}(V_{\text{reg}} \otimes V \otimes L) \cong \bigotimes_{a\in A} T_a^*\mathcal{I}(D^{-1}) \cong \mathcal{I}\left(\sum_a T_a^* D^{-1}\right). \tag{9.9}$$

In particular,

$$\mathbb{L}(V_{\text{reg}} \otimes L) \cong \mathcal{I}(\ell) = \bigotimes_{a\in A} T_a^*\mathcal{I}(0). \tag{9.10}$$

(11) Suppose that the map

$$\tilde{\chi} : (BA^*)_{\mathbf{E}} \to \underline{\mathrm{hom}}(A, G)$$

of (8.2) is an isomorphism. If

$$A_T \xrightarrow{\ell} i^*G \xrightarrow{q} G'$$

is a level structure with cokernel q over T, then changing base in (9.10) along

$$T \times G \xrightarrow{\chi_\ell} \underline{\mathrm{hom}}(A, G) \times G$$

(where χ_ℓ is the map classifying the homomorphism ℓ; see (8.4)) and using (12.6) gives

$$\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}} \otimes L) \cong q^* N_q \mathcal{I}_G(0) \cong q^* \mathcal{I}_{G'}(0). \tag{9.11}$$

(12) Restricting the above example to BA^* we find that

$$\begin{aligned}\chi_\ell^* \mathbb{L}(V_{\mathrm{reg}}) = 0_G^* q^* \mathcal{I}_{G'}(0) &= 0_{G'}^* \mathcal{I}_{G'}(0) \\ &= \omega_{G'}.\end{aligned}$$

This series of examples establishes the following results:

Proposition 9.12. *For a pointed topological space X, let $\mathbf{F}$ be the spectrum $\mathbf{E}^{X_+}$, and let*

$$G = G_{\mathbf{F}} = \mathbb{C}P^{\infty}{}_{\mathbf{F}}$$

be the associated formal group. Attached to each (virtual) complex vector bundle V over X is a divisor D_V on G, and an isomorphism

$$t_V : \mathbb{L}(V \otimes L) \cong \mathcal{I}_G(D^{-1}). \tag{9.13}$$

The map t_V restricts to an isomorphism

$$t_V : \mathbb{L}(V) \cong 0^* \mathcal{I}(D^{-1}). \quad \square$$

Proposition 9.14. *The correspondence $V \mapsto D_V$ and the isomorphism* (9.13) *are determined by the following properties*

i) *If $V = V_1 \oplus V_2$, then $D_V = D_{V_1} + D_{V_2}$, and with the identifications*

$$\mathbb{L}(V_1) \otimes \mathbb{L}(V_2) \cong \mathbb{L}(V)$$
$$\mathcal{I}(D_{V_1}^{-1}) \otimes \mathcal{I}(D_{V_2}^{-1}) \cong \mathcal{I}(D_V^{-1}),$$

there is an equality

$$t_V = t_{V_1} \otimes t_{V_2}.$$

ii) *If $f : Y \to X$ is a map of pointed spaces, and if $W = f^*V$, then $D_W = f^* D_V$, and $t_W = f^* t_V$.*

iii) *If X is a point, and V has dimension 1, then $D = \{0\}$, and the isomorphism*

$$t_L : \mathbb{L}(L) \cong \mathcal{I}(0) \tag{9.15}$$

is given by applying $\pi_0 \mathbf{E}^{(-)}$ to the inclusion of the zero section $\mathbb{C}P^\infty_+ \to \mathbb{C}P^{\infty L}$. $\square$

Part 3. Level structures and isogenies of formal groups

10. Level structures

10.1. **Homomorphisms.** Suppose that A is a finite abelian group and G is a formal group over a formal scheme S.

Definition 10.1. We write $\underline{\mathrm{hom}}(A, G)$ for the functor from formal schemes to groups defined by the formula

$$\underline{\mathrm{hom}}(A, G)(T) = \{\text{pairs } (u, \ell) \mid u : T \to S\ ,\ \ell \in \mathrm{hom}(A, u^*G(T))\}.$$

Remark 10.2. We shall use the notation

$$A_T \xrightarrow{\ell} u^*G$$

to indicate that T is a formal scheme and $(u, \ell) \in \underline{\mathrm{hom}}(A, G)(T)$.

Example 10.3. Let G be a formal group over R, and suppose that x is a coordinate on G. Let F be the resulting group law. The "n-series" of F is the power series $[n](t) \in R[\![t]\!]$ defined by the formula

$$[n](x) = n^* x,$$

where the right-hand-side refers to the pull-back of functions along the homomorphism $n : G \to G$. To give a homomorphism

$$\ell : \mathbb{Z}/n \to G(T)$$

is to give a topologically nilpotent element $x(\ell(1))$ of $\mathcal{O}(T)$, with the property that

$$[n](x(\ell(1))) = 0;$$

the homomorphism ℓ is then given by

$$x(\ell(j)) = [j](x(\ell(1))).$$

It follows that

$$\underline{\mathrm{hom}}(\mathbb{Z}/n, G) = \mathrm{spf}\Big(R[\![x(\ell(1))]\!]/\big([n](x(\ell(1)))\big)\Big).$$

It is clear from the definition that if $B \subset A$ there is a restriction map

$$\underline{\mathrm{hom}}(A, G) \to \underline{\mathrm{hom}}(B, G),$$

and if $A = B \times C$ then the resulting map

$$\underline{\mathrm{hom}}(A, G) \to \underline{\mathrm{hom}}(B, G) \times_S \underline{\mathrm{hom}}(C, G) \tag{10.4}$$

is an isomorphism. Also from the definition we see that if $j : S' \to S$ is a map of formal schemes, then the natural map

$$\underline{\mathrm{hom}}(A, j^* G) \to j^* \underline{\mathrm{hom}}(A, G)$$

is an isomorphism. Combining these observations with Example 10.3 and the structure of finite abelian groups gives the following.

Lemma 10.5. *The functor* $\underline{\mathrm{hom}}(A, G)$ *is represented by an affine formal scheme over* S*. If* $j : S' \to S$ *is a map of formal schemes, then the natural map*

$$\underline{\mathrm{hom}}(A, j^* G) \to j^* \underline{\mathrm{hom}}(A, G)$$

is an isomorphism of formal schemes over S'. □

For formal groups over p-local rings, only the p-groups give anything interesting.

Example 10.6. Returning to Example 10.3, the n-series is easily seen to be of the form

$$[n](t) = nt + o(2).$$

If n is a unit in R, then

$$R[\![x]\!]/([n](x)) \cong R$$

so $\underline{\mathrm{hom}}(\mathbb{Z}/n, G)$ is the trivial group scheme over R.

Example 10.7. If R is a complete local ring of residue characteristic p, then there is an h with $1 \le h \le \infty$ such that

$$[p^m](t) \equiv \epsilon t^{p^{mh}} + o(t^{p^h} + 1) \mod \mathfrak{m}_R.$$

This h is called the *height* of G. If h is finite, then the Weierstrass Preparation Theorem [Lan78, pp. 129–131] implies that there are monic polynomials $g_m(t)$ of degree p^{mh} such that

$$[p^m](t) = g_m(t) \cdot \epsilon,$$

where ϵ is a unit of $R[\![t]\!]$. It follows that $\mathcal{O}(\underline{\mathrm{hom}}(\mathbb{Z}/p^m, G))$ is finite and free of rank p^{hm} over R.

These examples generalize to give the following.

Proposition 10.8. *Let* G *be a formal group of finite height over a local formal scheme* S*. Then* $\underline{\mathrm{hom}}(A, G)$ *is a local formal scheme over* S*. For* $B \subseteq A$*, the forgetful map*

$$\underline{\mathrm{hom}}(A, G) \to \underline{\mathrm{hom}}(B, G)$$

is a map of formal schemes, finite and free of rank d^h*, where* d *is the order of the* p*-torsion subgroup of* A/B*.*

Proof. By the product formula (10.4) and Example 10.6 we are reduced to the case that A and B are p-groups.

By induction it suffices to treat the case that

$$0 \to B \to A \to C \to 0$$

is a short exact sequence, where C is cyclic of order p. Let c be a generator of C, a and element of A mapping to c, and let $b = pa$. Suppose that x is a coordinate on G.

If

$$\ell : B \to G(T)$$

is a homomorphism, then to give a homomorphism

$$\ell' : A \to G(T)$$

with $\ell'|_B = \ell$ is to give a topologically nilpotent element $x(\ell'(a))$ of $\mathcal{O}(T)$ with the property that

$$[p](x(\ell'(a))) = x(\ell(b)).$$

It is clear that the universal example of such a situation occurs over the ring

$$\mathcal{O}(T) = R[\![x(\ell'(a))]\!]/([p](x(\ell'(a))) - x(\ell(b))),$$

where $R = \mathcal{O}(\underline{\mathrm{hom}}(B,G))$ and $\ell : B \to G(R)$ is the tautological map. The Weierstrass Preparation Theorem implies that $\mathcal{O}(T)$ is finite and free over R, of rank p^h. $\square$

10.2. **Level structures.** The scheme $\underline{\mathrm{hom}}(A,G)$ has an important closed subscheme $\underline{\mathrm{level}}(A,G)$, which was introduced by Drinfel'd [Dri74]. Suppose that G is a formal group over a formal scheme S.

Definition 10.9. Let T be a formal scheme. A T-valued point

$$A_T \xrightarrow{\ell} i^*G$$

of $\underline{\mathrm{hom}}(A,G)$ is a *level A structure* (or *level structure* or *A-structure* for short) if for each prime q dividing $|A|$, the subgroup $i^*G[q] = \ker(q : i^*G \to i^*G)$ is a divisor on G/T, and there is an inequality of divisors

$$\sum_{\substack{a \in A \\ qa=0}} \{\ell(a)\} \leq i^*G[q]$$

in i^*G. The subfunctor of $\underline{\mathrm{hom}}(A,G)$ consisting of level structures will be denoted $\underline{\mathrm{level}}(A,G)$.

Remark 10.10. If we say that

$$A_T \xrightarrow{\ell} i^*G$$

"is a level structure," we mean that T is a formal scheme, and (i,ℓ) is a T-valued point of $\underline{\mathrm{level}}(A,G)$. We may omit one of T and i if it is clear from the context.

Here are some examples to give a feel for level structures. First of all, only p-groups of small rank can produce level structures.

Lemma 10.11. *If $|A|$ is not a power of p, then*

$$\underline{\mathrm{level}}(A,G) = \emptyset.$$

If the height of G is h and the p-rank of A is greater than h, then again $\underline{\mathrm{level}}(A,G) = \emptyset$.

Proof. If $|A|$ is not a power of p, then there is a prime $q \neq p$ such that the divisor

$$\sum_{qa=0} \{\ell(a)\}$$

has degree greater than 1. However, $q : G \to G$ is an isomorphism, so $G[q] = \{0\}$ has degree 1. Similarly, if the height of G is h then the degree of $G[p]$ is p^h. $\square$

A level structure is trying to be a monomorphism; for example if R is a domain in which $|A| \neq 0$, then a homomorphism

$$\ell : A \to G(R)$$

is a level structure if and only if it is a monomorphism (Corollary 10.20). However, naive monomorphisms from A to G can't in general be a representable functor.

Example 10.12. Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group with coordinate x so that the group law is

$$x \underset{F}{+} y = x + y - xy.$$

The p-series is

$$[p](x) = 1 - (1-x)^p.$$

The monomorphism

$$\mathbb{Z}/p \to \widehat{\mathbb{G}}_m(\mathbb{Z}[\![y]\!]/[p](y))$$

given by $j \mapsto [j](y)$ becomes the zero map under the base change

$$\begin{aligned} \mathbb{Z}[\![y]\!]/([p](y)) &\to \mathbb{Z}/p \\ y &\mapsto 0. \end{aligned}$$

On the other hand, the functor $\underline{\mathrm{level}}(A,G)$ is representable.

Lemma 10.13. *Let G be a formal group of finite height over a local formal scheme S, and let A be a finite abelian group. The functor $\underline{\mathrm{level}}(A,G)$ is a closed formal subscheme of $\underline{\mathrm{hom}}(A,G)$.*

Proof. See Katz and Mazur [KM85, 1.3.4] or [Str97] (the general assumption in [Str97] that S is Noetherian is not used for this result). $\square$

It is clear from the definition that if $j : S' \to S$ is a map of formal schemes, then the natural map

$$\underline{\mathrm{level}}(A, j^*G) \to j^*\underline{\mathrm{level}}(A,G)$$

is an isomorphism of formal schemes over S'.

10.3. **The Noetherian case.** If G is an *Noetherian* (2.3) formal group of finite height h over a local formal scheme S of residue characteristic $p > 0$, then we have the following.

Proposition 10.14. *Suppose that A is a p-group and $|A[p]| \leq p^h$.*

i) *The functor $\underline{\mathrm{level}}(A,G)$ is represented by a local formal scheme which is finite and flat over S: indeed $\mathcal{O}(\underline{\mathrm{level}}(A,G))$ is a finite free $\mathcal{O}(S)$-module.*

ii) *If G is the universal deformation of a formal group over a field (see §14) then $\underline{\mathrm{level}}(A,G)$ is the formal spectrum of a Noetherian complete local domain which is regular of dimension h.*

Proof. With our hypotheses, we may suppose that G/S is the universal deformation of a formal group of height h over a perfect field k of characteristic p; the general case follows by change of base.

If $A = A[p]$, then the result is precisely the Lemma of [Dri74, p. 572, in proof of Prop. 4.3]. The proof in the general case follows similar lines and is given in [Str97].

The general case for part i) can be given easily: by definition of $\underline{\mathrm{level}}(A,G)$, the diagram

$$\begin{array}{ccc} \underline{\mathrm{level}}(A,G) & \xrightarrow{j} & \underline{\mathrm{hom}}(A,G) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} \\ \underline{\mathrm{level}}(A[p],G) & \xrightarrow{l} & \underline{\mathrm{hom}}(A[p],G). \end{array}$$

is a pull-back. Proposition 10.8 implies that k is finite and flat, and so i is too. It follows that $\underline{\mathrm{level}}(A,G)$ is finite and flat over S; for Noetherian complete local rings, finite and flat is equivalent to finite and free. $\square$

10.4. **Level structures over p-regular schemes.** In this section, we suppose that G is a formal group of finite height over a complete local ring E of residue characteristic $p > 0$. The following description of the subscheme $\underline{\mathrm{level}}(A, G)$ was found by Hopkins in the course of his work on [HKR00].

Proposition 10.15. *Suppose that G is Noetherian, and that p is not a zero divisor in E. Let x be a coordinate on G. The scheme $\underline{\mathrm{level}}(A, G)$ is the closed subscheme of $\underline{\mathrm{hom}}(A, G)$ defined by the ideal of annihilators of $x(\ell(a))$, where a ranges over the non-zero elements of $A[p]$.*

The proof will be given at the end of this section. Note that the ideal in the Proposition is independent of the coordinate used to describe it.

For $n \geq 1$ let $A[n]$ denote the n-torsion in A. Let R be a complete local E-algebra, and consider the following conditions on a homomorphism

$$\ell : A \to G(R).$$

Again, they are phrased in terms of a choice of a coordinate x on G, but they are easily seen to be independent of that choice.

(A) If $0 \neq a \in A[p]$ then $x(\ell(a))$ is regular (i.e. not a divisor of zero).
(B) If $0 \neq a \in A[p]$ then $x(\ell(a))$ divides p.
(C) $\prod_{a \in A[p]}(x - x(\ell(a)))$ divides $[p](x)$.
(D) The natural map

$$R[\![x]\!] \Big/ \left(\prod_{a \in A[p]} (x - x(\ell(a))) \right) \to \prod_{a \in A[p]} (R[\![x]\!]/(x - x(\ell(a)))) \tag{10.16}$$

is a monomorphism.

Condition (C) says precisely that there is an inequality of Cartier divisors

$$\sum_{pa=0} \{\ell(a)\} \subseteq G[p].$$

Thus condition (C) is that ℓ is a level structure.

Proposition 10.17. *If R is p-torsion free, then these conditions are equivalent.*

First we prove the following result. It will be convenient to use the symbol ϵ to denote a generic unit. Its value may change from line to line.

Lemma 10.18. *Let $n = |A[p^m]|$. The discriminant of the set*

$$\{x(\ell(a)) \mid a \in A[p^m]\}$$

is

$$\Delta = \epsilon \prod_{0 \neq a \in A[p^m]} x(\ell(a))^n.$$

Proof. Let F be the group law associated to a coordinate on G. The formula

$$x \underset{F}{-} y = (x - y)\epsilon(x, y),$$

where $\epsilon(x,y) \in E[\![x,y]\!]^\times$, gives

$$\begin{aligned}\Delta &= \prod_{a\neq b\in A[p^m]} (x(\ell(a)) - x(\ell(b)))\\ &= \epsilon \prod (x(\ell(a)) \underset{F}{-} x(\ell(b)))\\ &= \epsilon \prod x(\ell(a)-\ell(b))\\ &= \epsilon \prod_{c\neq 0}\prod_{a-b=c} x(\ell(c))\\ &= \epsilon \prod_{c\neq 0} x(\ell(c))^n.\end{aligned}$$

□

Proof of Proposition 10.17. Under the hypothesis that p is regular in R, it is clear that (B) implies (A). Let's check that (A) implies (B). Note that

$$[p](x) = x(p + xe(x))$$

for some $e(x) \in E[\![x]\!]$. For $a \in A[p]$ we have

$$0 = [p](x(\ell(a))) = x(\ell(a))(p + x(\ell(a))e(x(\ell(a)))).$$

If $x(\ell(a))$ is not a zero-divisor in R, then we must have

$$p = -x(\ell(a))e(x(\ell(a))).$$

Next, let check that (C) implies (B). If (C) holds, then there is a power series $e(x) \in E[\![x]\!]$ such that

$$e(x) \prod_{a\in A[p]} (x - x(\ell(a))) = [p](x) = px + o(x^2)$$

The coefficient of x on the left is (up to a sign)

$$e(0) \prod_{0\neq a\in A[p]} x(\ell(a))$$

so (B) holds.

Next let's check that (A) implies (D). With respect to the basis of powers of x in the domain and the obvious basis in the range, the matrix of (10.16) is the Vandermonde matrix on the set $x(\ell(A[p]))$. Condition (A) and Lemma 10.18 together imply that (10.16) is a monomorphism.

Finally, let's check that (D) implies (C). Each $x(\ell(a))$ is a root of $[p](x)$, so the image of $[p](x)$ in the range of (10.16) is zero. If (D) holds then $[p](x)$ is zero in the domain, which implies (C). □

Lemma 10.19. *Condition (A) holds if and only if, for* all *non-zero $a \in A$, $x(\ell(a))$ is a regular element of R. Condition (B) holds if and only if, for* all *non-zero $a \in A$, $x(\ell(a))$ divides a power of p.*

Proof. Recall that the p-series $[p](x)$ is divisible by x: let $\langle p\rangle(x)$ be the power series such that

$$[p](x) = x\langle p\rangle(x).$$

Thus

$$x(\ell(pa)) = x(\ell(a))\langle p\rangle(x(\ell(a))).$$

so if $x(\ell(pa))$ divides zero (resp. a power of p), then so does $x(\ell(a))$. □

Corollary 10.20. *If R is a domain of characteristic 0, then the conditions (A)—(C) hold if and only if $\ell: A \to G(R)$ is a monomorphism.* □

Proof of Proposition 10.15. By Proposition 10.14, $R = \mathcal{O}(\underline{\mathrm{level}}(A,G))$ is a finite free E-module. It follows that p is not a zero divisor in R. Proposition 10.17 implies that R is initial among complete local E-algebras satisfying (A). □

Example 10.21. Let G be a Noetherian formal group of finite height over a p-regular complete local ring R of residue characteristic p. Suppose that x is a coordinate on G. Let $\langle p\rangle(t) \in R[\![t]\!]$ be the power series such that

$$t\langle p\rangle(t) = [p](t).$$

Proposition 10.15 implies that

$$\underline{\mathrm{level}}(\mathbb{Z}/p, G) \cong \mathrm{spf}\left(R[\![x(\ell(1))]\!]/\langle p\rangle\big(x(\ell(1))\big)\right). \tag{10.22}$$

This calculation occurs as part of the proof of the Lemma in the proof of Proposition 4.3 of [Dri74].

10.5. **Calculations in $\underline{\mathrm{hom}}(\mathbb{Z}/p, G)$ via level structures.** Let G be a Noetherian (2.3) formal group of finite height over a p-regular complete local ring R of residue characteristic $p > 0$. Let A be a finite abelian group. By construction there is a natural map

$$\mathcal{O}(\underline{\mathrm{hom}}(A, G)) \to \mathcal{O}(\underline{\mathrm{level}}(A, G)).$$

There is also a ring homomorphism

$$\mathcal{O}(\underline{\mathrm{hom}}(A, G)) \to R$$

classifying the zero homomorphism. The proof of Proposition 7.1 uses the following result.

Proposition 10.23. *The natural map*

$$\mathcal{O}(\underline{\mathrm{hom}}(\mathbb{Z}/p, G)) \to R \times \mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p, G)) \tag{10.24}$$

is injective.

Remark 10.25. This result is equivalent to the injectivity for the group $\mathbb{Z}/p$ of the character map of [HKR00].

Proof. Let h be the height of G. Let $\Lambda = (\mathbb{Z}/p)^h$. Let $g(x)$ be the monic polynomial of degree p^h such that

$$[p](x) = g(x)\epsilon$$

where $\epsilon \in R[\![x]\!]^\times$. Then

$$\underline{\mathrm{hom}}(\mathbb{Z}/p, G) = \mathrm{spf}\, R[\![x]\!]/[p](x) \cong \mathrm{spf}\, R[\![x]\!]/g(x),$$

and Proposition 10.15 (see Example 10.21) implies that

$$\mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p, G)) = R[\![x]\!]/\langle p\rangle(x).$$

Let

$$D = \mathcal{O}(\underline{\mathrm{level}}(\Lambda, G))$$

and let

$$\ell : \Lambda \to G(\underline{\mathrm{hom}}(\Lambda, G))$$

be the tautological homomorphism. By definition,

$$D = \mathcal{O}(\underline{\mathrm{hom}}(\Lambda, G)) \Big/ \left(\begin{array}{c} \text{Ideal obtained by} \\ \text{equating coefficients in} \\ \prod_{a\in\Lambda}(x - x(\ell(a))) = g(x) \end{array} \right).$$

By Proposition 10.14, D is finite and free over R. Therefore, letting χ denote the map (10.24), it suffices to show that $D\widehat{\otimes}\chi$ is injective.

Each non-zero $a \in \Lambda$ gives a monomorphism $\mathbb{Z}/p \hookrightarrow \Lambda$ and so a homomorphism

$$\mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p, G)) \xrightarrow{a} D$$
$$x \mapsto x(\ell(a)).$$

We may view these all together as a ring homomorphism

$$D\widehat{\otimes}\mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p, G)) \xrightarrow{M} \prod_{0\neq a\in\Lambda} D[\![x]\!]/(x - x(\ell(a))).$$

Note that the identity map of D may be written as

$$D \xrightarrow{F} D[\![x]\!]/(x - x(\ell(0))) = D.$$

With this notation, the diagram

$$\begin{array}{ccc} D\widehat{\otimes}\mathcal{O}(\underline{\mathrm{hom}}(\mathbb{Z}/p,G)) & \xrightarrow{D\widehat{\otimes}\chi} & D\widehat{\otimes}(R\times\mathcal{O}(\underline{\mathrm{level}}(\mathbb{Z}/p,G))) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle F\times M} \\ D[\![x]\!]/\left(\prod_{a\in\Lambda}(x-x(\ell(a)))\right) & \longrightarrow & \prod_{a\in\Lambda} D[\![x]\!]/(x-x(\ell(a))) \end{array}$$

commutes, where the map across the bottom is the evident map (10.16). It is a monomorphism by Proposition 10.17. □

11. Isogenies

In this section we suppose that G and G' are formal groups over a local formal scheme S of residue characteristic p.

Definition 11.1. An *isogeny* is a finite and free homomorphism $f : G \to G'$ of formal groups. In particular, $\ker f$ is a finite group scheme over S.

Isogenies are epimorphisms.

Lemma 11.2. *If $f : G \to G'$ and $g : G \to G''$ are two isogenies, such that*

$$\ker f \subseteq \ker g,$$

then there is a unique isogeny

$$h : G' \to G''$$

such that $g = hf$. □

A level structure has a cokernel which is an isogeny. Let A be a finite group, and let

$$A_S \xrightarrow{\ell} G$$

be a level structure on a Noetherian (2.3) formal group of finite height over a local formal scheme S of residue characteristic p.

Proposition 11.3. i) *The induced map of formal group schemes*

$$A_S \xrightarrow{\ell} G \tag{11.4}$$

is the inclusion of a sub-groupscheme.

ii) *The map* (11.4) *has a cokernel*

$$G \xrightarrow{q_\ell} G/\ell(A)$$

which is an isogeny of formal groups. The map q_ℓ gives an isomorphism

$$\mathcal{O}(G/\ell(A)) \cong \mathcal{O}(G)^A. \tag{11.5}$$

If x is a coordinate on G, then the isomorphism (11.5) *identifies*

$$\prod_{a\in A} T_a^* x \tag{11.6}$$

with a coordinate on $G/\ell(A)$.

Proof. It suffices to prove the Proposition in the case that G is the universal deformation a formal group of height h over a field of characteristic p, and $R = \mathcal{O}(\underline{\mathrm{level}}(A,G))$. In that case the result is essentially Proposition 4.4 of [Dri74]: Drinfel'd actually considers a level structure of the form

$$\Lambda = (\mathbb{Z}/p^n)^h \xrightarrow{\ell} G(R)$$

and a subgroup $A \subseteq \Lambda$, but his argument uses only the A-structure and the fact that $R_\Lambda(G)$ is a Noetherian p-regular complete local domain. (The existence of the quotient and the coordinate (11.6) in that case is due to Lubin [Lub67].) Strickland [Str97] gives a complete proof in the generality considered here. □

12. The norm map

Let A be a finite abelian group. Let G and G' be formal groups of finite height over a local formal scheme S of residue characteristic p. Let

$$A_S \xrightarrow{\ell} G$$

be a level structure with cokernel

$$G \xrightarrow{q} G'.$$

An $\mathcal{O}_{G'}$-module W' gives rise to an A-equivariant $\mathcal{O}_G$-module $W = q^*W'$, and

$$W' = (q_*W)^A.$$

Proposition 11.3 implies the following.

Proposition 12.1. *The functor $W' \mapsto q^*W'$ is an equivalence of categories from finite (resp. finite and free) $\mathcal{O}_{G'}$-modules to A-equivariant finite (resp. finite and free) $\mathcal{O}_G$-modules. The $\mathcal{O}_{G'}$-module corresponding to an equivariant $\mathcal{O}_G$-module W is $(q_*W)^A$. In particular q^*W' is a line bundle if and only if W' is.* □

Now let $\mathcal{L}$ be a line bundle over G. The line bundle

$$\bigotimes_{a \in A} T_a^*\mathcal{L}$$

is equivariant, and so Proposition 12.1 justifies the following.

Definition 12.2. The *norm* of $\mathcal{L}$ is the line bundle $N\mathcal{L} = N_\ell\mathcal{L}$ over G' determined by the equation

$$q^*N\mathcal{L} = \bigotimes_{a \in A} T_a^*\mathcal{L}.$$

Explicitly, we have

$$\Gamma(N\mathcal{L}) = (\Gamma(\bigotimes T_a^*\mathcal{L}))^A.$$

If s is a section of $\mathcal{L}$, then the norm of s is the section

$$Ns = \bigotimes_a T_a^*s \in \Gamma(N\mathcal{L}).$$

The norm map is not additive, but it is multiplicative, in the sense that if f is a function on G, then

$$N(fs) = Nf \cdot Ns.$$

Let μ be the map of line bundles

$$\mu : \bigotimes_{a \in A} T_a^*\mathcal{O}_G \to \mathcal{O}_G$$

given by the formula

$$\mu(f_1 \otimes \cdots \otimes f_m) = f_1 \ldots f_m.$$

Lemma 12.3. *The map μ is an isomorphism of A-equivariant line bundles over G.* □

Lemma 12.3 and Proposition 11.3 give the following.

Proposition 12.4. *The map*

$$f \mapsto \mu^{-1}(f \circ q)$$

induces an isomorphism of line bundles

$$\mathcal{O}_{G'} \cong N\mathcal{O}_G$$

which restricts to an isomorphism

$$\mathcal{I}_{G'}(0) \cong N\mathcal{I}_G(0). \tag{12.5}$$

If s is a coordinate on G, then Ns is a coordinate on G'.

Equivalently, q induces an isomorphism

$$q^*\mathcal{I}_{G'}(0) \cong q^*N\mathcal{I}_G(0) \cong \mathcal{I}_G(\ell) \tag{12.6}$$

of line bundles over G.

13. Descent for level structures

In Definition 3.1 we described "descent data for level structures" as they appear on the formal group of an H_∞ ring spectrum. In this section, we give an equivalent description (Proposition 13.14) which displays the relationship to the usual notion of descent data. In addition to justifying the terminology, the new formulation simplifies the task of showing that the Lubin-Tate formal groups have canonical descent data for level structures (Proposition 14.8).

13.1. **Composition of isogenies: the simplicial functor $\underline{\text{level}}_*$.** Let FGps be the functor from admissible local rings R to sets whose value on R is the set of formal groups $G/\operatorname{spf} R$. If $f : R \to R'$ is a map of admissible local rings, then FGps(f) sends $G/\operatorname{spf} R$ to $f^*G/\operatorname{spf} R'$.

Let

$$\underline{\text{level}}(A) \to \text{FGps}$$

be the functor *over* formal FGps whose value on R is the set of formal groups $G/\operatorname{spf} R$ equipped with a level structure

$$A \to G.$$

We define

$$\underline{\text{level}}_1 \stackrel{\text{def}}{=} \coprod_{A_0} \underline{\text{level}}(A_0);$$

the coproduct is over all finite abelian groups. We have adorned the $\underline{\text{level}}$ and the A with subscripts so that we can make the more general definition

$$\underline{\text{level}}_n = \coprod_{0=A_n \subseteq A_{n-1} \cdots \subseteq A_0} \underline{\text{level}}(A_0).$$

The coproduct is over all sequences of *inclusions* of finite abelian groups *with* $A_n = 0$. With this convention we also have

$$\text{FGps} = \underline{\text{level}}_0.$$

We write

$$d_0 : \underline{\text{level}}_1 \to \text{FGps} \tag{13.1}$$

for the structural map.

Over $\underline{\text{level}}_1(A)$ we have a level structure

$$A \xrightarrow{\ell} d_0^*G$$

and an isogeny

$$d_0^*G \xrightarrow{q_A} G/\ell(A)$$

with kernel A. These assemble to give a group G/ℓ and an isogeny

$$d_0^*G \xrightarrow{q} G/\ell$$

over $\underline{\text{level}}_1$. We write

$$d_1 : \underline{\text{level}}_1 \to \text{FGps} \tag{13.2}$$

for the map classifying G/ℓ.

Lemma 13.3. *Let*

$$A \xrightarrow{\ell} G$$

be a level structure. If $B \subseteq A$, then the induced map

$$B \xrightarrow{\ell|_B} G$$

is a level structure. If $q : G \to G'$ is an isogeny with kernel $\ell|_B$, then the induced map

$$\ell' : A/B \to G'$$

is a level structure.

Proof. The first part is clear from the definition of a level structure (10.9). For the second part, consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ell} & G \\ \downarrow & & \downarrow{\scriptstyle q} \\ A/B & \xrightarrow{\ell'} & G'. \end{array}$$

Let D be the divisor

$$D = \sum_{\substack{a \in A \\ pa=0}} \{\ell(a)\}$$

on G; by hypothesis we have an inequality of Cartier divisors

$$D \leq G[p].$$

It follows that

$$\sum_{b \in B} T_b^* D \leq \sum_{b \in B} T_b^* G[p].$$

The formula (11.6) for the coordinate on the quotient G' shows that the left side descends to the divisor

$$\sum_{\substack{c \in (A/B) \\ pc=0}} \{\ell(c)\},$$

while the right side descends to the divisor $G'[p]$. $\square$

The Lemma gives maps

$$d_j : \underline{\text{level}}_n \to \underline{\text{level}}_{n-1}$$

for $0 \leq j \leq n$ as follows. For $0 \leq j \leq n-1$, the map d_j sends a point

$$0 = A_n \subseteq \cdots \subseteq A_j \subseteq \ldots \subseteq A_0 \to G \tag{13.4}$$

of $\underline{\text{level}}_n$ to the point

$$0 = A_n \subseteq \cdots \widehat{A_j} \cdots \subseteq A_0 \to G$$

of $\underline{\text{level}}_{n-1}$ obtained by omitting A_j. The map d_n sends (13.4) to

$$0 = A_{n-1}/A_{n-1} \subseteq \cdots \subseteq A_0/A_{n-1} \to G/\ell(A_{n-1}).$$

In the case $n = 1$ these are just the maps (13.1) and (13.2). We also have for $0 \leq j \leq n$ a map

$$s_j : \underline{\text{level}}_n \to \underline{\text{level}}_{n+1}$$

which sends the sequence (13.4) to the sequence

$$A_n \subseteq \cdots \subseteq A_j \subseteq A_j \subseteq \ldots \subseteq A_0 \to G$$

obtained by repeating A_j. It is easy to check that

Lemma 13.5. $(\underline{\text{level}}_*, d_*, s_*)$ *is a simplicial functor.* $\square$

13.2. **Descent data for functors over formal groups.** Now suppose that

$$\mathcal{P} \to \mathrm{FGps}$$

is a functor over FGps, and if $x \in \mathcal{P}(R)$ is an R-valued point, let's write G_x for the resulting formal group over spf R. As in the previous section, we define

$$\begin{aligned} \underline{\mathrm{level}}(A, \mathcal{P}) &= \underline{\mathrm{level}}(A) \times_{\mathrm{FGps}} \mathcal{P} \\ \underline{\mathrm{level}}_n(\mathcal{P}) &= \underline{\mathrm{level}}_n \times_{\mathrm{FGps}} \mathcal{P} \end{aligned}$$

and so on. A point $(\ell, x) \in \underline{\mathrm{level}}(A, \mathcal{P})(R)$ is a point x of $\mathcal{P}(R)$ and a level structure

$$A \xrightarrow{\ell} G_x.$$

We write

$$d_0 : \underline{\mathrm{level}}_1(\mathcal{P}) \to \underline{\mathrm{level}}_0(\mathcal{P}) = \mathcal{P}. \tag{13.6}$$

for the forgetful map

$$d_0(\ell, x) = x.$$

We also always have degeneracies

$$s_j : \underline{\mathrm{level}}_n(\mathcal{P}) \to \underline{\mathrm{level}}_{n+1}(\mathcal{P})$$

for $0 \le j \le n$.

If (ℓ, x) is an R-valued point of $\underline{\mathrm{level}}(A, \mathcal{P})$, then we get an isogeny

$$G_x \to G_x/\ell.$$

Suppose that we have a natural transformation

$$d_1 : \underline{\mathrm{level}}_1(\mathcal{P}) \to \mathcal{P} \tag{13.7}$$

such that

$$G_{d_1(\ell,x)} = G_x/\ell, \tag{13.8}$$

or equivalently that the diagram

$$\begin{tikzcd} \underline{\mathrm{level}}_1(\mathcal{P}) \arrow[r] \arrow[d, "d_1"'] & \underline{\mathrm{level}}_1 \arrow[d, "d_1"] \\ \mathcal{P} \arrow[r] & \mathrm{FGps} \end{tikzcd} \tag{13.9}$$

commutes. Lemma 13.3 then gives maps

$$d_j : \underline{\mathrm{level}}_n(\mathcal{P}) \to \underline{\mathrm{level}}_{n-1}(\mathcal{P})$$

for $0 \le j \le n$.

Definition 13.10. *Descent data for level structures on the functor* $\mathcal{P}$ consist of a natural transformation (13.7) such that

(1) the diagram (13.9) commutes, and
(2) $(\underline{\mathrm{level}}_*(\mathcal{P}), d_*, s_*)$ is a simplicial functor.

Remark 13.11. It is equivalent to ask for natural transformations

$$d_j : \underline{\mathrm{level}}_n(\mathcal{P}) \to \underline{\mathrm{level}}_{n-1}(\mathcal{P})$$

for $n \ge 1$ and $0 \le j \le n$, such that $(\underline{\mathrm{level}}_*(\mathcal{P}), d_*, s_*)$ is a simplicial functor, and the levelwise natural transformation

$$\underline{\mathrm{level}}_*(\mathcal{P}) \to \underline{\mathrm{level}}_*$$

is a map of simplicial functors.

For example, let G be a formal group of finite height over a p-local formal scheme S. The formal scheme S has the structure of a functor over FGps: if $x : \operatorname{spf} R \to S$ is a point of S, then

$$G_x = x^* G;$$

We briefly write G/S for S, considered as a functor over FGps in this way. The functor $\underline{\mathrm{level}}(A, G/S)$ is just the functor called $\underline{\mathrm{level}}(A, G)$ in § 10; in particular it is represented by the S-scheme $\underline{\mathrm{level}}(A, G)$ of Lemma 10.13. To give maps ψ_ℓ and f_ℓ which satisfy condition (1) of Definition 3.1 amounts to giving a map

$$d_1 : \underline{\mathrm{level}}_1(G/S) \to S$$

and an isogeny

$$d_0^* G \xrightarrow{q} d_1^* G$$

whose kernel on $\underline{\mathrm{level}}(A, G/S)$ is A. Lemma 13.3 gives maps

$$d_j : \underline{\mathrm{level}}_n(G/S) \to \underline{\mathrm{level}}_{n-1}(G/S)$$

for $0 \le j \le n$ as explained above. With these definitions, parts (2) and (3) of Definition 3.1 are equivalent to asserting that

$$(\underline{\mathrm{level}}_*(G/S), d_*, s_*)$$

is a simplicial functor, and over $\underline{\mathrm{level}}_2(G)$ the diagram

$$\begin{tikzcd}
& d_0^* d_0^* G \arrow[dl, equal] \arrow[dr, "d_0^* q"] & \\
d_1^* d_0^* G \arrow[d, "d_1^* q"'] & & d_0^* d_1^* G \arrow[d, equal] \\
d_1^* d_1^* G \arrow[dr, equal] & & d_2^* d_0^* G \arrow[dl, "d_2^* q"] \\
& d_2^* d_1^* G &
\end{tikzcd} \tag{13.12}$$

commutes.

A more convenient formulation of Definition 3.1 is the following. Let $\underline{G/S}$ to be the functor over FGps whose value on R is the set of pull-back diagrams

$$\begin{tikzcd}
G' \arrow[r, "f"] \arrow[d] & G \arrow[d] \\
\operatorname{spf} R \arrow[r, "i"] & S.
\end{tikzcd}$$

such that the map

$$G' \to i^* G$$

induced by f is a homomorphism (hence isomorphism) of formal groups over $\operatorname{spf} R$. For a finite abelian group A, $\underline{\mathrm{level}}(A, \underline{G/S})(R)$ is the set of diagrams

$$\begin{tikzcd}
A_{\operatorname{spf} R} \arrow[r, "\ell"] \arrow[dr] & G' \arrow[r, "f"] \arrow[d] & G \arrow[d] \\
& \operatorname{spf} R \arrow[r, "i"] & S,
\end{tikzcd}$$

where the square part is a point of $\underline{G/S}(R)$ and ℓ is a level structure. To give a map of functors

$$\underline{\mathrm{level}}_1(\underline{G/S}) \xrightarrow{d_1} \underline{G/S} \tag{13.13}$$

making the diagram

$$\begin{array}{ccc} \underline{\mathrm{level}}_1(\underline{G/S}) & \xrightarrow{d_1} & \underline{G/S} \\ \downarrow & & \downarrow \\ \underline{\mathrm{level}}_1 & \xrightarrow{d_1} & \mathrm{FGps} \end{array}$$

commute is to give a pull-back diagram

$$\begin{array}{ccc} G/\ell & \longrightarrow & G \\ \downarrow & & \downarrow \\ \underline{\mathrm{level}}_1(G/S) & \longrightarrow & S; \end{array}$$

it is equivalent to give a map of formal schemes

$$d_1 : \underline{\mathrm{level}}_1(G/S) \to S$$

and an isogeny

$$d_0^* G \xrightarrow{q} d_1^* G$$

whose kernel on $\underline{\mathrm{level}}(A, G/S)$ is A.

Proposition 13.14. *Let G be a formal group over an admissible local ring R, and let $S = \operatorname{spf} R$. Descent data for level structures on the group G/S is equivalent to descent data for level structures on the functor $\underline{G/S}$.*

Proof. One checks that the commutativity of the diagram (13.12) has been incorporated in the structure of the functor $\underline{G/S}$. □

13.3. **Noetherian rings and artin rings.** Suppose that $\mathcal{D}$ is a subcategory of the category of admissible local rings. If $\mathcal{P}$ is a functor from complete local rings to sets, let $\mathcal{P}^{\mathcal{D}}$ denote its restriction to $\mathcal{D}$.

Definition 13.15. *Descent data for level structures on $\mathcal{P}^{\mathcal{D}}$* consists of a natural transformation

$$d_1 : \underline{\mathrm{level}}_1(\mathcal{P})^{\mathcal{D}} \xrightarrow{d_1} \mathcal{P}^{\mathcal{D}},$$

such that the restriction to $\mathcal{D}$ of the diagram (13.9) commutes, and such that the $(\underline{\mathrm{level}}_*(\mathcal{P}))^{\mathcal{D}}, d_*, s_*)$ is a simplicial functor.

For example, let $\mathcal{N}$ be the category of Noetherian complete local rings, and let $\mathcal{A}$ be the category of Artin local rings. If S and T are Noetherian local formal schemes, then the natural maps

$$(\text{formal schemes})(S,T) \to (\text{functors})(S^{\mathcal{N}}, T^{\mathcal{N}}) \to (\text{functors})(S^{\mathcal{A}}, T^{\mathcal{A}}) \tag{13.16}$$

are isomorphisms.

Proposition 13.17. *If G is a formal group over a Noetherian local formal scheme S, then the forgetful maps, from the set of descent data for level structures on $\underline{G/S}$ to the set of descent data for level structures on $\underline{G/S}^{\mathcal{N}}$ and on $\underline{G/S}^{\mathcal{A}}$, are isomorphisms.*

Proof. If G is a formal group over a Noetherian local formal scheme, then by Proposition 10.14, $\underline{\mathrm{level}}(A, G)$ is also a Noetherian local formal scheme. The result follows easily from the isomorphism (13.16). □

14. Lubin-Tate groups

Let k be a perfect field of characteristic $p > 0$, and let Γ be a formal group of finite height over k. In this section we shall prove that the universal deformation of Γ has descent for level structures.

14.1. **Frobenius.** Let k be a perfect field of characteristic $p > 0$, and let Γ be a formal group of finite height over k. The Frobenius map ϕ gives rise to a relative Frobenius F map as in the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "F"] \arrow[rr, bend left, "\phi_\Gamma"] \arrow[dr] & \phi_k^*\Gamma \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ & \operatorname{spec} k \arrow[r, "\phi_k"] & \operatorname{spec} k. \end{tikzcd}$$

The Frobenius map F is an isogeny of degree p.

14.2. **Deformations.** If T is a local formal scheme, then we write T_0 for its closed point.

Definition 14.1. Let T be a local formal scheme. A *deformation of* Γ *to* T is a triple $(H/T, f, j)$ consisting of a formal group H over T and a pull-back diagram

$$\begin{tikzcd} H_{T_0} \arrow[r, "f"] \arrow[d] & \Gamma \arrow[d] \\ T_0 \arrow[r, "j"] & \operatorname{spec} k, \end{tikzcd}$$

such that the induced map $H_{T_0} \to j^*\Gamma$ is a homomorphism (and so isomorphism) of formal groups over T_0. The functor from complete local rings to sets which assigns to R the set of deformations of Γ to $\operatorname{spf} R$ will be denoted $\operatorname{Def}(\Gamma)$.

From the definition it is clear that if $(H/T, f, j)$ is a deformation of Γ, then there is a natural transformation

$$\underline{H/T} \to \operatorname{Def}(\Gamma).$$

Lubin and Tate [LT66] construct a deformation $(G/S, f_{\text{univ}}, j_{\text{univ}})$ with an isomorphism

$$S \cong \operatorname{spf} \mathbb{W}k[\![u_1, \dots, u_{h-1}]\!] \tag{14.2}$$

inducing $j_{\text{univ}} : S_0 \cong \operatorname{spec} k$ such that the natural transformation

$$\underline{G/S}^{\mathcal{N}} \to \operatorname{Def}(\Gamma)^{\mathcal{N}} \tag{14.3}$$

is an isomorphism of functors over FGps.

14.3. **Descent for level structures on deformations.** We continue to fix a formal group Γ of finite height over a perfect field k of characteristic $p > 0$.

Let A be a finite abelian group. If R is a complete local ring, then a point of $\underline{\text{level}}(A, \operatorname{Def}(\Gamma))$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\ell"] & H \arrow[d] & H_{T_0} \arrow[l] \arrow[r, "f"] \arrow[d] & \Gamma \arrow[d] \\ & T & T_0 \arrow[l] \arrow[r, "j"] & \operatorname{spec} k, \end{tikzcd} \tag{14.4}$$

consisting of a deformation (H, f, j) of Γ to R and a level structure ℓ on H. The level structure in (14.4) of $\operatorname{Def}(A, \Gamma)(R)$ has a cokernel

$$H \xrightarrow{q} H'$$

If x is a coordinate on H, then $x(\ell(a))$ is topologically nilpotent in $\mathcal{O}(T)$. It follows that $x(\ell(a)) = 0$ on T_0, and so there is a canonical isomorphism $\overline{q}$ making the diagram

$$\begin{tikzcd} H_{T_0} \arrow[r, "F^r"] \arrow[d, "q_{T_0}"'] & (\phi^r)^* H_{T_0} \arrow[r, "\text{can}"] \arrow[d] & H_{T_0} \arrow[r, "f"] \arrow[d] & \Gamma \arrow[d] \\ H'_{T_0} \arrow[r] \arrow[ur, "\overline{q}"'] & T_0 \arrow[r, "\phi^r"] & T_0 \arrow[r, "j"] & \operatorname{spec} k \end{tikzcd}$$

commute. In other words $(H', f\mathrm{can}\overline{q}, j\phi^r)$ is a point of $\mathrm{Def}(\Gamma)(T)$, and we have constructed a natural transformation

$$\underline{\mathrm{level}}_1(\mathrm{Def}(\Gamma)) \xrightarrow{d_1} \mathrm{Def}(\Gamma). \tag{14.5}$$

satisfying (13.8). The fact that $\phi^r\phi^s = \phi^{r+s}$ then implies

Lemma 14.6. *The map d_1 is descent data for level structures on the functor* $\mathrm{Def}(\Gamma)$.

Now let $(G/S, f_{\mathrm{univ}}, j_{\mathrm{univ}})$ be Lubin and Tate's universal deformation of $\Gamma/\operatorname{spec} k$. Using the isomorphism (14.3) and Proposition 13.17 we may trade $\underline{G/S}$ for $\mathrm{Def}(\Gamma)$ in (14.5) to get a map

$$d_1 : \underline{\mathrm{level}}_1(\underline{G/S}) \to \underline{G/S} \tag{14.7}$$

such that

Proposition 14.8. *The natural transformation d_1 is descent data for level structures on the functor $\underline{G/S}$, and so gives descent data for level structures on the formal group G/S.* □

14.4. **Comparison to the descent data coming from the E_∞ structure of Goerss and Hopkins.** The construction of the descent data in Proposition 14.8 uses the equality

$$p^r\{0\} = \ker F^r : \Gamma \to (\phi^r)^*\Gamma \tag{14.9}$$

of divisors on Γ and the equation

$$F^{r+s} = ((\phi^r)^*F^s)F^r : \Gamma \to (\phi^{r+s})^*\Gamma \tag{14.10}$$

More generally, to give descent data for level structures on $\underline{G/S}$ is equivalent to giving a collection of isogenies

$$F_r : \Gamma \to (\phi^r)^*\Gamma$$

for $r \geq 1$ satisfying the analogues of (14.9) and (14.10). The descent data in the Proposition are uniquely determined by the choice $F_r = F^r$.

Now let $\mathbf{E}$ be the homogeneous ring spectrum such that $G = G_{\mathbf{E}}$ is Lubin and Tate's universal deformation of Γ, so

$$S_{\mathbf{E}} = S = \operatorname{spf} \mathbb{W}k[\![u_1, \ldots, u_{h-1}]\!].$$

In work in preparation, Goerss and Hopkins [GH02] have shown that $\mathbf{E}$ is an E_∞ ring spectrum; by Theorem 3.26 it follows that there is a map

$$\underline{\mathrm{level}}_1(\underline{G/S}) \xrightarrow{d_1^{\mathrm{hg}}} \underline{G/S}$$

giving descent data for level structures on $\underline{G/S}$.

Let A be a finite group of order p^r, and let

$$A_{\operatorname{spf} R} \xrightarrow{\ell} i^*G$$

be a level structure on G. Reducing modulo the maximal ideal in the construction of $\psi_\ell^{\mathbf{E}}$ (3.9), one sees that

$$\psi_\ell^{\mathbf{E}} = \phi^r : S_{\mathbf{E}} \to S_{\mathbf{E}}.$$

Examination of the construction (3.14) of $\psi_\ell^{G/\mathbf{E}}$ shows that

$$(\psi_\ell^{G/\mathbf{E}})_{S_0} = F^r : G_{S_0} \to (\phi^r)^*G_{S_0}.$$

Thus we have the following result.

Proposition 14.11. *If $\mathbf{E}$ is the spectrum associated to the universal deformation of a formal group Γ of finite height over a perfect field k, then the descent data for level structures on $G_{\mathbf{E}}$ provided by the E_∞ structure of Goerss and Hopkins coincide with the descent data in Proposition 14.8.* □

Remark 14.12. At the time of the writing of this paper, the result of Goerss and Hopkins is not published. The arguments of this section do not depend on their result beyond the existence of the H_∞ structure, so a cautious statement of the Proposition is that "the descent data for level structures on $G_{\mathbf{E}}$ provided by *any* H_∞ structure on $\mathbf{E}$ coincide with the descent data in Proposition 14.8."

Part 4. The sigma orientation

15. Θ^k-STRUCTURES

15.1. **The functors Θ^k.** Suppose that G is a formal group over a formal scheme S, and suppose that $\mathcal{L}$ is a line bundle over G.

Definition 15.1. A *rigid* line bundle over G is a line bundle $\mathcal{L}$ equipped with a specified trivialization of $0^*\mathcal{L}$. A *rigid section* of such a line bundle is a section s which extends the specified section at the identity. A *rigid isomorphism* between two rigid line bundles is an isomorphism which preserves the specified trivializations.

Definition 15.2. Suppose that $k \geq 1$. We define the line bundle $\Theta^k(\mathcal{L})$ over G^k by the formula

$$\Theta^k(\mathcal{L}) \stackrel{\text{def}}{=} \bigotimes_{I\subseteq\{1,\ldots,k\}} (\mu_I^*\mathcal{L})^{(-1)^{|I|}}. \tag{15.3}$$

If s is a section of $\mathcal{L}$, then we write $\Theta^k s$ for the section

$$\Theta^k s = \bigotimes_{I\subseteq\{1,\ldots,k\}} (\mu_I^* s)^{(-1)^{|I|}}.$$

of $\Theta^k(\mathcal{L})$. We define $\Theta^0(\mathcal{L}) = \mathcal{L}$ and $\Theta^0 s = s$.

For example we have

$$\begin{aligned}
\Theta^1(\mathcal{L}) &= \frac{\pi^*0^*\mathcal{L}}{\mathcal{L}} \\
\Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_0}{\mathcal{L}_a} \\
\Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b}}{\mathcal{L}_a \otimes \mathcal{L}_b} \\
\Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_0 \otimes \mathcal{L}_{a+b} \otimes \mathcal{L}_{a+c} \otimes \mathcal{L}_{b+c}}{\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c \otimes \mathcal{L}_{a+b+c}}.
\end{aligned}$$

We observe three facts about these bundles.

(1) $\Theta^k(\mathcal{L})$ has a natural rigid structure for $k > 0$.
(2) For each permutation $\sigma \in \Sigma_k$, there is a canonical isomorphism

$$\xi_\sigma : \pi_\sigma^*\Theta^k(\mathcal{L}) \cong \Theta^k(\mathcal{L}).$$

Moreover, these isomorphisms compose in the obvious way.
(3) There is a canonical identification (of rigid line bundles over X^{k+1})

$$\Theta^k(\mathcal{L})_{a_1,a_2,\ldots} \otimes \Theta^k(\mathcal{L})^{-1}_{a_0+a_1,a_2,\ldots} \otimes \Theta^k(\mathcal{L})_{a_0,a_1+a_2,\ldots} \otimes \Theta^k(\mathcal{L})^{-1}_{a_0,a_1,\ldots} \cong 1. \tag{15.4}$$

Definition 15.5. A Θ^k–structure on a line bundle $\mathcal{L}$ over G is a trivialization s of the line bundle $\Theta^k(\mathcal{L})$ such that

(1) for $k > 0$, s is a rigid section;
(2) s is symmetric in the sense that for each $\sigma \in \Sigma_k$, we have $\xi_\sigma\pi_\sigma^* s = s$;
(3) we have

$$s(a_1, a_2, \ldots) \otimes s(a_0 + a_1, a_2, \ldots)^{-1} \otimes s(a_0, a_1 + a_2, \ldots) \otimes s(a_0, a_1, \ldots)^{-1} = 1 \tag{15.6}$$

under the isomorphism (15.4).

A Θ^3–structure is known as a *cubical structure* [Bre83]. We write $C^k(G;\mathcal{L})$ for the set of Θ^k-structures on $\mathcal{L}$ over G. Note that $C^0(G;\mathcal{L})$ is just the set of trivializations of $\mathcal{L}$, and $C^1(G;\mathcal{L})$ is the set of rigid trivializations of $\Theta^1(\mathcal{L})$. We also define a functor from rings to sets by

$$\underline{C}^k(G;\mathcal{L})(R) = \{(u, f) \mid u : \operatorname{spec}(R) \to S\ ,\ f \in C^k_{\operatorname{spec}(R)}(u^*G; u^*\mathcal{L})\},$$

and we recall the following.

Proposition 15.7 ([AHS01]). *Let G be a formal group over a scheme S, and let $\mathcal{L}$ be a line bundle over G. The functor $\underline{C}^k(G;\mathcal{L})$ is represented by an affine scheme over S, and for $j : S' \to S$, the natural map*

$$\underline{C}^k(j^*G; j^*\mathcal{L}) \to j^*\underline{C}^k(G;\mathcal{L})$$

is an isomorphism. □

15.2. **Relations among the Θ^k: the functor Δ.**

Definition 15.8. If $\mathcal{M}$ is a line bundle over G^n, then we define $\Delta\mathcal{M}$ to be the rigid line bundle over G^{n+1} given fiberwise by the formula

$$\Delta\mathcal{M}_{a_1,a_2,\dots,a_{n+1}} = \frac{\mathcal{M}_{a_1,a_3,\dots,a_{n+1}} \otimes \mathcal{M}_{a_2,\dots,a_{n+1}}}{\mathcal{M}_{a_1+a_2,a_3,\dots,a_{n+1}} \otimes \mathcal{M}_{0,a_3,\dots,a_{n+1}}}.$$

If s is a section of $\mathcal{M}$ then we write Δs for the rigid section

$$\Delta s(a_1,\dots,a_{n+1}) = \frac{s(a_1,\dots) \otimes s(a_{2,\dots})}{s(a_1+a_2,\dots) \otimes s(0,a_3,\dots)}$$

of $\Delta\mathcal{M}$.

The following can be checked directly from the definitions.

Lemma 15.9. i) *Δ is multiplicative: if $\mathcal{M}$ is a line bundle over G^n then there is a canonical isomorphism of rigid line bundles*

$$\Delta(\mathcal{M}_1 \otimes \mathcal{M}_2) \cong \Delta(\mathcal{M}_1) \otimes \Delta(\mathcal{M}_2). \tag{15.10}$$

ii) *Under the identification* (15.10), *one has*

$$\Delta(s_1 \otimes s_2) = \Delta(s_1) \otimes \Delta(s_2).$$

iii) *If $\mathcal{L}$ is a line bundle over G then for $k \geq 2$ there is a canonical isomorphism of rigid line bundles*

$$\Theta^k\mathcal{L} \cong \Delta\Theta^{k-1}\mathcal{L}. \tag{15.11}$$

iv) *If s is a section of $\mathcal{L}$ then under the isomorphism* (15.11), *one has*

$$\Theta^k s = \Delta\Theta^{k-1}s$$

□

16. The norm map for Θ^k-structures

Given a line bundle $\mathcal{M}$ over G^n, $n > 0$, and a section b of $\mathcal{M}$, let $\tilde{T}_b\mathcal{M}$ be the line bundle whose fiber over $(a_1,\dots,a_n)$ is

$$\tilde{T}_b\mathcal{M}_{a_1,\dots,a_n} \overset{\text{def}}{=} \frac{\mathcal{M}_{(b+a_1,a_2,\dots,a_n)}}{\mathcal{M}_{(b,a_2,\dots)}}. \tag{16.1}$$

If s is a section of $\mathcal{M}$, define $\tilde{T}_b^*s$ by

$$\tilde{T}_b s(a_1,\dots,a_n) = \frac{s(b+a_1,\dots,a_n)}{s(b,a_2,\dots,a_n)}.$$

There is a canonical identifications

$$\tilde{T}_b(\mathcal{L}_1 \otimes \mathcal{L}_2) \cong \tilde{T}_b\mathcal{L}_1 \otimes \tilde{T}_b\mathcal{L}_2, \tag{16.2}$$

$$\tilde{T}_b\tilde{T}_c\mathcal{L} \cong \tilde{T}_{b+c}\mathcal{L}, \tag{16.3}$$

and with respect to these identifications one has

$$\tilde{T}_b(s_1s_2) = \tilde{T}_b(s_1)\tilde{T}_b(s_2)$$
$$\tilde{T}_b\tilde{T}_c(s) = \tilde{T}_{b+c}(s).$$

The operation $\tilde{T}_b$ commutes with Θ^k.

Proposition 16.4. *For a line bundle $\mathcal{L}$ over G and for $k \geq 1$ there are natural isomorphisms*

$$\tilde{T}_b \Theta^k \mathcal{L} \cong \Theta^k \tilde{T}_b \mathcal{L} \cong \Theta^k T_b^* \mathcal{L} \tag{16.5}$$

of rigid line bundles over G^k.

Proof. One checks directly that if $\mathcal{L}$ is a line bundle over G and $\mathcal{M}$ is a line bundle over G^n, then there are natural isomorphisms of rigid line bundles

$$\begin{aligned} \tilde{T}_b \Theta^1 \mathcal{L} &\cong \Theta^1 \tilde{T}_b \mathcal{L} \cong \Theta^1 T_b^* \mathcal{L} \\ \Delta \tilde{T}_b \mathcal{M} &\cong \tilde{T}_b \Delta \mathcal{M}. \end{aligned}$$

The proof follows by induction, using the isomorphism (15.11)

$$\Theta^k \mathcal{L} \cong \Delta \Theta^{k-1} \mathcal{L}.$$

□

Now suppose that A is a finite abelian group, and

$$A_S \xrightarrow{\ell} G \xrightarrow{q} G'$$

is a level structure with cokernel q. Suppose that $k \geq 1$, and that $\mathcal{L}$ is a line bundle over G.

Lemma 16.6. *The rigid line bundle $\bigotimes_{b \in A} \tilde{T}_b \Theta^k \mathcal{L}$ over G^k is equivariant with respect to the action of A^k.*

Proof. Suppose that $a \in A$ and $1 \leq i \leq k$. For $a \in A$ let

$$T_{i,a} : G^k \to G^k$$

denote translation by a in the i coordinate, and, for a line bundle $\mathcal{M}$ over G^k let

$$\tilde{T}_{i,a} \mathcal{M}_{a_1,\dots,a_k} = \frac{\mathcal{M}_{a_1,\dots,a_i+a,\dots,a_k}}{\mathcal{M}_{a_1,\dots,a,\dots,a_k}}.$$

The symmetries of the line bundles $\Theta^k \mathcal{L}$ together with the isomorphism (16.5) imply that there is a canonical isomorphism

$$\tilde{T}_{i,b} \Theta^k \mathcal{L} \cong \Theta^k (T_b^* \mathcal{L})$$

and so, again using (16.5),

$$\bigotimes_{b \in A} \tilde{T}_b \Theta^k (\mathcal{L}) \cong \bigotimes_{b \in A} \tilde{T}_{i,b} \Theta^k (\mathcal{L}).$$

Of course $\tilde{T}_{i,a+b} = \tilde{T}_{i,a} \tilde{T}_{i,b}$ and so for any line bundle $\mathcal{M}$ over G^k there is a canonical isomorphism

$$\tilde{T}_{i,a} \bigotimes_{b \in A} \tilde{T}_{i,b} \mathcal{M} \cong \bigotimes_{b \in A} \tilde{T}_{i,b} \mathcal{M}.$$

Finally

$$\begin{aligned} \left(T_{i,a}^* \bigotimes_{b \in A} \tilde{T}_{i,b} \mathcal{M} \right)_{a_1,\dots,a_k} &\cong \left(\tilde{T}_{i,a}^* \bigotimes_{b \in A} \tilde{T}_{i,b} \mathcal{M} \right)_{a_1,\dots,a_k} \bigotimes_{b \in A} (\tilde{T}_{i,b} \mathcal{M})_{a_1,\dots,a,\dots,a_k} \\ &\cong \left(\bigotimes_{b \in A} \tilde{T}_{i,b} \mathcal{M} \right)_{a_1,\dots,a_k} \otimes \bigotimes_{b \in A} \frac{\mathcal{M}_{a_1,\dots,a+b,\dots,a_k}}{\mathcal{M}_{a_1,\dots,b,\dots,a_k}}. \end{aligned}$$

The last factor has a canonical trivialization. □

The Lemma together with Proposition 12.1 justifies the following.

Definition 16.7. If $\mathcal{M}$ is a line bundle over the form $\Theta^k \mathcal{L}$, then we define $\tilde{N} \mathcal{M}$ to be the line bundle over ${G'}^k$ such that

$$q^* \tilde{N} \mathcal{M} = \bigotimes_{b \in A} \tilde{T}_b \mathcal{M}.$$

There is a canonical isomorphism

$$\tilde{N}(\mathcal{M}_1 \otimes \mathcal{M}_2) \cong \tilde{N}\mathcal{M}_1 \otimes \tilde{N}\mathcal{M}_2. \tag{16.8}$$

Proposition 16.4 implies

Proposition 16.9. *There is are natural isomorphisms*

$$\tilde{N}\Theta^k\mathcal{L} \cong \Theta^k\tilde{N}\mathcal{L} \cong \Theta^k N\mathcal{L}$$

of rigid line bundles over ${G'}^k$. □

As with the ordinary norm, the map $\tilde{N}$ can be extended to sections. If s is a section of $\mathcal{M}$, we define a section $\tilde{N}s$ of $\tilde{N}\mathcal{M}$ by

$$\tilde{N}s = \bigotimes_{a\in A} \tilde{T}_a^* s.$$

As with the ordinary norm, this is not linear, but satisfies

$$\tilde{N}(s_1 \otimes s_2) = \tilde{N}s_1 \otimes \tilde{N}s_2.$$

under the isomorphism (16.8). The norm map $\tilde{N}$ commutes with Δ in the sense that

$$\tilde{N}\Delta s = \Delta\tilde{N}s.$$

With all this it is straightforward if tedious to verify

Proposition 16.10. *If s is a Θ^k-structure on $\mathcal{L}$, then $\tilde{N}s$ is a Θ^k-structure on $N\mathcal{L}$.* □

17. Elliptic curves

Definition 17.1. An *elliptic curve* is a pointed proper smooth curve

$$\begin{tikzcd} C \arrow[r] & S \arrow[l, bend right, "0"'] \end{tikzcd}$$

whose geometric fibers are connected and of genus 1.

Much of the theory of level structures, isogenies, Θ^k-structures, which we have described in detail in this paper for formal groups, carries over to elliptic curves. In this section we briefly recall some results which we will need.

17.1. Abel's Theorem.

Note that the discussion of the line bundles $\Theta^k\mathcal{L}$ in §15 applies to abelian groups in any category where the notion of line bundle makes sense. The first result about elliptic curves is that they are group schemes.

Theorem 17.2 (Abel)**.** *An elliptic curve C/S has a unique structure of abelian group scheme such that the rigid line bundle $\Theta^3(\mathcal{I}_C(0))$ is trivial. The (necessarily unique) rigid trivialization $s(C/S)$ of $\Theta^3(\mathcal{I}(0))$ is a cubical structure.*

Proof. See for example [KM85, p. 63]. □

Remark 17.3. The *theorem of the cube* says that any line bundle over an abelian variety has a unique cubical structure. A general enough statement of the theorem of the cube, together with the group structure on elliptic curves, implies Theorem 17.2. We have stated Theorem 17.2 to emphasize that the group structure on an elliptic curve is *constructed* to trivialize $\Theta^3(\mathcal{I}(0))$, so that by the time you get around to applying the theorem of the cube, you already know the conclusion for $\mathcal{I}(0)$.

17.2. **Level structures on elliptic curves.** Abel's Theorem 17.2 means that it makes sense to study level structure on elliptic curves. Katz and Mazur do this in [KM85].

Let C be an elliptic curve over a scheme S, and let A be an abelian group.

Definition 17.4. A group homomorphism

$$A \xrightarrow{\ell} C(S)$$

is a *level A-structure* (or level structure or A-structure) if the Cartier divisor $\{\ell\}$ is a sub-groupscheme.

Lemma 17.5. *Let C be an elliptic curve over an admissible local ring R. If*

$$\ell : A \to \widehat{C}(R)$$

is a level structure on the formal group of C, then

$$A \to \widehat{C}(R) \to C(R)$$

is a level structure on C.

Proof. This follows from the definition and Proposition 11.3. □

17.3. **Isogenies.** The definition of isogeny (11.1) is the same if G and G' are taken to be elliptic curves over a scheme S.

Proposition 17.6. *Let $A \xrightarrow{\ell} C(S)$ be a level structure on an elliptic curve C over a scheme S. The induced map*

$$A_S \xrightarrow{\ell_S} C$$

of groups schemes has a cokernel

$$C \xrightarrow{q} C/\ell(A)$$

which is an isogeny of elliptic curves. The structure sheaf of $C/\ell(A)$ is $(q_*\mathcal{O}_C)^A$; that is,

$$\mathcal{O}_{C/\ell(A)}(U) = \mathcal{O}_C(q^{-1}U)^A$$

for an open set $U \subset C/\ell(A)$.

Proof. See for example [Mum70, p. 111]. □

Corollary 17.7. *Let C be an elliptic curve over an admissible local ring R of residue characteristic $p > 0$, and let*

$$A \xrightarrow{\ell} \widehat{C}$$

be an A-structure on its formal group. The natural map

$$\widehat{C}/\ell(A) \to \widehat{C/\ell(A)}$$

is an isomorphism of formal groups. □

17.4. **The norm map.** Proposition 12.1 carries over verbatim to the case that

$$A \xrightarrow{\ell} G \xrightarrow{q} G'$$

is an isogeny of elliptic curves with an A-structure on its kernel (see for example [Mum70, p. 111]). It follows that the discussion of norms including Proposition 12.4 carries over to this situation as well.

Explicitly, if $\mathcal{L}$ is a line bundle on G, then we get a line bundle $N\mathcal{L}$ on G' such that

$$q^* N\mathcal{L} = \bigotimes_{a \in A} T_a^* \mathcal{L};$$

if $U \subset G'$ is an open set then

$$\Gamma(N\mathcal{L}, U) = \left(\Gamma(\mathcal{L}, q^{-1}U)\right)^A.$$

The norm map applies to sections: if $U \in G'$ and

$$s \in \Gamma(\mathcal{L}, q^{-1}U),$$

then
$$Ns = \bigotimes T_a^* s \in \left(\Gamma(\mathcal{L}, q^{-1}U)\right)^A = \Gamma(N\mathcal{L}, U).$$
The isomorphism
$$N\mathcal{O}_G \cong \mathcal{O}_{G'}$$
of Proposition 11.3 induces an isomorphism
$$N\mathcal{I}_G(\ell) \cong \mathcal{I}_{G'}(0).$$

The discussion of the reduced norm $\tilde{N}$ of § 16 applies to elliptic curves as well. The main point is that, if $\mathcal{L}$ is a line bundle on the elliptic curve G, then the isogeny q gives isomorphisms of rigid line bundles
$$\tilde{N}\Theta^k\mathcal{L} \cong \Theta^k\tilde{N}\mathcal{L} \cong \Theta^k N\mathcal{L}$$
over G' as in Proposition 16.9, and if s is a Θ^k structure on $\mathcal{L}$, then $\tilde{N}s$ is a Θ^k-structure on $\tilde{N}\mathcal{L}$, as in Proposition 16.10.

17.5. **The Serre-Tate theorem.** Let C_0 be an elliptic curve over a field k of characteristic $p > 0$.

Definition 17.8. A *deformation of* C_0 is a triple $(D/T, f, j)$ consisting of an elliptic curve D over a local formal scheme T of residue characteristic $p > 0$ and a pull-back diagram
$$\begin{tikzcd} D_{T_0} \arrow[r, "f"] \arrow[d] & C_0 \arrow[d] \\ T_0 \arrow[r, "j"] & \operatorname{spec} k \end{tikzcd}$$
of elliptic curves. A *map* deformations
$$(\alpha, \beta) : (D, f, j) \to (D', f', j')$$
is a pull-back square
$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d] & D' \arrow[d] \\ T \arrow[r, "\beta"] & T' \end{tikzcd}$$
such that the diagram
$$\begin{tikzcd} D_{T_0} \arrow[r, "\alpha_{T_0}"] \arrow[d] \arrow[rr, bend left, "f"] & D'_{T'_0} \arrow[r, "f'"] \arrow[d] & C_0 \arrow[d] \\ T_0 \arrow[r, "\beta_0"] \arrow[rr, bend right, "j"'] & T'_0 \arrow[r, "j'"] & \operatorname{spec} k \end{tikzcd}$$
commutes.

Let C_0 be a supersingular elliptic curve over a field k of characteristic $p > 0$.

Theorem 17.9 (Serre-Tate)**.** *The natural transformation*
$$\operatorname{Def}(C_0)^{\mathcal{N}} \to \operatorname{Def}(\widehat{C_0})^{\mathcal{N}}$$
is an isomorphism of functors over $\mathrm{FGps}^{\mathcal{N}}$*. Suppose that k is perfect, and let G/S be the universal deformation of the formal group $\widehat{C_0}$. Then there is a deformation $(C/S, f_{\mathrm{univ}}, j_{\mathrm{univ}})$ of C_0 to S such that the natural maps*
$$\underline{C/S}^{\mathcal{N}} \to \operatorname{Def}(C_0)^{\mathcal{N}} \to \operatorname{Def}(\widehat{C_0})^{\mathcal{N}} \leftarrow \underline{G/S}^{\mathcal{N}}$$
are isomorphism of functors over $\mathrm{FGps}^{\mathcal{N}}$.

Proof. The Serre-Tate Theorem as stated in [Kat81] proves that the forgetful natural transformation induces an isomorphism

$$\mathrm{Def}(C_0)^{\mathcal{A}} \to \mathrm{Def}(\widehat{C_0})^{\mathcal{A}} \tag{17.10}$$

of functors of Artin local rings. On the other hand, the functor $\mathrm{Def}(C_0)$ is *effectively prorepresentable*: there is deformation $(C'/S', f', j')$ with $S' \cong \operatorname{spf} \mathbb{W}k[\![u]\!]$, such that the natural map

$$(\underline{C'/S'})^{\mathcal{A}} \to \mathrm{Def}(C_0)^{\mathcal{A}} \tag{17.11}$$

is an isomorphism (see for example [DR73]). It follows that

$$(\underline{C'/S'})^{\mathcal{N}} \cong \mathrm{Def}(C_0)^{\mathcal{N}}.$$

Combining the isomorphisms (17.10) and (17.11) with the isomorphism

$$\mathrm{Def}(\widehat{C_0})^{\mathcal{N}} \cong \underline{G/S}^{\mathcal{N}}$$

gives an isomorphism of formal schemes

$$S \xrightarrow{\cong} S'$$

and, if C is the elliptic curve over S obtained from C'/S' by pull-back, an isomorphism

$$\underline{C/S}^{\mathcal{N}} \to \underline{G/S}^{\mathcal{N}}.$$

□

Example 17.12. In characteristic 2 the elliptic curve C_0 given by the Weierstrass equation

$$y^2 + y = x^3$$

is supersingular (e.g. [Sil99]). The universal deformation of its formal group is a formal group G over $S \cong \operatorname{spf} \mathbb{Z}_2[\![u_1]\!]$. It is well-known (e.g. by the Exact Functor Theorem [Lan76]) that there is a spectrum $\mathbf{E}$ with

$$G_{\mathbf{E}}/S_{\mathbf{E}} = G/S :$$

it is a form of E_2. The Serre-Tate Theorem endows $\mathbf{E}$ with the structure of an elliptic spectrum: if C/S is the universal deformation of C_0 to S, then there is a canonical isomorphism

$$G_{\mathbf{E}} = G \cong \widehat{C}$$

of formal groups over $S_{\mathbf{E}}$.

17.6. **Descent for level structures on a Serre-Tate curve.** Since $\underline{C/S}$ is a functor over FGps, Definition 13.10 provides a notion of descent for level structures on $\underline{C/S}$. Proposition 13.17 and the descent data (14.5) give such descent data

$$d_1 : \underline{\mathrm{level}}_1(\underline{C/S}) \to \underline{C/S}$$

for $\underline{C/S}$. Explicitly, suppose that

$$A_T \xrightarrow{\ell} i^*\widehat{C}$$

is a level structure over a Noetherian local formal scheme T. The descent data provide an isogeny of formal groups

$$i^*\widehat{C} \xrightarrow{f_\ell} \psi_\ell^*\widehat{C} \tag{17.13}$$

over T with kernel ℓ.

It is natural to ask for an isogeny of elliptic curves

$$i^*C \xrightarrow{g_\ell} \psi_\ell^* C$$

extending f_ℓ. This corresponds, in the language of §13, to replacing the functor FGps with the functor Ell whose value on a ring R is the set of elliptic curves $C/\operatorname{spec} R$. Thus we shall refer to descent data for level structures on $\underline{C/S}$ together with isogenies g_ℓ extending f_ℓ as *descent data for level structures on* $\underline{C/S}$ *over* Ell.

Now by Proposition 17.6 we have an isogeny of elliptic curves over T

$$i^*C \xrightarrow{q} C'$$

with kernel ℓ. Corollary 17.7 gives a canonical isomorphism

$$\widehat{C'} \cong \psi_\ell^* \widehat{C} \tag{17.14}$$

of formal groups over T. Theorem 17.9 implies that there is a unique isomorphism of elliptic curves

$$C' \cong \psi_\ell^* C$$

extending (17.14); put another way, we have the following.

Corollary 17.15. *The functor $\underline{C/S}$ has descent data for level structures over Ell, whose restriction to $\underline{\widehat{C}/S} = \underline{G/S}$ are the descent data given by Proposition 14.8. In particular, for each level structure*

$$A_T \xrightarrow{\ell} i^*\widehat{C},$$

there is a canonical isogeny of elliptic curves g_ℓ making the diagram

$$\begin{tikzcd} i^*\widehat{C} \arrow[r] \arrow[d, "f_\ell"'] & i^*C \arrow[d, "g_\ell"] \\ \psi_\ell^*\widehat{C} \arrow[r] & \psi_\ell^* C \end{tikzcd}$$

commute. □

18. The cubical structure of an elliptic curve is compatible with descent

Proposition 18.1. *Let C be an elliptic curve, and let $s(C/S)$ be the cubical structure of Theorem 17.2. If $i^*C \to \psi^*C$ is an isogeny, then*

$$\psi^* s(C/S) = \tilde{N} i^* s(C/S).$$

Proof. Let $f : C \to C'$ be an isogeny of elliptic curves over S. The uniqueness of the cubical structure implies that

$$s(C'/S) = \tilde{N} s(C/S).$$

□

19. The sigma orientation

Suppose that $\mathbf{E}$ is a homogeneous ring spectrum and let $G = G_{\mathbf{E}}$. Let V be the line bundle

$$V = \prod_{j=1}^{k} (1 - L_i)$$

over $(\mathbb{C}P^\infty)^k$. In Lemma 6.1 we observed that Proposition 9.12 gives an isomorphism

$$t_V : \mathbb{L}(V) \cong \Theta^k(\mathcal{I}_{G_{\mathbf{E}}}(0)),$$

and if

$$g : MU\langle 2k\rangle \to \mathbf{E}$$

is an orientation, then the composition

$$(\mathbb{C}P^{\infty k})^V \to MU\langle 2k\rangle \xrightarrow{g} \mathbf{E}$$

represents a rigid section s of $\Theta^k(\mathcal{I}_G(0))$. In fact it is easily seen to be a Θ^k-structure, that is a $\pi_0\mathbf{E}$-valued point of $\underline{C}^k(G; \mathcal{I}_G(0))$. Similarly, if $g : BU\langle 2k\rangle_+ \to \mathbf{E}$ is a homotopy multiplicative map, then the composite

$$\mathbb{C}P^{\infty k} \to BU\langle 2k\rangle \to \mathbf{E}$$

represents a Θ^k-structure on the trivial line bundle $\mathcal{O}_G$, and so a point of $\underline{C}^k(G; \mathcal{O}_G)$. In [AHS01] we proved

Theorem 19.1. *If $\mathbf{E}$ is a homogeneous spectrum and $k \leq 3$, then these correspondences induce isomorphisms*

$$\text{RingSpectra}(MU\langle 2k\rangle, \mathbf{E}) \to \underline{C}^k(G; \mathcal{I}_G(0))(\pi_0\mathbf{E}) \tag{19.2}$$

and

$$\text{RingSpectra}(BU\langle 2k\rangle_+, \mathbf{E}) \to \underline{C}^k(G; \mathcal{O}_G)(\pi_0\mathbf{E}).$$

□

Now suppose that $(\mathbf{E}, C, t)$ is an *elliptic spectrum*: that is, $\mathbf{E}$ is a homogeneous ring spectrum, C is an elliptic curve over $S_{\mathbf{E}}$, and t is an isomorphism

$$t : G_{\mathbf{E}}\ \widehat{C}$$

of formal groups over $S_{\mathbf{E}}$. Abel's Theorem 17.2 gives a cubical structure $s(C/S)$ on C, which gives a cubical structure $t^*\widehat{s}(C/S)$ on $G_{\mathbf{E}}$.

Definition 19.3. [AHS01] The *sigma orientation* for $(\mathbf{E}, C, t)$ is the map of ring spectra

$$\sigma(\mathbf{E}, C, t) : MU\langle 6\rangle \to \mathbf{E}$$

which corresponds to $t^*\widehat{s}(C/S)$ under the isomorphism (19.2).

Now suppose that $\mathbf{E}$ is a homogeneous H_∞ spectrum, with the property that $\pi_0\mathbf{E}$ is an admissible local ring of residue characteristic $p > 0$. Let $S = S_{\mathbf{E}}$. Suppose that $(\mathbf{E}, C, t)$ is an elliptic spectrum. In particular, the $G = G_{\mathbf{E}}$ is Noetherian and of finite height. By Theorem 3.26, the H_∞ structure on $\mathbf{E}$ gives descent data for level structures on G.

Definition 19.4. An H_∞ *elliptic spectrum* is an elliptic spectrum $(\mathbf{E}, C, t)$ whose underlying spectrum $\mathbf{E}$ is a homogeneous H_∞ spectrum $\mathbf{E}$ as above, together with descent data for level structures on $\underline{C/S}$, considered as a functor over Ell as in §17.6, such that the diagram of functors over FGps

$$\begin{tikzcd}
\underline{\text{level}}_1(\underline{C/S}) \arrow[r, "t"] \arrow[d, "d_1"'] & \underline{\text{level}}_1(\underline{G/S}) \arrow[d, "d_1"] \\
\underline{C/S} \arrow[r, "t"] & \underline{G/S}
\end{tikzcd}$$

commutes.

Proposition 19.5. *Let $(\mathbf{E}, C, t)$ be an H_∞ elliptic spectrum, and suppose in addition that p is regular in $\pi_0\mathbf{E}$. Then the σ-orientation*

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{E}, C, t)} \mathbf{E}$$

is an H_∞ map.

Proof. By Proposition 7.1, it suffices to show that, for each level structure

$$A_T \xrightarrow{\ell} i^*\widehat{C},$$

we have

$$\tilde{N}_{g_\ell} s(C/S_{\mathbf{E}}) = \left(\psi_\ell^{\mathbf{E}}\right)^* s(C/S_{\mathbf{E}}),$$

where g_ℓ is the isogeny of elliptic curves making the diagram

$$\begin{tikzcd}
i^*G_{\mathbf{E}} \arrow[r] \arrow[d, "\psi_\ell^{G/\mathbf{E}}"'] & i^*C \arrow[d, "g_\ell"] \\
\left(\psi_\ell^{\mathbf{E}}\right)^* G_{\mathbf{E}} \arrow[r] & \left(\psi_\ell^{\mathbf{E}}\right)^* C
\end{tikzcd}$$

commute. Proposition 18.1 gives the result. □

Now let $(\mathbf{E}, C, t)$ be the elliptic spectrum associated to the universal deformation of a supersingular elliptic curve C_0 over a perfect field k of characteristic $p > 0$. For example, we may take C_0 to be the Weierstrass curve

$$y^2 + y = x^3$$

over $\mathbb{F}_2$ (Example 17.12). Applying the Proposition, Corollary 17.15, and Proposition 14.11 gives the

Corollary 19.6. *The orientation*

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{E},C,t)} \mathbf{E}$$

is an H_∞ map. □

Appendix A. H_∞-ring spectra

Given an integer $n \geq 0$, let $D_n : \mathbb{S}_U \to \mathbb{S}_U$ be the functor

$$\mathbf{E} \mapsto \mathcal{L}(n) \underset{\Sigma_n}{\wedge} \mathbf{E}^{(n)},$$

where $\mathcal{L}(n) = \mathcal{L}(U^n, U)$ is the space of linear isometric embeddings from U^n to U.

An E_∞-ring spectrum is a spectrum with maps

$$D_n(\mathbf{E}) \to \mathbf{E}, \quad n \geq 0,$$

making the following diagrams commute:

$$\begin{array}{ccccccc} \{1_U\} \rtimes \mathbf{E} & \longrightarrow & D_1\mathbf{E} & & D_nD_m\mathbf{E} & \longrightarrow & D_{n+m}\mathbf{E} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{E} & = & \mathbf{E} & & D_n\mathbf{E} & \longrightarrow & \mathbf{E}. \end{array} \tag{A.1}$$

An H_∞-ring spectrum is a spectrum $\mathbf{E}$ together with maps $D_n\mathbf{E} \to \mathbf{E}$ such that the diagrams (A.1) commute up to homotopy.

The category of $\mathbf{E}_\infty$-ring spectra is naturally enriched over topological spaces. The *space* of E_∞-maps from $\mathbf{E}$ to $\mathbf{F}$ is the subspace of all maps consisting of those which make the diagrams

$$\begin{array}{ccc} D_n\mathbf{E} & \longrightarrow & D_n\mathbf{F} \\ \downarrow & & \downarrow \\ \mathbf{E} & \longrightarrow & \mathbf{F} \end{array}$$

commute. For a topological space X, the spectrum which underlies the "function object" is simply the spectrum $\mathbf{E}^{X_+}$. The spectrum which underlies $\mathbf{E} \otimes X$ is more difficult to describe. If $\mathbf{E}$ is only an H_∞-ring spectrum, the spectrum $\mathbf{E}^{X_+}$ is still H_∞.

These remarks actually depend very little on the construction of the functor D_n and are mostly matters of pure category theory. Indeed, the map $D_n\mathbf{E} \to \mathbf{E}$ can be regarded as a natural transformation of functors

$$\mathbb{S}_U(\mathbf{F}, \mathbf{E}) \to \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}).$$

Given a topological space X, we can use (2.4) to define a transformation

$$\mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X_+}) \to \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}^{X_+}). \tag{A.2}$$

as the composite

$$\begin{aligned} \mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X_+}) &\cong \mathrm{Spaces}(X, \mathbb{S}_U(\mathbf{F}, \mathbf{E})) \\ &\to \mathrm{Spaces}(X, \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E})) \\ &\cong \mathbb{S}_U(D_n\mathbf{F}, \mathbf{E}^{X_+}). \end{aligned}$$

A more subtle property is that the transformation (A.2) is also given by

$$\begin{aligned} \mathbb{S}_U(\mathbf{F}, \mathbf{E}^{X_+}) \cong \mathbb{S}_U(\mathbf{F} \wedge X_+, \mathbf{E})) &\longrightarrow \mathbb{S}_U(D_n(\mathbf{F} \wedge X_+), \mathbf{E})) \\ &\xrightarrow{\text{diag}} \mathbb{S}_U(D_n(\mathbf{F}) \wedge X_+), \mathbf{E}) \\ &\cong \mathbb{S}_U(D_n \mathbf{F}, \mathbf{E}^{X_+}). \end{aligned} \tag{A.3}$$

An important property of the functors is summarized in the following result of [BMMS86].

Proposition A.4. *There is a natural weak equivalence*

$$\bigvee_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{F}) \to D_n(\mathbf{E} \vee \mathbf{F}).$$

Furthermore, the ij-component of

$$D_n(\mathbf{E}) \xrightarrow{D_n(\nabla)} D_n(\mathbf{E} \vee \mathbf{E}) \to \bigvee_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{E}) \to \prod_{i+j=n} \mathcal{L}(2) \wedge D_i(\mathbf{E}) \wedge D_j(\mathbf{E})$$

is the transfer map Tr_{ij} with respect to the inclusion $\Sigma_i \times \Sigma_j \subset \Sigma_n$.

Note also that if W is a virtual bundle of dimension 0 over a space X, then $D_A(X^W)$ is the Thom spectrum the virtual bundle $V_{\text{reg}} \otimes W$ over $D_A(X)$, where V_{reg} is the regular representation of A^*.

References

[Ada74] J. Frank Adams. *Stable homotopy and generalised homology.* Univ. of Chicago Press, 1974.

[AHS01] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. Elliptic spectra, the Witten genus, and the theorem of the cube. *Inventiones Mathematicae*, 146:595–687, 2001. DOI 10.1007/s002220100175.

[And95] Matthew Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Math. J.*, 79(2):423–485, 1995.

[BMMS86] R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[Bre83] Lawrence Breen. *Fonctions thêta et théorème du cube*, volume 980 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[DR73] P Deligne and M Rapoport. Les schemas de modules de courbes elliptiques. In *Modular functions of one variable II*, volume 349 of *Springer lecture notes*, 1973.

[Dri74] V. G. Drinfeld. Elliptic modules. *Math. USSR-Sb.*, 23(4):561–592, 1974.

[GH02] Paul Goerss and Michael J. Hopkins. Realizing commutative ring spectra as E_∞ ring spectra, 2002. Preprint.

[EGAI] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594 (electronic), 2000.

[Hop95] Michael J. Hopkins. Topological modular forms, the Witten genus, and the theorem of the cube. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 554–565, Basel, 1995. Birkhäuser.

[Kat81] Nicholas M. Katz. Serre-Tate local moduli. In *Surfaces algébriques: séminaire de géométrie algébrique d'Orsay 1976–1978*, volume 868 of *Lecture Notes in Mathematics*, pages 138–202. Springer, 1981.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves.* Princeton University Press, Princeton, NJ, 1985.

[Lan76] Peter S. Landweber. Homological properties of comodules over MU_*MU and BP_*BP. *Amer. J. Math.*, 98:591–610, 1976.

[Lan78] Serge Lang. *Cyclotomic fields.* Springer-Verlag, New York, 1978. Graduate Texts in Mathematics, Vol. 59.

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal lie groups. *Bull. Soc. math. France*, 94:49–60, 1966.

[Lub67] Jonathan Lubin. Finite subgroups and isogenies of one-parameter formal Lie groups. *Ann. of Math. (2)*, 85:296–302, 1967.

[McC86] James E. McClure. Power operations in H_∞^d ring theories. In *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[Mum70] David Mumford. *Abelian varieties.* Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.

[Qui71] Daniel Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Math.*, 7:29–56 (1971), 1971.

[Sil99] Joseph H. Silverman. *The arithmetic of elliptic curves*. Springer-Verlag, New York, 199? Corrected reprint of the 1986 original.
[Sin68] William M. Singer. Connective fiberings over BU and U. *Topology*, 7, 1968.
[Str97] Neil P. Strickland. Finite subgroups of formal groups. *J. Pure and Applied Algebra*, 121:161–208, 1997.
[Str98] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

Department of Mathematics, The University of Illinois at Urbana-Champaign, Urbana IL 61801, USA

E-mail address: mando@math.uiuc.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

E-mail address: mjh@math.mit.edu

Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH, England

E-mail address: N.P.Strickland@sheffield.ac.uk